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in Hybrid Analysis for General Circuits**

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Structural Characterization on Index of DAEs in Hybrid Analysis for General Circuits

Satoru Iwata*, Mizuyo Takamatsu[†] and Caren Tischendorf[‡]

August 2009

Abstract

Modern modeling approaches for circuit simulation such as the modified nodal analysis (MNA) lead to differential-algebraic equations (DAEs). The index of a DAE is a measure of the degree of numerical difficulty. In general, the higher the index is, the more difficult it is to solve the DAE. The index of DAEs arising from MNA has been characterized by the network structure.

In this paper, we consider a broader class of analysis method called the hybrid analysis. For nonlinear time-varying circuits with general dependent sources, we give a structural characterization for the index of DAEs arising from the hybrid analysis. This enables us to determine the index efficiently, which helps to avoid solving higher index DAEs in circuit simulation.

1 Introduction

In circuit simulation, we set up a system of equations by using circuit analysis methods such as the modified nodal analysis (MNA), the loop analysis, the cutset analysis, and the hybrid analysis. MNA is the most popular method, because it allows an automatic setup of model equations. In contrast, the hybrid analysis retains flexibility, which can be exploited to find a model description that reduces the numerical difficulties.

Circuit analysis methods lead to differential-algebraic equations (DAEs), which consist of algebraic equations and differential operations. DAEs present numerical and analytical difficulties which do not occur with ordinary differential equations (ODEs). The numerical difficulty of DAEs is measured by the *index*. In general, the higher the index is, the more difficult it is

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to solve the DAE. While many different concepts exist to assign an index to a DAE such as the *differentiation index* [4, 5, 9], the *perturbation index* [9], and the *strangeness index* [19], this paper focuses on the *tractability index* [20, 29]. The index more than one is called the *higher index*. The difficulties of DAEs with higher index are much greater than DAEs with index zero or one.

For nonlinear time-invariant electric circuits which are composed of independent voltage/current sources, resistors, inductors, and capacitors, it is shown in [31] that the index of a DAE arising from MNA does not exceed two. It is also proved that MNA leads to a DAE with index at most one if and only if a circuit contains neither L-I cutsets nor C-V loops, where an *L-I cutset* means a cutset consisting only of inductors and/or current sources, and a *C-V loop* means a cycle consisting only of capacitors and voltage sources. This means that the index of a DAE arising from MNA is determined uniquely by the network. Furthermore, these results in [31] are generalized for nonlinear time-varying electric circuits that may contain a wide class of dependent sources [29]. The results in [29, 31] suggest that DAEs arising from MNA often have higher index.

The hybrid analysis is a common generalization of the loop analysis and the cutset analysis. Kron [18] proposed the hybrid analysis in 1939, and Amari [1] and Branin [3] developed it further in 1960s. While the procedure of MNA is uniquely determined, the hybrid analysis starts with selecting a *partition* of elements and a *reference tree* in the network. This selection determines DAEs, called the *hybrid equations*, to be solved numerically. Thus it is natural to seek for an optimal selection that makes the hybrid equations easy to solve. In fact, the problem of obtaining the minimum size hybrid equations was solved [12, 17, 26] in 1968. This turned out to be an application of matroid intersection [13]. See also [25, 27] for matroid theoretic approach to circuit analysis.

Recently, the analysis of the index of the hybrid equations has been developed. For linear time-invariant circuits which are composed of resistors, inductors, capacitors, independent voltage/current sources, and dependent voltage/current sources, an algorithm is proposed in [14] for finding an optimal hybrid analysis which minimizes the index of the hybrid equations. For linear time-invariant RLC circuits, it is proved in [30] that the index of the hybrid equations is at most one. A structural characterization of circuits with index zero is also given in [30]. These results indicate that the index of the hybrid equations never exceeds the index of DAEs arising from MNA.

The results in [30] are extended to nonlinear time-varying circuits which may contain a certain restricted class of dependent sources in [15, 16]. In particular, it is proved that the index of the hybrid equations is at most one. From a practical point of view, however, it is important to deal with general dependent sources, which often result in higher index DAEs.

In this paper, extending the results in [15, 16] to the circuits with general dependent sources, we give structural characterizations of circuits with index zero and at most one. This enables us to determine efficiently whether the hybrid equations of a circuit have higher index or not. The proof exploits properties of skew-symmetric matrices.

The organization of this paper is as follows. In Section 2, we recapitulate some lemmas from linear algebra, which are useful in the proof of our main theorem. In Section 3, we describe nonlinear time-varying circuits. We present the procedure of the hybrid analysis in Section 4.

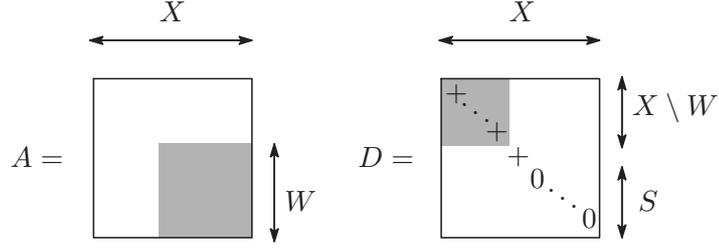


Figure 1: A skew-symmetric matrix A and a diagonal matrix D .

Section 5 is devoted to the definition of the tractability index of DAEs. We analyze the hybrid equations in Section 6. Section 7 gives structural characterizations of the hybrid equations with index zero and at most one. Finally, Section 8 concludes this paper.

2 Preliminaries

For a matrix A , we denote the submatrix of A with row set W_R and column set W_C by $A[W_R, W_C]$. For a square matrix A , we denote by $A[W]$ the principal submatrix of A with row/column set W . A square matrix A is said to be *skew-symmetric* if $A = -A^\top$. The following lemma is a well-known fact.

Lemma 2.1 ([24, Proposition 7.3.6]). Let A be a skew-symmetric matrix. Then the rank of A is equal to the maximum size of a nonsingular principal submatrix.

We will use the following two lemmas concerning skew-symmetric matrices in the proof of our main result.

Lemma 2.2. Let A be a skew-symmetric matrix, and D be a diagonal matrix with nonnegative entries. Then $A + D$ is nonsingular if and only if there exists a nonsingular principal submatrix of A containing $A[S]$, where S is a row/column set of D corresponding to zero diagonals.

Proof. Let X be the row/column set of A and D . Since D is a diagonal matrix, we have

$$\begin{aligned} \det(A + D) &= \sum_{W \subseteq X} \det A[W] \cdot \det D[X \setminus W] \\ &= \sum_{W \supseteq S} \det A[W] \cdot \det D[X \setminus W] \end{aligned} \quad (1)$$

by the definition of S . Figure 1 shows submatrices $A[W]$ and $D[X \setminus W]$. Since $A[W]$ is skew-symmetric, $\det A[W] \geq 0$ holds. Moreover, we have $\det D[X \setminus W] > 0$ for $W \supseteq S$. Hence each term of (1) is nonnegative. Thus, $A + D$ is nonsingular if and only if there exists $W \supseteq S$ such that $A[W]$ is nonsingular. \square

Lemma 2.3. Let A be a skew-symmetric matrix with row/column set X . For a subset S of X , there exists a nonsingular principal submatrix containing $A[S]$ if and only if $A[S, X]$ is of full row rank.

Proof. If there exists a nonsingular principal matrix containing $A[S]$, it is obvious that $A[S, X]$ is of full row rank. To show the converse, we now assume that $A[S, X]$ is of full row rank. Consider a subset $S^\circ \subseteq S$ such that $B = A[S^\circ]$ is a maximum size nonsingular principal submatrix of $A[S]$. By Lemma 2.1, there exists a nonsingular matrix D such that

$$D^\top A[S]D = \begin{pmatrix} B & O \\ O & O \end{pmatrix}.$$

Using this D with row set indexed by S , we transform A into

$$\tilde{A} = \begin{pmatrix} I & O \\ O & D^\top \end{pmatrix} A \begin{pmatrix} I & O \\ O & D \end{pmatrix}.$$

Since $A[S, X]$ is of full row rank, so is $\tilde{A}[S, X]$. Then there exists a subset $X^\circ \subseteq X \setminus S$ such that $C = \tilde{A}[S \setminus S^\circ, X^\circ]$ is nonsingular. Thus \tilde{A} is in the form of

$$\tilde{A} = \left(\begin{array}{cc|cc} * & * & * & -C^\top \\ * & * & * & * \\ * & * & B & O \\ C & * & O & O \end{array} \right).$$

Therefore, $\tilde{A}[X^\circ \cup S]$ is a nonsingular principal submatrix, and so is $A[X^\circ \cup S]$. \square

We close this section with the following well-known lemma.

Lemma 2.4 (Schur complement). Suppose that A is a nonsingular matrix. Then a square matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is nonsingular if and only if $D - CA^{-1}B$ is nonsingular.

Proof. Since we have

$$\begin{pmatrix} I & O \\ -CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ O & D - CA^{-1}B \end{pmatrix},$$

the matrix M is nonsingular if and only if $D - CA^{-1}B$ is nonsingular by the nonsingularity of A . \square

3 Nonlinear Time-Varying Circuits

In this paper, we consider nonlinear time-varying circuits composed of resistors, inductors, capacitors, independent voltage/current sources, and dependent voltage/current sources.

We denote the vector of currents through all branches of the circuit by \mathbf{i} , and the vector of voltages across all branches by \mathbf{u} . Let V , J , C , L , and R denote the sets of independent voltage sources, independent current sources, capacitors, inductors, and resistors, respectively. Dependent voltage/current sources, denoted by S_U and S_I , are controlled by voltages across or currents through other branches.

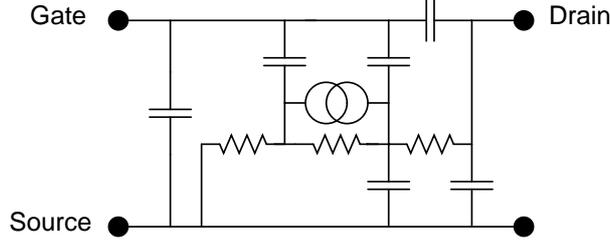


Figure 2: A MOSFET model.

The vectors of currents through V , J , C , L , R , S_U , and S_I are denoted by \mathbf{i}_V , \mathbf{i}_J , \mathbf{i}_C , \mathbf{i}_L , \mathbf{i}_R , \mathbf{i}_U , and \mathbf{i}_I . Similarly, the vectors of voltages are denoted by \mathbf{u}_V , \mathbf{u}_J , \mathbf{u}_C , \mathbf{u}_L , \mathbf{u}_R , \mathbf{u}_U , and \mathbf{u}_I . The physical characteristics of elements determine *constitutive equations*. Independent voltage and current sources simply read as

$$\mathbf{u}_V = \mathbf{v}_s(t) \quad \text{and} \quad \mathbf{i}_J = \mathbf{j}_s(t). \quad (2)$$

We assume that the constitutive equations of capacitors and inductors are described by

$$\mathbf{i}_C = \frac{d}{dt} \mathbf{q}(\mathbf{u}_C, t) \quad \text{and} \quad \mathbf{u}_L = \frac{d}{dt} \phi(\mathbf{i}_L, t). \quad (3)$$

Resistors are given in the form of

$$\mathbf{i}_R = \boldsymbol{\sigma}(\mathbf{u}_R, t).$$

Moreover, dependent current sources and dependent voltage sources are modeled by

$$\mathbf{i}_I = \mathbf{j}_I(\mathbf{u}_C, \mathbf{u}_L, \mathbf{u}_V, \mathbf{i}_C, \mathbf{i}_L, \mathbf{i}_J, t) \quad \text{and} \quad \mathbf{u}_U = \mathbf{v}_U(\mathbf{u}_C, \mathbf{u}_L, \mathbf{u}_V, \mathbf{i}_C, \mathbf{i}_L, \mathbf{i}_J, t). \quad (4)$$

This includes a wide class of dependent current/voltage sources.

