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A Mixed Integer Programming for Robust Truss Topology Optimization with Stress Constraints

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Abstract

This paper presents a formulation of robust topology optimization of trusses subjected to the stress constraints under the load uncertainty. A design-dependent uncertainty model of external loads is proposed for considering the variation of truss topology in the course of optimization. For a truss with the discrete member cross-sectional areas, it is shown that the robust topology optimization problem can be reduced to a mixed integer programming problem, which is solved globally. Numerical examples illustrate that the globally optimal topology of robust truss depends on the magnitude of uncertainty.

Keywords

Topology optimization; Robust optimization; Mixed integer program; Global optimization; Stress constraints.

1 Introduction

This paper develops a global optimization method for the topology optimization of trusses subjected to the uncertain external loads. We rigorously deal with the stress constraint conditions in the worst cases, as well as the variation of truss topology. It is shown that the optimization problem presented can be formulated as a mixed integer programming problem under several assumptions such as the discreteness of member cross-sectional areas.

For considering the uncertain property of structural systems, there exist two alternative approaches; probabilistic and non-probabilistic approaches for representing uncertainty. Based on

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the probabilistic uncertainty model, various methods have been well-developed for reliability-based optimization (see, e.g., [7, 21], and the references therein). On the other hand, a non-probabilistic uncertainty model is often less information-intensive than a probabilistic model, because no stochastic distributions of uncertain parameters is required.

The *convex model* approach [4] is one of well-known approaches using a non-probabilistic uncertainty model. The robust optimization of structures was performed by using the convex model [1, 18]. Lee and Park [16] presented a robust structural optimization based on the first-order approximation of the extremal response. Note that the variation of topology of a structure was not considered in the literature [1, 16, 18] cited above, and all the members in the initial solution remain at the obtained solution.

A unified methodology of robust counterpart was presented for a broader class of convex optimization problems including non-probabilistic uncertainty [6]. This methodology was applied to the compliance minimization problem of a truss subjected to the uncertain load [5]. A min-max formulation of a robust compliance design was presented for continua [9]. In the info-gap decision theory [3], the *robustness function* plays a key role which represents the greatest level of uncertainty at which any failure of the mechanical performance cannot occur. By using the robustness function, a robustness maximization problem of a truss was investigated under the load uncertainty [14]. In the uncertainty models considered in [5, 14], it is supposed that uncertain external forces are possibly applied to all the nodes of a truss. Therefore all the nodes of the initial ground structure remain at the obtained optimal solution, and hence the truss topology cannot change drastically.

In this paper we consider the variation of truss topology rigorously in the process of robust structural optimization under the load uncertainty. For dealing with the variation of topology, we develop a design-dependent uncertainty model of the external load. In our uncertainty model, uncertain external forces are possibly applied to all the existing nodes, while no uncertain force is applied to the nodes which vanish as a result of optimization. We also consider the stress constraints in the extremal cases rigorously: the stress constraint must be satisfied in the worst case if the corresponding member exists, while such a robust constraint condition on the stress should be removed if the corresponding member vanishes; see section 5 for more account. We next suppose that the member cross-sectional areas are chosen from among a finite number of candidates, and that uncertain external forces are bounded in the sense of the maximum-norm. Under this restrictive situation we show that the presented robust truss topology optimization problem can be reformulated as a *mixed integer programming* (MIP) problem, which is solved globally (see section 9 for our MIP formulation). The basic idea for reduction of the truss topology optimization, without uncertainty, to an MIP problem can be found in [19, 22]. We extend this idea to the robust truss topology optimization. See, e.g., [23] for basics of MIP.

This paper is organized as follows. In section 2, we recall the conventional, or nominal, structural optimization problem for the preparation of formulating the robust optimization problem. A topology-dependent uncertainty model of the external load is proposed in section 3. Section 4 describes the relation between the robust constraint condition and the worst-case detection problem. In section 5, we present a rigorous formulation of the robust truss topology optimization problem. Sections 6–8 prepare the MIP reformulation of the robust topology optimization problem; in section 6 we introduce the binary variables to represent the existence of members, as well as the candidates of member cross-sectional areas; the optimality conditions of the worst-case detection problem are dealt with by introducing some auxiliary binary variables in section 7; a heuristic treatment of the kinematical determinacy condition is presented in section 8. The MIP problems, which are our goal formulations to be solved globally, are presented in section 9. Numerical experiments are presented in section 10. Some conclusions are drawn in section 11.

A few words regarding our notation: all vectors are assumed to be column vectors. The $(m+n)$-dimensional column vector $(\boldsymbol{u}^{\mathrm{T}}, \boldsymbol{v}^{\mathrm{T}})^{\mathrm{T}}$ consisting of $\boldsymbol{u} \in \mathbb{R}^m$ and $\boldsymbol{v} \in \mathbb{R}^n$ is often written simply as $(\boldsymbol{u}, \boldsymbol{v})$. For vectors $\boldsymbol{p} = (p_i) \in \mathbb{R}^n$ and $\boldsymbol{q} = (q_i) \in \mathbb{R}^n$, we write $\boldsymbol{p} \geq \boldsymbol{q}$ if $p_i \geq q_i$ $(i = 1, \dots, n)$. Particularly, by $\boldsymbol{p} \geq \boldsymbol{0}$ we mean $p_i \geq 0$ $(i = 1, \dots, n)$. For any n, I_n denotes the $n \times n$ identity matrix, and $\mathbf{1}_n$ denotes the vector $(1, \dots, 1)^{\mathrm{T}} \in \mathbb{R}^n$. We denote by $A \otimes B$ the Kronecker product of $A = (A_{ij}) \in \mathbb{R}^{m_1 \times n_1}$ and $B \in \mathbb{R}^{m_2 \times n_2}$, i.e. $A \otimes B$ is the $m_1 m_2 \times n_1 n_2$ matrix defined by

$$A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n_1}B \\ A_{21}B & A_{22}B & \cdots & A_{2n_1}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{m_1 1}B & A_{m_1 2}B & \cdots & A_{m_1 n_1}B \end{bmatrix}.$$

2 Preliminaries: Nominal structural optimization problem

Consider a linear elastic truss consisting of m members in the dim-dimensional space, where $\mathrm{dim} \in \{2, 3\}$. We denote by d the the number of degrees of freedom of displacements. Small displacements and small strains are assumed.

Let $\boldsymbol{x} \in \mathbb{R}^m$ denote the vector of member cross-sectional areas, which are considered as design variables. We denote by $K(\boldsymbol{x}) \in \mathbb{R}^{d \times d}$ the stiffness matrix. The displacements vector $\boldsymbol{u} \in \mathbb{R}^d$ is found from the system of equilibrium equations,

$$K(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{f}, \tag{1}$$

where $\boldsymbol{f} \in \mathbb{R}^d$ is the nodal loads vector.

Consider the mechanical performance of structures written as

$$g_q(\boldsymbol{u}) \leq 0, \qquad q = 1, \dots, \ell, \tag{2}$$

where $g_q : \mathbb{R}^d \to \mathbb{R}$ $(q = 1, \dots, \ell)$. Let $\mathcal{X} \subseteq \mathbb{R}^m$ denote the set of the admissible design variables, e.g. $\mathcal{X} = \{\boldsymbol{x} \in \mathbb{R}^m \mid x_i \geq 0 \ (1 \leq i \leq m)\}$. We denote by l_i the length of the ith member. The conventional structural optimization problem, which attempts to minimize the structural volume over the constraint conditions in (2), is formulated as

$$\left. \begin{array}{ll} \min & \displaystyle\sum_{1 \leq i \leq m} l_i x_i \\ \mathrm{s.t.} & \boldsymbol{x} \in \mathcal{X}, \\ & K(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{f}, \\ & g_q(\boldsymbol{u}) \leq 0, \qquad q = 1, \dots, \ell, \end{array} \right\} \tag{3}$$

where $\boldsymbol{x}$ and $\boldsymbol{u}$ are the variables.

3 Design-dependent uncertainty model

A design-dependent model for uncertainty in the external load is presented.

