MATHEMATICAL ENGINEERING TECHNICAL REPORTS

Cooperative Gain Output Feedback Stabilization for Multi-agent Dynamical Systems

Shinji HARA, Masaaki KANNO and Hideaki TANAKA

(Communicated by Kazuo MUROTA)

METR 2009–37 September 2009

DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

WWW page: http://www.keisu.t.u-tokyo.ac.jp/research/techrep/index.html

Cooperative Gain Output Feedback Stabilization for Multi-agent Dynamical Systems

Shinji Hara*, Masaaki Kanno†, and Hideaki Tanaka*

September 1st, 2009

Abstract

This paper is concerned with cooperative stabilization for LTI homogeneous multi-agent dynamical systems. We first formulate the cooperative stabilization problem by constant output feedback and show that it can be reduced to a stabilization problem with complex gain feedback. We then present several classes of systems in which the system is cooperatively stabilizable if and only if it can be stabilized alone. We also show a multi-agent system with even number of agents whose dynamics is represented by a 4th order transfer function, which can be stabilized by cooperation even if any single agent alone is not stabilizable.

1 Introduction

Due to the insatiable growth of computation speed of the computer and the increasing demand for complex networking, modern engineering systems have become more and more complex, hierarchical, and subject to a multitude of system dimensions. This also motivates us to analyze complex dynamical systems in nature such as bio systems and atmosphere.

To cope with these challenges, many studies of different approaches in a variety of areas have been made. One of the bulk flows in these studies is the decentralized autonomous control of the multi-agent systems (see e.g., [7] and references therein.). There have been many researches in the form of proposing a specific approach within an individual problem formulation, but very few results are available so far to provide a unifying theoretical framework and most of researches are focused on the analysis

*Department of Informattion Physics and Computing, Graduate School of Information Science and Engineering, The University of Tokyo, 7-3-1 Hongo, Bunkyo, Tokyo 113-8656, Japan, E-mail: {shinji_hara, hideaki_tanaka}@ipc.t.u-tokyo.ac.jp

†Institute of Science and Technology, Academic Assembly, Niigata University, 8050 Ikarashi 2-no-cho, Nishi-ku, Niigata-shi 950-2181, Japan, E-mail: kanno@ie.niigata-u.ac.jp

such as stability rather than synthesis issues including stabilization. In analyzing such complex, large-scale systems from the control-theoretic point of view, it is possible to address these issues by applying existing methods to high-dimensional systems in its standard form. However, since the system dimension of the overall system can grow considerably fast as the dimensions of its subsystems become large, it is important to establish a unified approach which provides us with a methodology for analyzing and designing (large-scale) multi-agent dynamical systems *in which agents autonomously cooperate through mutual information exchange.*

Recently, the authors and their research group introduced a new framework as one of the unified expressions of multi-agent systems. It is called *a linear time-invariant system with a generalized frequency variable* [3, 4]. Specifically, the transfer function $\mathcal{G}(s)$ representing the overall dynamics of a multi-agent system is described by simply replacing s by a rational function $\phi(s)$ in a transfer function $G(s)$, i.e., $\mathcal{G}(s) := G(\phi(s))$. We call $\phi(s)$ the generalized frequency variable, although the similar form were introduced in [8] as one of general extensions and the robust stability was investigated. This class of system descriptions has a potential to provide a theoretical foundation for analyzing and designing large-scale dynamical systems in a variety of areas. For example, the framework of the generalized frequency variable can be applied to the analysis and synthesis of central pattern generators (CPGs) [6] and gene-protein regulatory networks [1, 9] as well as consensus and formation problems as surveyed in [7].

There are two motivations for focusing on stabilization rather than stability in this paper.

- **Theoretical Point of View:** There exists a very fundamental natural question stated as "Is there any advantage of cooperation?", or "Is there any case where $h(s)$ is not stabilizable alone (not "solely stabilizable") but it is stabilizable by cooperation of multiple agents ("cooperatively stabilizable") ?"

- **Application Point of View:** In gene regulatory networks, the linearized model $h(s)$ should be unstable for the existence of limit cycle oscillations, but the total system should be stable for synchronization.

We here only investigate the most fundamental case, i.e., *constant output feedback stabilization* by focusing on the gap between the sole stabilizability and cooperative stabilizability.

The organization of this paper is as follows. In Section 2, we introduce the class of linear systems with generalized frequency variables and provide their dynamical equations in the frequency and time domains. Section 3 is devoted to the stability condition. In Section 4, we define the cooperative stabilization by gain output feedback and explain related notions and the several properties. The investigations for 2nd order subsystems and two classes of 3rd order subsystems are made in Section 5. Section 6 derives a result on the gap between the cooperative stabilization and the sole stabilization for three class of higher order subsystems and applies it to an inverted pendulum system. In Section 7, we give a numerical example of 4th order system which is cooperatively stabilizable but not solely stabilizable. Finally, we provide concluding remarks in Section 8.