Example 3.1. Consider a MOSFET model [7] depicted in Figure 2. The dependent current source is controlled by voltages across other branches. Since this circuit has a spanning tree which consists only of capacitors, the dependent current source can be described by a constitutive equation with argument \mathbf{u}_C , that is to say, equation (4).

Example 3.2. Consider another MOSFET model [29] depicted in Figure 3. The dependent current source is controlled by u_{GS} , u_{DS} , and u_{BS} , where u_{GS} is a branch voltage between G and S, u_{DS} is a branch voltage between D and S, and u_{BS} is a branch voltage between B and S. Since these voltages are expressed by voltages of capacitors, the dependent current source can be described by equation (4).

Remark 3.3. In the case of circuits with dependent sources controlled by voltage/current variables for resistors, we can replace them with dependent sources controlled only by artificial

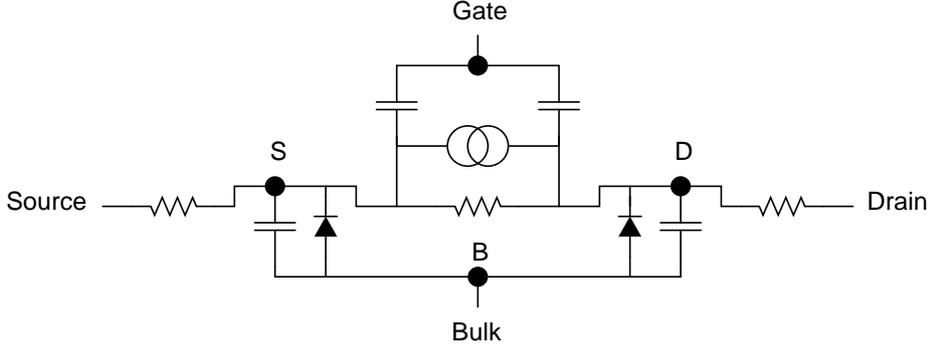


Figure 3: Another MOSFET model.

independent sources as follows. For dependent source controlled by a voltage variable for resistor e , we put an independent current source J in parallel to e with $i_J = 0$. Then, the voltage of e is equal to the voltage of the independent current source J . Analogously, for a dependent source controlled by a current variable for resistor e , we put an independent voltage source V in series to e with $u_V = 0$. Then, the current of e is equal to the current of the independent voltage source V .

For a matrix A , we denote the (i, j) entry of A by $(A)_{ij}$. For a vector valued function \mathbf{f} , we denote the i th component of \mathbf{f} by $(\mathbf{f})_i$. The capacitance matrix C , the inductance matrix L , and the conductance matrix K are given by

$$(C)_{ij} = \frac{\partial(\mathbf{q})_i}{\partial(\mathbf{u}_C)_j}, \quad (L)_{ij} = \frac{\partial(\phi)_i}{\partial(\mathbf{i}_L)_j}, \quad \text{and} \quad (K)_{ij} = \frac{\partial(\boldsymbol{\sigma})_i}{\partial(\mathbf{u}_R)_j}.$$

A square matrix A is called *positive definite* if $\mathbf{x}^\top A \mathbf{x} > 0$ for all $\mathbf{x} \neq 0$. In this paper, we assume the following conditions.

Assumption 3.4. The capacitance matrix C and the inductance matrix L are positive definite.

Assumption 3.5. The conductance matrix K is symmetric and positive definite.

Assumption 3.4 means that capacitors and inductors are strictly passive elements. Assumption 3.5 indicates that resistors are reciprocal and strictly passive [6].

Let $\Gamma = (W, E)$ be the network graph with vertex set W and edge set E . An edge in Γ corresponds to a branch that contains one element in the circuit. For a consistent model description, Γ contains no cycles consisting only of independent voltage sources and no cutsets consisting only of independent current sources. We denote the set of edges corresponding to independent voltage sources and independent current sources by E_v and E_j , respectively. We split $E_* := E \setminus (E_v \cup E_j)$ into E_y and E_z , i.e., $E_y \cup E_z = E_*$ and $E_y \cap E_z = \emptyset$. A partition (E_y, E_z) is called an *admissible partition*, if E_y includes all the capacitors and all the dependent current sources, and E_z includes all the inductors and all the dependent voltage sources.

We now rewrite constitutive equations with respect to an admissible partition (E_y, E_z) . We split \mathbf{i} and \mathbf{u} into

$$\mathbf{i} = (\mathbf{i}_V, \mathbf{i}_C, \mathbf{i}_I, \mathbf{i}_Y, \mathbf{i}_Z, \mathbf{i}_U, \mathbf{i}_L, \mathbf{i}_J)^\top \quad \text{and} \quad \mathbf{u} = (\mathbf{u}_V, \mathbf{u}_C, \mathbf{u}_I, \mathbf{u}_Y, \mathbf{u}_Z, \mathbf{u}_U, \mathbf{u}_L, \mathbf{u}_J)^\top,$$

where the subscripts Y and Z correspond to the resistors in E_y and E_z . Resistors are modeled by constitutive equations in the form of

$$\mathbf{i}_Y = \mathbf{g}(\mathbf{i}_Z, \mathbf{u}_Y, t) \quad \text{and} \quad \mathbf{u}_Z = \mathbf{h}(\mathbf{i}_Z, \mathbf{u}_Y, t). \quad (5)$$

The matrices Z, H, G, Y are defined by

$$(Z)_{ij} = \frac{\partial(\mathbf{h})_i}{\partial(\mathbf{i}_Z)_j}, \quad (H)_{ij} = \frac{\partial(\mathbf{h})_i}{\partial(\mathbf{u}_Y)_j}, \quad (G)_{ij} = \frac{\partial(\mathbf{g})_i}{\partial(\mathbf{i}_Z)_j}, \quad (Y)_{ij} = \frac{\partial(\mathbf{g})_i}{\partial(\mathbf{u}_Y)_j}.$$

With the aid of the conductance matrix in the form of $K = \begin{pmatrix} K_Y & K_G \\ K_H & K_Z \end{pmatrix}$, where the row/column sets of K_Y and K_Z are the sets of resistors in E_y and E_z , the above four Jacobian matrices are expressed by

$$Z = K_Z^{-1}, \quad H = -K_Z^{-1}K_H, \quad G = K_GK_Z^{-1}, \quad Y = K_Y - K_GK_Z^{-1}K_H.$$

Then, by Assumption 3.5, one can show that the hybrid immittance matrix $\begin{pmatrix} Z & H \\ G & Y \end{pmatrix}$ satisfies the following conditions:

- (i) The hybrid immittance matrix $\begin{pmatrix} Z & H \\ G & Y \end{pmatrix}$ is positive definite.
- (ii) The principal submatrices Z and Y are symmetric.
- (iii) $H = -G^\top$ holds.

We call a spanning tree T of Γ a *reference tree* if T contains all edges in E_v , no edges in E_j , and as many edges in E_y as possible. Note that a reference tree T may contain some edges in E_z . A reference tree is called *normal* if it contains as many edges as possible in the order corresponding to V, C, S_I, Y, Z, S_U , and L . The cotree of T is denoted by $\bar{T} = E \setminus T$. The hybrid equations are determined by an admissible partition (E_y, E_z) and a reference tree T , which is not necessarily normal. In this paper, we adopt a normal reference tree.

With respect to a normal reference tree T , we further split \mathbf{i} and \mathbf{u} into

$$\mathbf{i} = (\mathbf{i}_V, \mathbf{i}_C^\tau, \mathbf{i}_I^\tau, \mathbf{i}_Y^\tau, \mathbf{i}_Z^\tau, \mathbf{i}_U^\tau, \mathbf{i}_L^\tau, \mathbf{i}_C^\lambda, \mathbf{i}_I^\lambda, \mathbf{i}_Y^\lambda, \mathbf{i}_Z^\lambda, \mathbf{i}_U^\lambda, \mathbf{i}_L^\lambda, \mathbf{i}_J)^\top$$

and

$$\mathbf{u} = (\mathbf{u}_V, \mathbf{u}_C^\tau, \mathbf{u}_I^\tau, \mathbf{u}_Y^\tau, \mathbf{u}_Z^\tau, \mathbf{u}_U^\tau, \mathbf{u}_L^\tau, \mathbf{u}_C^\lambda, \mathbf{u}_I^\lambda, \mathbf{u}_Y^\lambda, \mathbf{u}_Z^\lambda, \mathbf{u}_U^\lambda, \mathbf{u}_L^\lambda, \mathbf{u}_J)^\top,$$

where the superscripts τ and λ designate the tree T and the cotree \bar{T} . With respect to a normal reference tree T , the vector valued function \mathbf{g} is also split into \mathbf{g}^τ and \mathbf{g}^λ . This means

$\mathbf{i}_Y^\tau = \mathbf{g}^\tau(\mathbf{i}_Z, \mathbf{u}_Y, t)$ and $\mathbf{i}_Y^\lambda = \mathbf{g}^\lambda(\mathbf{i}_Z, \mathbf{u}_Y, t)$. Similarly, we split \mathbf{h} , \mathbf{q} , and ϕ . The matrix Y is written in the form of

$$\begin{pmatrix} Y_\tau^\tau & Y_\lambda^\tau \\ Y_\tau^\lambda & Y_\lambda^\lambda \end{pmatrix},$$

where

$$(Y_\tau^\tau)_{ij} = \frac{\partial(\mathbf{g}^\tau)_i}{\partial(\mathbf{u}_Y^\tau)_j}, \quad (Y_\lambda^\tau)_{ij} = \frac{\partial(\mathbf{g}^\tau)_i}{\partial(\mathbf{u}_Y^\lambda)_j}, \quad (Y_\tau^\lambda)_{ij} = \frac{\partial(\mathbf{g}^\lambda)_i}{\partial(\mathbf{u}_Y^\tau)_j}, \quad (Y_\lambda^\lambda)_{ij} = \frac{\partial(\mathbf{g}^\lambda)_i}{\partial(\mathbf{u}_Y^\lambda)_j}.$$

In a similar way, the matrices C , L , Z , H , G are written in the form of

$$\begin{pmatrix} C_\tau^\tau & C_\lambda^\tau \\ C_\tau^\lambda & C_\lambda^\lambda \end{pmatrix}, \quad \begin{pmatrix} L_\tau^\tau & L_\lambda^\tau \\ L_\tau^\lambda & L_\lambda^\lambda \end{pmatrix}, \quad \begin{pmatrix} Z_\tau^\tau & Z_\lambda^\tau \\ Z_\tau^\lambda & Z_\lambda^\lambda \end{pmatrix}, \quad \begin{pmatrix} H_\tau^\tau & H_\lambda^\tau \\ H_\tau^\lambda & H_\lambda^\lambda \end{pmatrix}, \quad \begin{pmatrix} G_\tau^\tau & G_\lambda^\tau \\ G_\tau^\lambda & G_\lambda^\lambda \end{pmatrix}.$$