We first introduce a binary variable s_j indicating the existence of the jth free node, in order to develop a design-dependent uncertainty model of the external load. If the current design $\boldsymbol{x}$ includes no members connected to the jth node, then we put $s_j = 0$ and suppose that no uncertain force can be applied to the jth node. On the other hand, if there exists a remaining member connected to the jth node, then we put $s_j = 1$ and suppose that uncertain forces are possibly applied to the jth node. In summary, s_j is related to the set of remaining members as

$$s_j = \begin{cases} 1 & \text{if at least one remaining member is connected to the } j\text{th node,} \\ 0 & \text{if all the members connected to the } j\text{th node are removed.} \end{cases} \tag{4}$$

Let $\mathcal{I}_j \subseteq \{1, \dots, m\}$ denote the set of indices of members which are connected to the jth node. We see that $x_i = 0$ for any $i \in \mathcal{I}_j$ if and only if all the members connected to the jth node vanish at the current design $\boldsymbol{x}$. Hence, the condition (4) is equivalently rewritten as

$$s_j = \begin{cases} 1 & \text{if } \exists i \in \mathcal{I}_j : x_i > 0, \\ 0 & \text{if } x_i = 0 \ (\forall i \in \mathcal{I}_j). \end{cases} \tag{5}$$

Thus, s_j is a function of $\boldsymbol{x}$.

Let $\tilde{\boldsymbol{f}} \in \mathbb{R}^d$ denote the nominal value, or the best estimate, of the external load $\boldsymbol{f}$. We describe the uncertainty of $\boldsymbol{f}$ by using an unknown vector $\boldsymbol{\zeta} \in \mathbb{R}^n$. We call $\boldsymbol{\zeta}$ the vector of *uncertain parameters*, or *unknown-but-bounded parameters*. Assume that $\boldsymbol{f}$ depends on $\boldsymbol{\zeta}$ affinely as

$$\boldsymbol{f} = \tilde{\boldsymbol{f}} + \mathrm{diag}(\boldsymbol{f}_0)\boldsymbol{\zeta}, \tag{6}$$

where $\boldsymbol{f}_0 = (f_{0j}) \in \mathbb{R}^d$ is a constant vector. Note that f_{0j} represents the relative magnitude of the uncertainty of f_j.

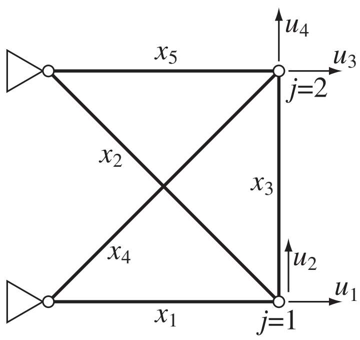
For each $j = 1, \dots, n$, define the constant matrix $T_j \in \mathbb{R}^{\dim \times d}$ so that the vector $T_j \boldsymbol{u}$ corresponds to the nodal displacement of the jth free node. Here, each element of T_j is either 0 or 1, and each row of T_j includes only one nonzero element (which is equal to 1). Suppose that $\boldsymbol{\zeta}$ satisfies

$$\alpha s_j \geq \|T_j \boldsymbol{\zeta}\|, \qquad j = 1, \dots, n, \tag{7}$$

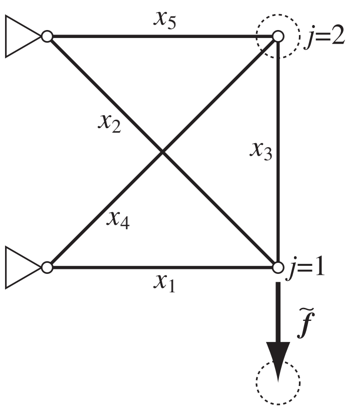
where $\alpha \in \mathbb{R}$ ($\alpha > 0$) is a constant, s_j is defined by (5), and $\|\cdot\|$ denotes an appropriate vector norm. By using (6) and (7), we define the uncertainty set of $\boldsymbol{f}$ by

$$\mathcal{F}(\boldsymbol{s}) = \{\tilde{\boldsymbol{f}} + \mathrm{diag}(\boldsymbol{f}_0)\boldsymbol{\zeta} \mid \alpha s_j \geq \|T_j \boldsymbol{\zeta}\| \ (1 \leq j \leq n)\}. \tag{8}$$

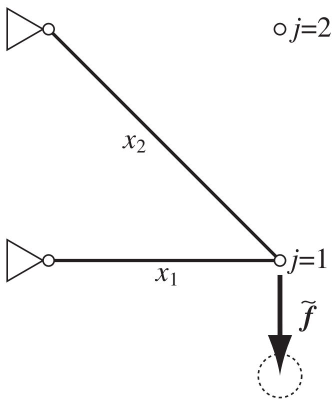
Note that $\mathrm{diag}(\boldsymbol{f}_0)\boldsymbol{\zeta} \in \mathbb{R}^d$ corresponds to the external load vector in the generalized coordinate system, while $T_j \,\mathrm{diag}(\boldsymbol{f}_0)\boldsymbol{\zeta} \in \mathbb{R}^{\dim}$ corresponds to the nodal load applied to the jth free node. Hence, (7) implies that no uncertain force is applied to the node satisfying $s_j = 0$. Consequently, in (8), uncertain forces are removed from the vanishing nodes. It is emphasized that the uncertainty set $\mathcal{F}(\boldsymbol{s})$ defined in (8) depends on the design variables $\boldsymbol{x}$, because $\boldsymbol{s}$ is a function of $\boldsymbol{x}$. Hence, we call (8) the *design-dependent uncertainty model* of the external load.



(a) The ground structure.



(b) x_1, x_2, x_3, x_4, $x_5 > 0$.



(c) x_1, $x_2 > 0$, $x_3 = x_4 = x_5 = 0$.

Figure 1: Schematic representation of the definition of a design-dependent uncertainty set $\mathcal{F}(\boldsymbol{s})$ in (8). Figure 1(a): The ground structure with $d = 4$, $m = 5$ and $n = 2$. Figure 1(b): $\mathcal{F}(\boldsymbol{s})$ corresponding to $s_1 = s_2 = 1$; Figure 1(c): $\mathcal{F}(\boldsymbol{s})$ corresponding to $s_1 = 1$ and $s_2 = 0$. Uncertain forces are running through the circles depicted with the dotted lines.

Example 3.1. Consider the plane-truss example shown in Figure 1. The ground structure illustrated in Figure 1(a) consists of 5 members and 2 free nodes, i.e. $m = 5$, $n = 2$, and $d = 4$. The sets of member indices connected to the free nodes are $\mathcal{I}_1 = \{1, 2, 3\}$ and $\mathcal{I}_2 = \{3, 4, 5\}$. The matrices T_j $(j = 1, 2)$ in (7) are explicitly written as

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We consider the ℓ^2-norm, i.e. the standard Euclidean norm, in (7).

Consider the two designs shown in Figure 1(b) and Figure 1(c). In Figure 1(b), the cross-sectional areas of all the members are supposed to be positive. From (5) we obtain $s_1 = s_2 = 1$, which represents that the two free nodes exist at the corresponding design. The uncertainty set of $\boldsymbol{f}$ defined by (8) reads

$$\mathcal{F}(\boldsymbol{s}) = \{\tilde{\boldsymbol{f}} + \mathrm{diag}(\boldsymbol{f}_0)\boldsymbol{\zeta} \mid \alpha \geq \|(\zeta_1, \zeta_2)\|_2,\ \alpha \geq \|(\zeta_3, \zeta_4)\|_2\},$$

i.e. uncertain forces may possibly exist at both nodes. In contrast, we suppose that the members 3, 4, and 5 are vanishing in Figure 1(c). Then (5) yields $s_1 = 1$ and $s_2 = 0$. The corresponding uncertainty set of $\boldsymbol{f}$ defined by (8) is reduced to

$$\mathcal{F}(\boldsymbol{s}) = \{\tilde{\boldsymbol{f}} + \mathrm{diag}(\boldsymbol{f}_0)\boldsymbol{\zeta} \mid \alpha \geq \|(\zeta_1, \zeta_2)\|_2,\ \zeta_3 = \zeta_4 = 0\}.$$

Thus uncertain forces are supposed to be applied only at the remaining node. ■

4 Worst-case detection and kinematical determinacy

When the external load $\boldsymbol{f}$ takes all the values in the uncertainty set $\mathcal{F}(\boldsymbol{s})$ defined in (8), the displacements vector $\boldsymbol{u}$ is running through the set $\{\boldsymbol{u} \mid K(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{f},\ \boldsymbol{f} \in \mathcal{F}(\boldsymbol{s})\}$. We require that the constraints on the mechanical performance should be satisfied by all the possible realizations of $\boldsymbol{u}$. Thus the robust counterpart to the constraint conditions (2) is introduced as

$$g_q(\boldsymbol{u}) \leq 0\ (\forall \boldsymbol{u} : K(\boldsymbol{x})\boldsymbol{u} \in \mathcal{F}(\boldsymbol{s})), \qquad q = 1, \dots, \ell. \tag{9}$$

Alternatively, the condition (9) is rewritten as

$$\max_{\boldsymbol{u}} \{g_q(\boldsymbol{u}) \mid K(\boldsymbol{x})\boldsymbol{u} \in \mathcal{F}(\boldsymbol{s})\} \leq 0, \qquad q = 1, \dots, \ell. \tag{10}$$

However, if the truss is kinematically indeterminate at $\boldsymbol{x}$, then the condition (10) does not correspond to the constraint in the most critical case, as shown in the example below.