We use the following notation. The sets of real numbers and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. The complex conjugate of $z \in \mathbb{C}$ is denoted by $\bar{z}$. For a square matrix A, the set of eigenvalues is denoted by $\sigma(A)$. For matrices A and B, $A \otimes B$ means their Kronecker product. Let the open left-half complex plane be denoted by $\mathbb{C}_- := \{s \in \mathbb{C} \mid \mathrm{Re}(s) < 0\}$.

2 Linear Systems with Generalized Frequency Variable

In this section, we introduce linear systems with generalized frequency variables and provide their dynamical equations in the frequency and time domains, or the transfer function description and the state-space realization [3, 4]. Consider the linear time-invariant system described by the transfer function

$$\mathcal{G}(s) = C\left(\frac{1}{h(s)}I_n + A\right)^{-1} B + D = \mathcal{F}_u\left(\begin{bmatrix} -A & B \\ C & D \end{bmatrix}, h(s)I_n\right), \qquad (1)$$

where $h(s)$ is a single-input single-output, k-th order, strictly proper transfer function, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, and $\mathcal{F}_u$ denotes the upper linear fractional transformation.

The system $\mathcal{G}(s)$ can be viewed as an interconnection of n identical agents, each of which has the internal dynamics $h(s)$. As depicted in Fig. 1, the interconnection structure is specified by A, and the input-output structure for the whole system is specified by B, C, and D. Defining the standard transfer function as

$$G(s) = C(sI_n + A)^{-1}B + D, \qquad (2)$$

the system $\mathcal{G}(s)$ can be described as

$$\mathcal{G}(s) = G(\phi(s)), \quad \phi(s) := 1/h(s). \qquad (3)$$

Note that the variable 's' in (2) characterizes frequency properties of the transfer function $G(s)$ and that $\mathcal{G}(s)$ is generated by simply replacing 's' by '$\phi(s)$' in G. Hence, we say that the system (3) is described by the transfer function G with the *generalized frequency variable* $\phi(s)$ [3, 4].

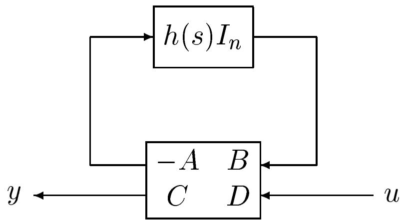

Figure 1: LTI Systems with Generalized Frequency Variable

Let $h(s)$ have a minimal realization $h(s) \sim (A_h, b_h, c_h, 0)$, where $A_h \in \mathbb{R}^{k\times k}$, $b_h \in \mathbb{R}^k$, $c_h \in \mathbb{R}^{1\times k}$. It can be shown [3, 4] that a realization of $\mathcal{G}(s)$ is given by $\mathcal{G}(s) \sim (\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$, where

$$\begin{aligned} \mathcal{A} &= I_n \otimes A_h - A \otimes (b_h c_h) \in \mathbb{R}^{nk\times nk}, \\ \mathcal{B} &= B \otimes b_h \in \mathbb{R}^{nk\times m}, \quad \mathcal{C} = C \otimes c_h \in \mathbb{R}^{p\times nk}, \\ \mathcal{D} &= D \in \mathbb{R}^{p\times m}, \end{aligned} \tag{4}$$

or

$$\begin{aligned} \mathcal{A} &= A_h \otimes I_n - (b_h c_h) \otimes A \in \mathbb{R}^{nk\times nk}, \\ \mathcal{B} &= b_h \otimes B \in \mathbb{R}^{nk\times m}, \quad \mathcal{C} = c_h \otimes C \in \mathbb{R}^{p\times nk}, \\ \mathcal{D} &= D \in \mathbb{R}^{p\times m}. \end{aligned} \tag{5}$$

As seen in the formulae of the state-space realizations, the size of the state vector is nk which is normally quite large. Hence, it is not our approach to directly deal with the realization of $\mathcal{G}(s)$ ignoring the structural information, but rather, we will aim to investigate the fundamental properties such as stability of $\mathcal{G}(s)$ in terms of $G(s)$ and $h(s)$.