By the definition of a normal reference tree, the *fundamental cutset matrix* F is given by

$$F = \begin{pmatrix} \mathbf{i}_V & \mathbf{i}_C^\tau & \mathbf{i}_I^\tau & \mathbf{i}_Y^\tau & \mathbf{i}_Z^\tau & \mathbf{i}_U^\tau & \mathbf{i}_L^\tau & \mathbf{i}_C^\lambda & \mathbf{i}_I^\lambda & \mathbf{i}_Y^\lambda & \mathbf{i}_Z^\lambda & \mathbf{i}_U^\lambda & \mathbf{i}_L^\lambda & \mathbf{i}_J \\ I & O & O & O & O & O & O & A_{VC} & A_{VI} & A_{VY} & A_{VZ} & A_{VU} & A_{VL} & A_{VJ} \\ O & I & O & O & O & O & O & A_{CC} & A_{CI} & A_{CY} & A_{CZ} & A_{CU} & A_{CL} & A_{CJ} \\ O & O & I & O & O & O & O & O & A_{II} & A_{IY} & A_{IZ} & A_{IU} & A_{IL} & A_{IJ} \\ O & O & O & I & O & O & O & O & O & A_{YY} & A_{YZ} & A_{YU} & A_{YL} & A_{YJ} \\ O & O & O & O & I & O & O & O & O & O & A_{ZZ} & A_{ZU} & A_{ZL} & A_{ZJ} \\ O & O & O & O & O & I & O & O & O & O & O & A_{UU} & A_{UL} & A_{UJ} \\ O & O & O & O & O & O & I & O & O & O & O & O & A_{LL} & A_{LJ} \end{pmatrix}.$$

Then *Kirchhoff's current law* (KCL), which states that the sum of currents entering each node is equal to zero, is written as

$$F\mathbf{i} = \mathbf{0}.$$

This is rewritten as

$$\mathbf{i}_V + A_{VC}\mathbf{i}_C^\lambda + A_{VI}\mathbf{i}_I^\lambda + A_{VY}\mathbf{i}_Y^\lambda + A_{VZ}\mathbf{i}_Z^\lambda + A_{VU}\mathbf{i}_U^\lambda + A_{VL}\mathbf{i}_L^\lambda + A_{VJ}\mathbf{i}_J = \mathbf{0}, \quad (6)$$

$$\mathbf{i}_C^\tau + A_{CC}\mathbf{i}_C^\lambda + A_{CI}\mathbf{i}_I^\lambda + A_{CY}\mathbf{i}_Y^\lambda + A_{CZ}\mathbf{i}_Z^\lambda + A_{CU}\mathbf{i}_U^\lambda + A_{CL}\mathbf{i}_L^\lambda + A_{CJ}\mathbf{i}_J = \mathbf{0}, \quad (7)$$

$$\mathbf{i}_I^\tau + A_{II}\mathbf{i}_I^\lambda + A_{IY}\mathbf{i}_Y^\lambda + A_{IZ}\mathbf{i}_Z^\lambda + A_{IU}\mathbf{i}_U^\lambda + A_{IL}\mathbf{i}_L^\lambda + A_{IJ}\mathbf{i}_J = \mathbf{0}, \quad (8)$$

$$\mathbf{i}_Y^\tau + A_{YY}\mathbf{i}_Y^\lambda + A_{YZ}\mathbf{i}_Z^\lambda + A_{YU}\mathbf{i}_U^\lambda + A_{YL}\mathbf{i}_L^\lambda + A_{YJ}\mathbf{i}_J = \mathbf{0}, \quad (9)$$

$$\mathbf{i}_Z^\tau + A_{ZZ}\mathbf{i}_Z^\lambda + A_{ZU}\mathbf{i}_U^\lambda + A_{ZL}\mathbf{i}_L^\lambda + A_{ZJ}\mathbf{i}_J = \mathbf{0}, \quad (10)$$

$$\mathbf{i}_U^\tau + A_{UU}\mathbf{i}_U^\lambda + A_{UL}\mathbf{i}_L^\lambda + A_{UJ}\mathbf{i}_J = \mathbf{0}, \quad (11)$$

$$\mathbf{i}_L^\tau + A_{LL}\mathbf{i}_L^\lambda + A_{LJ}\mathbf{i}_J = \mathbf{0}. \quad (12)$$

Kirchhoff's voltage law (KVL), which states that the sum of voltages in each loop of the network is equal to zero, provides

$$F^\perp \mathbf{u} = \mathbf{0}$$

with the *fundamental loop matrix*

$$F^\perp = \begin{pmatrix} \mathbf{u}_V & \mathbf{u}_C^\tau & \mathbf{u}_I^\tau & \mathbf{u}_Y^\tau & \mathbf{u}_Z^\tau & \mathbf{u}_U^\tau & \mathbf{u}_L^\tau & \mathbf{u}_C^\lambda & \mathbf{u}_I^\lambda & \mathbf{u}_Y^\lambda & \mathbf{u}_Z^\lambda & \mathbf{u}_U^\lambda & \mathbf{u}_L^\lambda & \mathbf{u}_J \\ -A_{VC}^\top & -A_{CC}^\top & O & O & O & O & O & I & O & O & O & O & O & O \\ -A_{VI}^\top & -A_{CI}^\top & -A_{II}^\top & O & O & O & O & O & I & O & O & O & O & O \\ -A_{VY}^\top & -A_{CY}^\top & -A_{IY}^\top & -A_{YY}^\top & O & O & O & O & O & I & O & O & O & O \\ -A_{VZ}^\top & -A_{CZ}^\top & -A_{IZ}^\top & -A_{YZ}^\top & -A_{ZZ}^\top & O & O & O & O & O & I & O & O & O \\ -A_{VU}^\top & -A_{CU}^\top & -A_{IU}^\top & -A_{YU}^\top & -A_{ZU}^\top & -A_{UU}^\top & O & O & O & O & O & I & O & O \\ -A_{VL}^\top & -A_{CL}^\top & -A_{IL}^\top & -A_{YL}^\top & -A_{ZL}^\top & -A_{UL}^\top & -A_{LL}^\top & O & O & O & O & O & I & O \\ -A_{VJ}^\top & -A_{CJ}^\top & -A_{IJ}^\top & -A_{YJ}^\top & -A_{ZJ}^\top & -A_{UJ}^\top & -A_{LJ}^\top & O & O & O & O & O & O & I \end{pmatrix}.$$

This is rewritten as

$$-A_{VC}^\top \mathbf{u}_V - A_{CC}^\top \mathbf{u}_C^\tau + \mathbf{u}_C^\lambda = \mathbf{0}, \quad (13)$$

$$-A_{VI}^\top \mathbf{u}_V - A_{CI}^\top \mathbf{u}_C^\tau - A_{II}^\top \mathbf{u}_I^\tau + \mathbf{u}_I^\lambda = \mathbf{0}, \quad (14)$$

$$-A_{VY}^\top \mathbf{u}_V - A_{CY}^\top \mathbf{u}_C^\tau - A_{IY}^\top \mathbf{u}_I^\tau - A_{YY}^\top \mathbf{u}_Y^\tau + \mathbf{u}_Y^\lambda = \mathbf{0}, \quad (15)$$

$$-A_{VZ}^\top \mathbf{u}_V - A_{CZ}^\top \mathbf{u}_C^\tau - A_{IZ}^\top \mathbf{u}_I^\tau - A_{YZ}^\top \mathbf{u}_Y^\tau - A_{ZZ}^\top \mathbf{u}_Z^\tau + \mathbf{u}_Z^\lambda = \mathbf{0}, \quad (16)$$

$$-A_{VU}^\top \mathbf{u}_V - A_{CU}^\top \mathbf{u}_C^\tau - A_{IU}^\top \mathbf{u}_I^\tau - A_{YU}^\top \mathbf{u}_Y^\tau - A_{ZU}^\top \mathbf{u}_Z^\tau - A_{UU}^\top \mathbf{u}_U^\tau + \mathbf{u}_U^\lambda = \mathbf{0}, \quad (17)$$

$$-A_{VL}^\top \mathbf{u}_V - A_{CL}^\top \mathbf{u}_C^\tau - A_{IL}^\top \mathbf{u}_I^\tau - A_{YL}^\top \mathbf{u}_Y^\tau - A_{ZL}^\top \mathbf{u}_Z^\tau - A_{UL}^\top \mathbf{u}_U^\tau - A_{LL}^\top \mathbf{u}_L^\tau + \mathbf{u}_L^\lambda = \mathbf{0}, \quad (18)$$

$$-A_{VJ}^\top \mathbf{u}_V - A_{CJ}^\top \mathbf{u}_C^\tau - A_{IJ}^\top \mathbf{u}_I^\tau - A_{YJ}^\top \mathbf{u}_Y^\tau - A_{ZJ}^\top \mathbf{u}_Z^\tau - A_{UJ}^\top \mathbf{u}_U^\tau - A_{LJ}^\top \mathbf{u}_L^\tau + \mathbf{u}_J = \mathbf{0}. \quad (19)$$

4 Hybrid Analysis

In this section, we describe the procedure of the hybrid analysis. The idea of its derivation is to use all constitutive equations so that $F\mathbf{i} = \mathbf{0}$ and $F^\perp \mathbf{u} = \mathbf{0}$ provide a system depending only on \mathbf{u}_C^τ , \mathbf{u}_I^τ , \mathbf{u}_Y^τ and \mathbf{i}_Z^λ , \mathbf{i}_U^λ , \mathbf{i}_L^λ . Performing the hybrid analysis presented in [14], we obtain the *hybrid equations* (or *hybrid equation system*)

$$\begin{aligned} -A_{VL}^\top \mathbf{v}_s(t) - A_{CL}^\top \mathbf{u}_C^\tau - A_{IL}^\top \mathbf{u}_I^\tau - A_{YL}^\top \mathbf{u}_Y^\tau - A_{ZL}^\top \mathbf{h}^\tau - A_{UL}^\top \mathbf{v}_U^\tau(\cdot) - A_{LL}^\top \frac{d}{dt} \phi^\tau + \frac{d}{dt} \phi^\lambda &= \mathbf{0}, \\ -A_{VU}^\top \mathbf{v}_s(t) - A_{CU}^\top \mathbf{u}_C^\tau - A_{IU}^\top \mathbf{u}_I^\tau - A_{YU}^\top \mathbf{u}_Y^\tau - A_{ZU}^\top \mathbf{h}^\tau - A_{UU}^\top \mathbf{v}_U^\tau(\cdot) + \mathbf{v}_U^\lambda(\cdot) &= \mathbf{0}, \\ -A_{VZ}^\top \mathbf{v}_s(t) - A_{CZ}^\top \mathbf{u}_C^\tau - A_{IZ}^\top \mathbf{u}_I^\tau - A_{YZ}^\top \mathbf{u}_Y^\tau - A_{ZZ}^\top \mathbf{h}^\tau + \mathbf{h}^\lambda &= \mathbf{0}, \\ \mathbf{g}^\tau + A_{YY} \mathbf{g}^\lambda + A_{YZ} \mathbf{i}_Z^\lambda + A_{YU} \mathbf{i}_U^\lambda + A_{YL} \mathbf{i}_L^\lambda + A_{YJ} \mathbf{j}_s(t) &= \mathbf{0}, \\ \mathbf{j}_I^\tau(\cdot) + A_{II} \mathbf{j}_I^\lambda(\cdot) + A_{IY} \mathbf{g}^\lambda + A_{IZ} \mathbf{i}_Z^\lambda + A_{IU} \mathbf{i}_U^\lambda + A_{IL} \mathbf{i}_L^\lambda + A_{IJ} \mathbf{j}_s(t) &= \mathbf{0}, \\ \frac{d}{dt} \mathbf{q}^\tau + A_{CC} \frac{d}{dt} \mathbf{q}^\lambda + A_{CI} \mathbf{j}_I^\lambda(\cdot) + A_{CY} \mathbf{g}^\lambda + A_{CZ} \mathbf{i}_Z^\lambda + A_{CU} \mathbf{i}_U^\lambda + A_{CL} \mathbf{i}_L^\lambda + A_{CJ} \mathbf{j}_s(t) &= \mathbf{0}, \end{aligned}$$