Example 4.1. Consider a three-bar truss example illustrated in Figure 2. As the robust constraint condition (10), we consider the upper bound constraint on the displacement in the y-direction, i.e.

$$\max_{\boldsymbol{u}=(u_x, u_y)} \{u_y - \bar{u} \mid K(\boldsymbol{x})\boldsymbol{u} \in \mathcal{F}(\boldsymbol{s})\} \leq 0. \tag{11}$$

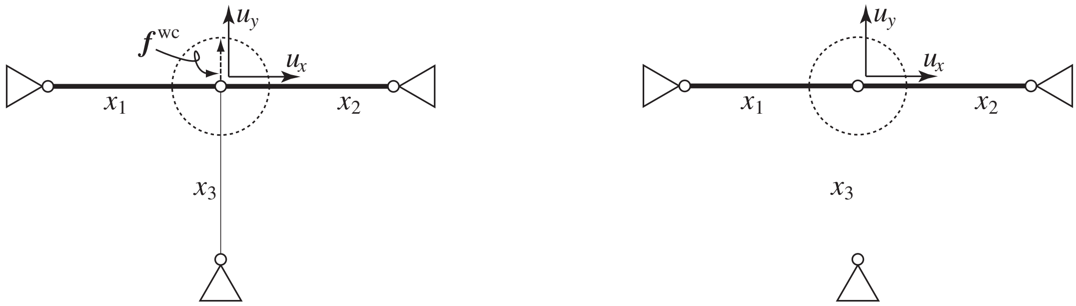
Suppose that x_3 has a small positive value in Figure 2(a). Then the optimal solution of the problem on the left-hand side of (11) corresponds to the external force indicated as $\boldsymbol{f}^{\mathrm{wc}}$. Thus, the constraint (11) corresponds to the most critical case.

In contrast, suppose $x_3 = 0$ in Figure 2(b). Since the truss is kinematically indeterminate, u_y corresponding to $\boldsymbol{f}^{\mathrm{wc}}$ (in Figure 2(a)) takes an infinite value. However, $\boldsymbol{f}^{\mathrm{wc}}$ is not feasible for the optimization problem on the left-hand side of (11). In fact, the optimal value of the left-hand side of (11) vanishes, because only horizontal loads are feasible. In this case the constraint (11) does not correspond to the most critical case, which happens if we allow that the truss can become kinematically indeterminate. ■

As discussed in Example 4.1, it is required that any $\boldsymbol{f} \in \mathcal{F}(\boldsymbol{s})$ is feasible for the problem on the left-hand side of (10), in order to guarantee that the condition (10) corresponds to the constraint in the most critical external load. This motivates us to consider the constraint condition

$$\mathrm{Im}\, K(\boldsymbol{x}) \supseteq \mathcal{F}(\boldsymbol{s}), \tag{12}$$

where $\mathrm{Im}\, K(\boldsymbol{x})$ is the image of $K(\boldsymbol{x})$. Note that $\boldsymbol{s}$ in (12) is a function of $\boldsymbol{x}$ as shown in (5), and hence (12) is regarded as a constraint condition on $\boldsymbol{x}$.



(a) $x_3 = \varepsilon$.

(b) $x_3 = 0$.

Figure 2: Worst-case detection and kinematical determinacy of a truss. The optimal solution of the maximization problem of u_y is denoted by $\boldsymbol{f}_{\mathrm{wc}}$. Figure 2(a): $\boldsymbol{f}_{\mathrm{wc}}$ corresponds to the most critical load. Figure 2(b): the most critical load is not feasible for the worst-case detection problem.

5 Robust truss topology optimization

We define the robust structural optimization as a robust counterpart of the problem (3) by replacing the constraint of the mechanical performance, (2), with its robust counterpart, (10). From (5), (8), (9), and (12), the robust optimization problem is formulated as

$$\left.\begin{array}{lll}
\min & \displaystyle\sum_{1 \leq i \leq m} l_i x_i & \\
\mathrm{s.t.} & \boldsymbol{x} \in \mathcal{X}, & \\
& s_j = \begin{cases} 1 & \text{if } \exists i \in \mathcal{I}_j : x_i > 0, \\ 0 & \text{otherwise}, \end{cases} & j = 1, \dots, n, \\
& \mathcal{F}(\boldsymbol{s}) = \{\tilde{\boldsymbol{f}} + \mathrm{diag}(\boldsymbol{f}_0)\boldsymbol{\zeta} \mid \alpha s_j \geq \|T_j \boldsymbol{\zeta}\| \ (1 \leq j \leq n)\}, & \\
& \mathrm{Im}\, K(\boldsymbol{x}) \supseteq \mathcal{F}(\boldsymbol{s}), & \\
& g_q(\boldsymbol{u}) \leq 0 \ (\forall \boldsymbol{u} : K(\boldsymbol{x})\boldsymbol{u} \in \mathcal{F}(\boldsymbol{s})), & q = 1, \dots, \ell.
\end{array}\right\} \tag{13}$$

Since it is very difficult to solve the problem (13), we consider only the following restrictive situation. Firstly, we consider only the ℓ^∞-norm in the definition (8) of $\mathcal{F}(\boldsymbol{s})$, i.e. we suppose that $\boldsymbol{\zeta}$ is bounded as

$$\alpha s_j \geq \|T_j \boldsymbol{\zeta}\|_\infty, \qquad j = 1, \dots, n. \tag{14}$$

For simplicity, we also write (14) as

$$\alpha s_{j(r)} \geq |\zeta_r|, \qquad r = 1, \dots, d, \tag{15}$$

where $j(r)$ is the index of node associated with u_r. Secondly, we suppose that the member cross-sectional areas can be chosen only from finitely many candidates. In other words, x_i is considered as a discrete variable. Finally, we restrict ourselves to the case in which g_q is an affine function, i.e.

$$g_q(\boldsymbol{u}) = \boldsymbol{a}_q^{\mathrm{T}} \boldsymbol{u} - b_q, \qquad q = 1, \dots, \ell \tag{16}$$

with constant $\boldsymbol{a}_q$ and b_q. Since we consider only truss structures, the member stress, denoted by $\sigma_i(\boldsymbol{u})$, is a linear function of $\boldsymbol{u}$. However, the stress constraints require a particular treatment [8, 12, 17, 20], because they should be imposed only to the existing members. By incorporating these restrictions, the robust truss topology optimization problem considering the stress constraints is formulated as

$$\begin{aligned}
\min \quad & \sum_{1 \le i \le m} l_i x_i && && \text{(17a)} \\
\text{s.t.} \quad & \boldsymbol{x} \in \{0, \xi_1, \dots, \xi_k\}^m, && && \text{(17b)} \\
& s_j = \begin{cases} 1 & \text{if } \exists i \in \mathcal{I}_j : x_i > 0, \\ 0 & \text{otherwise,} \end{cases} && j = 1, \dots, n, && \text{(17c)} \\
& \mathcal{F}(\boldsymbol{s}) = \{\tilde{\boldsymbol{f}} + \mathrm{diag}(\boldsymbol{f}_0)\boldsymbol{\zeta} \mid \alpha s_j \ge \|T_j \boldsymbol{\zeta}\|_\infty \ (1 \le j \le n)\}, && && \text{(17d)} \\
& \mathrm{Im}\, K(\boldsymbol{x}) \supseteq \mathcal{F}(\boldsymbol{s}), && && \text{(17e)} \\
& |\sigma_i(\boldsymbol{u})| \le \bar{\sigma} \ (\forall \boldsymbol{u} : K(\boldsymbol{x})\boldsymbol{u} \in \mathcal{F}(\boldsymbol{s})), && \forall i : x_i > 0, && \text{(17f)}
\end{aligned}$$

where $\bar{\sigma}$ is the upper bound of the stress, and $\xi_1, \dots, \xi_k$ are the available candidates of member cross-sectional area. Note that we consider the upper-bound constraint on the modulus of $\sigma_i(\boldsymbol{u})$ in (17f). Since $\sigma_i(\boldsymbol{u})$ is a linear function in terms of $\boldsymbol{u}$, the constraint conditions $|\sigma_i(\boldsymbol{u})| \le \bar{\sigma}$ $(i = 1, \dots, m)$ can be written in the form of (16) by putting

$$\begin{aligned}
a_q^{\mathrm{T}} \boldsymbol{u} &= \begin{cases} \sigma_i(\boldsymbol{u}) & \text{for } q = i, \\ -\sigma_i(\boldsymbol{u}) & \text{for } q = m + i, \end{cases} \\
b_q &= \bar{\sigma}_i \qquad \text{for } q = i, m + i,
\end{aligned}$$

and $\ell = 2m$. The following three sections prepare a MIP reformulation of the problem (17).

