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Abstract

In this paper, two improved versions of an existing Sinc-collocation scheme for the Fredholm integral equations of the second kind are presented. The first version is obtained by improving the existing scheme so that it becomes more practical, and natural from a theoretical view point. Then it is rigorously proved that the convergence rate of the modified scheme is $\mathrm{O}(\exp(-c\sqrt{N}))$, as suggested in the literature. In the second version, the variable transformation employed in the original scheme, the "tanh transformation," is replaced with the "double exponential transformation." It is proved that the replacement improves the convergence rate to $\mathrm{O}(\exp(-cN/\log N))$. Numerical examples which support the theoretical results are also given.

1 Introduction

Fredholm integral equations of the second kind take the form

$$u(x) - \lambda \int_a^b k(x,t)u(t)\,\mathrm{d}t = g(x), \quad a \leq x \leq b, \tag{1.1}$$

where λ is a given constant, $g(x)$ and $k(x,t)$ are given continuous functions, and $u(x)$ is the solution to be determined. Since the equations are important in application, theory and numerical methods for the equations have been studied by many authors, which are reviewed in some books [2–7]. Most numerical methods in the literature have been focused on the cases where the functions to be approximated are differentiable on the whole interval $[a,\, b]$, and do not work well when the functions behave badly at the endpoints (see, for example, Delves–Mohamed [3, Example 4.2.5]).

The Sinc-collocation method recently reported by Rashidinia–Zarebnia [12] surmounts the difficulty. Based on the Sinc approximation, they have derived their scheme without assuming the smoothness at the endpoints, and they intuitively argued that the scheme can converge *exponentially*, i.e., $\mathrm{O}(\exp(-c_1\sqrt{N}))$. In fact, they confirmed the exponential convergence in some cases by numerical experiments.

The main purpose of this paper is to improve the Rashidinia–Zarebnia scheme (RZ scheme) by proposing two alternative schemes. The first scheme is obtained by modifying the RZ scheme in the following two points (we call it the "modified RZ scheme" throughout this paper). First, we reformulate the basis functions of the approximate solution and collocation points in the scheme. In the RZ scheme, the basis functions are selected depending on the values of the solution u at the endpoints, i.e. $u(a)$ and $u(b)$. In a practical situation, however, it is hard to obtain the values in prior to the computation, since the solution u is an *unknown* function to be determined. To remedy the issue, we take the same approach as Stenger [16], where basis functions are fixed in any cases. In connection with this modification, we also modify the collocation points. Second, we give a concrete and optimal way to select the mesh size h. In the RZ scheme, it is selected based on d and α, which are the smoothness parameters of the unknown solution u (thus, again, they are essentially unknown before the computation). Furthermore, granted that they are somehow known, the choice of h in the RZ scheme (based on d and α) is not optimal. In this improvement, we utilize the same approach as in [11], which have dealt with weakly-singular cases, and then show that d and α can be determined in prior to the computation based on the given functions g and k. Based on these two improvement, we finally prove in a rigorous manner that the convergence rate of the modified RZ scheme is in fact $\mathrm{O}(\exp(-c_1\sqrt{N}))$.

Based on the modified RZ scheme, we propose the second scheme that can achieve faster convergence. The scheme is derived by replacing the variable transformation employed in the (modified) RZ scheme, which is the standard "tanh transformation," with the so-called "double exponential transformation." In a wide range of numerical analysis (such as function approximations and quadratures) it is known that such a replacement can accelerate the rate of convergence (see, for example, [8,18]), and the observation encourages us to employ the transformation here as well. In fact, it turns out both theoretically and numerically that the scheme with the double exponential transformation enjoys the notably faster rate of convergence: $\mathrm{O}(\exp(-c_2 N/\log N))$. We call this scheme the DE-Sinc scheme.

This paper is organized as follows. We first summarize theoretical results of Sinc methods in Section 2. In Section 3, we briefly review the RZ scheme and sketch our ideas of improvement. Section 4 is a preliminary section where the smoothness properties of the solution u are investigated; the information is required in the subsequent sections. The modified RZ scheme is derived in Section 5, and its convergence analysis is given in Section 6. The results are extended to the DE-Sinc scheme in Section 7 and 8. Numerical experiments are shown in Section 9. Finally in Section 10 we conclude this paper.

2 Basic definitions and theorems of Sinc methods

2.1 Sinc approximation and Sinc quadrature on the whole real line

Sinc methods are based on the Sinc approximation on the whole real line, expressed as

$$F(\xi) \approx \sum_{j=-N}^{N} F(jh) S(j,h)(\xi), \quad \xi \in \mathbb{R}, \tag{2.1}$$

where h is the mesh size, suitably selected depending on properties of the function F and a given positive integer N. Here $S(j,h)(x)$ denotes the so-called Sinc function defined by

$$S(j,h)(\xi) = \frac{\sin \pi(\xi/h - j)}{\pi(\xi/h - j)}. \tag{2.2}$$

The quadrature rule (Sinc quadrature) can be derived by integrating both sides of (2.1) as follows:

$$\int_{-\infty}^{\infty} F(\xi)\,\mathrm{d}\xi \approx \sum_{j=-N}^{N} F(jh) \int_{-\infty}^{\infty} S(j,h)(\xi)\,\mathrm{d}\xi = h \sum_{j=-N}^{N} F(jh), \tag{2.3}$$

which is actually nothing but the (truncated) trapezoidal formula.

2.2 SE-Sinc approximation and SE-Sinc quadrature on the finite interval

The variable x in the equation (1.1) is limited to the finite interval $(a,\, b)$, whereas ξ in (2.1) moves on $\mathbb{R}$. In such a case, the tanh transformation is frequently used [16, 17]:

$$x = \psi^{\mathrm{SE}}(\xi) = \frac{b-a}{2}\tanh\left(\frac{\xi}{2}\right) + \frac{b+a}{2}, \tag{2.4}$$

which maps $\mathbb{R}$ onto $(a,\, b)$. This transformation is also called the single exponential transformation, and accordingly we call it the SE transformation in what follows. The inverse map is

$$\xi = \{\psi^{\mathrm{SE}}\}^{-1}(x) = \log\left(\frac{x-a}{b-x}\right). \tag{2.5}$$

Incorporated with the SE transformation, the Sinc approximation (2.1) can be applied to the function f defined on the finite interval $(a,\, b)$ as follows:

$$f(\psi^{\mathrm{SE}}(\xi)) = \sum_{j=-N}^{N} f(\psi^{\mathrm{SE}}(jh)) S(j,h)(\xi), \quad \xi \in \mathbb{R}, \tag{2.6}$$

which is equivalent to:

$$f(x) = \sum_{j=-N}^{N} f(\psi^{\mathrm{SE}}(jh)) S(j,h)(\{\psi^{\mathrm{SE}}\}^{-1}(x)), \quad x \in (a,\, b). \tag{2.7}$$

We call this approximation the SE-Sinc approximation. Besides, the Sinc quadrature (2.3) can be applied to the integral over $(a,\, b)$ by using the SE transformation as follows:

$$\int_a^b f(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} f(\psi^{\mathrm{SE}}(\xi))\{\psi^{\mathrm{SE}}\}'(\xi)\,\mathrm{d}\xi \approx h \sum_{j=-N}^{N} f(\psi^{\mathrm{SE}}(jh))\{\psi^{\mathrm{SE}}\}'(jh). \tag{2.8}$$

We call this approximation the SE-Sinc quadrature. In order to state the error analysis of these approximations, we here introduce the following function space.

Definition 2.1. Let α be a positive constant, and let $\mathscr{D}$ be a bounded and simply-connected domain which satisfies $(a,\, b) \subset \mathscr{D}$. Then $\mathbf{L}_{\alpha}(\mathscr{D})$ denotes the family of functions f that satisfy the following conditions: (i) f is analytic in $\mathscr{D}$; (ii) there exists a constant C such that for all z in $\mathscr{D}$

$$|f(z)| \leq C|Q(z)|^{\alpha}, \tag{2.9}$$

where the function Q is defined by $Q(z) = (z-a)(b-z)$.

When the SE transformation is utilized, the domain $\mathscr{D}$ in Definition 2.1 should be eye-shaped region: $\psi^{\mathrm{SE}}(\mathscr{D}_d) = \{z = \psi^{\mathrm{SE}}(\zeta) : \zeta \in \mathscr{D}_d\}$, where $\mathscr{D}_d$ is a strip domain defined by $\mathscr{D}_d = \{\zeta \in \mathbb{C} : |\operatorname{Im}\zeta| < d\}$ for a positive constant d (cf. Stenger [16, Figure 1.7.4c]). Then convergence theorems of the above approximations are described as follows.

Theorem 2.2 (Stenger [16, Theorem 4.2.5])**.** Let $f \in \mathbf{L}_\alpha(\psi^{\mathrm{SE}}(\mathscr{D}_d))$ for d with $0 < d < \pi$, let N be a positive integer, and let h be selected by the formula

$$h = \sqrt{\frac{\pi d}{\alpha N}}. \tag{2.10}$$

Then there exists a constant C which is independent of N, such that

$$\max_{a \le x \le b} \left| f(x) - \sum_{j=-N}^{N} f(\psi^{\mathrm{SE}}(jh)) S(j,h)(\{\psi^{\mathrm{SE}}\}^{-1}(x)) \right| \le C\sqrt{N}\,\mathrm{e}^{-\sqrt{\pi d \alpha N}}. \tag{2.11}$$

Remark 2.3. The definition of $\mathbf{L}_\alpha(\mathscr{D})$ in Definition 2.1 is slightly different from the one in Stenger [16, Definition 4.1.1], but they are equivalent if $\mathscr{D} = \psi^{\mathrm{SE}}(\mathscr{D}_d)$ and $0 < d < \pi$ [16, p. 189]. In this paper we use Definition 2.1 since it is also convenient for the case of the DE transformation, which is described later.