where

$$\begin{aligned} \mathbf{q}^\tau &= \mathbf{q}^\tau(\mathbf{u}_C^\tau, A_{VC}^\top \mathbf{v}_s(t) + A_{CC}^\top \mathbf{u}_C^\tau, t), \\ \mathbf{q}^\lambda &= \mathbf{q}^\lambda(\mathbf{u}_C^\tau, A_{VC}^\top \mathbf{v}_s(t) + A_{CC}^\top \mathbf{u}_C^\tau, t), \\ \mathbf{g}^\tau &= \mathbf{g}^\tau(-A_{ZZ} \mathbf{i}_Z^\lambda - A_{ZU} \mathbf{i}_U^\lambda - A_{ZL} \mathbf{i}_L^\lambda - A_{ZJ} \mathbf{j}_s(t), \mathbf{i}_Z^\lambda, \mathbf{u}_Y^\tau, A_{VY}^\top \mathbf{v}_s(t) + A_{CY}^\top \mathbf{u}_C^\tau + A_{IY}^\top \mathbf{u}_I^\tau + A_{YY}^\top \mathbf{u}_Y^\tau, t), \\ \mathbf{g}^\lambda &= \mathbf{g}^\lambda(-A_{ZZ} \mathbf{i}_Z^\lambda - A_{ZU} \mathbf{i}_U^\lambda - A_{ZL} \mathbf{i}_L^\lambda - A_{ZJ} \mathbf{j}_s(t), \mathbf{i}_Z^\lambda, \mathbf{u}_Y^\tau, A_{VY}^\top \mathbf{v}_s(t) + A_{CY}^\top \mathbf{u}_C^\tau + A_{IY}^\top \mathbf{u}_I^\tau + A_{YY}^\top \mathbf{u}_Y^\tau, t), \\ \mathbf{h}^\tau &= \mathbf{h}^\tau(-A_{ZZ} \mathbf{i}_Z^\lambda - A_{ZU} \mathbf{i}_U^\lambda - A_{ZL} \mathbf{i}_L^\lambda - A_{ZJ} \mathbf{j}_s(t), \mathbf{i}_Z^\lambda, \mathbf{u}_Y^\tau, A_{VY}^\top \mathbf{v}_s(t) + A_{CY}^\top \mathbf{u}_C^\tau + A_{IY}^\top \mathbf{u}_I^\tau + A_{YY}^\top \mathbf{u}_Y^\tau, t), \\ \mathbf{h}^\lambda &= \mathbf{h}^\lambda(-A_{ZZ} \mathbf{i}_Z^\lambda - A_{ZU} \mathbf{i}_U^\lambda - A_{ZL} \mathbf{i}_L^\lambda - A_{ZJ} \mathbf{j}_s(t), \mathbf{i}_Z^\lambda, \mathbf{u}_Y^\tau, A_{VY}^\top \mathbf{v}_s(t) + A_{CY}^\top \mathbf{u}_C^\tau + A_{IY}^\top \mathbf{u}_I^\tau + A_{YY}^\top \mathbf{u}_Y^\tau, t), \\ \phi^\tau &= \phi^\tau(-A_{LL} \mathbf{i}_L^\lambda - A_{LJ} \mathbf{j}_s(t), \mathbf{i}_L^\lambda, t), \\ \phi^\lambda &= \phi^\lambda(-A_{LL} \mathbf{i}_L^\lambda - A_{LJ} \mathbf{j}_s(t), \mathbf{i}_L^\lambda, t). \end{aligned}$$

The derivation of the hybrid equations is given in Appendix A. The procedure of the hybrid analysis is as follows.

1. The values of \mathbf{u}_V and \mathbf{i}_J are obvious from (2).
2. Compute the values of $\mathbf{i}_Z^\lambda, \mathbf{i}_U^\lambda, \mathbf{i}_L^\lambda$ and $\mathbf{u}_C^\tau, \mathbf{u}_I^\tau, \mathbf{u}_Y^\tau$ by solving the hybrid equations.
3. Compute the values of $\mathbf{i}_Z^\tau, \mathbf{i}_U^\tau, \mathbf{i}_L^\tau$ and $\mathbf{u}_C^\lambda, \mathbf{u}_I^\lambda, \mathbf{u}_Y^\lambda$ by substituting the values obtained in Steps 1–2 into the equations (10)–(12), and (13)–(15).
4. Compute the values of $\mathbf{u}_Z^\tau, \mathbf{u}_Z^\lambda, \mathbf{u}_L^\tau, \mathbf{u}_L^\lambda$, and $\mathbf{i}_C^\tau, \mathbf{i}_C^\lambda, \mathbf{i}_Y^\tau, \mathbf{i}_Y^\lambda$ by substituting the values obtained in Steps 1–3 into (3) and (5).
5. Compute the values of $\mathbf{u}_U^\tau, \mathbf{u}_U^\lambda$, and $\mathbf{i}_I^\tau, \mathbf{i}_I^\lambda$ by substituting the values obtained in Steps 1–4 into (4).
6. Compute the values of \mathbf{i}_V and \mathbf{u}_J by substituting the values obtained in Steps 1–5 into (6) and (19).

All operations in Steps 3–6 are substitutions and differentiations of the obtained solutions. Consequently, the numerical difficulty is determined by the index of the hybrid equation system. Higher index variables as known from MNA do not appear in the hybrid equation system.

5 DAEs with Properly Stated Leading Term

In this section, we explain DAEs with properly stated leading term, and define the tractability index. Consider a DAE in the form of

$$A(\mathbf{x}(t), t) \frac{d}{dt} \mathbf{d}(\mathbf{x}(t), t) + \mathbf{b}(\mathbf{x}(t), t) = \mathbf{0}. \quad (20)$$

Let $A(\mathbf{x}(t), t)$ be an $m \times n$ matrix. We define

$$D(\mathbf{x}, t) = \frac{\partial \mathbf{d}(\mathbf{x}, t)}{\partial \mathbf{x}}, \quad B(\mathbf{x}, t) = \frac{\partial \mathbf{b}(\mathbf{x}, t)}{\partial \mathbf{x}}, \quad \text{and} \quad M(\mathbf{x}, t) = A(\mathbf{x}, t)D(\mathbf{x}, t).$$

A matrix $Q(\mathbf{x}, t)$ satisfying $Q(\mathbf{x}, t)^2 = Q(\mathbf{x}, t)$ is called a *projector*. Moreover, a projector $Q(\mathbf{x}, t)$ is called a *projector onto* a subspace Π if $\text{im } Q(\mathbf{x}, t) = \Pi$.

Definition 5.1 ([11, Definition 26]). The equation (20) is a DAE with properly stated leading term if the size of $D(\mathbf{x}, t)$ is $n \times m$,

$$\ker A(\mathbf{x}, t) \oplus \text{im } D(\mathbf{x}, t) = \mathbb{R}^n \quad (21)$$

holds for all \mathbf{x} and t from the definition domain, and there is an $n \times n$ projector function $P(t)$ continuously differentiable with respect to t such that $\ker P(t) = \ker A(\mathbf{x}, t)$, $\text{im } P(t) = \text{im } D(\mathbf{x}, t)$, and $\mathbf{d}(\mathbf{x}, t) = P(t)\mathbf{d}(\mathbf{x}, t)$.

A DAE with properly stated leading term (20) arises in circuit simulation via circuit analysis methods such as MNA [11]. Since this concept was first introduced in [2], the analysis of such DAEs has been developed in [10, 11, 22, 23, 28].

Lemma 5.2 ([11, Lemma A.1]). Let $A(\mathbf{x}, t)$ be an $m \times n$ matrix and $D(\mathbf{x}, t)$ be an $n \times m$ matrix. Then, the relation $\ker A(\mathbf{x}, t) \oplus \operatorname{im} D(\mathbf{x}, t) = \mathbb{R}^n$ is equivalent to the following three conditions:

$$\operatorname{im} M(\mathbf{x}, t) = \operatorname{im} A(\mathbf{x}, t), \quad \ker M(\mathbf{x}, t) = \ker D(\mathbf{x}, t), \quad \ker A(\mathbf{x}, t) \cap \operatorname{im} D(\mathbf{x}, t) = \{\mathbf{0}\},$$

where $M(\mathbf{x}, t) = A(\mathbf{x}, t)D(\mathbf{x}, t)$.

Obviously, the DAE (20) represents a regular ODE if and only if the matrix $M(\mathbf{x}, t)$ is nonsingular for all \mathbf{x} and t of the definition domain. In this case we say that the DAE (20) has index zero. In the case of a singular matrix $M(\mathbf{x}, t)$, the DAE (20) contains algebraic equations. Furthermore, one may have to differentiate certain part of the system to get a solution. A simple criteria for the absence of this problem is given by the tractability index one condition.

Definition 5.3 ([21, Definition 3.3]). The DAE (20) has the *tractability index one* on the definition domain if $M(\mathbf{x}, t)$ is singular and

$$\ker D(\mathbf{x}, t) \cap \{\mathbf{z} \in \mathbb{R}^m \mid B(\mathbf{x}, t)\mathbf{z} \in \operatorname{im} M(\mathbf{x}, t)\} = \{\mathbf{0}\}$$

for all (\mathbf{x}, t) of the definition domain.

Remark 5.4 ([32, Remark 4.6]). The index of a DAE (20) is at most one if and only if the matrix $M(\mathbf{x}, t) + B(\mathbf{x}, t)Q(\mathbf{x}, t)$ is nonsingular for all \mathbf{x} and t with a projector $Q(\mathbf{x}, t)$ onto $\ker M(\mathbf{x}, t)$.

The following lemma is useful in the derivation of a necessary and sufficient condition for the index at most one.

Lemma 5.5. The following three conditions (a)–(c) are equivalent.

- (a) For some projector $Q(\mathbf{x}, t)$ onto $\ker M(\mathbf{x}, t)$, $M(\mathbf{x}, t) + B(\mathbf{x}, t)Q(\mathbf{x}, t)$ is nonsingular for all \mathbf{x} and t .
- (b) If $\mathbf{w} \in \ker M(\mathbf{x}, t)$ and $B(\mathbf{x}, t)\mathbf{w} \in \operatorname{im} M(\mathbf{x}, t)$ hold, then we have $\mathbf{w} = \mathbf{0}$.
- (c) For any projector $Q(\mathbf{x}, t)$ onto $\ker M(\mathbf{x}, t)$, $M(\mathbf{x}, t) + B(\mathbf{x}, t)Q(\mathbf{x}, t)$ is nonsingular for all \mathbf{x} and t .

Proof. It clearly holds that (c) \Rightarrow (a). Let us prove (a) \Rightarrow (b) \Rightarrow (c).

First we show that if (b) does not hold, then $M(\mathbf{x}, t) + B(\mathbf{x}, t)Q(\mathbf{x}, t)$ is singular for any projector $Q(\mathbf{x}, t)$ onto $\ker M(\mathbf{x}, t)$. Let us assume that there exists $\mathbf{w} \neq \mathbf{0}$ satisfying $\mathbf{w} \in \ker M(\mathbf{x}, t)$ and $B(\mathbf{x}, t)\mathbf{w} = M(\mathbf{x}, t)\mathbf{z}$ for some \mathbf{z} . Then, for any projector $Q(\mathbf{x}, t)$ onto $\ker M(\mathbf{x}, t)$,

$$\mathbf{w} \in \ker M(\mathbf{x}, t) = \operatorname{im} Q(\mathbf{x}, t) = \ker(I - Q(\mathbf{x}, t))$$

holds. Hence we have $(I - Q(\mathbf{x}, t))\mathbf{w} = \mathbf{0}$, which implies $\mathbf{w} = Q(\mathbf{x}, t)\mathbf{w}$. For $\mathbf{y} = -(I - Q(\mathbf{x}, t))\mathbf{z} + Q(\mathbf{x}, t)\mathbf{w}$, it follows from $\mathbf{y} \in \operatorname{im}(I - Q(\mathbf{x}, t)) \oplus \operatorname{im} Q(\mathbf{x}, t)$ that

$$\mathbf{y} = \mathbf{0} \quad \Leftrightarrow \quad (I - Q(\mathbf{x}, t))\mathbf{z} = \mathbf{0} \quad \text{and} \quad Q(\mathbf{x}, t)\mathbf{w} = \mathbf{0}.$$

By $Q(\mathbf{x}, t)\mathbf{w} = \mathbf{w} \neq \mathbf{0}$, we obtain $\mathbf{y} \neq \mathbf{0}$.