3 Stability Condition

As shown in [3, 4], $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ is minimal if (A, B, C, D) and $(A_h, b_h, c_h, 0)$ are both minimal realizations. This is a distinct feature in comparison with corresponding results for the standard cascade, parallel and feedback connections of two systems, where we need to care pole/zero cancellations. Since there is no chance of occurrences of pole/zero cancellations, the linear time-invariant system with the generalized frequency variable $\mathcal{G}(s)$ given by (3) is stable, or all the poles of $\mathcal{G}(s)$ are in $\mathbb{C}_-$, if and only if the feedback system $\Sigma(h(s), A)$ depicted in Fig 2 is internally stable. In other words, the stability of $\mathcal{G}(s)$ is equivalent to that of

$$\mathcal{H}_A(s) := \left(\frac{1}{h(s)} I + A\right)^{-1} = (\phi(s) I + A)^{-1}, \tag{6}$$

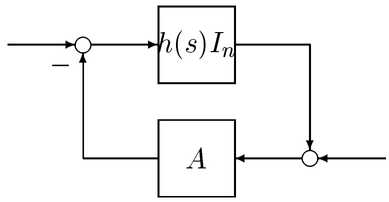

Figure 2: Feedback system $\Sigma(h(s), A)$

or $\mathcal{H}_A(s)$ is proper and analytic in the closed right half complex plane.

Thus, the problem is now to find a necessary and sufficient condition for stability of the linear time-invariant system (6) in terms of the generalized frequency variable $\phi(s) := 1/h(s)$ and the interconnection matrix A. There are two fundamental results on dynamical system theory, namely a Hurwitz-type stability test for characteristic polynomials with complex coefficients in [2] and a generalized Lyapunov inequality in [5, 10], which play key roles in deriving a systematic way of checking the stability.

The following proposition together with the Hurwitz-type stability test in [2] leads to a necessary and sufficient condition for the stability of $\mathcal{H}_A(s)$ [10].

Proposition 1 *Let a matrix $A \in \mathbb{R}^{n\times n}$ and a strictly proper rational function $h(s) = n(s)/d(s)$ be given. Define $\mathcal{H}_A(s)$ by (6) and $p(\lambda, s)$ by*

$$p(\lambda, s) := d(s) + \lambda n(s)\ . \tag{7}$$

Suppose that $n(s)$ and $d(s)$ are coprime. The following statements are equivalent:

(i) $\mathcal{H}_A(s)$ is stable.

(ii) $\sigma(A) \subset \Lambda := \{\ \lambda \in \mathbb{C} \mid p(\lambda, s)$ is Hurwitz $\}$.

4 Cooperative Stabilization

4.1 Notions of stabilizability

We here define several notions of stabilizability for $h(s)$. We say that $h(s)$ is *solely stabilizable* if there exists a constant output feedback gain so that the closed-loop system is stable, or all the poles are in the open left half complex plane $\mathbb{C}_-$. We say that $h(s)$ is *cooperatively stabilizable* if there exists a matrix $A \in \mathbb{R}^{n\times n}$ such that all the poles of $\mathcal{H}_A(s)$ are stable.

There are two motivations for considering the stabilization problem as stated in the introduction section. One is a purely theoretical motivation, and the other is from the view point of application for synchronization of a bunch of oscillatory elements.

Let us introduce the following two related notions of stabilizability for $h(s)$:

- **real gain output feedback stabilizability** (*Real-GOFS*): $h(s)$ is *Real-GOFS* if there exists a real scalar gain $k \in \mathbb{R}$ so that $p(\lambda, s) := d(s) + kn(s) = 0$ has no roots in the closed right half complex plane.
- **complex gain output feedback stabilizability** (*Complex-GOFS*): $h(s)$ is *Complex-GOFS* if there exists a complex scalar gain $\lambda \in \mathbb{C}$ so that $p(\lambda, s) := d(s) + \lambda n(s) = 0$ has no roots in the closed right half complex plane.

It is clear that *Real-GOFS* implies *Complex-GOFS*, and we have a very interesting question in engineering: "Is there a gap between *Real-GOFS* and *Complex-GOFS*?" This actually corresponds to our original question from the view point of control theory, because we can readily see the following facts:

Proposition 2

- *$h(s)$ is solely stabilizable if and only if it is Real-GOFS.*
- *$h(s)$ is cooperatively stabilizable if and only if it is Complex-GOFS.*

4.2 Properties on cooperative stabilization

This subsection is devoted to several properties on the cooperative stabilization.

It is obvious that any stable $h(s)$ is solely stabilizable and hence it is cooperatively stabilizable. Although [3] claimed that any minimum phase system is cooperatively stabilizable, it is not true as seen in the following counterexample.

Let us consider a 2nd order system expressed as

$$h(s) = \frac{1}{s(s-1)}.$$

Since the characteristic polynomial for output feedback with constant gain k is $s^2 - s + k$, we can see that the system cannot be stabilized for any choice of k. In other words, the system is not solely stabilizable (*Real-GOFS*). Let us examine the cooperative stabilizability for $n = 2$. Define

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

and calculate the characteristic polynomial. Then we have

$$\begin{aligned}\left|\frac{1}{h(s)}I + A\right| &= \begin{vmatrix} s(s-1)+a_1 & a_2 \\ a_3 & s(s-1)+a_4 \end{vmatrix} \\ &= s^4 - 2s^3 + (a_1 + a_4 + 1)s^2 - (a_1 + a_4)s + a_1a_4 - a_2a_3 \ .\end{aligned}$$

Since the coefficient of the s^3 term is always negative no matter how we choose A, we can see the system is not cooperatively stabilizable for $n = 2$. Indeed, we can show that the system is not cooperatively stabilizable for any n by applying Proposition 3 given below.