Theorem 2.4 (Stenger [16, Theorem 4.2.6])**.** Let $(fQ) \in \mathbf{L}_\alpha(\psi^{\mathrm{SE}}(\mathscr{D}_d))$ for d with $0 < d < \pi$, let N be a positive integer, and let $\tilde{h}$ be selected by the formula

$$\tilde{h} = \sqrt{\frac{2\pi d}{\alpha N}}. \tag{2.12}$$

Then there exists a constant C which is independent of N, such that

$$\left| \int_a^b f(t)\,\mathrm{d}t - \tilde{h} \sum_{j=-N}^{N} f(\psi^{\mathrm{SE}}(j\tilde{h})) \{\psi^{\mathrm{SE}}\}'(j\tilde{h}) \right| \le C\,\mathrm{e}^{-\sqrt{2\pi d \alpha N}}. \tag{2.13}$$

The mesh sizes h and $\tilde{h}$ in Theorem 2.2 and 2.4 are optimal in each approximation (see the references for this optimality). Notice that they are different. In the Sinc-collocation method described later, however, they should coincide in order to utilize the same collocation points. In order to realize this we have two options: either use h of (2.10) in common, or $\tilde{h}$ of (2.12).

If we take the first option, Theorem 2.4 should be modified as follows. The convergence order becomes worse, but it is still better than the order in (2.11), and thus the latter becomes dominant when two approximations are combined.

Corollary 2.5 (Okayama et al. [11, Corollary 2.8])**.** Let $(fQ) \in \mathbf{L}_\alpha(\psi^{\mathrm{SE}}(\mathscr{D}_d))$ for d with $0 < d < \pi$, let N be a positive integer, and let h be selected by the formula (2.10). Then there exists a constant C which is independent of N, such that

$$\left| \int_a^b f(t)\,\mathrm{d}t - h \sum_{j=-N}^{N} f(\psi^{\mathrm{SE}}(jh)) \{\psi^{\mathrm{SE}}\}'(jh) \right| \le C\,\mathrm{e}^{-\sqrt{\pi d \alpha N}}. \tag{2.14}$$

In contrast, in the second option we need a modification of Theorem 2.2 as follows; it turns out that the convergence rate is the worst of all the preceding results. The proof is straightforward by taking $\tilde{h}$ of (2.12) in the proof of Theorem 2.2. Thus we see that the first option is clearly better.

Corollary 2.6. Let $f \in \mathbf{L}_\alpha(\psi^{\mathrm{SE}}(\mathscr{D}_d))$ for d with $0 < d < \pi$, let N be a positive integer, and let $\tilde{h}$ be selected by the formula (2.12). Then there exists a constant C which is independent of N, such that

$$\max_{a \le x \le b} \left| f(x) - \sum_{j=-N}^{N} f(\psi^{\mathrm{SE}}(j\tilde{h})) S(j,\tilde{h})(\{\psi^{\mathrm{SE}}\}^{-1}(x)) \right| \le C\,\mathrm{e}^{-\sqrt{\pi d \alpha N/2}}. \tag{2.15}$$

Remark 2.7. It is stated in Rashidinia–Zarebnia [12, Theorem 1] that the convergence rate is $\mathrm{O}(\mathrm{e}^{-\sqrt{2\pi d\alpha N}})$ with $\tilde{h}$ of (2.12), but we believe the actual order is a bit worse. As it is clarified in the above discussion, the dominant error would be $\mathrm{O}(\mathrm{e}^{-\sqrt{\pi d\alpha N/2}})$ when $\tilde{h}$ of (2.12) is employed. This can be confirmed by numerical experiments in Section 9.

2.3 DE-Sinc approximation and DE-Sinc quadrature on the finite interval

In order to improve the convergence rate of the approximations above, the so-called double exponential transformation has been proposed (see, for example, [8, 18]):

$$x = \psi^{\mathrm{DE}}(\xi) = \frac{b-a}{2}\tanh\left(\frac{\pi}{2}\sinh(\xi)\right) + \frac{b+a}{2}, \tag{2.16}$$

which also maps $\mathbb{R}$ onto (a, b). We abbreviate it as the DE transformation. The inverse map is

$$\xi = \{\psi^{\mathrm{DE}}\}^{-1}(x) = \log\left[\frac{1}{\pi}\log\left(\frac{x-a}{b-x}\right) + \sqrt{1+\left\{\frac{1}{\pi}\log\left(\frac{x-a}{b-x}\right)\right\}^2}\right]. \tag{2.17}$$

If the SE transformation is replaced with the DE transformation in (2.7) and (2.8), we call them the DE-Sinc approximation and the DE-Sinc quadrature, respectively. In these cases, the domain $\mathscr{D}$ in Definition 2.1 should be $\psi^{\mathrm{DE}}(\mathscr{D}_d) = \{z = \psi^{\mathrm{DE}}(\zeta) : \zeta \in \mathscr{D}_d\}$ (cf. Tanaka et al. [19, Figure 5]), and their convergence has been analyzed as stated below.

Theorem 2.8 (Tanaka et al. [19, Theorem 3.1])**.** Let $f \in \mathbf{L}_\alpha(\psi^{\mathrm{DE}}(\mathscr{D}_d))$ for d with $0 < d < \pi/2$, let N be a positive integer, and let h be selected by the formula

$$h = \frac{\log(2dN/\alpha)}{N}. \tag{2.18}$$

Then there exists a constant C which is independent of N, such that

$$\max_{a\leq x\leq b}\left|f(x) - \sum_{j=-N}^{N} f(\psi^{\mathrm{DE}}(jh))S(j,h)(\{\psi^{\mathrm{DE}}\}^{-1}(x))\right| \leq C\exp\left\{\frac{-\pi dN}{\log(2dN/\alpha)}\right\}. \tag{2.19}$$

Theorem 2.9 (Tanaka et al. [20, Theorem 3.1])**.** Let $(fQ) \in \mathbf{L}_\alpha(\psi^{\mathrm{DE}}(\mathscr{D}_d))$ for d with $0 < d < \pi/2$, let N be a positive integer, and let $\tilde{h}$ be selected by the formula

$$\tilde{h} = \frac{\log(4dN/\alpha)}{N}. \tag{2.20}$$

Then there exists a constant C which is independent of N, such that

$$\left|\int_a^b f(t)\,\mathrm{d}t - \tilde{h}\sum_{j=-N}^{N} f(\psi^{\mathrm{DE}}(j\tilde{h}))\{\psi^{\mathrm{DE}}\}'(j\tilde{h})\right| \leq \tilde{C}\exp\left\{\frac{-2\pi dN}{\log(4dN/\alpha)}\right\}. \tag{2.21}$$

The mesh size in Theorem 2.9 is also different from the one in Theorem 2.8. In the case of the same mesh size, the next assertion holds.

Corollary 2.10 (Okayama et al. [11, Corollary 2.9])**.** Let $(fQ) \in \mathbf{L}_\alpha(\psi^{\mathrm{DE}}(\mathscr{D}_d))$ for d with $0 < d < \pi/2$, let N be a positive integer, and let h be selected by the formula (2.18). Then there exists a constant C which is independent of N, such that

$$\left|\int_a^b f(t)\,\mathrm{d}t - h\sum_{j=-N}^{N} f(\psi^{\mathrm{DE}}(jh))\{\psi^{\mathrm{DE}}\}'(jh)\right| \leq C\exp\left\{\frac{-2\pi dN}{\log(2dN/\alpha)}\right\}. \tag{2.22}$$

2.4 Generalized SE/DE-Sinc approximation on the finite interval

According to Theorem 2.2 and 2.8, the function to be approximated by the SE/DE-Sinc approximation should belong to $\mathbf{L}_\alpha(\mathscr{D})$ in order to achieve exponential convergence. By the condition (2.9), such a function is required to be zero at the endpoints, $x = a$ and $x = b$, which seems to be an impractical assumption. However, actually it can be relaxed to the following function space $\mathbf{M}_\alpha(\mathscr{D})$.

Definition 2.11. Let $\mathscr{D}$ be a bounded and simply-connected domain. Then we denote by $\mathbf{HC}(\mathscr{D})$ the family of all functions that are analytic in $\mathscr{D}$ and continuous on $\overline{\mathscr{D}}$. The function space is complete with the norm $\|\cdot\|_{\mathbf{HC}(\mathscr{D})}$ defined by $\|f\|_{\mathbf{HC}(\mathscr{D})} = \max_{z\in\overline{\mathscr{D}}}|f(z)|$.