Now we have

$$\begin{aligned} (M(\mathbf{x}, t) + B(\mathbf{x}, t)Q(\mathbf{x}, t))\mathbf{y} &= (M(\mathbf{x}, t) + B(\mathbf{x}, t)Q(\mathbf{x}, t))\{- (I - Q(\mathbf{x}, t))\mathbf{z} + Q(\mathbf{x}, t)\mathbf{w}\} \\ &= -(M(\mathbf{x}, t) + B(\mathbf{x}, t)Q(\mathbf{x}, t))\mathbf{z} + B(\mathbf{x}, t)Q(\mathbf{x}, t)\mathbf{z} + B(\mathbf{x}, t)Q(\mathbf{x}, t)\mathbf{w} \\ &= -M(\mathbf{x}, t)\mathbf{z} + B(\mathbf{x}, t)Q(\mathbf{x}, t)\mathbf{w}, \end{aligned}$$

because $M(\mathbf{x}, t)Q(\mathbf{x}, t) = O$ and $Q(\mathbf{x}, t)^2 = Q(\mathbf{x}, t)$. It follows from $\mathbf{w} = Q(\mathbf{x}, t)\mathbf{w}$ and $B(\mathbf{x}, t)\mathbf{w} = M(\mathbf{x}, t)\mathbf{z}$ that

$$-M(\mathbf{x}, t)\mathbf{z} + B(\mathbf{x}, t)Q(\mathbf{x}, t)\mathbf{w} = -M(\mathbf{x}, t)\mathbf{z} + B(\mathbf{x}, t)\mathbf{w} = \mathbf{0}.$$

Thus $(M(\mathbf{x}, t) + B(\mathbf{x}, t)Q(\mathbf{x}, t))\mathbf{y} = \mathbf{0}$ holds for $\mathbf{y} \neq \mathbf{0}$. This means that $M(\mathbf{x}, t) + B(\mathbf{x}, t)Q(\mathbf{x}, t)$ is singular for any $Q(\mathbf{x}, t)$, which implies that (a) does not hold. Hence we obtain (a) \Rightarrow (b).

Secondly, we show that if (c) does not hold, then there exists $\mathbf{w} \neq \mathbf{0}$ satisfying $\mathbf{w} \in \ker M(\mathbf{x}, t)$ and $B(\mathbf{x}, t)\mathbf{w} \in \text{im } M(\mathbf{x}, t)$. Let us assume that there exists a projector $Q(\mathbf{x}, t)$ onto $\ker M(\mathbf{x}, t)$ such that $M(\mathbf{x}, t) + B(\mathbf{x}, t)Q(\mathbf{x}, t)$ is singular. Then it holds that

$$(M(\mathbf{x}, t) + B(\mathbf{x}, t)Q(\mathbf{x}, t))\mathbf{y} = \mathbf{0} \tag{22}$$

for some $\mathbf{y} \neq \mathbf{0}$. By setting $\mathbf{w} = Q(\mathbf{x}, t)\mathbf{y}$ and $\mathbf{z} = -(I - Q(\mathbf{x}, t))\mathbf{y}$, we have

$$\begin{aligned} M(\mathbf{x}, t)\mathbf{w} &= M(\mathbf{x}, t)Q(\mathbf{x}, t)\mathbf{y} = \mathbf{0}, \\ B(\mathbf{x}, t)\mathbf{w} - M(\mathbf{x}, t)\mathbf{z} &= B(\mathbf{x}, t)Q(\mathbf{x}, t)\mathbf{y} + M(\mathbf{x}, t)\mathbf{y} - M(\mathbf{x}, t)Q(\mathbf{x}, t)\mathbf{y} \\ &= (M(\mathbf{x}, t) + B(\mathbf{x}, t)Q(\mathbf{x}, t))\mathbf{y} = \mathbf{0}, \end{aligned}$$

because $M(\mathbf{x}, t)Q(\mathbf{x}, t) = O$ and (22). This means that

$$\mathbf{w} \in \ker M(\mathbf{x}, t) \quad \text{and} \quad B(\mathbf{x}, t)\mathbf{w} \in \text{im } M(\mathbf{x}, t). \tag{23}$$

Now let us assume $\mathbf{w} = \mathbf{0}$. By $Q(\mathbf{x}, t)\mathbf{y} = \mathbf{w} = \mathbf{0}$, we have $\mathbf{y} \in \ker Q(\mathbf{x}, t)$. Moreover, since

$$M(\mathbf{x}, t)\mathbf{y} = M(\mathbf{x}, t)\mathbf{y} + B(\mathbf{x}, t)Q(\mathbf{x}, t)\mathbf{y} = (M(\mathbf{x}, t) + B(\mathbf{x}, t)Q(\mathbf{x}, t))\mathbf{y} = \mathbf{0}$$

holds, it follows that $\mathbf{y} \in \ker M(\mathbf{x}, t) = \text{im } Q(\mathbf{x}, t)$. Thus we obtain $\mathbf{y} \in \ker Q(\mathbf{x}, t) \cap \text{im } Q(\mathbf{x}, t) = \{\mathbf{0}\}$, which contradicts $\mathbf{y} \neq \mathbf{0}$. Hence there exists $\mathbf{w} \neq \mathbf{0}$ satisfying (23). This implies that (b) does not hold. \square

By Lemma 5.5, we have the following proposition concerning DAEs with index at most one.

Proposition 5.6. Let $Q(\mathbf{x}, t)$ be a projector onto $\ker M(\mathbf{x}, t)$. The index of the DAE (20) is at most one if and only if $M(\mathbf{x}, t) + B(\mathbf{x}, t)Q(\mathbf{x}, t)$ is nonsingular for all \mathbf{x} and t .

Proof. If $M(\mathbf{x}, t) + B(\mathbf{x}, t)Q(\mathbf{x}, t)$ is nonsingular for all \mathbf{x} and t , the index of the DAE (20) is at most one by Remark 5.4.

If $M(\mathbf{x}, t) + B(\mathbf{x}, t)Q(\mathbf{x}, t)$ is singular for all \mathbf{x} and t , (c) in Lemma 5.5 does not hold, which implies that (a) in Lemma 5.5 does not hold. Hence the index of the DAE (20) is more than one by Remark 5.4. \square

6 Hybrid Equations with Properly Stated Leading Term

In this section, we rewrite the hybrid equation system as a DAE with properly stated leading term. We first define a reflexive generalized inverse.

Definition 6.1. A reflexive generalized inverse of a matrix A is a matrix A^- which satisfies $AA^-A = A$ and $A^-AA^- = A^-$.

A reflexive generalized inverse A^- satisfies

$$\dim \operatorname{im} A^-A = \dim \operatorname{im} A. \quad (24)$$

We now define

$$A = \begin{pmatrix} O & -A_{LL}^\top & I & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & I & A_{CC} & O \end{pmatrix}, \quad \mathbf{x}(t) = \begin{pmatrix} \mathbf{i}_L^\lambda \\ \mathbf{i}_U^\lambda \\ \mathbf{i}_Z^\lambda \\ \mathbf{u}_Y^\tau \\ \mathbf{u}_I^\tau \\ \mathbf{u}_C^\tau \end{pmatrix}, \quad \mathbf{d}(\mathbf{x}, t) = A^-A \begin{pmatrix} \mathbf{0} \\ \phi^\tau \\ \phi^\lambda \\ \mathbf{q}^\tau \\ \mathbf{q}^\lambda \\ \mathbf{0} \end{pmatrix}, \quad (25)$$

and

$$\mathbf{b}(\mathbf{x}, t) = \begin{pmatrix} -A_{VL}^\top \mathbf{v}_s(t) - A_{CL}^\top \mathbf{u}_C^\tau - A_{IL}^\top \mathbf{u}_I^\tau - A_{YL}^\top \mathbf{u}_Y^\tau - A_{ZL}^\top \mathbf{h}^\tau - A_{UL}^\top \mathbf{v}_U^\tau(\cdot) \\ -A_{VU}^\top \mathbf{v}_s(t) - A_{CU}^\top \mathbf{u}_C^\tau - A_{IU}^\top \mathbf{u}_I^\tau - A_{YU}^\top \mathbf{u}_Y^\tau - A_{ZU}^\top \mathbf{h}^\tau - A_{UU}^\top \mathbf{v}_U^\tau(\cdot) + \mathbf{v}_U^\lambda(\cdot) \\ -A_{VZ}^\top \mathbf{v}_s(t) - A_{CZ}^\top \mathbf{u}_C^\tau - A_{IZ}^\top \mathbf{u}_I^\tau - A_{YZ}^\top \mathbf{u}_Y^\tau - A_{ZZ}^\top \mathbf{h}^\tau + \mathbf{h}^\lambda \\ \mathbf{g}^\tau + A_{YY} \mathbf{g}^\lambda + A_{YZ} \mathbf{i}_Z^\lambda + A_{YU} \mathbf{i}_U^\lambda + A_{YL} \mathbf{i}_L^\lambda + A_{YJ} \mathbf{j}_s(t) \\ \mathbf{j}_I^\tau(\cdot) + A_{II} \mathbf{j}_I^\lambda(\cdot) + A_{IY} \mathbf{g}^\lambda + A_{IZ} \mathbf{i}_Z^\lambda + A_{IU} \mathbf{i}_U^\lambda + A_{IL} \mathbf{i}_L^\lambda + A_{IJ} \mathbf{j}_s(t) \\ A_{CI} \mathbf{j}_I^\lambda(\cdot) + A_{CY} \mathbf{g}^\lambda + A_{CZ} \mathbf{i}_Z^\lambda + A_{CU} \mathbf{i}_U^\lambda + A_{CL} \mathbf{i}_L^\lambda + A_{CJ} \mathbf{j}_s(t) \end{pmatrix}.$$

By $A = AA^-A$, this gives the hybrid equation system in the form of (20).

Remark 6.2. The matrix A and the vector valued function $\mathbf{d}(\mathbf{x}, t)$ coincide with the case for the circuits without S_I and S_U [15, 16], while $\mathbf{b}(\mathbf{x}, t)$ does not.