We can see from the stability condition derived in [3, 4, 10] that stabilizability for a given $h(s)$ is equivalent to the non-emptiness of Ω_+^c, which is defined as the complement of the image of $\phi(s)$ for $\mathrm{Re}(s) \geq 0$ in the complex plane. It is also trivial that $h(s)$ is solely stabilizable if and only if the corresponding Ω_+^c includes a segment of the real axis. Note that

- If n is odd, the matrix A has at least one real eigenvalue, and we can set A so that all the eigenvalues of A are real.
- If n is even, we can choose a matrix A of which all the eigenvalues are purely complex numbers.

Then, we have the following proposition.

Proposition 3

- *When n is odd, $h(s)$ is cooperatively stabilizable if and only if $h(s)$ is Real-GOFS.*
- *When n is even, $h(s)$ is cooperatively stabilizable if and only if $h(s)$ is Complex-GOFS.*

The second statement of the above proposition yields a purely mathematical problem defined as follows:

For a given strictly proper transfer function $h(s)$, the problem is to find a necessary and sufficient condition for the existence of 2 by 2 real constant matrix A so that the numerator of $|I_2 + h(s)A|$ is a Hurwitz polynomial.

We will focus on the gap between *Real-GOFS* and *Complex-GOFS* in the subsequent sections.

5 Cooperative Stabilizability for Low Order Systems

5.1 Stability test

We first show a useful stability condition to check the condition (ii) in Proposition 1. It is based on an interlacing property of even and odd parts of the characteristic polynomial.

Lemma 1 *[5, p. 334]*
For a given polynomial of order n with complex coefficients expressed as

$$p(s) = \sum_{i=0}^{n} (\alpha_i + \beta_i j) s^i \ , \tag{8}$$

define two real polynomials $\phi_R(s)$ and $\phi_I(s)$ as

$$p(js) := \phi_R(s) + \phi_I(s) j \ . \tag{9}$$

Then, $p(s)$ is Hurwitz, or all the roots of $p(s) = 0$ lie in the open left half complex plane, if and only if the following conditions hold:

(i) $\alpha_{n-1}\alpha_n + \beta_{n-1}\beta_n > 0$.

(ii) All the roots of $\phi_R(s)$ and $\phi_I(s)$ are real, simple, and interlacing.

5.2 Stabilizability for 2nd order systems

We will investigate the cooperative stabilizability for strictly proper 2nd order systems based on Lemma 1. Since any strictly proper 2nd order system can be represented by

$$h_{21}(s) = \frac{cs + 1}{s^2 + as + b} \ , \tag{10}$$

or

$$h_{22}(s) = \frac{s}{s^2 + as + b} \tag{11}$$

with appropriate constant gain, we only treat the above two classes of systems. The goal of this subsection is to show the equivalence of *Real-GOFS* and *Complex-GOFS* for both $h_{21}(s)$ and $h_{22}(s)$. In other words, we will prove for any strictly proper 2nd order system $h(s)$ that it is cooperatively stabilizable if and only if it is solely stabilizable, or $h(s)$ is not cooperatively stabilizable if it cannot be stabilized by itself.

Example 1: Consider a system $h_{21}(s)$ expressed as (10). Since the corresponding $p(\lambda, s)$ is given by

$$p(\lambda, s) = s^2 + as + b + \lambda(cs + 1) = s^2 + (a + cx + cyj)s + (b + x + yj) \ ,$$

we have

$$\phi_R(s) = -s^2 - cys + (b + x), \ \ \phi_I(s) = (a + cx)s + y$$

We first see that x should be chosen so that

$$a + cx > 0 \tag{12}$$

holds in order to satisfy the condition (i) in Lemma 1. Since the root of $\phi_I(s) = 0$ is given by $z_1 := -y/(a + cx)$,

$$\phi_R(z_1) = -\frac{y^2}{(a+cx)^2} + \frac{cy^2}{a+cx} + (b+x) > 0$$

should hold for satisfying the interlacing property. It is noted that the above condition is equivalent to

$$(b+x)(a+cx)^2 - \{1 - c(a+cx)\}y^2 > 0 \ . \tag{13}$$

We make a detailed investigation by considering two cases depending on the sign of c.