Definition 2.12. Let α be a constant with $0 < \alpha \leq 1$ and let $\mathscr{D}$ be a bounded and simply-connected domain which satisfies $(a, b) \subset \mathscr{D}$. Then the space $\mathbf{M}_\alpha(\mathscr{D})$ consists of all functions f that satisfy the following conditions: (i) $f \in \mathbf{HC}(\mathscr{D})$; (ii) there exists a constant C for all z in $\mathscr{D}$ such that

$$|f(z) - f(a)| \leq C|z - a|^\alpha, \tag{2.23}$$

$$|f(b) - f(z)| \leq C|b - z|^\alpha. \tag{2.24}$$

As pointed out in Stenger [16, p. 190], the translated function

$$\mathcal{T}[f](x) = f(x) - \frac{(b - x)f(a) + (x - a)f(b)}{b - a} \tag{2.25}$$

belongs to $\mathbf{L}_\alpha(\mathscr{D})$ if $f \in \mathbf{M}_\alpha(\mathscr{D})$. Then, if $f \in \mathbf{M}_\alpha(\psi^{\mathrm{SE}}(\mathscr{D}_d))$, we can apply the SE-Sinc approximation to the function $\mathcal{T}f$ as:

$$\mathcal{T}[f](x) \approx \sum_{j=-N}^{N} \mathcal{T}[f](\psi^{\mathrm{SE}}(jh))S(j, h)(\{\psi^{\mathrm{SE}}\}^{-1}(x)). \tag{2.26}$$

It can also be represented as

$$f(x) \approx \mathcal{P}_N^{\mathrm{SE}}[f](x) = f(a)w_a(x) + \sum_{j=-N}^{N} \mathcal{T}[f](\psi^{\mathrm{SE}}(jh))S(j, h)(\{\psi^{\mathrm{SE}}\}^{-1}(x)) + f(b)w_b(x), \tag{2.27}$$

where w_a and w_b are auxiliary basis functions defined by

$$w_a(x) = \frac{b - x}{b - a}, \quad w_b(x) = \frac{x - a}{b - a}. \tag{2.28}$$

In this paper we call the approximation (2.27) the generalized SE-Sinc approximation. In the same manner, if $f \in \mathbf{M}_\alpha(\psi^{\mathrm{DE}}(\mathscr{D}_d))$, we can derive the generalized DE-Sinc approximation:

$$f(x) \approx \mathcal{P}_N^{\mathrm{DE}}[f](x) = f(a)w_a(x) + \sum_{j=-N}^{N} \mathcal{T}[f](\psi^{\mathrm{DE}}(jh))S(j, h)(\{\psi^{\mathrm{DE}}\}^{-1}(x)) + f(b)w_b(x). \tag{2.29}$$

We can obtain the convergence theorems for each case, which correspond to Theorem 2.2 and Theorem 2.8.

Theorem 2.13. Let $f \in \mathbf{M}_\alpha(\psi^{\mathrm{SE}}(\mathscr{D}_d))$ for d with $0 < d < \pi$, let N be a positive integer, and let h be selected by the formula (2.10). Then there exists a constant C which is independent of N, such that

$$\|f - \mathcal{P}_N^{\mathrm{SE}} f\|_{C([a,b])} \leq C\sqrt{N}\,\mathrm{e}^{-\sqrt{\pi d\alpha N}}. \tag{2.30}$$

Theorem 2.14. Let $f \in \mathbf{M}_\alpha(\psi^{\mathrm{DE}}(\mathscr{D}_d))$ for d with $0 < d < \pi/2$, let N be a positive integer, and let h be selected by the formula (2.18). Then there exists a constant C which is independent of N, such that

$$\|f - \mathcal{P}_N^{\mathrm{DE}} f\|_{C([a,\,b])} \leq C \exp\left\{\frac{-\pi d N}{\log(2dN/\alpha)}\right\}. \tag{2.31}$$

3 Review of the Rashidinia–Zarebnia scheme and ideas to improve it

After briefly describing the Rashidinia–Zarebnia (RZ) scheme, we sketch our ideas for improvement.

3.1 The RZ scheme

Rashidinia–Zarebnia [12] have assumed the solution of (1.1) belongs to $\mathbf{L}_\alpha(\psi^{\mathrm{SE}}(\mathscr{D}_d))$, and considered the following four cases[1]:

Case 1. $\lim_{x\to a} u(x) = \lim_{x\to b} u(x) = 0$.

Case 2. $\lim_{x\to a} u(x) \neq 0, \quad \lim_{x\to b} u(x) = 0$.

Case 3. $\lim_{x\to a} u(x) = 0, \quad \lim_{x\to b} u(x) \neq 0$.

Case 4. $\lim_{x\to a} u(x) \neq 0, \quad \lim_{x\to b} u(x) \neq 0$.

The solution u is approximated in different manners in each case. In the case 1, the approximate solution u_N^{RZ} is set as

$$u(x) \approx u_N^{\mathrm{RZ}}(x) = \sum_{j=-N}^{N} u_j S(j, \tilde{h})(\{\psi^{\mathrm{SE}}\}^{-1}(x)), \tag{3.1}$$

and in the case 2,

$$u(x) \approx u_N^{\mathrm{RZ}}(x) = u_{-N} w_a(x) + \sum_{j=-N+1}^{N} u_j S(j, \tilde{h})(\{\psi^{\mathrm{SE}}\}^{-1}(x)), \tag{3.2}$$

and in the case 3,

$$u(x) \approx u_N^{\mathrm{RZ}}(x) = \sum_{j=-N}^{N-1} u_j S(j, \tilde{h})(\{\psi^{\mathrm{SE}}\}^{-1}(x)) + u_N w_b(x), \tag{3.3}$$

and in the case 4,

$$u(x) \approx u_N^{\mathrm{RZ}}(x) = u_{-N} w_a(x) + \sum_{j=-N+1}^{N-1} u_j S(j, \tilde{h})(\{\psi^{\mathrm{SE}}\}^{-1}(x)) + u_N w_b(x), \tag{3.4}$$

where the mesh size $\tilde{h}$ is selected as (2.12). For the sake of simplicity, we describe their method for the case 1 here. Let $k(x, \cdot)$ satisfy the assumptions in Theorem 2.4 for all x in (a, b) uniformly. In order to obtain the unknown coefficients u_j in (3.1), consider substituting u_N^{RZ} into

[1] The cases 2–4 contradict the condition (2.9), but this is what the original paper says.

the equation (1.1), and approximating the integral by Theorem 2.4 as

$$\lambda \int_a^b k(x,t)u_n(t)\,\mathrm{d}t \approx \lambda\tilde{h}\sum_{j=-N}^{N} k(x,\psi^{\mathrm{SE}}(j\tilde{h}))u_n(\psi^{\mathrm{SE}}(j\tilde{h}))\cdot\{\psi^{\mathrm{SE}}\}'(j\tilde{h})$$
$$= \lambda\tilde{h}\sum_{j=-N}^{N} k(x,\psi^{\mathrm{SE}}(j\tilde{h}))u_j\cdot\{\psi^{\mathrm{SE}}\}'(j\tilde{h}). \tag{3.5}$$

The last equality holds since $S(j,\tilde{h})(i\tilde{h}) = \delta_{ij}$, where δ_{ij} is Kronecker's delta. Then an approximated equation is obtained:

$$\sum_{j=-N}^{N}\left\{S(j,\tilde{h})(\{\psi^{\mathrm{SE}}\}^{-1}(x)) - \lambda\tilde{h}k(x,\psi^{\mathrm{SE}}(j\tilde{h}))\{\psi^{\mathrm{SE}}\}'(j\tilde{h})\right\}u_j = g(x). \tag{3.6}$$

Discretizing this equation at the collocation points:

$$x_i^{\mathrm{RZ}} = \psi^{\mathrm{SE}}(i\tilde{h}),\quad i=-N,\,\ldots,\,N, \tag{3.7}$$

leads to a system of linear equations

$$\sum_{j=-N}^{N}\left\{\delta_{ij} - \lambda\tilde{h}k(x_i^{\mathrm{RZ}},x_j^{\mathrm{RZ}})\{\psi^{\mathrm{SE}}\}'(j\tilde{h})\right\}u_j = g(x_i^{\mathrm{RZ}}),\quad i=-N,\,\ldots,\,N. \tag{3.8}$$

Here $S(j,\tilde{h})(i\tilde{h}) = \delta_{ij}$ is used again. By solving this system, we obtain coefficients u_j in (3.1).

In the cases 2–4, they have derived respective Sinc-collocation methods by the same procedure:

1. Substitute the approximate solution u_N^{RZ} into the equation (1.1),
2. Approximate the integral based on Theorem 2.4,
3. Set collocation points as (3.7) and obtain a system of linear equations.

3.2 Discussions: Ideas to improve and reinforce the RZ scheme

From a practical and theoretical point of view, the RZ scheme can be further improved. We here summarize the main ideas of the improvement.

3.2.1 How to determine the smoothness parameters of the solution (in Section 4)

In the RZ scheme, the parameters α and d of the solution u are required to determine the mesh size $\tilde{h}$ by (2.12). However, since the solution u is an *unknown* function to be determined, the scheme cannot be launched unless an way for finding (or at least estimating) α and d is equipped. As a remedy, in Section 4 we show that the values of α and d can be found from the known functions g and k.

3.2.2 Change of basis functions, collocation points, and mesh size (in Section 5)

Firstly, we modify the basis functions. In the RZ scheme, basis functions of the approximate solution u_N^{RZ} are selected from the four cases depending on the behavior of the solution u at endpoints. It seems to be, however, quite hard to know it *before* solving the problem. Furthermore, in the cases 2–4, the solutions clearly would not belong to $\mathbf{L}_\alpha(\psi^{\mathrm{SE}}(\mathscr{D}_d))$ (see the footnote in Section 3.1). In the present paper, we suppose $u \in \mathbf{M}_\alpha(\psi^{\mathrm{SE}}(\mathscr{D}_d))$, and set the approximate solution u_N^{SE} based on the approximations (2.27) as

$$u_N^{\mathrm{SE}}(x) = u_{-N-1} w_a(x) + \sum_{j=-N}^{N} u_j S(j,h)(\{\psi^{\mathrm{SE}}\}^{-1}(x)) + u_{N+1} w_b(x), \tag{3.9}$$

whose basis functions are fixed in any cases. In connection to this, we modify the collocation points as

$$x_i^{\mathrm{SE}} = \begin{cases} a & (i = -N-1), \\ \psi^{\mathrm{SE}}(ih) & (i = -N, \ldots, N), \\ b & (i = N+1), \end{cases} \tag{3.10}$$

in order that all the collocation points would be the support abscissas when we regard the generalized SE-Sinc approximation (2.27) as an interpolation; i.e., $f(x_i^{\mathrm{SE}}) = \mathcal{P}_N^{\mathrm{SE}}[f](x_i^{\mathrm{SE}})$ holds. This is a useful property for the subsequent theoretical analysis. In the RZ scheme, however, the collocation points (3.7) coincide the abscissas for all $i = -N+1, \ldots, N-1$, but not for $i = -N, N$ (strictly speaking, in the cases 2–4).