Let us define

$$\Omega(\mathbf{x}, t) = \begin{pmatrix} O & O & O & O & O & O \\ O & L_\tau^\tau & L_\lambda^\tau & O & O & O \\ O & L_\tau^\lambda & L_\lambda^\lambda & O & O & O \\ O & O & O & C_\tau^\tau & C_\lambda^\tau & O \\ O & O & O & C_\tau^\lambda & C_\lambda^\lambda & O \\ O & O & O & O & O & O \end{pmatrix}.$$

The matrices $D(\mathbf{x}, t)$ and $M(\mathbf{x}, t)$ are given by

$$D(\mathbf{x}, t) = A^-A \begin{pmatrix} O & O & O & O & O & O \\ -L_\tau^\tau A_{LL} + L_\lambda^\tau & O & O & O & O & O \\ -L_\tau^\lambda A_{LL} + L_\lambda^\lambda & O & O & O & O & O \\ O & O & O & O & C_\tau^\tau + C_\lambda^\tau A_{CC}^\top & O \\ O & O & O & O & C_\tau^\lambda + C_\lambda^\lambda A_{CC}^\top & O \\ O & O & O & O & O & O \end{pmatrix} = A^-A \Omega(\mathbf{x}, t) A^\top \quad (26)$$

and

$$M(\mathbf{x}, t) = A\Omega(\mathbf{x}, t)A^\top = \begin{pmatrix} M_L(\mathbf{x}, t) & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & M_C(\mathbf{x}, t) \end{pmatrix}, \quad (27)$$

where

$$\begin{aligned} M_L(\mathbf{x}, t) &= A_{LL}^\top L_\tau^\top A_{LL} - A_{LL}^\top L_\lambda^\top - L_\tau^\top A_{LL} + L_\lambda^\top, \\ M_C(\mathbf{x}, t) &= C_\tau^\top + C_\lambda^\top A_{CC}^\top + A_{CC} C_\tau^\lambda + A_{CC} C_\lambda^\lambda A_{CC}^\top. \end{aligned}$$

Lemma 6.3. Under Assumption 3.4, $M_L(\mathbf{x}, t)$ and $M_C(\mathbf{x}, t)$ are positive definite.

Proof. The matrices $M_L(\mathbf{x}, t)$ and $M_C(\mathbf{x}, t)$ are expressed by

$$\begin{aligned} M_L(\mathbf{x}, t) &= \begin{pmatrix} -A_{LL}^\top & I \end{pmatrix} \begin{pmatrix} L_\tau^\top & L_\lambda^\top \\ L_\tau^\lambda & L_\lambda^\lambda \end{pmatrix} \begin{pmatrix} -A_{LL} \\ I \end{pmatrix} \quad \text{and} \\ M_C(\mathbf{x}, t) &= \begin{pmatrix} I & A_{CC} \end{pmatrix} \begin{pmatrix} C_\tau^\top & C_\lambda^\top \\ C_\tau^\lambda & C_\lambda^\lambda \end{pmatrix} \begin{pmatrix} I \\ A_{CC}^\top \end{pmatrix}. \end{aligned}$$

Since $\begin{pmatrix} L_\tau^\top & L_\lambda^\top \\ L_\tau^\lambda & L_\lambda^\lambda \end{pmatrix}$ is positive definite and $\begin{pmatrix} -A_{LL} \\ I \end{pmatrix}$ is of full column rank, $M_L(\mathbf{x}, t)$ is positive definite. Similarly, $M_C(\mathbf{x}, t)$ is positive definite. \square

Lemma 6.4. Under Assumption 3.4, $\text{im } M(\mathbf{x}, t) = \text{im } A$ and $\ker M(\mathbf{x}, t) = \ker D(\mathbf{x}, t) = \ker A^\top$ hold.

Proof. It follows from (25) and (27) that $\text{im } M(\mathbf{x}, t) = \text{im } A$, because Lemma 6.3 ensures that $M_L(\mathbf{x}, t)$ and $M_C(\mathbf{x}, t)$ are nonsingular. We now have

$$\ker M(\mathbf{x}, t) = \ker AD(\mathbf{x}, t) \supseteq \ker D(\mathbf{x}, t) = \ker A^- A\Omega(\mathbf{x}, t)A^\top \supseteq \ker A^\top.$$

Let \mathbf{z} be an element in $\ker M(\mathbf{x}, t)$. Since $M_L(\mathbf{x}, t)$ and $M_C(\mathbf{x}, t)$ are nonsingular by Lemma 6.3, \mathbf{z} is in the form of

$$\mathbf{z} = \begin{pmatrix} \mathbf{0} \\ * \\ * \\ * \\ * \\ \mathbf{0} \end{pmatrix}.$$

Hence $A^\top \mathbf{z} = \mathbf{0}$ holds, which implies $\mathbf{z} \in \ker A^\top$. Thus we obtain $\ker M(\mathbf{x}, t) = \ker D(\mathbf{x}, t) = \ker A^\top$. \square

Lemma 6.5. Under Assumption 3.4, $\ker A \cap \text{im } D(\mathbf{x}, t) = \{\mathbf{0}\}$ holds.

Proof. Let \mathbf{z} be an element in $\ker A \cap \operatorname{im} D(\mathbf{x}, t)$. Then we have $A\mathbf{z} = \mathbf{0}$ and $\mathbf{z} = D(\mathbf{x}, t)\mathbf{y}$ for some \mathbf{y} . Hence $AD(\mathbf{x}, t)\mathbf{y} = \mathbf{0}$ holds, which implies that $\mathbf{y} \in \ker AD(\mathbf{x}, t) = \ker D(\mathbf{x}, t)$ by Lemma 6.4. Thus we obtain $\mathbf{z} = D(\mathbf{x}, t)\mathbf{y} = \mathbf{0}$. \square

With the use of a reflexive generalized inverse, we define a constant projector $P = A^-A$. Then the projector P has the following property.

Lemma 6.6. Under Assumption 3.4, we have $\ker P = \ker A$ and $\operatorname{im} P = \operatorname{im} D(\mathbf{x}, t)$ for a projector $P = A^-A$.

Proof. We first prove $\ker P = \ker A$. It clearly holds that $\ker P = \ker A^-A \supseteq \ker A$. Since we have $\ker A = \ker AA^-A \supseteq \ker A^-A = \ker P$, it holds that $\ker P = \ker A$.

Secondly, we prove $\operatorname{im} P = \operatorname{im} D(\mathbf{x}, t)$. It clearly holds that $\operatorname{im} D(\mathbf{x}, t) = \operatorname{im} A^-A\Omega(\mathbf{x}, t)A^\top \subseteq \operatorname{im} P$. By Lemma 6.4, $\ker D(\mathbf{x}, t) = \ker A^\top$ holds. Hence we have

$$\dim \operatorname{im} D(\mathbf{x}, t) = m - \dim \ker D(\mathbf{x}, t) = m - \dim \ker A^\top = \dim \operatorname{im} A^\top.$$

It follows from (24) that

$$\dim \operatorname{im} A^\top = \dim \operatorname{im} A = \dim \operatorname{im} A^-A,$$

which implies that $\dim \operatorname{im} D(\mathbf{x}, t) = \dim \operatorname{im} P$. Thus we obtain $\operatorname{im} P = \operatorname{im} D(\mathbf{x}, t)$. \square

By Lemmas 6.4–6.6, we obtain the following proposition.

Proposition 6.7. Under Assumption 3.4, the hybrid equation system in the form of (20) is a DAE with properly stated leading term.

Proof. We obtain (21) by Lemmas 5.2, 6.4, and 6.5. Moreover, $P\mathbf{d}(\mathbf{x}, t) = A^-A\mathbf{d}(\mathbf{x}, t) = \mathbf{d}(\mathbf{x}, t)$ holds. Thus, by Lemma 6.6, P is a projector satisfying the conditions in Definition 5.1. \square

Let us define

$$Q = \begin{pmatrix} O & O & O & O & O & O \\ O & I & O & O & O & O \\ O & O & I & O & O & O \\ O & O & O & I & O & O \\ O & O & O & O & I & O \\ O & O & O & O & O & O \end{pmatrix}.$$

In fact, Q is a projector satisfying the condition in Proposition 5.6 as follows.

Lemma 6.8. Under Assumption 3.4, $\operatorname{im} Q = \ker M(\mathbf{x}, t)$ holds.

Proof. Since $M_L(\mathbf{x}, t)$ and $M_C(\mathbf{x}, t)$ are nonsingular by Lemma 6.3, we obtain $\ker M(\mathbf{x}, t) = \operatorname{im} Q$ by (27). \square

7 Index of Hybrid Equations

This section gives two main theorems concerning the index of the hybrid equations. We present a structural characterization for index zero in Section 7.1, and for index at most one in Section 7.2. These characterizations lead to an algorithm for determining the index of the hybrid equations, which is given in Section 7.3.

7.1 Necessary and Sufficient Condition for Index Zero

We now introduce the *Resistor-Acyclic condition* for admissible partition (E_y, E_z) , which is proved in Theorem 7.2 to be a necessary and sufficient condition for the hybrid equations with index zero.

[Resistor-Acyclic condition]

- Each resistor in Y and each dependent current source in S_I belong to a cycle consisting of independent voltage sources, capacitors, and itself.
- Each resistor in Z and each dependent voltage source in S_U belong to a cutset consisting of inductors, independent current sources, and itself.

This is an extension of the Resistor-Acyclic condition discussed in [15, 16] for the circuits without S_I and S_U . The Resistor-Acyclic condition can be expressed as follows.

Lemma 7.1. An admissible partition (E_y, E_z) satisfies the Resistor-Acyclic condition if and only if there exists a normal reference tree T such that $S_I \cup Y \subseteq \bar{T}$ and $Z \cup S_U \subseteq T$. \square

We obtain the necessary and sufficient condition for index zero as follows.

Theorem 7.2. Under Assumption 3.4, the index of the hybrid equations is zero if and only if the admissible partition (E_y, E_z) satisfies the Resistor-Acyclic condition.

Proof. The index of the hybrid equations is zero if and only if $M(\mathbf{x}, t)$ is nonsingular. Since $M_L(\mathbf{x}, t)$ and $M_C(\mathbf{x}, t)$ are nonsingular by Lemma 6.3, this is equivalent to the condition that we have no variables \mathbf{i}_Z^λ , \mathbf{i}_U^λ , and \mathbf{u}_I^τ , \mathbf{u}_Y^τ . In other words, $S_I \cup Y \subseteq \bar{T}$ and $Z \cup S_U \subseteq T$ hold. This is equivalent to the Resistor-Acyclic condition by Lemma 7.1. \square

7.2 Necessary and Sufficient Condition for Index at Most One

We now derive a necessary and sufficient condition for index at most one. Let us define

$$A_Z = \begin{pmatrix} -A_{ZU}^\top & O \\ -A_{ZZ}^\top & I \end{pmatrix}, \quad A_Y = \begin{pmatrix} I & A_{YY} \\ O & A_{IY} \end{pmatrix}, \quad \text{and} \quad N = \begin{pmatrix} A_{YU} & A_{YZ} \\ A_{IU} & A_{IZ} \end{pmatrix}.$$

Moreover, we define

$$\Lambda = \begin{pmatrix} O & -N^\top \\ N & O \end{pmatrix} - \begin{pmatrix} A_Z & O \\ O & A_Y \end{pmatrix} \begin{pmatrix} Z & H \\ G & Y \end{pmatrix} \begin{pmatrix} -A_Z^\top & O \\ O & -A_Y^\top \end{pmatrix}.$$

Proposition 5.6 leads to the following lemma.

Lemma 7.3. Under Assumption 3.4, the index of the hybrid equations is at most one if and only if Λ is nonsingular.