- **Case 1 ($c > 0$) :** Let us set $y = 0$ and choose any x satisfying $x > \max\{-a/c, -b\}$. Then, we see that both the conditions (12) and (13) hold, and hence $h(s)$ is *Real-GOFS*, which implies that it is *Complex-GOFS*.
- **Case 2 ($c \leq 0$) :** Since $1 - c(a + cx) \geq 1$, the inequality condition (13) is rewritten as

$$y^2 < \frac{(b+x)(a+cx)^2}{1 - c(a+cx)} \ .$$

The condition always holds for $y = 0$ if it holds for non-zero y. This implies that $h_{21}(s)$ is *Real-GOFS* if it is *Complex-GOFS*.

Example 2: $p(\lambda, s)$ for $h_{22}(s)$ defined by (11) is expressed as

$$p(\lambda, s) = s^2 + as + b + \lambda s = s^2 + (a + x + yj)s + b \ .$$

Hence, we have

$$\phi_R(s) = -s^2 - ys + b, \ \ \phi_I(s) = (a+x)s \ .$$

In order to satisfy the condition (i) in Lemma 1, x should be chosen so that the inequality (12) holds. Since $\phi_I(s) = 0$ has a root at the origin regardless of the choice of y, $\phi_R(s) = 0$ has to have one positive root and one negative root. Hence, $b > 0$ is required. It is clear that any real number x satisfying (12) stabilizes $h_{22}(s)$, which implies the equivalence of *Real-GOFS* and *Complex-GOFS*.

5.3 Stabilizability for 3rd order systems

We here investigate the cooperative stabilizability for two classes of 3rd order systems using Lemma 1. The target classes are

$$h_{31}(s) = \frac{1}{s^3 + as^2 + bs + c} \ , \tag{14}$$

and

$$h_{32}(s) = \frac{s}{s^3 + as^2 + bs + c} \ . \tag{15}$$

We will show that *Complex-GOFS* is equivalent to *Real-GOFS* for any $h(s)$ in the classes, or $h(s)$ is not cooperatively stabilizable if it cannot be stabilized alone.

Example 3: Noting that $p(\lambda, s)$ for $h_{31}(s)$ defined by (14) is expressed as

$$p(\lambda, s) = s^3 + as^2 + bs + c + \lambda = s^3 + as^2 + bs + (c + x + yj) ,$$

we have

$$\phi_R(s) = -as^2 + (x + c), \quad \phi_I(s) = -s^3 + bs + y .$$

It is obvious from the condition (i) in Lemma 1 that a should be positive. Moreover, x has to be chosen so that

$$c + x > 0$$

holds in order for $\phi_R(s) = 0$ to have two real roots. Since $\phi_I(s)$ is a monotonically decreasing function with respect to a real number s for $b \leq 0$, b should be positive no matter how we choose y. Hence, $a > 0$ and $b > 0$ are necessary conditions for the cooperative stabilizability.

Let us suppose $y = 0$ under the assumptions of $a > 0$ and $b > 0$. We can see that $\phi_I(s) = 0$ has three real roots at $s = 0$ and $s = \pm\sqrt{b}$. Therefore, the interlacing property holds for any x satisfying

$$x + c < ab .$$

This implies that the positivities of $a > 0$ and $b > 0$ are the necessary and sufficient condition for the sole stabilizability, and hence we can conclude that *Real-GOFS* and *Complex-GOFS* are equivalent.

Example 4: Since $p(\lambda, s)$ for $h_{32}(s)$ defined by (15) is expressed as

$$p(\lambda, s) = s^3 + as^2 + (b + x + yj)s + c ,$$

we have

$$\phi_R(s) = -as^2 - ys + c, \quad \phi_I(s) = -s^3 + (b + x)s .$$

Similar to Example 3, a should be positive for the condition (i) in Lemma 1. We can also see that $\phi_I(s) = 0$ has three real roots at $s = 0$ and $s = \pm\sqrt{b + x}$ for any choice of y, if we choose x so that $b + x$ is positive. Note that the necessary and sufficient condition for $\phi_R(s) = 0$ having one positive and one negative roots is $c > 0$ and $y^2 + 4ac > 0$. Under this condition with $y > 0$, the interlacing property can be written as

$$2\sqrt{x + b} > y + \sqrt{y^2 + 4ac} .$$

Since the right hand side of the inequality is a monotonically increasing function in the non-negative interval $y \geq 0$, the inequality holds for $y = 0$ if it is satisfied for a positive number y. Since the similar discussion is valid for $y < 0$, we can prove the equivalence of *Real-GOFS* and *Complex-GOFS*.

6 Cooperative Stabilizability for Higher Order Systems

6.1 Cooperative stabilizability vs sole stabilizability

We can generalize the results for the two classes of 3rd order systems shown in the previous section to three classes of higher order systems. The main theorem of this section is as follows.