6 Discrete design variables

The conditions (17b) and (17c) are rewritten by using some binary variables.

For the reformulation of (17b), we introduce binary variables $t_{i1}, \dots, t_{ik} \in \{0, 1\}$ for each member, where $t_{ip} = 1$ implies that x_i is equal to ξ_p. More precisely, we see that $x_i \in \{0, \xi_1, \dots, \xi_k\}$ if and only if

$$x_i = \sum_{1 \le p \le k} \xi_p t_{ip}, \tag{18}$$

$$\sum_{1 \le p \le k} t_{ip} \le 1, \tag{19}$$

$$t_{ip} \in \{0, 1\}, \qquad p = 1, \dots, k. \tag{20}$$

Note that (19) implies that at most one of $t_{i1}, \dots, t_{ik}$ becomes one, and the others vanish. Hence, if $t_{ip} = 1$ in (18)–(20), then $x_i = \xi_p$. Moreover, $t_{i1} = \dots = t_{ik} = 0$ implies $x_i = 0$.

By using (18)–(20), the condition (17c) is equivalently rewritten as

$$t_{ip} \le s_j \le 1, \qquad \forall i \in \mathcal{I}_j;\ \forall p = 1, \dots, k, \tag{21}$$

$$s_j \le \sum_{i \in \mathcal{I}_j} \sum_{1 \le p \le k} t_{ip} \tag{22}$$

for each $j = 1, \dots, n$. Note that (21) and (22) are linear inequalities, which are more tractable than (17c).

7 Reduction of robust constraint conditions

A tractable reformulation of the condition (17f) is presented.

7.1 Reduction of complementarity conditions

We start with the design-independent constraint condition (16), instead of (17f). The corresponding worst-case determination problem is formulated as

$$\left.\begin{array}{ll} \max & \boldsymbol{a}_q^{\mathrm{T}}\boldsymbol{u} \\ \mathrm{s.t.} & K(\boldsymbol{x})\boldsymbol{u} = \tilde{\boldsymbol{f}} + F_0\boldsymbol{\zeta}, \\ & \alpha s_{j(r)} \geq |\zeta_r|, \quad r = 1, \dots, d. \end{array}\right\} \tag{23}$$

Note that the problem (23) is an LP (*linear programming*) problem. Since we assume (17e), the problem (23) has a feasible solution. Hence, from the KKT conditions, $(\boldsymbol{u}_q^*, \boldsymbol{\zeta}_q^*)$ is an optimal solution of (23) if and only if there exists a Lagrange multipliers vector $(\boldsymbol{\mu}_q^*, \boldsymbol{\lambda}_q^{+*}, \boldsymbol{\lambda}_q^{-*}) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ satisfying

$$\begin{aligned} & K(\boldsymbol{x})\boldsymbol{u}_q^* = \tilde{\boldsymbol{f}} + F_0\boldsymbol{\zeta}_q^*, \\ & K(\boldsymbol{x})\boldsymbol{\mu}_q^* = -\boldsymbol{a}_q, \\ & \boldsymbol{\lambda}_q^{+*} - \boldsymbol{\lambda}_q^{-*} = -F_0^{\mathrm{T}}\boldsymbol{\mu}_q^*, \\ & \alpha s_{j(r)} \geq |\zeta_{rq}^*|, && r = 1, \dots, d, \\ & \lambda_{rq}^{+*} \geq 0, \quad \lambda_{rq}^{-*} \geq 0, && r = 1, \dots, d, \\ & \lambda_{rq}^{+*}(\alpha s_{j(r)} - \zeta_{rq}^*) = 0, \quad \lambda_{rq}^{-*}(\alpha s_{j(r)} + \zeta_{rq}^*) = 0, && r = 1, \dots, d. \end{aligned}$$

Consequently, $\boldsymbol{x} \in \mathcal{X}$ satisfies the robust constraint condition

$$\boldsymbol{a}_q^{\mathrm{T}}\boldsymbol{u} \leq b_q, \qquad \forall \boldsymbol{u} : K(\boldsymbol{x})\boldsymbol{u} \in \mathcal{F}(\boldsymbol{s}) \tag{24}$$

if and only if there exist $\boldsymbol{u}_q$, $\boldsymbol{\zeta}_q$, $\boldsymbol{\mu}_q$, $\boldsymbol{\lambda}_q^+$ and $\boldsymbol{\lambda}_q^-$ satisfying

$$\boldsymbol{a}_q^{\mathrm{T}}\boldsymbol{u}_q \leq b_q, \tag{25}$$
$$K(\boldsymbol{x})\boldsymbol{u}_q = \tilde{\boldsymbol{f}} + F_0\boldsymbol{\zeta}_q, \tag{26}$$
$$K(\boldsymbol{x})\boldsymbol{\mu}_q = -\boldsymbol{a}_q, \tag{27}$$
$$\boldsymbol{\lambda}_q^+ - \boldsymbol{\lambda}_q^- = -F_0^{\mathrm{T}}\boldsymbol{\mu}_q, \tag{28}$$
$$\alpha s_{j(r)} \geq |\zeta_{rq}|, \qquad r = 1, \dots, d, \tag{29}$$
$$\lambda_{rq}^+ \geq 0, \quad \lambda_{rq}^- \geq 0, \qquad r = 1, \dots, d, \tag{30}$$
$$\lambda_{rq}^+(\alpha s_{j(r)} - \zeta_{rq}) = 0, \quad \lambda_{rq}^-(\alpha s_{j(r)} + \zeta_{rq}) = 0, \qquad r = 1, \dots, d. \tag{31}$$

Note that (26), (27), and (31) are nonlinear constraint conditions, while (25), (28), (29), and (30) are reduced to linear inequality constraint conditions. We shall deal with (26) and (27) in section 7.2.

We introduce the binary variables τ_{rq}^{+} and τ_{rq}^{-} in order to deal with the complementarity conditions (31). Then the conditions (29)–(31) are equivalently rewritten as

$$
\begin{aligned}
& 0 \le \alpha s_{j(r)} - \zeta_{rq} \le M\tau_{rq}^{+}, && (32)\\
& 0 \le \lambda_{rq}^{+} \le M(1-\tau_{rq}^{+}), && (33)\\
& 0 \le \alpha s_{j(r)} + \zeta_{rq} \le M\tau_{rq}^{-}, && (34)\\
& 0 \le \lambda_{rq}^{-} \le M(1-\tau_{rq}^{-}), && (35)\\
& \tau_{rq}^{+} \in \{0,1\}, \quad \tau_{rq}^{-} \in \{0,1\} && (36)
\end{aligned}
$$

for each $r = 1,\dots,d$, where $M \gg 0$ is a constant. We easily see in (32)–(36) that $\tau_{rq}^{+} = 1$ implies that the constraint condition $\lambda_{rq}^{+} \ge 0$ in (30) is active, while $\tau_{rq}^{+} = 0$ implies that the constraint condition $\alpha s_{j(r)} - \zeta_{rq} \ge 0$ in (29) is active. Similarly, τ_{rq}^{-} indicates whether $\lambda_{rq}^{-} \ge 0$ or $\alpha s_{j(r)} + \zeta_{rq} \ge 0$ should be active. It should be clear that (32)–(35) are linear inequality constraint conditions.

7.2 Reduction of equilibrium equations

In this section we reformulate (26) into a tractable form, where $\boldsymbol{x}$ is supposed to satisfy (18)–(20). Note that (27) can also be dealt with in a manner similar to (26).