Secondly, we modify the mesh size. In the RZ scheme, the mesh size $\tilde{h}$ of (2.12) is selected. In this case, the convergence rate of the SE-Sinc quadrature is $\mathrm{O}(\mathrm{e}^{-\sqrt{2\pi d\alpha N}})$ according to Theorem 2.4. But according to Corollary 2.6 the convergence rate of the SE-Sinc approximation is worse: $\mathrm{O}(\mathrm{e}^{-\sqrt{\pi d\alpha N/2}})$, which reduces the performance of the scheme (recall the discussion in Section 2). In the present paper, the mesh size h is selected as (2.10); this will allow us to obtain the optimal rate: $\mathrm{O}(\sqrt{N}\,\mathrm{e}^{-\sqrt{\pi d\alpha N}})$. This is because of Theorem 2.2 and Corollary 2.5.

3.2.3 Convergence analysis (in Section 6)

In their paper [12], it is claimed that the convergence rate of the RZ scheme is $\mathrm{O}(\exp(-c_1\sqrt{N}))$ based on the intuitive discussion and numerical results, but rigorous theoretical analysis is not given. On the other hand, they have given a certain error analysis of the Sinc-Nyström method [13], which gives us a clue to analyze the present case (Sinc-collocation method). It will be shown that the error analysis of the Sinc-collocation method can be reduced to that of the Sinc-Nyström method, when the interpolation property $f(x_i^{\mathrm{SE}}) = \mathcal{P}_N^{\mathrm{SE}}[f](x_i^{\mathrm{SE}})$ holds (note that this is guaranteed by the appropriate choice of the collocation points above). Then we can show theoretically that the convergence rate of the modified RZ scheme is $\mathrm{O}(\exp(-c_1\sqrt{N}))$.

3.2.4 Replacement of the variable transformation (in Section 7 and 8)

The variable transformation utilized in the RZ and the modified RZ schemes above is the SE transformation, but we replace it with the DE transformation in Section 7. It is then shown in Section 8 that the replacement improves the convergence rate to $\mathrm{O}(\exp(-c_2 N/\log N))$, which is much faster than the case of the SE transformation.

4 Analysis of the smoothness properties of the solution

Throughout this section, let $\mathscr{D}$ represent either $\psi^{\mathrm{SE}}(\mathscr{D}_d)$ or $\psi^{\mathrm{DE}}(\mathscr{D}_d)$. We give below a sufficient condition for $u \in \mathbf{M}_\alpha(\mathscr{D})$ using the known functions g and k.

First we show $u \in \mathbf{HC}(\mathscr{D})$ under some assumptions (recall Definition 2.12). Suppose $k(z,\cdot) \in \mathbf{HC}(\mathscr{D})$ and $k(\cdot,w) \in \mathbf{HC}(\mathscr{D})$ for all $z, w \in \overline{\mathscr{D}}$, and let us introduce the integral operator

$\mathcal{K} : \mathbf{HC}(\mathscr{D}) \to \mathbf{HC}(\mathscr{D})$ defined by

$$\mathcal{K}[f](x) = \lambda \int_a^b k(x,t) f(t)\,\mathrm{d}t. \tag{4.1}$$

Note that the equation (1.1) can be rewritten as $(I - \mathcal{K})u = g$. The operator $\mathcal{K}$ is compact on $\mathbf{HC}(\mathscr{D})$ as shown below.

Lemma 4.1. Let $k(z, \cdot) \in \mathbf{HC}(\mathscr{D})$ and $k(\cdot, w) \in \mathbf{HC}(\mathscr{D})$ for all z, $w \in \overline{\mathscr{D}}$. Then the operator $\mathcal{K} : \mathbf{HC}(\mathscr{D}) \to \mathbf{HC}(\mathscr{D})$ is compact.

Proof. It is easily seen that the operators $\mathcal{K}$ map the set $\{f : \|f\|_{\mathbf{HC}(\mathscr{D})} \leq 1\}$ onto a uniformly bounded and equicontinuous set. Therefore the claim follows from the Arzelà–Ascoli theorem for complex functions (cf. Rudin [15, Theorem 11.28]). ■

Then the existence and uniqueness of the solution are immediately shown by the Fredholm alternative theorem.

Theorem 4.2. Suppose that the assumptions of Lemma 4.1 are fulfilled, and the homogeneous equation $(I - \mathcal{K})f = 0$ has only the trivial solution $f \equiv 0$. Then the operator $(I - \mathcal{K}) : \mathbf{HC}(\mathscr{D}) \to \mathbf{HC}(\mathscr{D})$ has a bounded inverse, $(I - \mathcal{K})^{-1} : \mathbf{HC}(\mathscr{D}) \to \mathbf{HC}(\mathscr{D})$. Furthermore, if $g \in \mathbf{HC}(\mathscr{D})$, then the equation (1.1) has a unique solution $u \in \mathbf{HC}(\mathscr{D})$.

Next we show $u \in \mathbf{M}_\alpha(\mathscr{D})$. For this purpose the next theorem is useful.

Theorem 4.3 (Stenger [17, Theorem 6.1])**.** Let $k(x, \cdot) \in L^1(a, b)$ and $k(\cdot, t) \in \mathbf{M}_\alpha(\mathscr{D})$ for all x, $t \in (a, b)$, and $g \in \mathbf{M}_\alpha(\mathscr{D})$. If the equation (1.1) has a unique solution u, then $u \in \mathbf{M}_\alpha(\mathscr{D})$.

Combining Theorem 4.2 with 4.3, we obtain the desired result as stated below.

Theorem 4.4. Let $k(z, \cdot) \in \mathbf{HC}(\mathscr{D})$ and $k(\cdot, w) \in \mathbf{M}_\alpha(\mathscr{D})$ for all z, $w \in \overline{\mathscr{D}}$, and let also $g \in \mathbf{M}_\alpha(\mathscr{D})$. Furthermore, assume that the homogeneous equation $(I - \mathcal{K})f = 0$ has only the trivial solution $f \equiv 0$. Then the equation (1.1) has a unique solution $u \in \mathbf{M}_\alpha(\mathscr{D})$.

5 Derivation of the modified RZ scheme

In this section we show a concrete form of the modified RZ scheme roughly described in §3.2.2. Suppose that the assumptions of Theorem 4.4 are fulfilled with $\mathscr{D} = \psi^{\mathrm{SE}}(\mathscr{D}_d)$. Then $u \in \mathbf{M}_\alpha(\psi^{\mathrm{SE}}(\mathscr{D}_d))$, and accordingly we can set the approximate solution u_N^{SE} as (3.9). Let us substitute u_N^{SE} into the equation (1.1), and approximate the integral operator $\mathcal{K}$ by $\mathcal{K}_N^{\mathrm{SE}}$:

$$\mathcal{K}_N^{\mathrm{SE}}[f](x) = \lambda h \sum_{j=-N}^{N} k(x, x_j^{\mathrm{SE}}) f(x_j^{\mathrm{SE}}) \{\psi^{\mathrm{SE}}\}'(jh). \tag{5.1}$$

In view of Theorem 2.13 and Corollary 2.5, the mesh size h is selected by the formula (2.10). Finally, by setting the collocation points as $x = x_i^{\mathrm{SE}}$ defined by (3.10), we obtain $(2N + 3) \times (2N + 3)$ system of linear equations:

$$\begin{aligned}
&\{w_a(x_i^{\mathrm{SE}}) - \mathcal{K}_N^{\mathrm{SE}}[w_a](x_i^{\mathrm{SE}})\} u_{-N-1} \\
&+ \sum_{j=-N}^{N} \left\{\delta_{ij} - \lambda h k(x_i^{\mathrm{SE}}, x_j^{\mathrm{SE}}) \{\psi^{\mathrm{SE}}\}'(jh)\right\} u_j \\
&+ \{w_b(x_i^{\mathrm{SE}}) - \mathcal{K}_N^{\mathrm{SE}}[w_b](x_i^{\mathrm{SE}})\} u_{N+1} = g(x_i^{\mathrm{SE}}), \quad i = -N-1, -N, \ldots, N, N+1.
\end{aligned} \tag{5.2}$$