Proof. With $Q = \begin{pmatrix} O & O & O & O & O & O \\ O & I & O & O & O & O \\ O & O & I & O & O & O \\ O & O & O & I & O & O \\ O & O & O & O & I & O \\ O & O & O & O & O & O \end{pmatrix}$, which is a projector onto $\ker M(\mathbf{x}, t)$ by Lemma 6.8,

the computation of $M(\mathbf{x}, t) + B(\mathbf{x}, t)Q$ gives

$$M(\mathbf{x}, t) + B(\mathbf{x}, t)Q = \begin{pmatrix} M_L(\mathbf{x}, t) & * & * & O \\ O & B_{ZZ}(\mathbf{x}, t) & -N^\top + B_{ZY}(\mathbf{x}, t) & O \\ O & N + B_{YZ}(\mathbf{x}, t) & B_{YY}(\mathbf{x}, t) & O \\ O & * & * & M_C(\mathbf{x}, t) \end{pmatrix},$$

where

$$\begin{aligned} B_{ZZ}(\mathbf{x}, t) &= A_Z Z A_Z^\top, & B_{ZY}(\mathbf{x}, t) &= A_Z H A_Y^\top, \\ B_{YZ}(\mathbf{x}, t) &= A_Y G A_Z^\top, & B_{YY}(\mathbf{x}, t) &= A_Y Y A_Y^\top. \end{aligned}$$

By Proposition 5.6, the index of the hybrid equations is at most one if and only if $M(\mathbf{x}, t) + B(\mathbf{x}, t)Q$ is nonsingular. The matrix $M(\mathbf{x}, t) + B(\mathbf{x}, t)Q$ is nonsingular if and only if

$$\begin{aligned} \begin{pmatrix} B_{ZZ}(\mathbf{x}, t) & -N^\top + B_{ZY}(\mathbf{x}, t) \\ N + B_{YZ}(\mathbf{x}, t) & B_{YY}(\mathbf{x}, t) \end{pmatrix} &= \begin{pmatrix} O & -N^\top \\ N & O \end{pmatrix} + \begin{pmatrix} A_Z Z A_Z^\top & A_Z H A_Y^\top \\ A_Y G A_Z^\top & A_Y Y A_Y^\top \end{pmatrix} \\ &= \begin{pmatrix} O & -N^\top \\ N & O \end{pmatrix} - \begin{pmatrix} A_Z & O \\ O & A_Y \end{pmatrix} \begin{pmatrix} Z & H \\ G & Y \end{pmatrix} \begin{pmatrix} -A_Z^\top & O \\ O & -A_Y^\top \end{pmatrix} \end{aligned}$$

is nonsingular by Lemma 6.3. \square

We now obtain the following lemma.

Lemma 7.4. Under Assumptions 3.4 and 3.5, the index of the hybrid equations is at most one if and only if $\begin{pmatrix} A_Z & N^\top \end{pmatrix}$ and $\begin{pmatrix} N & A_Y \end{pmatrix}$ are of full row rank.

Proof. By Lemma 7.3, the index is at most one if and only if Λ is nonsingular. Under Assumption 3.5, there exists an orthogonal matrix Θ such that $\Sigma = \Theta^\top \begin{pmatrix} Z & O \\ O & Y \end{pmatrix} \Theta$ is a diagonal matrix. Then all the diagonal entries of Σ are positive. By setting

$$\tilde{A} = \begin{pmatrix} A_Z & O \\ O & A_Y \end{pmatrix} \Theta \quad \text{and} \quad \tilde{N} = \begin{pmatrix} O & -N^\top \\ N & O \end{pmatrix} + \tilde{A} \Theta^\top \begin{pmatrix} O & H \\ G & O \end{pmatrix} \Theta \tilde{A}^\top,$$

we can express Λ by $\Lambda = \tilde{N} - \tilde{A} \Sigma (-\tilde{A}^\top)$. Then Λ is nonsingular if and only if $\begin{pmatrix} \Sigma^{-1} & -\tilde{A}^\top \\ \tilde{A} & \tilde{N} \end{pmatrix}$ is nonsingular by Lemma 2.4.

In the rest of the proof, we find a necessary and sufficient condition for the nonsingularity of the matrix

$$\begin{pmatrix} \Sigma^{-1} & -\tilde{A}^\top \\ \tilde{A} & \tilde{N} \end{pmatrix} = \begin{pmatrix} O & -\tilde{A}^\top \\ \tilde{A} & \tilde{N} \end{pmatrix} + \begin{pmatrix} \Sigma^{-1} & O \\ O & O \end{pmatrix}, \quad (28)$$

which is the sum of a skew-symmetric matrix and a diagonal matrix with nonnegative entries. Let X be the row/column set of the matrix in (28), and $S \subseteq X$ be the row/column set of \tilde{N} . It follows from Lemma 2.2 that the matrix in (28) is nonsingular if and only if there exists a nonsingular principal submatrix of $\begin{pmatrix} O & -\tilde{A}^\top \\ \tilde{A} & \tilde{N} \end{pmatrix}$ containing $\begin{pmatrix} O & -\tilde{A}^\top \\ \tilde{A} & \tilde{N} \end{pmatrix} [S] = \tilde{N}$. By Lemma 2.3, this is equivalent to the condition that

$$\begin{pmatrix} O & -\tilde{A}^\top \\ \tilde{A} & \tilde{N} \end{pmatrix} [S, X] = \begin{pmatrix} \tilde{A} & \tilde{N} \end{pmatrix}$$

is of full row rank. Since we have

$$\begin{aligned} \left(\tilde{A} \mid \tilde{N} \right) &= \left(\tilde{A} \mid \begin{pmatrix} O & -N^\top \\ N & O \end{pmatrix} + \tilde{A}\Theta^\top \begin{pmatrix} O & H \\ G & O \end{pmatrix} \Theta \tilde{A}^\top \right) \xrightarrow[\text{operations}]{\text{column}} \left(\tilde{A} \mid \begin{pmatrix} O & -N^\top \\ N & O \end{pmatrix} \right) \\ &\xrightarrow[\text{operations}]{\text{column}} \left(\begin{array}{cc|cc} A_Z & O & O & -N^\top \\ O & A_Y & N & O \end{array} \right) \xrightarrow{\text{permutations}} \left(\begin{array}{cc|cc} A_Z & -N^\top & O & O \\ O & O & N & A_Y \end{array} \right), \end{aligned}$$

$\left(\tilde{A} \mid \tilde{N} \right)$ is of full row rank if and only if $\begin{pmatrix} A_Z & N^\top \end{pmatrix}$ and $\begin{pmatrix} N & A_Y \end{pmatrix}$ are of full row rank. \square

For the network graph $\Gamma = (W, E)$, *contracting* $e \in E$ means deleting e and identifying its end-vertices. Let Γ° denote the graph obtained by contracting all edges in $V \cup C$ and deleting all edges in $L \cup J$. The fundamental cutset matrix F° is given by

$$F^\circ = \begin{pmatrix} \mathbf{i}_I^\tau & \mathbf{i}_Y^\tau & \mathbf{i}_Z^\tau & \mathbf{i}_U^\tau & \mathbf{i}_I^\lambda & \mathbf{i}_Y^\lambda & \mathbf{i}_Z^\lambda & \mathbf{i}_U^\lambda \\ I & O & O & O & A_{II} & A_{IY} & A_{IZ} & A_{IU} \\ O & I & O & O & O & A_{YY} & A_{YZ} & A_{YU} \\ O & O & I & O & O & O & A_{ZZ} & A_{ZU} \\ O & O & O & I & O & O & O & A_{UU} \end{pmatrix}.$$

Then Lemma 7.4 leads to the following main theorem.

Theorem 7.5. Under Assumptions 3.4 and 3.5, the index of the hybrid equations is at most one if and only if Γ° contains neither a cycle consisting of dependent voltage sources nor a cutset consisting of dependent current sources.

Proof. By Lemma 7.4, the index is at most one if and only if $\begin{pmatrix} A_Z & N^\top \end{pmatrix}$ and $\begin{pmatrix} N & A_Y \end{pmatrix}$ are of full row rank. The condition that

$$\begin{pmatrix} A_Z & N^\top \end{pmatrix} = \begin{pmatrix} -A_{ZU}^\top & O & A_{YU}^\top & A_{IU}^\top \\ -A_{ZZ}^\top & I & A_{YZ}^\top & A_{IZ}^\top \end{pmatrix}$$

is of full row rank is equivalent to that

$$\begin{pmatrix} O & A_{IU} \\ O & A_{YU} \\ O & A_{ZU} \\ I & A_{UU} \end{pmatrix} \tag{29}$$

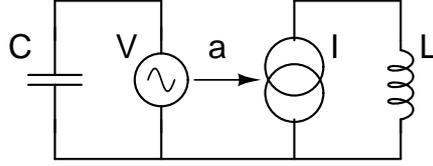


Figure 4: A circuit with a dependent current source.

is of full column rank. This is a submatrix of F° with the column set corresponding to S_U , and hence it is of full column rank if and only if Γ° contains no cycles that consist of dependent voltage sources. The condition that

$$\begin{pmatrix} N & A_Y \end{pmatrix} = \begin{pmatrix} A_{YU} & A_{YZ} & I & A_{YY} \\ A_{IU} & A_{IZ} & O & A_{IY} \end{pmatrix}$$

is of full row rank is equivalent to that

$$\begin{pmatrix} O & O & O & A_{IY} & A_{IZ} & A_{IU} \\ I & O & O & A_{YY} & A_{YZ} & A_{YU} \\ O & I & O & O & A_{ZZ} & A_{ZU} \\ O & O & I & O & O & A_{UU} \end{pmatrix} \quad (30)$$

is of full row rank. This is a submatrix of F° with the column set corresponding to $Y \cup Z \cup S_U$, and hence it is of full row rank if and only if Γ° has a spanning forest consisting of edges in Y , Z , and S_U , which condition is equivalent to that Γ° contains no cutsets consisting of dependent current sources. \square

Theorem 7.2 implies that an admissible partition that leads to the hybrid equations with index zero is unique if exists. On the other hand, Theorem 7.5 indicates that it does not depend on the choice of an admissible partition whether the index exceeds one or not. Moreover, Theorem 7.5 leads to the statement that the index of hybrid equations is at most one for the circuits without S_I and S_U , which is proved in [15, 16].

Example 7.6 ([8]). Consider a circuit depicted in Figure 4, which contains a dependent current source I . While MNA results in a DAE with index three [8], the hybrid analysis with admissible partition

$$E_g = \{V\}, \quad E_h = \emptyset, \quad E_y = \{C, I\}, \quad E_z = \{L\}$$

results in a DAE with index two [14].

Let us check that this circuit does not satisfy the condition in Theorem 7.5. Figure 5 shows the network graph Γ of this circuit. After deleting the edge L and contracting V and C , we obtain the graph Γ° given in Figure 6. Since the resulting graph has the cutset I , Γ° contains a cutset consisting of a dependent current source.

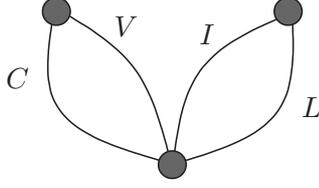


Figure 5: The graph Γ of Example 7.6.

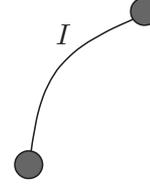


Figure 6: The graph Γ° of Example 7.6.

7.3 Algorithm for Index Determination

We now provide an algorithm for determining the index of the hybrid equations. The correctness of this algorithm follows from Theorems 7.2 and 7.5.

A *coloop* is an edge whose deletion increases the number of connected components in the graph. The following algorithm first determines whether the minimum index ν is at most one or not. If it turns out to be at most one, the algorithm further checks whether it can be zero for some choice of an admissible partition. If it can, the algorithm also finds such an admissible partition (E_y, E_z) .

Algorithm for index determination

Step 1 Set $E_y \leftarrow \{e \mid e \in C \cup S_I\}$ and $E_z \leftarrow \{e \mid e \in S_U \cup L\}$.

Step 2 Contract all edges in $V \cup C$ and delete all edges in $L \cup J$ from $\Gamma = (W, E)$. Then we obtain graph Γ° .

Step 3 If Γ° contains a cycle consisting of dependent voltage sources S_U or a cutset consisting of dependent current sources S_I , then return $\nu \geq 2$ and halt.

Step 4 If Γ° satisfies at least one of the following three conditions, then return $\nu = 1$ and halt:

- S_I does not consist of selfloops,
- S_U does not consist of coloops,
- resistors form a cycle except selfloops.