Theorem 1 *Real-GOFS and Complex-GOFS are equivalent, or $h(s)$ is not cooperatively stabilizable if it cannot be stabilized alone, for any $h(s)$ which belongs to one of the following three classes:*

$$\mathcal{H}_0(s) \triangleq \{\ h(s) = \frac{k}{d(s)} \mid k \neq 0\ \}\ , \tag{16}$$

$$\mathcal{H}_1(s) \triangleq \{\ h(s) = \frac{ks}{d(s)} \mid k \neq 0,\ a_0 \neq 0\ \}\ , \tag{17}$$

$$\mathcal{H}_2(s) \triangleq \{\ h(s) = \frac{k(s^2-b^2)}{d(s)} \mid k \neq 0,\ d(\pm b) \neq 0\}\ , \tag{18}$$

where

$$d(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0\ . \tag{19}$$

(Proof) Let us express the numerator of $h(s)$ as

$$n(s) = \alpha s^2 + \beta s + \gamma \tag{20}$$

and define $d_R(s)$ and $d_I(s)$ by

$$d(js) = d_R(s) + d_I(s)j\ .$$

Then, we have

$$\begin{aligned} \Phi_R(s) &= d_R(s) + (-\alpha x s^2 - \beta y s + \gamma x)\ , && (21)\\ \Phi_I(s) &= d_I(s) + (-\alpha y s^2 + \beta x s + \gamma y)\ , && (22) \end{aligned}$$

since

$$\lambda n(s) = (\alpha x s^2 + \beta x s + \gamma x) + (\alpha y s^2 + \beta y s + \gamma y)j\ .$$

We first assume $\beta = 0$ in order to prove the classes for $\mathcal{H}_0(s)$ and $\mathcal{H}_2(s)$. It is clear that $\Phi_R(s)$ does not depend on the value of y. Let $s_1 < s_2 < \cdots < s_\ell$ be real roots of $\Phi_R(s) = 0$, where $\ell = n$ and $\ell = n-1$ when n is even and odd, respectively.

• n **is even:** Set $z_1 = s_i$ and $z_2 = s_{i+1}$ for any $i = 1, 2, \cdots, n-1$, and define q_1 and q_2 as $q_1 = d_I(z_1)$ and $q_2 = d_I(z_2)$. Then, the interlacing property can be expressed as

$$\Phi_I(z_1) \cdot \Phi_I(z_2) < 0\ .$$

In other words, if $h(s)$ is *Complex-GOFS* then there exists a $\lambda = x_* + y_* j$ which satisfies

$$\begin{aligned}\Phi_I(z_1)\cdot\Phi_I(z_2) &= \{q_1 - (\alpha z_1^2 - \gamma)y_*\}\cdot\{q_2 - (\alpha z_2^2 - \gamma)y_*\} \\ &= q_1 q_2 - \{q_1(\alpha z_2^2 - \gamma) + q_2(\alpha z_1^2 - \gamma)\}y_* \\ &\quad +(\alpha z_1^2 - \gamma)(\alpha z_2^2 - \gamma)y_*^2 \ < \ 0\ . \end{aligned} \tag{23}$$

Noting that the condition holds for $\lambda = x_* - y_* j$ if it is satisfied for $\lambda = x_* + y_* j$, we have

$$q_1 q_2 + \{q_1(\alpha z_2^2 - \gamma) + q_2(\alpha z_1^2 - \gamma)\}y_* + (\alpha z_1^2 - \gamma)(\alpha z_2^2 - \gamma)y_*^2 \ < \ 0\ . \tag{24}$$

Adding two inequalities (23) and (24) leads to

$$q_1 q_2 + (\alpha z_1^2 - \gamma)(\alpha z_2^2 - \gamma)y_*^2 \ < \ 0\ . \tag{25}$$

Suppose that $\alpha\gamma \le 0$ holds. Then, we can readily see that $\lambda = x_*$ also satisfies the condition, which implies the equivalence of *Real-GOFS* and *Complex-GOFS*.

• n **is odd:** The difference we should pay attention to in comparison with the even case is that the order of $\Phi_R(s)$ is smaller than that of $\Phi_I(s)$ by 1. Therefore, the number of inequalities for checking the interlacing property is $n-2$, and hence we have to prove that $\Phi_I(s) = 0$ has a real root smaller than s_1.

Suppose $n = 4m+1$. Then the condition can be written as

$$q_1 - (\alpha s_1^2 - \gamma)y_* > 0\ ,$$

since $\Phi_I(-\infty) < 0$ holds. Noting that the same inequality should be satisfied for $-y_*$, we see $q_1 > 0$ has to hold, and hence it is satisfied even for $y = 0$.

Since a similar discussion is valid for $n = 4m-1$, the proofs for $\mathcal{H}_0(s)$ and $\mathcal{H}_2(s)$ are now complete.