Let us define n by $n = 2N+3$, and let E_n^{SE} and K_n^{SE} be $n \times n$ matrices defined by

$$E_n^{\mathrm{SE}} = \left[\begin{array}{c|ccc|c} 1 & 0 & \cdots & 0 & 0 \\ \hline w_a(x_{-N}^{\mathrm{SE}}) & 1 & & \mathrm{O} & w_b(x_{-N}^{\mathrm{SE}}) \\ \vdots & & \ddots & & \vdots \\ w_a(x_N^{\mathrm{SE}}) & \mathrm{O} & & 1 & w_b(x_N^{\mathrm{SE}}) \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array}\right], \tag{5.3}$$

$$K_n^{\mathrm{SE}} = \left[\begin{array}{c|ccc|c} \mathcal{K}_N^{\mathrm{SE}}[w_a](a) & \cdots & \lambda h k(a, x_j^{\mathrm{SE}})\{\psi^{\mathrm{SE}}\}'(jh) & \cdots & \mathcal{K}_N^{\mathrm{SE}}[w_b](a) \\ \hline \mathcal{K}_N^{\mathrm{SE}}[w_a](x_{-N}^{\mathrm{SE}}) & \cdots & \lambda h k(x_{-N}^{\mathrm{SE}}, x_j^{\mathrm{SE}})\{\psi^{\mathrm{SE}}\}'(jh) & \cdots & \mathcal{K}_N^{\mathrm{SE}}[w_b](x_{-N}^{\mathrm{SE}}) \\ \vdots & & \vdots & & \vdots \\ \mathcal{K}_N^{\mathrm{SE}}[w_a](x_N^{\mathrm{SE}}) & \cdots & \lambda h k(x_N^{\mathrm{SE}}, x_j^{\mathrm{SE}})\{\psi^{\mathrm{SE}}\}'(jh) & \cdots & \mathcal{K}_N^{\mathrm{SE}}[w_b](x_N^{\mathrm{SE}}) \\ \hline \mathcal{K}_N^{\mathrm{SE}}[w_a](b) & \cdots & \lambda h k(b, x_j^{\mathrm{SE}})\{\psi^{\mathrm{SE}}\}'(jh) & \cdots & \mathcal{K}_N^{\mathrm{SE}}[w_b](b) \end{array}\right]. \tag{5.4}$$

Furthermore let $\boldsymbol{g}_n^{\mathrm{SE}} = [g(a),\, g(x_{-N}^{\mathrm{SE}}),\, \ldots,\, g(x_N^{\mathrm{SE}}),\, g(b)]^{\mathrm{T}}$. Then the resulting system of linear equations (5.2) can be written in the matrix-vector form:

$$(E_n^{\mathrm{SE}} - K_n^{\mathrm{SE}})\boldsymbol{u}_n = \boldsymbol{g}_n^{\mathrm{SE}}, \tag{5.5}$$

where $\boldsymbol{u}_n = [u_{-N-1},\, u_{-N},\, \ldots,\, u_N,\, u_{N+1}]^{\mathrm{T}}$. By solving this for the coefficients $\boldsymbol{u}_n$, the approximate solution u_N^{SE} is obtained by (3.9).

6 Convergence analysis of the modified RZ scheme

In this section, the convergence rate of the modified RZ scheme is rigorously given. Note that in this section the norm $\|\cdot\|_{C([a,\,b])}$ is used for evaluating the error instead of $\|\cdot\|_{\mathbf{HC}(\mathscr{D})}$. This is because we are interested in the error on the interval $[a,\,b]$, not on the complex domain $\overline{\mathscr{D}}$. We write $X = C([a,\,b])$ for short. We also regard $\mathcal{K}$, $\mathcal{K}_N^{\mathrm{SE}}$, $\mathcal{P}_N^{\mathrm{SE}}$ as operators from X onto X.

6.1 Sketch of the proof

In addition to the equations (1.1) and (5.5), we consider the following two equations:

$$(I - \mathcal{K}_N^{\mathrm{SE}})[v](x) = g(x), \quad a \le x \le b, \tag{6.1}$$

$$(I - \mathcal{P}_N^{\mathrm{SE}}\mathcal{K}_N^{\mathrm{SE}})[w](x) = \mathcal{P}_N^{\mathrm{SE}}[g](x), \quad a \le x \le b. \tag{6.2}$$

Then in § 6.2 it is shown that $u_N^{\mathrm{SE}} = w = \mathcal{P}_N^{\mathrm{SE}} v$, and the error in the modified RZ scheme can be evaluated as

$$\|u - u_N^{\mathrm{SE}}\|_X \le \|u - \mathcal{P}_N^{\mathrm{SE}} u\|_X + \|\mathcal{P}_N^{\mathrm{SE}}\|_{\mathcal{L}(X,X)}\|u - v\|_X. \tag{6.3}$$

We already have the estimate of the first term in Theorem 2.13. The terms $\|u - v\|_X$ and $\|\mathcal{P}_N^{\mathrm{SE}}\|_{\mathcal{L}(X,X)}$ are estimated in § 6.3 and in § 6.4, respectively. Combining them, we obtain the desired estimate.

6.2 Relations between the solutions u_N^{SE}, v and w

Following Atkinson [2, § 4.3], we easily see that $w = u_N^{\mathrm{SE}}$. The proof is omitted.

Proposition 6.1. If the equation (6.2) has a unique solution $w \in X$, then the equation (5.5) is uniquely solvable, and vice versa. Furthermore, $w = u_N^{\mathrm{SE}}$ holds.

We also see $u_N^{\mathrm{SE}} = \mathcal{P}_N^{\mathrm{SE}} v$ as shown below.

Proposition 6.2. The following two statements are equivalent:

(A) the equation (6.1) has a unique solution $v \in X$.

(B) the equation (6.2) has a unique solution $w \in X$.

Furthermore, $w = \mathcal{P}_N^{\text{SE}} v$ and $v = g + \mathcal{K}_N^{\text{SE}} w$ hold if the solutions exist.

Proof. We only show (A) $\Rightarrow$ (B) since (B) $\Rightarrow$ (A) can be shown in the same manner. The key here is the interpolation property $f(x_i^{\text{SE}}) = \mathcal{P}_N^{\text{SE}}[f](x_i^{\text{SE}})$, from which it follows that

$$\mathcal{K}_N^{\text{SE}} \mathcal{P}_N^{\text{SE}} f = \mathcal{K}_N^{\text{SE}} f. \tag{6.4}$$

Assume (A). By applying $\mathcal{P}_N^{\text{SE}}$ to both sides of (6.1), we have

$$\mathcal{P}_N^{\text{SE}} g = \mathcal{P}_N^{\text{SE}} v - \mathcal{P}_N^{\text{SE}} \mathcal{K}_N^{\text{SE}} v = (\mathcal{P}_N^{\text{SE}} v) - \mathcal{P}_N^{\text{SE}} \mathcal{K}_N^{\text{SE}} (\mathcal{P}_N^{\text{SE}} v) = (I - \mathcal{P}_N^{\text{SE}} \mathcal{K}_N^{\text{SE}})(\mathcal{P}_N^{\text{SE}} v). \tag{6.5}$$

This equation implies that there exists a solution $w = \mathcal{P}_N^{\text{SE}} v \in X$ in the equation (6.2). Next we show the uniqueness. Suppose that there exists another solution $\tilde{w} \in X$ in the equation (6.2), and define a function $\tilde{v}$ as $\tilde{v} = g + \mathcal{K}_N^{\text{SE}} \tilde{w}$. Then $\tilde{w} = \mathcal{P}_N^{\text{SE}} \tilde{v}$ holds since $\tilde{w}$ is a solution of the equation (6.2). Therefore $\tilde{v} = g + \mathcal{K}_N^{\text{SE}}(\mathcal{P}_N^{\text{SE}} \tilde{v}) = g + \mathcal{K}_N^{\text{SE}} \tilde{v}$, which means $\tilde{v}$ is a solution of the equation (6.1), and then $v = \tilde{v}$ from the uniqueness of the equation. Thus we have $w = \mathcal{P}_N^{\text{SE}} v = \mathcal{P}_N^{\text{SE}} \tilde{v} = \tilde{w}$, which shows the desired uniqueness. ■

Using the propositions above and the following equality

$$u - u_N^{\text{SE}} = u - w = (u - \mathcal{P}_N^{\text{SE}} u) + (\mathcal{P}_N^{\text{SE}} u - \mathcal{P}_N^{\text{SE}} v), \tag{6.6}$$

we immediately have the next result.

Lemma 6.3. Assume that the equation (6.1) has a unique solution $v \in X$. Then the equation (5.5) is uniquely solvable, and the inequality (6.3) holds.

6.3 Error analysis of the Sinc-Nyström method

Next we show the existence and uniqueness of the solution v in (6.1), and estimate the term $\|u - v\|_X$. Actually, v is an approximate solution obtained by the so-called Sinc-Nystoröm method that has been developed by Rashidinia–Zarebnia [13]. They have pointed out a certain way of the analysis of the method, which utilizes the tool called "collectively compact operators."

Definition 6.4. Let $\mathcal{W}_m : X \to X$ $(m = 1, 2, \dots)$ be linear operators and let the set

$$\{\mathcal{W}_m f : m \geq 1, \|f\|_X \leq 1\} \tag{6.7}$$

be relatively compact on X. Then the set $\{\mathcal{W}_m : m \geq 1\}$ is collectively compact.

The following theorem summarizes the important tools for the error analysis (see also, for example, Kress [6] and Atkinson [2]).

Theorem 6.5 (Rashidinia–Zarebnia [13, Theorem IV])**.** Assume the following conditions:

1. $k \in C([a, b] \times [a, b])$, and $\mathcal{K}$ is a linear compact operator on X.
2. The operator $(I - \mathcal{K})$ is injective[2].

[2]The original paper says $I - \mathcal{K}$ is "invertible," but from their proof "injective" makes more sense.

3. The set $\{\mathcal{K}_N^{\mathrm{SE}} : N \geq 1\}$ is collectively compact on X.

4. For all $f \in X$ it holds that $\|\mathcal{K}f - \mathcal{K}_N^{\mathrm{SE}} f\|_X \to 0 \quad (N \to \infty)$.

5. For sufficiently large N it holds that $\|(I - \mathcal{K})^{-1}(\mathcal{K} - \mathcal{K}_N^{\mathrm{SE}})\mathcal{K}_N^{\mathrm{SE}}\|_{\mathcal{L}(X,X)} < 1$.