It is known that the stiffness matrix for a truss can be written in the form of

$$K(\boldsymbol{x}) = \sum_{1 \le i \le m} x_i \boldsymbol{h}_i \boldsymbol{h}_i^{\mathrm{T}}, \tag{37}$$

where $\boldsymbol{h}_1,\dots,\boldsymbol{h}_m \in \mathbb{R}^d$ are constant vectors. Alternatively, (37) is rewritten as

$$K(\boldsymbol{x}) = H \,\mathrm{diag}(\boldsymbol{x}) H^{\mathrm{T}}, \tag{38}$$

where $H = \begin{bmatrix} \boldsymbol{h}_1 & \cdots & \boldsymbol{h}_m \end{bmatrix} \in \mathbb{R}^{d\times m}$. By using (37) and (38), the condition (26) is reduced to

$$
\begin{aligned}
& H\boldsymbol{v}_q = \tilde{\boldsymbol{f}} + F_0 \boldsymbol{\zeta}_q, && (39)\\
& v_{iq} = x_i \hat{c}_{iq}, \qquad i = 1,\dots,m, && (40)\\
& \hat{c}_{iq} = \boldsymbol{h}_i^{\mathrm{T}} \boldsymbol{u}_q, \qquad i = 1,\dots,m, && (41)
\end{aligned}
$$

where $\boldsymbol{v}_q$ and $\hat{\boldsymbol{c}}_q$ are the auxiliary variables.

Define c_{ipq} $(p = 1,\dots,k)$ by

$$c_{ipq} = \begin{cases} \hat{c}_{iq} & \text{if } t_{ip} = 1, \\ 0 & \text{if } t_{ip} = 0, \end{cases} \tag{42}$$

from which we can decompose $\hat{c}_{iq}$ as

$$\hat{c}_{iq} = \sum_{1 \le p \le k} c_{ipq}. \tag{43}$$

It follows from (18), (42), and (43) that (40) is reduced to

$$\begin{aligned} v_{iq} &= \sum_{1 \le p \le k} x_i c_{ipq} \\ &= \sum_{1 \le p \le k} \Big(\sum_{1 \le l \le k} \xi_l t_{il} \Big) c_{ipq} \\ &= \sum_{1 \le p \le k} \xi_p c_{ipq} \end{aligned} \tag{44}$$

for each $i = 1, \dots, m$, because at most one of $t_{i1}, \dots, t_{ik}$ is equal to one, while the others are vanishing. For simplicity, define $\boldsymbol{c}_q$ and $\boldsymbol{t}$ by

$$\boldsymbol{c}_q = \begin{bmatrix} (c_{i1q})_{i=1}^m \\ \vdots \\ (c_{ikq})_{i=1}^m \end{bmatrix} \in \mathbb{R}^{km}, \qquad \boldsymbol{t} = \begin{bmatrix} (t_{i1})_{i=1}^m \\ \vdots \\ (t_{ik})_{i=1}^m \end{bmatrix} \in \mathbb{R}^{km}.$$

Then, from (44) we see that (40) is reduced to

$$\boldsymbol{v}_q = (\boldsymbol{\xi}^{\mathrm{T}} \otimes I_m) \boldsymbol{c}_q. \tag{45}$$

By substituting (45) into (39), we obtain

$$H(\boldsymbol{\xi}^{\mathrm{T}} \otimes I_m) \boldsymbol{c}_q = \tilde{\boldsymbol{f}} + F_0 \boldsymbol{\zeta}_q. \tag{46}$$

On the other hand, we can rewrite (44) as

$$M(1 - t_{ip}) \ge |c_{ipq} - \hat{c}_{iq}|, \quad M t_{ip} \ge |c_{ipq}|, \qquad p = 1, \dots, k, \tag{47}$$

where M is a sufficiently large constant. The substitution of (41) into (47) yields

$$-M(\boldsymbol{1}_{km} - \boldsymbol{t}) \le \boldsymbol{c}_q - (\boldsymbol{1}_k \otimes H^{\mathrm{T}}) \boldsymbol{u}_q \le M(\boldsymbol{1}_{km} - \boldsymbol{t}), \tag{48}$$

$$-M\boldsymbol{t} \le \boldsymbol{c}_q \le M\boldsymbol{t}. \tag{49}$$

As a consequence, when $\boldsymbol{x}$ and $\boldsymbol{t}$ satisfy (18)–(20), then the condition (26) is equivalently rewritten as (46), (48), and (49). Similarly, we can see that the condition (27) is reduced to

$$H(\boldsymbol{\xi}^{\mathrm{T}} \otimes I_m) \boldsymbol{\gamma}_q = -\boldsymbol{a}_q, \tag{50}$$

$$-M(\boldsymbol{1}_{km} - \boldsymbol{t}_p) \le \boldsymbol{\gamma}_q - (\boldsymbol{1}_k \otimes H^{\mathrm{T}}) \boldsymbol{\mu}_q \le M(\boldsymbol{1}_{km} - \boldsymbol{t}), \tag{51}$$

$$-M\boldsymbol{t} \le \boldsymbol{\gamma}_q \le M\boldsymbol{t}. \tag{52}$$

7.3 Reduction of stress constraints

We reformulate the design-dependent constraint condition (17f) by using the results obtained in sections 7.1 and 7.2.

We utilize the decomposition of the equilibrium equations introduced in (39)–(41). For the existing member, i.e. $x_i > 0$, we see that the condition $|\sigma_i(\boldsymbol{u})| \le \bar{\sigma}$ can be rewritten as

$$|\hat{c}_{iq}| \le \bar{c}_i \quad (q = i), \tag{53}$$

where $\bar{c}_i = \sqrt{E/l_i}\bar{\sigma}$ and E is the elastic modulus. By using (42) and (53), the condition (17f) is equivalently rewritten as

$$\Big| \sum_{1 \le p \le k} c_{ipq} \Big| \le \bar{c}_i \quad (q = i), \qquad i = 1, \dots, m. \tag{54}$$

For each $i = 1, \dots, m$, we can decompose (54) into two linear inequalities. Hence, for the stress constraints, the condition (25) in (25)–(31) should be replaced with

$$\sum_{1 \le p \le k} c_{ipq} \le \bar{c}_i \quad (q = i), \qquad i = 1, \dots, m, \tag{55}$$

$$\sum_{1 \le p \le k} c_{ipq} \ge -\bar{c}_i \quad (q = m + i), \qquad i = 1, \dots, m. \tag{56}$$

Note that we put

$$\boldsymbol{a}_q^{\mathrm{T}} \boldsymbol{u} = \begin{cases} \sigma_i(\boldsymbol{u}) & \text{for } q = i, \\ -\sigma_i(\boldsymbol{u}) & \text{for } q = m + i, \end{cases} \qquad i = 1, \dots, m.$$

in order to embed (54) into the form of (25)–(31).

It should be noted that in (25)–(31) we require the existence of the Lagrange multipliers corresponding to the worst case. However, if a member is removed, then the corresponding Lagrange multipliers do not necessarily exist. Thus, in consideration of the stress constraints, not only (53) but also the constraints on the Lagrange multipliers should be treated as the vanishing constraints. Among (27)–(31), it is sufficient to remove only (27), because $\boldsymbol{\lambda}_q^+ = \boldsymbol{\lambda}_q^- = \boldsymbol{u}_q = \boldsymbol{0}$ always becomes feasible to (28)–(31) for any $\boldsymbol{\zeta}_q$ satisfying (29). In other words, (27) is required to be satisfied only for the existing members, which is realized by

$$|\boldsymbol{a}_q + H(\boldsymbol{\xi}^{\mathrm{T}} \otimes I_m)\boldsymbol{\gamma}_q| \le M\Big(1 - \sum_{1 \le p \le k} t_{ip}\Big)\mathbf{1}_d \quad (q = i, m + i), \qquad i = 1, \dots, m. \tag{57}$$

Note that if $x_i = 0$, then $t_{i1} = \dots = t_{ik} = 0$, and hence (57) is reduced to $|\boldsymbol{a}_q + H(\boldsymbol{\xi}^{\mathrm{T}} \otimes I)\boldsymbol{\gamma}_q| \le M$ as expected.

8 Kinematical determinacy constraint

A tractable sufficient condition for (17e) is presented.