We focus on the case for $\mathcal{H}_1(s)$. Since $n(s) = s$, we have

$$\Phi_R(s) = d_R(s) - ys,\ \ \Phi_I(s) = d_I(s) + xs\ .$$

Therefore, we see that roots of $\Phi_I(s) = 0$ are independent of the choice of y and that $\Phi_I(s)$ is an odd function with a root at the origin $s = 0$. Noting this fact, we define z_1 and z_2 as consecutive real positive roots of $\Phi_I(s) = 0$. Then, the interlacing property can be expressed as

$$\begin{aligned}\Phi_R(z_1)\cdot\Phi_R(z_2) &= \{d_R(z_1) - z_1 y_*\}\cdot\{d_R(z_1) - z_1 y_*\} \\ &= d_R(z_1)d_R(z_2) - \{d_R(z_1)z_2 + d_R(z_2)z_1\}y_* \\ &\quad +z_1 z_2 y_*^2 \ < \ 0\ .\end{aligned}$$

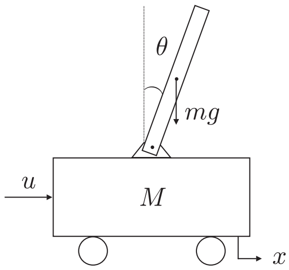

Figure 3: The inverted pendulum system.

Adding the corresponding inequality for $-y_*$ to the above inequality, we have

$$d_R(z_1)d_R(z_2) + z_1 z_2 y_*^2 < 0 \ . \tag{26}$$

We can see from this together with $z_1 z_2 > 0$ that the interlacing property holds even for $y = 0$. It is trivial that the interlacing property for $s = 0$ holds, because $\Phi_R(s) = d_R(s)$ does not depend on the value of y. Consequently, the claim is true for odd numbers of n. We can prove the odd case by a similar treatment done for the cases of $\mathcal{H}_0(s)$ and $\mathcal{H}_2(s)$. $\square$

6.2 Example: inverted pendulum

For an illustration of our results on tracking performance limitations we consider the inverted pendulum system shown in Fig. 3, where an inverted pendulum is mounted on a motor driven-cart. We assume that the pendulum moves only in the vertical plane, i.e., we consider a two dimensional control problem. We here assume that M, m, and 2ℓ respectively denote the mass of the cart, the mass of the pendulum, and the length of the pendulum. We also assume that the friction between the track and the cart is $\mu_{\rm t}$ and that between the pendulum and the cart is $\mu_{\rm p}$. We consider an uniform pendulum so that its inertia is given by $I = \frac{1}{3}m\ell^2$.

The equations of motion between the control input u, or the force to the cart, and the position of the cart x and the angle of the pendulum θ are represented by

$$\begin{array}{l} \frac{4}{3}m\ell^2\ddot{\theta} + \mu_{\rm p}\dot{\theta} - mg\ell\theta = -m\ell\ddot{x} \ , \\ (M+m)\ddot{x} + \mu_{\rm t}\dot{x} + m\ell\ddot{\theta} = u \ , \end{array}$$

under the assumption that the angle θ is small. Taking the Laplace transform of the system equations yields

$$\begin{array}{l} \frac{4}{3}m\ell^2\Theta(s)s^2 + \mu_{\rm p}\Theta(s)s - mg\ell\Theta(s) = -m\ell X(s)s^2, \\ (M+m)X(s)s^2 + \mu_{\rm t}X(s)s + m\ell\Theta(s)s^2 = U(s) \ . \end{array}$$

Then, the transfer functions from u to θ (denoted by P_θ) and from u to x

(denoted by P_x) are, respectively, expressed as

$$P_\theta(s) = \frac{-m\ell s}{D(s)}\ ,\ \ P_x(s) = \frac{\frac{4}{3}m\ell^2 s^2 + \mu_{\rm p} s - mg\ell}{sD(s)}\ ,$$

where

$$D(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

with $a_0 := -\mu_{\rm t} mg\ell$, $a_1 := -(M+m)mg\ell + \mu_{\rm p}\mu_{\rm t}$, $a_2 := (M+m)\mu_{\rm p} + \frac{4}{3}\mu_{\rm t} m\ell^2$, and $a_3 := \frac{1}{3}(4M+m)m\ell^2$.

Note that the constant term of the denominator of $P_\theta(s)$, or a_0, is negative and cannot be changed by any real constant feedback, since the numerator has a zero at $s = 0$. Similarly, we see that the term $a_0 s$ in the denominator of $P_x(s)$ is irrelevant to the choice of the real feedback gain. These investigates show that both $P_\theta(s)$ and $P_x(s)$ are not *Real-GOFS*.

Now apply Theorem 1 to $P_\theta(s)$. Since $P_\theta(s)$ belongs to $\mathcal{H}_1(s)$ defined in (18), we see that $P_\theta(s)$ is not *Complex-GOFS*, or it is not cooperatively stabilizable. We also see that $P_x(s)$ with $\mu_{\rm p}$ is in $\mathcal{H}_2(s)$, and hence it is not cooperatively stabilizable. This together with the continuity property with respect to $\mu_{\rm p}$ yields that $P_x(s)$ is not cooperatively stabilizable for sufficiently small $\mu_{\rm p} > 0$.