Then the approximate inverses $(I - \mathcal{K}_N^{\mathrm{SE}})^{-1}$ exist and are uniformly bounded,

$$\|(I - \mathcal{K}_N^{\mathrm{SE}})^{-1}\|_{\mathcal{L}(X,X)} \leq \frac{1 + \|(I - \mathcal{K})^{-1}\mathcal{K}_N^{\mathrm{SE}}\|_{\mathcal{L}(X,X)}}{1 - \|(I - \mathcal{K})^{-1}(\mathcal{K} - \mathcal{K}_N^{\mathrm{SE}})\mathcal{K}_N^{\mathrm{SE}}\|_{\mathcal{L}(X,X)}}. \tag{6.8}$$

If the five conditions in Theorem 6.5 are fulfilled, then based on this theorem, we can easily obtain the estimate for $\|u - v\|_X$, as will be shown in Theorem 6.10 (this has not been done in [13] in a rigorous manner).

In what follows we show that they are in fact fulfilled under the assumptions in Theorem 4.4. The conditions 1 and 2 are clear from the assumptions in Theorem 4.4. For the conditions 3 and 4, let us introduce some notations here. Let $Q : X \to \mathbb{R}$ be an integral operator defined by $\mathcal{Q}f = \int_a^b f(t)\,\mathrm{d}t$, and $\mathcal{Q}_m : X \to \mathbb{R}$ be a quadrature rule defined by

$$\mathcal{Q}f \approx \mathcal{Q}_m f = \sum_{j=1}^{m} w_j f(t_j), \tag{6.9}$$

for some weights w_j and quadrature node points t_j. Furthermore, let $\mathcal{K}_m$ be an approximate operator of $\mathcal{K}$ based on $\mathcal{Q}_m$, defined by

$$\mathcal{K}[f](x) \approx \mathcal{K}_m[f](x) = \lambda \mathcal{Q}_m[k(x,\cdot)f(\cdot)] = \lambda \sum_{j=1}^{m} w_j k(x, t_j) f(t_j). \tag{6.10}$$

Under this abstract setting, the next proposition holds.

Proposition 6.6 (Anselone [1, Proposition 2.1 and 2.2])**.** Suppose that $\mathcal{Q}_m f \to \mathcal{Q}f$ for all $f \in X$. Furthermore, let $k \in C([a, b] \times [a, b])$. Then the next assertions hold:

3′. The set $\{\mathcal{K}_m : m \geq 1\}$ is collectively compact on X.

4′. For all $f \in X$ it holds that $\|\mathcal{K}f - \mathcal{K}_m f\|_X \to 0 \quad (m \to \infty)$.

If we define $\mathcal{Q}_N^{\mathrm{SE}} : X \to \mathbb{R}$ by

$$\mathcal{Q}f \approx \mathcal{Q}_N^{\mathrm{SE}} f = h \sum_{j=-N}^{N} f(x_j^{\mathrm{SE}}) \{\psi^{\mathrm{SE}}\}'(jh), \tag{6.11}$$

which is nothing but the approximation (2.8), then we can use Proposition 6.6 by replacing $\mathcal{Q}_m$ with $\mathcal{Q}_N^{\mathrm{SE}}$, and $\mathcal{K}_m$ with $\mathcal{K}_N^{\mathrm{SE}}$ (note that h is selected as (2.10) here). In this regard, we have to show the assumption of Proposition 6.6: $\mathcal{Q}_N^{\mathrm{SE}} f \to \mathcal{Q}f$ for all $f \in X$. For that purpose, the Banach–Steinhaus theorem is useful.

Theorem 6.7 (cf. Atkinson [2, Corollary A.1])**.** Let X and Y be Banach spaces, and let $\mathcal{Q}, \mathcal{Q}_m : X \to Y$ be bounded linear operators. Let E be a dense subspace of X. Then in order that $\mathcal{Q}_m f \to \mathcal{Q}f$ for all $f \in X$, it is necessary and sufficient that

(a) $\mathcal{Q}_m f \to \mathcal{Q}f$ for all $f \in E$.

(b) $\sup_m \|\mathcal{Q}_m\|_{\mathcal{L}(X,Y)} < \infty$.

We can prove the statements (a) and (b) as follows.

Lemma 6.8. For all $f \in X$ it holds that $\mathcal{Q}_N^{\mathrm{SE}} f \to \mathcal{Q} f$ as $N \to \infty$.

Proof. Let E be a family of polynomials, which is a dense subspace of $X = C([a, b])$. Since polynomials are analytic functions on the whole complex plain, clearly $\mathcal{Q}_N^{\mathrm{SE}} f$ converges to $\mathcal{Q} f$ for all $f \in E$ (cf. Stenger [16, Corollary 4.2.7]). This implies (a) in Theorem 6.7. We prove (b) next. It holds that for all $f \in X$

$$\frac{|\mathcal{Q}_N^{\mathrm{SE}} f|}{\|f\|_X} \leq h \sum_{j=-N}^{N} \{\psi^{\mathrm{SE}}\}'(jh), \tag{6.12}$$

and the right hand side converges to $\int_{-\infty}^{\infty} \{\psi^{\mathrm{SE}}\}'(t)\,\mathrm{d}t$ as $N \to \infty$. Thus it is uniformly bounded, and we have $\|\mathcal{Q}_N^{\mathrm{SE}}\|_{\mathcal{L}(X,\mathbb{R})} < \infty$. This completes the proof. ∎

Finally, the condition 5 shall be shown by the next lemma.

Lemma 6.9 (Atkinson [2, Lemma 4.1.2])**.** Let $\mathcal{K}$, $\mathcal{K}_m : X \to X$ be linear operators, and assume the following two statements:

3′. The set $\{\mathcal{K}_m : m \geq 1\}$ is collectively compact on X.

4′. For all $f \in X$ it holds that $\|\mathcal{K} f - \mathcal{K}_m f\|_X \to 0 \quad (m \to \infty)$.

Then it holds that $\|(\mathcal{K} - \mathcal{K}_m)\mathcal{K}_m\|_{\mathcal{L}(X,X)} \to 0 \quad (m \to \infty)$.

The assumptions of this lemma have already been shown (i.e. conditions 1, 3 and 4), and thus we immediately have

$$\|(I - \mathcal{K})^{-1}(\mathcal{K} - \mathcal{K}_N^{\mathrm{SE}})\mathcal{K}_N^{\mathrm{SE}}\|_{\mathcal{L}(X,X)} \leq \|(I - \mathcal{K})^{-1}\|_{\mathcal{L}(X,X)} \|(\mathcal{K} - \mathcal{K}_N^{\mathrm{SE}})\mathcal{K}_N^{\mathrm{SE}}\|_{\mathcal{L}(X,X)} \to 0. \tag{6.13}$$

Thus the all conditions 1–5 are satisfied, which allows us to obtain the error analysis of the Sinc-Nyström method.

Theorem 6.10. Suppose that the assumptions in Theorem 4.4 are fulfilled with $\mathscr{D} = \psi^{\mathrm{SE}}(\mathscr{D}_d)$ for $d \in (0, \pi)$. Then there exists a positive integer N_0 such that for all $N \geq N_0$, the equation (6.1) has a unique solution $v \in X$. Furthermore, there exists a constant C for all $N \geq N_0$ such that

$$\|u - v\|_X \leq C\,\mathrm{e}^{-\sqrt{\pi d \alpha N}}. \tag{6.14}$$

Proof. Since $(I - \mathcal{K}) : X \to X$ and $(I - \mathcal{K}_N^{\mathrm{SE}}) : X \to X$ have bounded inverses, it holds that

$$\begin{aligned} u - v &= (I - \mathcal{K})^{-1} g - (I - \mathcal{K}_N^{\mathrm{SE}})^{-1} g \\ &= (I - \mathcal{K}_N^{\mathrm{SE}})^{-1}\{(I - \mathcal{K}_N^{\mathrm{SE}}) - (I - \mathcal{K})\}(I - \mathcal{K})^{-1} g \\ &= (I - \mathcal{K}_N^{\mathrm{SE}})^{-1}\{\mathcal{K} u - \mathcal{K}_N^{\mathrm{SE}} u\}. \end{aligned} \tag{6.15}$$

Clearly $k(x, \cdot)u(\cdot)Q(\cdot) \in \mathbf{L}_\alpha(\psi^{\mathrm{SE}}(\mathscr{D}_d))$ holds from $k(x, \cdot) \in \mathbf{HC}(\psi^{\mathrm{SE}}(\mathscr{D}_d))$ and $u \in \mathbf{M}_\alpha(\psi^{\mathrm{SE}}(\mathscr{D}_d))$. Therefore we can apply Corollary 2.5 as follows:

$$\|u - v\|_X \leq \|(I - \mathcal{K}_N^{\mathrm{SE}})^{-1}\|_{\mathcal{L}(X,X)} \|\mathcal{K} u - \mathcal{K}_N^{\mathrm{SE}} u\|_X \leq \|(I - \mathcal{K}_N^{\mathrm{SE}})^{-1}\|_{\mathcal{L}(X,X)} C\,\mathrm{e}^{-\sqrt{\pi d \alpha N}}. \tag{6.16}$$

Theorem 6.5 claims $\|(I - \mathcal{K}_N^{\mathrm{SE}})^{-1}\|_{\mathcal{L}(X,X)}$ is uniformly bounded, which completes the proof. ∎

Remark 6.11. In a precise sense, this theorem does not show the convergence rate of the Sinc-Nyström method in Rashidinia–Zarebnia [13]. This is because the way of selecting the mesh size is different; the formula (2.10) is used in this paper, whereas the formula (2.12) in [13]. Accordingly the rate of convergence should be different.