By $\boldsymbol{f} \in \mathcal{F}(\boldsymbol{s})$ we suppose that uncertain forces can be applied to all the existing nodes. Let $\boldsymbol{f}_j \in \mathbb{R}^{\mathrm{dim}}$ denote the external nodal load applied to the jth node. From the definition (4) of s_j, we see that any external load applied to the existing nodes can be written as $\sum_{1 \le j \le n} s_j T_j^{\mathrm{T}} \boldsymbol{f}_j$. Hence, the condition (17e) is satisfied if and only if there exists a vector $\check{\boldsymbol{u}} \in \mathbb{R}^d$ satisfying

$$K(\boldsymbol{x})\check{\boldsymbol{u}} = \sum_{1 \le j \le n} s_j T_j^{\mathrm{T}} \boldsymbol{f}_j, \quad \forall \boldsymbol{f}_j \in \mathbb{R}^{\mathrm{dim}} \ (1 \le j \le n). \tag{58}$$

Note that the equilibrium equations in (58) are nonlinear conditions in terms of $\boldsymbol{x}$ and $\check{\boldsymbol{u}}$. On the other hand, we do not need to consider any performance constraint condition on $\check{\boldsymbol{u}}$. Hence, instead of (58) it is sufficient to consider the existence of the set of axial forces of the existing members which satisfies the equilibrium conditions against the external loads. Thus, we see that (58) is satisfied if and only if there exists $\check{\boldsymbol{v}} \in \mathbb{R}^m$ satisfying

$$\check{v}_i = 0 \ (\forall i : x_i = 0), \quad H\check{\boldsymbol{v}} = \sum_{1 \le j \le n} s_j T_j^{\mathrm{T}} \boldsymbol{f}_j, \tag{59}$$

for any $\boldsymbol{f}_j \in \mathbb{R}^{\mathrm{dim}}$ $(1 \le j \le n)$. Note that (59) consists of the linear equations in terms of $\check{\boldsymbol{v}}$, which is the reason why we prefer (59) to (58).

Since $\sum_{1 \le p \le k} t_{ip} = 0$ holds if and only if $x_i = 0$, the condition $\check{v}_i = 0$ $(\forall i : x_i = 0)$ in (59) can be written as linear inequalities in terms of $\check{\boldsymbol{v}}$ and $\boldsymbol{t}$. In contrast, it is difficult deal with infinitely many external loads $\boldsymbol{f}_j \in \mathbb{R}^{\mathrm{dim}}$ $(1 \le j \le n)$. In this paper we consider a sufficient condition for (59) by considering only some samples $\check{\boldsymbol{f}}_j \in \mathbb{R}^{\mathrm{dim}}$ $(j = 1, \dots, n)$ which are randomly generated. Consequently, a sufficient condition for (59) is obtained as

$$H\check{\boldsymbol{v}} = \sum_{1 \le j \le n} s_j T_j^{\mathrm{T}} \check{\boldsymbol{f}}_j, \tag{60}$$

$$-M \sum_{1 \le p \le k} t_{ip} \le \check{v}_i \le M \sum_{1 \le p \le k} t_{ip}, \qquad i = 1, \dots, m. \tag{61}$$

As a heuristic way to deal with (17e), we consider (60) and (61) as the constraint conditions in the robust truss topology optimization problem.

9 Mixed integer programming formulations

MIP formulations of the problem (17) are presented.

9.1 Linear inequality constraints

Instead of the design-dependent constraint conditions (17f) in the problem (17), we first consider the linear inequality constraint conditions (24) as the constraints on the mechanical performance.

In section 6 we have shown that the constraint condition (17b) is equivalent to (18)–(20), while the constraint condition (17c) has been reduced to (21) and (22). In section 7, the robust constraint conditions consisting of (17d) and (24) have been shown to be equivalent to (25)–(28) and (32)–(36). Moreover, we have shown that (26) is equivalent to (46), (48), and (49), while (27) is equivalent to (50)–(52). As a sufficient condition for (17e), we have presented (60) and (61) in section 8.

As a consequence, if we consider (24) instead of (17f), we can see that the problem (17) is reduced to the following MIP problem:

$$
\left.
\begin{array}{ll}
\min & \displaystyle\sum_{1 \le i \le m} l_i \sum_{1 \le p \le k} \xi_p t_{ip} \\
\mathrm{s.t.} & \displaystyle\sum_{1 \le p \le k} t_{ip} \le 1, \qquad i = 1, \dots, m, \\
& t_{ip} \le s_j \le 1 \quad (\forall i \in \mathcal{I}_j;\ \forall p), \qquad j = 1, \dots, n, \\
& \displaystyle s_j \le \sum_{i \in \mathcal{I}_j} \sum_{1 \le p \le k} t_{ip}, \qquad j = 1, \dots, n, \\
& \displaystyle H \check{\boldsymbol{v}} = \sum_{1 \le j \le n} s_j T_j^{\mathrm{T}} \check{\boldsymbol{f}}_j, \\
& \displaystyle -M \sum_{1 \le p \le k} t_{ip} \le \check{v}_i \le M \sum_{1 \le p \le k} t_{ip}, \qquad i = 1, \dots, m, \\
& \boldsymbol{t} \in \{0,1\}^{km}, \\
& \forall q = 1, \dots, \ell : \left\{
\begin{array}{l}
\boldsymbol{a}_q^{\mathrm{T}} \boldsymbol{u}_q \le b_q, \\
H(\boldsymbol{\xi}^{\mathrm{T}} \otimes I_m) \boldsymbol{c}_q = \tilde{\boldsymbol{f}} + F_0 \boldsymbol{\zeta}_q, \\
-M(\mathbf{1}_{km} - \boldsymbol{t}) \le \boldsymbol{c}_q - \mathbf{1}_k \otimes H^{\mathrm{T}} \boldsymbol{u}_q \le M(\mathbf{1}_{km} - \boldsymbol{t}), \\
-M\boldsymbol{t} \le \boldsymbol{c}_q \le M\boldsymbol{t}, \\
H(\boldsymbol{\xi}^{\mathrm{T}} \otimes I_m) \boldsymbol{\gamma}_q = -\boldsymbol{a}_q, \\
-M(\mathbf{1}_{km} - \boldsymbol{t}) \le \boldsymbol{\gamma}_q - \mathbf{1}_k \otimes H^{\mathrm{T}} \boldsymbol{\mu}_q \le M(\mathbf{1}_{km} - \boldsymbol{t}), \\
-M\boldsymbol{t} \le \boldsymbol{\gamma}_q \le M\boldsymbol{t}, \\
\boldsymbol{\lambda}_q^{+} - \boldsymbol{\lambda}_q^{-} = -F_0^{\mathrm{T}} \boldsymbol{\mu}_q, \\
0 \le \alpha s_{j(r)} - \zeta_{rq} \le M \tau_{rq}^{+}, \quad r = 1, \dots, d, \\
\mathbf{0} \le \boldsymbol{\lambda}_q^{+} \le M(\mathbf{1}_d - \boldsymbol{\tau}_q^{+}), \\
0 \le \alpha s_{j(r)} + \zeta_{rq} \le M \tau_{rq}^{-}, \quad r = 1, \dots, d, \\
\mathbf{0} \le \boldsymbol{\lambda}_q^{-} \le M(\mathbf{1}_d - \boldsymbol{\tau}_q^{-}), \\
\boldsymbol{\tau}_q^{+} \in \{0,1\}^d, \quad \boldsymbol{\tau}_q^{-} \in \{0,1\}^d.
\end{array}
\right.
\end{array}
\right\} \qquad (62)
$$

In the problem (62), the continuous variables are $\boldsymbol{s}$, $\check{\boldsymbol{v}}$, $\boldsymbol{u}_q$, $\boldsymbol{\zeta}_q$, $\boldsymbol{c}_q$, $\boldsymbol{\gamma}_q$, $\boldsymbol{\mu}_q$, $\boldsymbol{\lambda}_q^{+}$, and $\boldsymbol{\lambda}_q^{-}$, while the binary variables are $\boldsymbol{t}$, $\boldsymbol{\tau}_q^{+}$, and $\boldsymbol{\tau}_q^{-}$ $(q = 1, \dots, \ell)$. Since (62) is a 0–1 mixed integer programming problem, it can be solved by using available software packages based on branch-and-cut algorithms, e.g., CPLEX [13].