Step 5 Set $E_y \leftarrow E_y \cup \{e \mid e : \text{selfloop of a resistor}\}$ and $E_z \leftarrow E_z \setminus E_y$. Return $\nu = 0$ and (E_y, E_z) , and halt.

This is an extension of *Algorithm for index minimization in RLC circuit* given in [30]. The algorithm runs in linear time in $|E|$, i.e., the number of elements in the circuit.

8 Conclusion

For nonlinear time-varying circuits composed of resistors, inductors, capacitors, independent voltage/current sources, and dependent voltage/current sources, we have given structural characterizations for the index of the hybrid equations. This enables us to determine efficiently whether the hybrid equations have higher index or not, which helps to avoid solving higher index DAEs in circuit simulation.

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A Derivation of Hybrid Equations

We derive the hybrid equations. For ease in notation, let us omit the time argument t . Moreover, we use

$$\begin{aligned}
\mathbf{q}^\tau &= \mathbf{q}^\tau(\mathbf{u}_C^\tau, A_{VC}^\top \mathbf{v}_s(t) + A_{CC}^\top \mathbf{u}_C^\tau, t), \\
\mathbf{q}^\lambda &= \mathbf{q}^\lambda(\mathbf{u}_C^\tau, A_{VC}^\top \mathbf{v}_s(t) + A_{CC}^\top \mathbf{u}_C^\tau, t), \\
\mathbf{g}^\tau &= \mathbf{g}^\tau(-A_{ZZ} \mathbf{i}_Z^\lambda - A_{ZU} \mathbf{i}_U^\lambda - A_{ZL} \mathbf{i}_L^\lambda - A_{ZJ} \mathbf{j}_s(t), \mathbf{i}_Z^\lambda, \mathbf{u}_Y^\tau, A_{VY}^\top \mathbf{v}_s(t) + A_{CY}^\top \mathbf{u}_C^\tau + A_{IY}^\top \mathbf{u}_I^\tau + A_{YY}^\top \mathbf{u}_Y^\tau, t), \\
\mathbf{g}^\lambda &= \mathbf{g}^\lambda(-A_{ZZ} \mathbf{i}_Z^\lambda - A_{ZU} \mathbf{i}_U^\lambda - A_{ZL} \mathbf{i}_L^\lambda - A_{ZJ} \mathbf{j}_s(t), \mathbf{i}_Z^\lambda, \mathbf{u}_Y^\tau, A_{VY}^\top \mathbf{v}_s(t) + A_{CY}^\top \mathbf{u}_C^\tau + A_{IY}^\top \mathbf{u}_I^\tau + A_{YY}^\top \mathbf{u}_Y^\tau, t), \\
\mathbf{h}^\tau &= \mathbf{h}^\tau(-A_{ZZ} \mathbf{i}_Z^\lambda - A_{ZU} \mathbf{i}_U^\lambda - A_{ZL} \mathbf{i}_L^\lambda - A_{ZJ} \mathbf{j}_s(t), \mathbf{i}_Z^\lambda, \mathbf{u}_Y^\tau, A_{VY}^\top \mathbf{v}_s(t) + A_{CY}^\top \mathbf{u}_C^\tau + A_{IY}^\top \mathbf{u}_I^\tau + A_{YY}^\top \mathbf{u}_Y^\tau, t), \\
\mathbf{h}^\lambda &= \mathbf{h}^\lambda(-A_{ZZ} \mathbf{i}_Z^\lambda - A_{ZU} \mathbf{i}_U^\lambda - A_{ZL} \mathbf{i}_L^\lambda - A_{ZJ} \mathbf{j}_s(t), \mathbf{i}_Z^\lambda, \mathbf{u}_Y^\tau, A_{VY}^\top \mathbf{v}_s(t) + A_{CY}^\top \mathbf{u}_C^\tau + A_{IY}^\top \mathbf{u}_I^\tau + A_{YY}^\top \mathbf{u}_Y^\tau, t), \\
\phi^\tau &= \phi^\tau(-A_{LL} \mathbf{i}_L^\lambda - A_{LJ} \mathbf{j}_s(t), \mathbf{i}_L^\lambda, t), \\
\phi^\lambda &= \phi^\lambda(-A_{LL} \mathbf{i}_L^\lambda - A_{LJ} \mathbf{j}_s(t), \mathbf{i}_L^\lambda, t)
\end{aligned}$$

for convenience.

We first transform (7). By substituting (3), (4), and (5), we obtain

$$\begin{aligned}
\frac{d}{dt} \mathbf{q}^\tau(\mathbf{u}_C^\tau, \mathbf{u}_C^\lambda) + A_{CC} \frac{d}{dt} \mathbf{q}^\lambda(\mathbf{u}_C^\tau, \mathbf{u}_C^\lambda) + A_{CI} \mathbf{j}_I^\lambda(\cdot) + A_{CY} \mathbf{g}^\lambda(\mathbf{i}_Z^\tau, \mathbf{i}_Z^\lambda, \mathbf{u}_Y^\tau, \mathbf{u}_Y^\lambda) \\
+ A_{CZ} \mathbf{i}_Z^\lambda + A_{CU} \mathbf{i}_U^\lambda + A_{CL} \mathbf{i}_L^\lambda + A_{CJ} \mathbf{j}_s = \mathbf{0}.
\end{aligned}$$

By substituting (2), (10), (13), and (15), we have

$$\frac{d}{dt} \mathbf{q}^\tau + A_{CC} \frac{d}{dt} \mathbf{q}^\lambda + A_{CI} \mathbf{j}_I^\lambda(\cdot) + A_{CY} \mathbf{g}^\lambda + A_{CZ} \mathbf{i}_Z^\lambda + A_{CU} \mathbf{i}_U^\lambda + A_{CL} \mathbf{i}_L^\lambda + A_{CJ} \mathbf{j}_s(t) = \mathbf{0}. \quad (31)$$

By transforming (8)–(9) in a similar way, we obtain

$$\mathbf{j}_I^\tau(\cdot) + A_{II} \mathbf{j}_I^\lambda(\cdot) + A_{IY} \mathbf{g}^\lambda + A_{IZ} \mathbf{i}_Z^\lambda + A_{IU} \mathbf{i}_U^\lambda + A_{IL} \mathbf{i}_L^\lambda + A_{IJ} \mathbf{j}_s(t) = \mathbf{0}, \quad (32)$$

$$\mathbf{g}^\tau + A_{YY} \mathbf{g}^\lambda + A_{YZ} \mathbf{i}_Z^\lambda + A_{YU} \mathbf{i}_U^\lambda + A_{YL} \mathbf{i}_L^\lambda + A_{YJ} \mathbf{j}_s(t) = \mathbf{0}. \quad (33)$$

Next, we transform (18). By substituting (3), (4), and (5), we obtain

$$\begin{aligned}
-A_{VL}^\top \mathbf{u}_V - A_{CL}^\top \mathbf{u}_C^\tau - A_{IL}^\top \mathbf{u}_I^\tau - A_{YL}^\top \mathbf{u}_Y^\tau - A_{ZL}^\top \mathbf{h}^\tau(\mathbf{i}_Z^\tau, \mathbf{i}_Z^\lambda, \mathbf{u}_Y^\tau, \mathbf{u}_Y^\lambda) \\
- A_{UL}^\top \mathbf{v}_U^\tau(\cdot) - A_{LL}^\top \frac{d}{dt} \phi^\tau(\mathbf{i}_L^\tau, \mathbf{i}_L^\lambda) + \frac{d}{dt} \phi^\lambda(\mathbf{i}_L^\tau, \mathbf{i}_L^\lambda) = \mathbf{0}.
\end{aligned}$$

By substituting (2), (10), (12), and (15), we have

$$-A_{VL}^\top \mathbf{v}_s(t) - A_{CL}^\top \mathbf{u}_C^\tau - A_{IL}^\top \mathbf{u}_I^\tau - A_{YL}^\top \mathbf{u}_Y^\tau - A_{ZL}^\top \mathbf{h}^\tau - A_{UL}^\top \mathbf{v}_U^\tau(\cdot) - A_{LL}^\top \frac{d}{dt} \phi^\tau + \frac{d}{dt} \phi^\lambda = \mathbf{0}. \quad (34)$$

By transforming (16)–(17) in a similar way, we obtain

$$-A_{VZ}^\top \mathbf{v}_s(t) - A_{CZ}^\top \mathbf{u}_C^\tau - A_{IZ}^\top \mathbf{u}_I^\tau - A_{YZ}^\top \mathbf{u}_Y^\tau - A_{ZZ}^\top \mathbf{h}^\tau + \mathbf{h}^\lambda = \mathbf{0}, \quad (35)$$

$$-A_{VU}^\top \mathbf{v}_s(t) - A_{CU}^\top \mathbf{u}_C^\tau - A_{IU}^\top \mathbf{u}_I^\tau - A_{YU}^\top \mathbf{u}_Y^\tau - A_{ZU}^\top \mathbf{h}^\tau - A_{UU}^\top \mathbf{v}_U^\tau(\cdot) + \mathbf{v}_U^\lambda(\cdot) = \mathbf{0}. \quad (36)$$

Thus, by (31)–(36), we obtain the hybrid equations

$$\begin{aligned}
\frac{d}{dt}\mathbf{q}^\tau + A_{CC}\frac{d}{dt}\mathbf{q}^\lambda + A_{CI}\mathbf{j}_I^\lambda(\cdot) + A_{CY}\mathbf{g}^\lambda + A_{CZ}\mathbf{i}_Z^\lambda + A_{CU}\mathbf{i}_U^\lambda + A_{CL}\mathbf{i}_L^\lambda + A_{CJ}\mathbf{j}_s(t) &= \mathbf{0}, \\
\mathbf{j}_I^\tau(\cdot) + A_{II}\mathbf{j}_I^\lambda(\cdot) + A_{IY}\mathbf{g}^\lambda + A_{IZ}\mathbf{i}_Z^\lambda + A_{IU}\mathbf{i}_U^\lambda + A_{IL}\mathbf{i}_L^\lambda + A_{IJ}\mathbf{j}_s(t) &= \mathbf{0}, \\
\mathbf{g}^\tau + A_{YY}\mathbf{g}^\lambda + A_{YZ}\mathbf{i}_Z^\lambda + A_{YU}\mathbf{i}_U^\lambda + A_{YL}\mathbf{i}_L^\lambda + A_{YJ}\mathbf{j}_s(t) &= \mathbf{0}, \\
-A_{VL}^\top\mathbf{v}_s(t) - A_{CL}^\top\mathbf{u}_C^\tau - A_{IL}^\top\mathbf{u}_I^\tau - A_{YL}^\top\mathbf{u}_Y^\tau - A_{ZL}^\top\mathbf{h}^\tau - A_{UL}^\top\mathbf{v}_U^\tau(\cdot) - A_{LL}^\top\frac{d}{dt}\phi^\tau + \frac{d}{dt}\phi^\lambda &= \mathbf{0}, \\
-A_{VZ}^\top\mathbf{v}_s(t) - A_{CZ}^\top\mathbf{u}_C^\tau - A_{IZ}^\top\mathbf{u}_I^\tau - A_{YZ}^\top\mathbf{u}_Y^\tau - A_{ZZ}^\top\mathbf{h}^\tau + \mathbf{h}^\lambda &= \mathbf{0}, \\
-A_{VU}^\top\mathbf{v}_s(t) - A_{CU}^\top\mathbf{u}_C^\tau - A_{IU}^\top\mathbf{u}_I^\tau - A_{YU}^\top\mathbf{u}_Y^\tau - A_{ZU}^\top\mathbf{h}^\tau - A_{UU}^\top\mathbf{v}_U^\tau(\cdot) + \mathbf{v}_U^\lambda(\cdot) &= \mathbf{0}.
\end{aligned}$$