6.4 Convergence theorem of the modified RZ scheme

What is left is to estimate $\|\mathcal{P}_N^{\mathrm{SE}}\|_{\mathcal{L}(X,X)}$ in (6.3); the following estimate is essential for that.

Lemma 6.12 (Stenger [16, p. 142])**.** Let $h > 0$. Then it holds that

$$\sup_{\xi\in\mathbb{R}} \sum_{j=-N}^{N} |S(j,h)(\xi)| \leq \frac{2}{\pi}(3+\log N). \tag{6.17}$$

Based on this estimate, we can deduce the next lemma immediately.

Lemma 6.13. There exists a constant C for all N such that $\|\mathcal{P}_N^{\mathrm{SE}}\|_{\mathcal{L}(X,X)} \leq C\log(N+1)$.

Thus the desired estimate is obtained as follows.

Theorem 6.14. Suppose that the assumptions in Theorem 4.4 are fulfilled with $\mathscr{D} = \psi^{\mathrm{SE}}(\mathscr{D}_d)$ for $d \in (0,\pi)$. Then there exists a positive integer N_0 such that for all $N \geq N_0$, the equation (5.5) is uniquely solvable. Furthermore, there exists a constant C for all $N \geq N_0$ such that

$$\|u - u_N^{\mathrm{SE}}\|_X \leq C\sqrt{N}\,\mathrm{e}^{-\sqrt{\pi d\alpha N}}. \tag{6.18}$$

7 Derivation of the DE-Sinc scheme

In this section, the DE-Sinc scheme is described, by replacing the SE transformation in the modified RZ scheme with the DE transformation. Suppose that the assumptions of Theorem 4.4 are fulfilled with $\mathscr{D} = \psi^{\mathrm{DE}}(\mathscr{D}_d)$. Then $u \in \mathbf{M}_\alpha(\psi^{\mathrm{DE}}(\mathscr{D}_d))$, and accordingly we can set the approximate solution u_N^{DE} as

$$u_N^{\mathrm{DE}}(x) = u_{-N-1}w_a(x) + \sum_{j=-N}^{N} u_j S(j,h)(\{\psi^{\mathrm{DE}}\}^{-1}(x)) + u_{N+1}w_b(x). \tag{7.1}$$

Let us substitute u_N^{DE} into the equation (1.1), and approximate the integral operator $\mathcal{K}$ by $\mathcal{K}_N^{\mathrm{DE}}$:

$$\mathcal{K}_N^{\mathrm{DE}}[f](x) = \lambda h \sum_{j=-N}^{N} k(x, x_j^{\mathrm{DE}}) f(x_j^{\mathrm{DE}})\{\psi^{\mathrm{DE}}\}'(jh). \tag{7.2}$$

In view of Theorem 2.14 and Corollary 2.10, the mesh size h is selected by the formula (2.18). Finally, by setting the collocation points as $x = x_i^{\mathrm{DE}}$ defined by

$$x_i^{\mathrm{DE}} = \begin{cases} a & (i = -N-1), \\ \psi^{\mathrm{DE}}(ih) & (i = -N, \ldots, N), \\ b & (i = N+1), \end{cases} \tag{7.3}$$

we obtain $n \times n$ system of linear equations (recall $n = 2N+3$):

$$(E_n^{\mathrm{DE}} - K_n^{\mathrm{DE}})\boldsymbol{u}_n = \boldsymbol{g}_n^{\mathrm{DE}}, \tag{7.4}$$

where $\boldsymbol{g}_n^{\rm DE} = [g(a),\, g(x_{-N}^{\rm DE}),\, \ldots,\, g(x_N^{\rm DE}),\, g(b)]^{\rm T}$, and $E_n^{\rm DE}$ and $K_n^{\rm DE}$ are $n\times n$ matrices defined by

$$E_n^{\rm DE} = \left[\begin{array}{c|ccc|c} 1 & 0 & \cdots & 0 & 0 \\ \hline w_a(x_{-N}^{\rm DE}) & 1 & & \mathrm{O} & w_b(x_{-N}^{\rm DE}) \\ \vdots & & \ddots & & \vdots \\ w_a(x_N^{\rm DE}) & \mathrm{O} & & 1 & w_b(x_N^{\rm DE}) \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array}\right], \tag{7.5}$$

$$K_n^{\rm DE} = \left[\begin{array}{l|ccc|l} \mathcal{K}_N^{\rm DE}[w_a](a) & \cdots & \lambda h k(a, x_j^{\rm DE})\{\psi^{\rm DE}\}'(jh) & \cdots & \mathcal{K}_N^{\rm DE}[w_b](a) \\ \hline \mathcal{K}_N^{\rm DE}[w_a](x_{-N}^{\rm DE}) & \cdots & \lambda h k(x_{-N}^{\rm DE}, x_j^{\rm DE})\{\psi^{\rm DE}\}'(jh) & \cdots & \mathcal{K}_N^{\rm DE}[w_b](x_{-N}^{\rm DE}) \\ \vdots & & \vdots & & \vdots \\ \mathcal{K}_N^{\rm DE}[w_a](x_N^{\rm DE}) & \cdots & \lambda h k(x_N^{\rm DE}, x_j^{\rm DE})\{\psi^{\rm DE}\}'(jh) & \cdots & \mathcal{K}_N^{\rm DE}[w_b](x_N^{\rm DE}) \\ \hline \mathcal{K}_N^{\rm DE}[w_a](b) & \cdots & \lambda h k(b, x_j^{\rm DE})\{\psi^{\rm DE}\}'(jh) & \cdots & \mathcal{K}_N^{\rm DE}[w_b](b) \end{array}\right]. \tag{7.6}$$

By solving this for the coefficients $\boldsymbol{u}_n$ in (7.4), the approximate solution $u_N^{\rm DE}$ is obtained by (7.1).

8 Convergence analysis of the DE-Sinc scheme

In this section, the convergence rate of the DE-Sinc scheme is given under the same settings as in Section 6. Since the proof goes almost in the same way as in the modified RZ scheme, we only show the results here.

8.1 Sketch of the proof

In addition to the equations (1.1) and (7.4), we consider the following two equations:

$$(I - \mathcal{K}_N^{\rm DE})[v](x) = g(x), \quad a \leq x \leq b, \tag{8.1}$$

$$(I - \mathcal{P}_N^{\rm DE}\mathcal{K}_N^{\rm DE})[w](x) = \mathcal{P}_N^{\rm DE}[g](x), \quad a \leq x \leq b. \tag{8.2}$$

Then in § 8.2 it is shown that $u_N^{\rm DE} = w = \mathcal{P}_N^{\rm DE} v$, and the error in the DE-Sinc scheme can be evaluated as

$$\|u - u_N^{\rm DE}\|_X \leq \|u - \mathcal{P}_N^{\rm DE} u\|_X + \|\mathcal{P}_N^{\rm DE}\|_{\mathcal{L}(X,X)} \|u - v\|_X. \tag{8.3}$$

We already have the estimate of the first term in Theorem 2.14. The terms $\|u - v\|_X$ and $\|\mathcal{P}_N^{\rm DE}\|_{\mathcal{L}(X,X)}$ are estimated in § 8.3 and in § 8.4, respectively. Combining them, we obtain the desired estimate.

8.2 Relations between the solutions $u_N^{\rm DE}$, v and w

The following propositions and lemma hold good.

Proposition 8.1. If the equation (8.2) has a unique solution $w \in X$, then the equation (7.4) is uniquely solvable, and vice versa. Furthermore, $w = u_N^{\rm DE}$ holds.

Proposition 8.2. The following two statements are equivalent:

(A) the equation (8.1) has a unique solution $v \in X$.

(B) the equation (8.2) has a unique solution $w \in X$.

Furthermore, $w = \mathcal{P}_N^{\rm DE} v$ and $v = g + \mathcal{K}_N^{\rm DE} w$ hold if the solutions exist.

Lemma 8.3. Assume that the equation (8.1) has a unique solution $v \in X$. Then the equation (7.4) is uniquely solvable, and the inequality (8.3) holds.

8.3 Error analysis of the Sinc-Nyström method

If we define $\mathcal{Q}_N^{\mathrm{DE}} : X \to \mathbb{R}$ by

$$\mathcal{Q}f \approx \mathcal{Q}_N^{\mathrm{DE}} f = h \sum_{j=-N}^{N} f(x_j^{\mathrm{DE}})\{\psi^{\mathrm{DE}}\}'(jh), \tag{8.4}$$

we have the following lemma using Corollary 2.10.

Lemma 8.4. For all $f \in X$ it holds that $\mathcal{Q}_N^{\mathrm{DE}} f \to \mathcal{Q}f$ as $N \to \infty$.

Besides, by the same arguments in § 8.3, the next theorem follows.

Theorem 8.5. Suppose that the assumptions in Theorem 4.4 are fulfilled with $\mathscr{D} = \psi^{\mathrm{DE}}(\mathscr{D}_d)$ for $d \in (0, \pi/2)$. Then there exists a positive integer N_0 such that for all $N \geq N_0$, the equation (8.1) has a unique solution $v \in X$. Furthermore, there exists a constant C for all $N \geq N_0$ such that

$$\|u - v\|_X \leq C \exp\left\{\frac{-2\pi d N}{\log(2\pi d/\alpha)}\right\}. \tag{8.5}$$

8.4 Convergence theorem of the DE-Sinc scheme

We can deduce the next lemma immediately from Lemma 6.12.