9.2 Stress constraints

In this section we present a MIP formulation of the robust truss topology optimization considering the stress constraints. In addition to the conditions considered in section 9.1, we have shown in section 7.3 that the stress constraints (17f) are reduced to (55)–(57). Consequently, the problem (17) is equivalently rewritten as the following MIP problem:

$$
\left.
\begin{array}{ll}
\min & \displaystyle\sum_{1 \le i \le m} l_i \sum_{1 \le p \le k} \xi_p t_{ip} \\
\mathrm{s.t.} & \displaystyle\sum_{1 \le p \le k} t_{ip} \le 1, \qquad i = 1, \dots, m, \\
& t_{ip} \le s_j \le 1 \quad (\forall i \in \mathcal{I}_j;\ \forall p), \qquad j = 1, \dots, n, \\
& \displaystyle s_j \le \sum_{i \in \mathcal{I}_j} \sum_{1 \le p \le k} t_{ip}, \qquad j = 1, \dots, n, \\
& \displaystyle H \check{\boldsymbol{v}} = \sum_{1 \le j \le n} s_j T_j^{\mathrm{T}} \check{\boldsymbol{f}}_j, \\
& \displaystyle -M \sum_{1 \le p \le k} t_{ip} \le \check{v}_i \le M \sum_{1 \le p \le k} t_{ip}, \qquad i = 1, \dots, m, \\
& \boldsymbol{t} \in \{0,1\}^{km}, \\
& \forall q = 1, \dots, 2m : \left\{
\begin{array}{l}
H(\boldsymbol{\xi}^{\mathrm{T}} \otimes I_m) \boldsymbol{c}_q = \tilde{\boldsymbol{f}} + F_0 \boldsymbol{\zeta}_q, \\
-M(\mathbf{1}_{km} - \boldsymbol{t}) \le \boldsymbol{c}_q - \mathbf{1}_k \otimes H^{\mathrm{T}} \boldsymbol{u}_q \le M(\mathbf{1}_{km} - \boldsymbol{t}), \\
-M\boldsymbol{t} \le \boldsymbol{c}_q \le M\boldsymbol{t}, \\
-M(\mathbf{1}_{km} - \boldsymbol{t}) \le \boldsymbol{\gamma}_q - \mathbf{1}_k \otimes H^{\mathrm{T}} \boldsymbol{\mu}_q \le M(\mathbf{1}_{km} - \boldsymbol{t}), \\
-M\boldsymbol{t} \le \boldsymbol{\gamma}_q \le M\boldsymbol{t}, \\
\boldsymbol{\lambda}_q^{+} - \boldsymbol{\lambda}_q^{-} = -F_0^{\mathrm{T}} \boldsymbol{\mu}_q, \\
0 \le \alpha s_{j(r)} - \zeta_{rq} \le M \tau_{rq}^{+}, \quad r = 1, \dots, d, \\
\mathbf{0} \le \boldsymbol{\lambda}_q^{+} \le M(\mathbf{1}_d - \boldsymbol{\tau}_q^{+}), \\
0 \le \alpha s_{j(r)} + \zeta_{rq} \le M \tau_{rq}^{-}, \quad r = 1, \dots, d, \\
\mathbf{0} \le \boldsymbol{\lambda}_q^{-} \le M(\mathbf{1}_d - \boldsymbol{\tau}_q^{-}), \\
\boldsymbol{\tau}_q^{+} \in \{0,1\}^d, \quad \boldsymbol{\tau}_q^{-} \in \{0,1\}^d.
\end{array}
\right. \\
& \displaystyle \sum_{1 \le p \le k} c_{ipq} \le \bar{c}_i \quad (q = i), \qquad i = 1, \dots, m, \\
& \displaystyle \sum_{1 \le p \le k} c_{ipq} \ge -\bar{c}_i \quad (q = m + i), \qquad i = 1, \dots, m, \\
& \displaystyle -M\Big(1 - \sum_{1 \le p \le k} t_{ip}\Big) \mathbf{1}_d \le \boldsymbol{a}_q + H(\boldsymbol{\xi}^{\mathrm{T}} \otimes I) \boldsymbol{\gamma}_q \\
& \displaystyle \qquad \le M\Big(1 - \sum_{1 \le p \le k} t_{ip}\Big) \mathbf{1}_d \quad (q = i, m + i), \qquad i = 1, \dots, m.
\end{array}
\right\} \quad (63)
$$

10 Numerical experiments

The robust optimal topologies are found for various trusses by solving the proposed MIP formulation. Computation has been carried out on Quad-Core Xeon E5450 (3 GHz) with 16 GB RAM. We solve the problem (63) by using CPLEX Ver.11.2 [13] with the default settings.

10.1 12-bar truss

Consider a 12-bar plane truss illustrated in Figure 3, where $m = 12$, $W = 1\,\mathrm{m}$, and $H = 0.6\,\mathrm{m}$. The two nodes on the left-hand side are pin-supported, and hence $d = 8$.

As the nominal external load $\tilde{\boldsymbol{f}}$, the vertical force of 5 kN is applied to the bottom-right node. The uncertainty model of the external load is defined by (8) with $f_0 = 0.5\,\mathrm{kN}$. The elastic modulus is $E = 2\,\mathrm{GPa}$. The stress constraint of each member is considered, where the upper bound of the

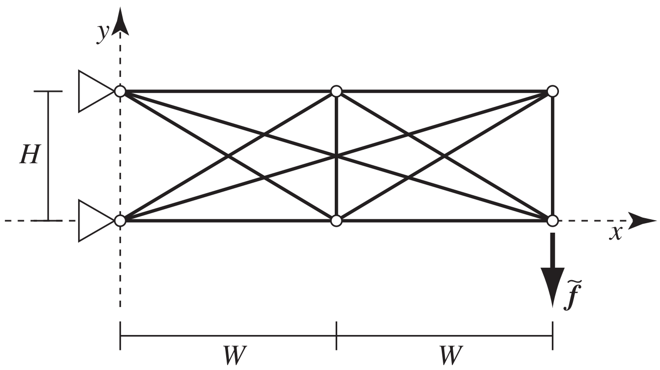


Figure 3: A 12-bar truss.

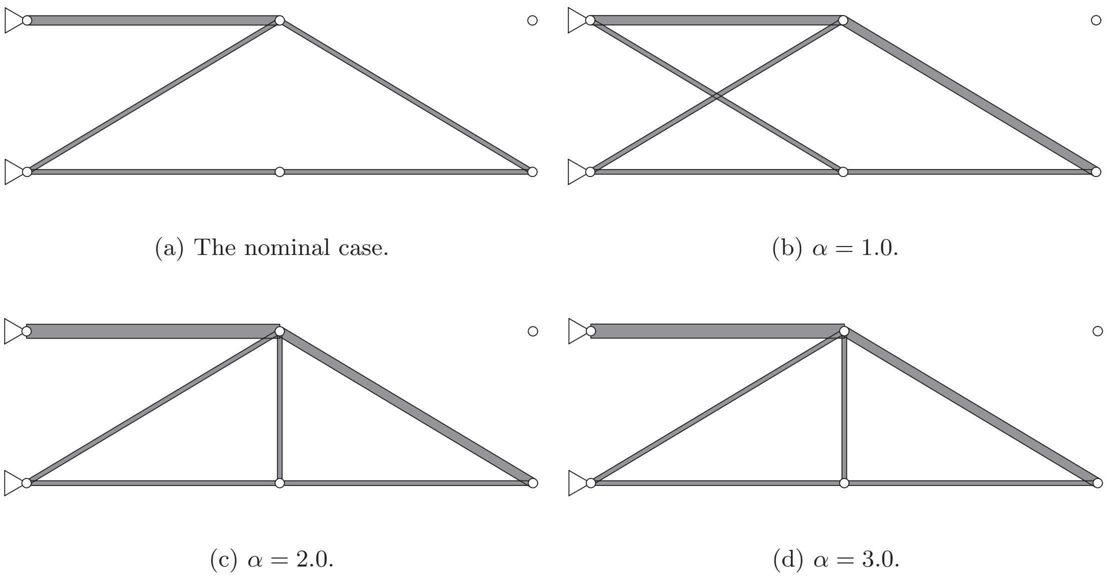


Figure 4: The optimal solutions of the 12-bar truss example.

Table 1: Computational results of the 12-member truss example.

α	Volume (cm^3)	CPU (sec)	Nodes
nominal	3166.19	0.1	72
1.0	4332.38	663.7	9453
2.0	4549.29	2111.4	42797
3.0	4549.29	5690.1	137492

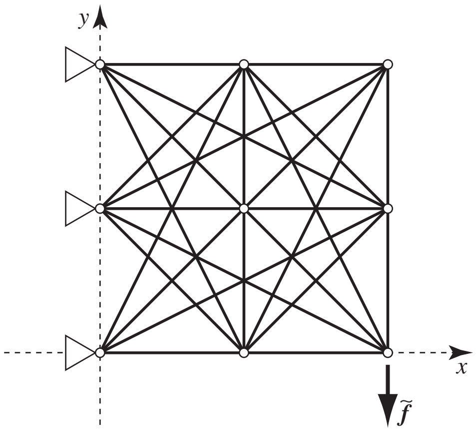


Figure 5: A 2×2 truss.

modulus of stress is $\bar{\sigma} = 20\,\mathrm{MPa}$. The member cross-sectional area of each member is chosen from the set $\{0, 5, 10, 15\}$ in cm^2, i.e. $\mathcal{X} = \{0, 5, 10, 15\}^{12}$ with $k = 3$ in (17b).