Lemma 8.6. There exists a constant C for all N such that $\|\mathcal{P}_N^{\mathrm{DE}}\|_{\mathcal{L}(X,X)} \leq C \log(N + 1)$.

Thus the desired estimate is obtained as follows.

Theorem 8.7. Suppose that the assumptions in Theorem 4.4 are fulfilled with $\mathscr{D} = \psi^{\mathrm{DE}}(\mathscr{D}_d)$ for $d \in (0, \pi/2)$. Then there exists a positive integer N_0 such that for all $N \geq N_0$, the equation (7.4) is uniquely solvable. Furthermore, there exists a constant C for all $N \geq N_0$ such that

$$\|u - u_N^{\mathrm{DE}}\|_X \leq C \exp\left\{\frac{-\pi d N}{\log(2\pi d/\alpha)}\right\}. \tag{8.6}$$

9 Numerical examples

In this section, we show numerical results that illustrate the improvement achieved in the present paper. All programs for computation were written in C++ with double-precision floating-point arithmetic.

Let us first consider the following problem which is also conducted in Rashidinia–Zarebnia [12].

Example 9.1 (Rashidinia–Zarebnia [12, Example 1])**.** Consider

$$u(x) - \int_0^1 (3t - 6x^2) u(t)\,\mathrm{d}t = \frac{1}{4} - x, \quad 0 \leq x \leq 1, \tag{9.1}$$

whose solution is $u(x) = x(x - 1)$.

According to Theorem 4.4, $u \in \mathbf{M}_1(\psi^{\mathrm{SE}}(\mathscr{D}_{\pi-\epsilon}))$ and $u \in \mathbf{M}_1(\psi^{\mathrm{DE}}(\mathscr{D}_{(\pi-\epsilon)/2})$ in this case. Here ϵ denotes an arbitrary small positive number, and we choose $\epsilon = \pi - 3.14$ in what follows. We select h by the formula (2.10) with $\alpha = 1$ and $d = \pi - \epsilon$ in the modified RZ scheme, and by the formula (2.18) with $\alpha = 1$ and $d = (\pi - \epsilon)/2$ in the DE-Sinc scheme. In contrast, Rashidinia–Zarebnia [12] have chosen $\alpha = 1/2$ and $d = \pi/2$, and selected $\tilde{h}$ by the formula (2.12). The

Table 1. Computational results of Example 9.1.

N	$\tilde{h}$ (2.12)	E_N^{RZ}	h (2.10)	E_N^{SE}	h (2.18)	E_N^{DE}
5	1.98692	7.16816e-03	1.40461	6.21548e-04	0.550732	1.37362e-03
10	1.40496	3.11497e-04	0.993207	5.07910e-06	0.344681	4.92750e-07
15	1.14715	2.39353e-05	0.810950	1.06313e-07	0.256818	1.59098e-10
20	0.993459	2.55171e-06	0.702303	3.77483e-09	0.206998	5.55627e-14
25	0.888577	3.40053e-07	0.628159	1.90739e-10	0.174524	1.72236e-16
30	0.811156	5.34663e-08	0.573428	1.24724e-11	0.151514	1.77636e-16
35	0.750984	9.56543e-09	0.530891	9.95601e-13	0.134273	2.09766e-16
40	0.702481	1.90005e-09	0.496603	9.33555e-14	0.120828	1.66610e-16
45	0.662306	4.11757e-10	0.468202	1.00659e-14	0.110020	3.10862e-16
50	0.628319	9.60836e-11	0.444176	1.03095e-15	0.101125	5.32907e-16

values of the mesh sizes are presented in Table 1. In the table, E_N^{RZ}, E_N^{SE} and E_N^{DE} mean the maximum of absolute errors on the respective collocation points, defined by

$$E_N^{\mathrm{RZ}} = \max_{i=-N,\,\dots,\,N} |u(x_i^{\mathrm{RZ}}) - u_N^{\mathrm{RZ}}(x_i^{\mathrm{RZ}})|, \tag{9.2}$$

$$E_N^{\mathrm{SE}} = \max_{i=-N-1,\,-N,\,\dots,\,N,\,N+1} |u(x_i^{\mathrm{SE}}) - u_N^{\mathrm{SE}}(x_i^{\mathrm{SE}})|, \tag{9.3}$$

$$E_N^{\mathrm{DE}} = \max_{i=-N-1,\,-N,\,\dots,\,N,\,N+1} |u(x_i^{\mathrm{DE}}) - u_N^{\mathrm{DE}}(x_i^{\mathrm{DE}})|. \tag{9.4}$$

In addition, let E_{1001} be the maximum of absolute errors on 1001 equally-spaced points, defined by

$$E_{1001} = \max_{i=0,\,1,\,\dots,\,999,\,1000} |u(x_i) - u_N(x_i)|, \tag{9.5}$$

where $x_i = a + (b-a)i/1000$, and u_N denotes one of u_N^{RZ}, u_N^{SE}, u_N^{DE}. The errors are shown in Figure 1. From both of the table and figure, we can conclude that the modified RZ scheme is more accurate than the original RZ scheme, and the DE-Sinc scheme further improves the convergence profile. Furthermore, the rates of convergence in the graph correspond to the theoretical results in Theorem 6.14 and Theorem 8.7, i.e. $\mathrm{O}(\sqrt{N}\,\mathrm{e}^{-\sqrt{\pi d\alpha N}})$ and $\mathrm{O}(\exp(-\pi dN/\log(2dN/\alpha)))$. In the case of the original RZ scheme, $\mathrm{O}(\,\mathrm{e}^{-\sqrt{\pi d\alpha N/2}})$ is observed from the graph, which coincides with the discussion in Remark 2.7.

Next we consider the case where there is derivative singularity at the endpoint $x = a$.

Example 9.2 (Delves–Mohamed [3, Example 4.2.5]). Consider

$$u(x) - \int_0^{\pi/2} (xt)^{3/4} u(t)\,\mathrm{d}t = x^{1/2}\left\{1 - \frac{\pi^2}{9}\left(\frac{\pi x}{2}\right)^{1/4}\right\}, \quad 0 \leq x \leq \pi/2, \tag{9.6}$$

whose solution is $u(x) = x^{1/2}$.

In this case $u \in \mathbf{M}_{1/2}(\psi^{\mathrm{SE}}(\mathscr{D}_{\pi-\epsilon}))$ and $u \in \mathbf{M}_{1/2}(\psi^{\mathrm{DE}}(\mathscr{D}_{(\pi-\epsilon)/2}))$, and we select h using these parameters α and d. It is not straightforward to select $\tilde{h}$ in the RZ scheme since the strategy is not given to choose the parameters α and d appearing in (2.12). For the experiment, we choose $\alpha = 1/2$ and $d = \pi - \epsilon$, which are the same values as the modified RZ scheme. The results are plotted in Figure 2, and we can observe similar results to Example 9.1.

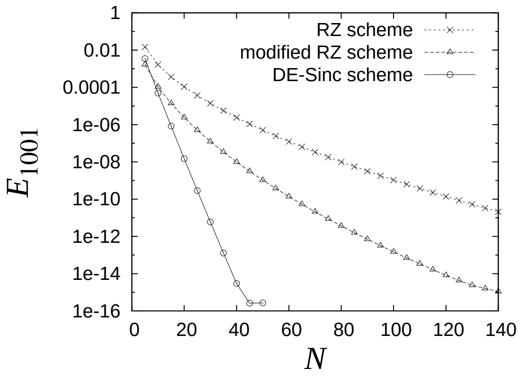

Figure 1. Errors of Example 9.1.

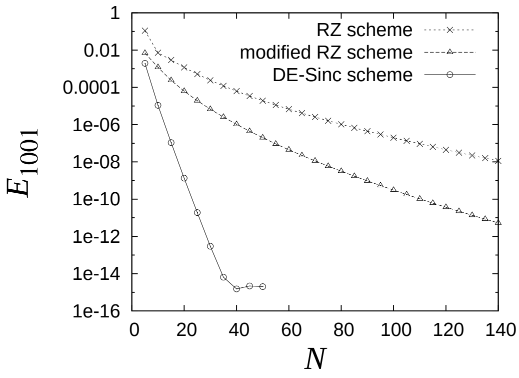

Figure 2. Errors of Example 9.2.

10 Concluding remarks

In this paper, two improved versions of the Sinc-collocation method by Rashidinia–Zarebnia [12] are derived: the modified RZ scheme (in Section 5) and the DE-Sinc scheme (in Section 7). The modified RZ scheme has the following advantages compared to the original one: (1) a concrete and optimal strategy to choose parameters α and d are given in Section 4; (2) the exponential convergence $\mathrm{O}(\exp(-c_1\sqrt{N}))$ is guaranteed in a rigorous manner in Section 6, and in fact the rate of convergence is better than the original one as seen in Section 9. In addition to the advantages, the DE-Sinc scheme gets much faster rate of convergence: $\mathrm{O}(\exp(-c_2 N/\log N))$; this is also confirmed theoretically and numerically in Section 8 and 9, respectively. It should be also noted that the schemes enjoy exponential convergence whether the functions to be approximated have endpoint singularities or not.

Future works include followings. Firstly, the Sinc-collocation method for Volterra integral equations has also been proposed by Rashidinia–Zarebnia [14], and we can establish similar results to the present paper. Secondly, as described in Remark 6.11, the issue of rigorous convergence analysis has been left for the Sinc-Nyström methods, in both cases of the SE transformation [13] and the DE transformation [9]. Thirdly, error estimates with explicit constants that users can compute are desired for verified numerical computation. This can be done by examining the constants appearing in the convergence theorems, and using the results in [10]. We are now working on these matters, and the results will be reported somewhere else soon.

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