The obtained optimal solutions are shown in Figure 4, where the width of each member is proportional to its cross-sectional area. Figure 4(a) depicts the conventional optimal solution without considering the uncertainties, which is kinematically indeterminate. In Figure 4(b)–Figure 4(d) we show the robust optimal solutions for various values of the magnitude of uncertainty, α. The topology of robust optimal solution depends on α in general, although the solutions at $\alpha = 2.0$ and at $\alpha = 3.0$ coincide. The computational results are listed in Table 1.

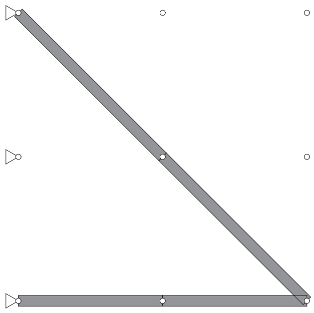
10.2 2×2 truss

We next consider a 2×2 grid truss illustrated in Figure 5. Each node is connected with each other node by a member, unless it corresponds to an overlapping member, i.e. $m = 26$. The lengths of horizontal and vertical members are $1\,\mathrm{m}$. The three nodes on the left-hand side are pin-supported, and hence $d = 12$. A vertical force of $10\,\mathrm{kN}$ is applied to the bottom-right node as the nominal load $\tilde{\boldsymbol{f}}$, while the coefficient in the uncertainty model (8) is $f_0 = 1\,\mathrm{kN}$. The upper bound of the modulus of the stress is $\bar{\sigma} = 10\,\mathrm{MPa}$ for each member . The cross-sectional are of each member is chosen from the set $\{0, 20\}$ in cm^2.

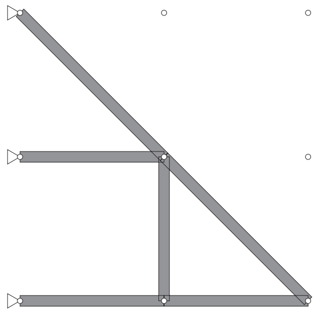
The obtained optimal solutions are shown in Figure 6. Figure 6(a) illustrates the conventional optimal solution considering only $\tilde{\boldsymbol{f}}$. Figure 6(b)–Figure 6(e) show the variation of the robust optimal topology with respect to the magnitude of uncertainty, α. It is emphasized that the nominal optimal solution in Figure 6(a) is kinematically indeterminate, while the robust optimal solutions in Figure 6(b)–Figure 6(e) are kinematically determinate.

10.3 29-bar truss

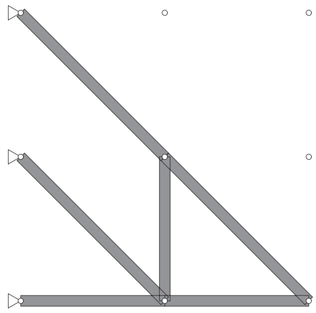
Consider a 29-bar truss illustrated in Figure 7. The nodes (a) and (b) are pin-supported, i.e. $m = 29$ and $d = 20$. The lengths of horizontal and vertical members are $1\,\mathrm{m}$. As the nominal external load $\tilde{\boldsymbol{f}}$, the vertical forces of $2.5\,\mathrm{kN}$ are applied to the nodes (c) and (d). The coefficient in the uncertainty model (8) is $f_0 = 0.25\,\mathrm{kN}$. The upper bound of the modulus of the stress is $\bar{\sigma} = 20\,\mathrm{MPa}$. The



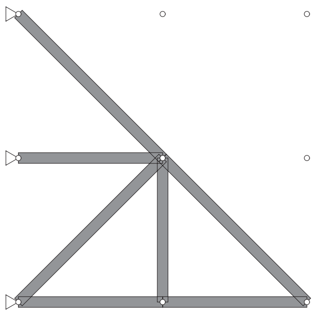
(a) The nominal case.



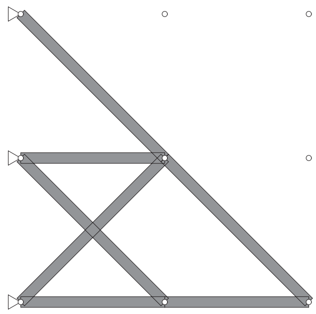
(b) $\alpha = 1.0$.



(c) $\alpha = 2.0$.



(d) $\alpha = 3.0$.



(e) $\alpha = 4.0$.

Figure 6: The optimal solutions of the 2×2 truss example.

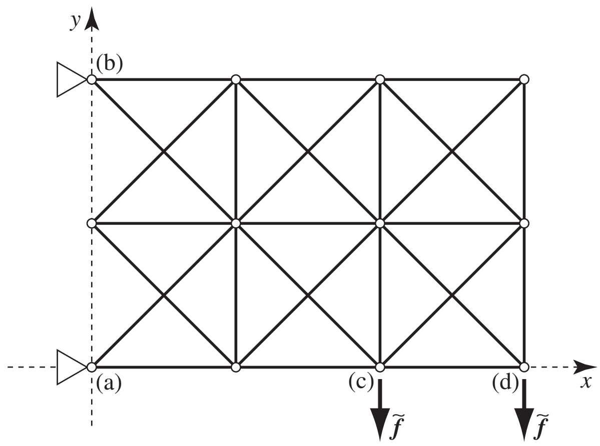
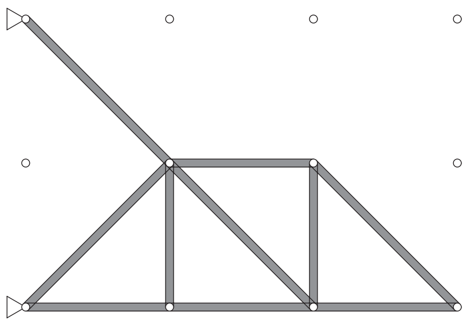
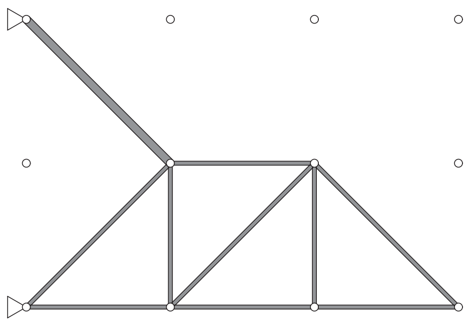


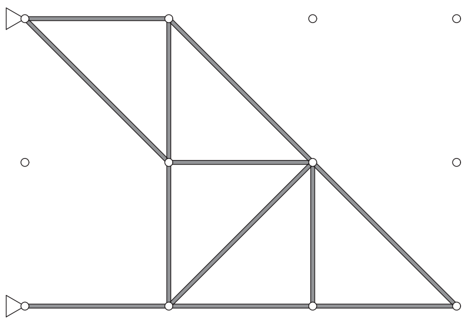
Figure 7: A 29-bar truss.



(a) The robust optimal solution for $\mathcal{X} = \{0, 10\}^m$ (in cm^2).



(b) The robust optimal solution for $\mathcal{X} = \{0, 5, 10\}^m$ (in cm^2).



(c) The robust optimal solution for $\mathcal{X} = \{0, 5, 15\}^m$ (in cm^2).

Figure 8: The optimal solutions of the 29-bar truss example for $\alpha = 1.0$.

obtained robust optimal solutions are shown in Figure 8. It is observed from Figure 8 that the robust truss topology depends on the set of candidates of member cross-sectional areas.

11 Conclusions

A rigorous formulation, as well as a global optimization method, has been presented for the robust truss topology optimization problem considering the stress constraints under the load uncertainty. We have proposed a design-dependent model of uncertainty in the external load in order to deal with the variation of truss topology in the course of optimization. Under several assumptions, e.g. the discreteness of member cross-sectional areas, it has been shown that the robust optimization problem can be reduced to a mixed integer programming (MIP) problem, which is solved globally. It is noted that the constraints on the Lagrange multipliers, as well as those on the member stress, should vanish if the corresponding member disappears.

Our MIP formulation inevitably includes large numbers of variables and constraint conditions, which makes it difficult to apply the presented approach to large-scale structures. It remains as our future work to develop an efficient algorithm which is applicable to large-scale robust topology optimization problems considering the stress constraint conditions. On the other hand, as a distinguished property of the presented approach, it is emphasized that the global optimal solution of our formulation can be found by using a well-developed MIP solver. Hence, the results obtained by our method can be used as benchmark examples for evaluating the performance of any other algorithm, e.g. an optimization method based on the local optimality or heuristic strategy, which is regarded as one of important contributions of our paper.

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