7 An Example: *Real-GOFS* $\neq$ *Complex-GOFS*

It seems non-trivial to find a transfer function $h(s)$ with order less than or equal to three that has no gap between *Real-GOFS* and *Complex-GOFS*. We now show an example of 4th order system which has the gap, i.e., it is cooperatively stabilizable (*Complex-GOFS*) but not solely stabilizable (*Real-GOFS*).

7.1 System description

The transfer function $h(s)$ of an agent is given by

$$h(s) = -\frac{100(s+2)(\frac{19}{10}s^2 - \frac{1}{500000}s + \frac{21}{10})}{(s-1)^2(s+1)(s+100)}\ , \tag{27}$$

and the corresponding $p(\lambda, s)$ is expressed as

$$p(\lambda, s) = (s-1)^2(s+1)(s+100) - \lambda \cdot 100(s+2)(\frac{19}{10}s^2 - \frac{1}{500000}s + \frac{21}{10})\ .$$

The even and odd parts of

$$p(\lambda, js) = \phi_R(s) + j\phi_I(s)$$

are expressed as follows:

$$\phi_R(s) = s^4 - 190ys^3 + (101 + \frac{1899999}{5000}x)s^2 + \frac{524999}{2500}ys - 420x + 100 ,$$

$$\phi_I(s) = (190x - 99)s^3 + \frac{1899999}{5000}ys^2 + (-\frac{524999}{2500}x - 99)s - 420y .$$

7.2 Stabilizability

We first investigate the stabilizability by real gain feedback (*Real-GOFS*). For $y = 0$, $\phi_R(s)$ can be written as

$$\phi_R(s) = s^4 + (101 + \frac{1899999}{5000}x)s^2 - 420x + 100 .$$

Hence, the condition for $\phi_R(s)$ having four real roots is given by

$$\begin{aligned} & \phi_R(0) > 0 \text{ and } -(101 + \frac{1899999}{5000}x) > 0 \\ & \text{and } (101 + \frac{1899999}{5000}x)^2 - 4(-420x + 100) > 0 \\ \Leftrightarrow \quad & x < 0.238095\cdots \text{ and } x < -0.265789\cdots \\ & \text{and } x > -0.194819\cdots, x < -0.348393\cdots \\ \Leftrightarrow \quad & x < -0.348393\cdots . \end{aligned} \tag{28}$$

Since $\phi_I(s)$ is represented by

$$\phi_I(s) = s\{(190x - 99)s^2 + (-\frac{524999}{2500}x - 99)\} ,$$

the condition for $\phi_I(s)$ having three real roots is given by

$$\frac{(\frac{524999}{2500}x + 99)}{(190x - 99)} > 0 \;\Leftrightarrow\; x > 0.521052\cdots, x < -0.471429\cdots .$$

Let $s = 0, z_+, z_-$ be the three roots of $\phi_I(s) = 0$. Then, the interlacing property among real roots of $\phi_R(s)$ and $\phi_I(s)$ is expressed as

$$\phi_R(z_+) < 0 \;\Leftrightarrow\; x > 6.167\cdots \times 10^5,\ 0 < x < 0.5174\cdots .$$

This inequality clearly violates inequality (28), which implies that this system is not *Real-GOFS*.

7.3 Stability region

This section is devoted to the derivation of the stability region for cooperative feedback. To this end, we consider the cooperative feedback control law represented by

$$\mathbf{u} = F\mathbf{y} = -A\mathbf{y} \tag{29}$$

where $\mathbf{u} := [u_1, \cdots, u_n]^T$, $\mathbf{y} := [y_1, \cdots, y_n]^T$, and $F := -A \in \mathbb{R}^{n\times n}$. The stability region Ω_+^c, or the allocatable region in the complex plane in which all the eigenvalues of F should lie to guarantee the feedback stability, can be represented by a series of inequalities which are systematically derived by a Hurwitz-type stability test developed in [10]. The sequence of inequality conditions $\Delta_i > 0$ $(i = 1, 2, 3, 4)$ any eigenvalue $\lambda = x + jy$ of F should satisfy are given by

$$\Delta_1 > 0 \Leftrightarrow x > -99/190$$

and the shaded regions in Figs. 4 $\sim$ 6. The region which satisfies all four inequality conditions are illustrated in Fig. 7. We can see that the region does not include the real axis in the complex plane. Hence, there exists no real gain which stabilizes $h(s)$. However, the stability region Ω_+^c is non-empty as seen in Fig. 7, and hence $h(s)$ can be stabilized by choosing F so that all the eigenvalues lie in the shaded area of Fig. 7.

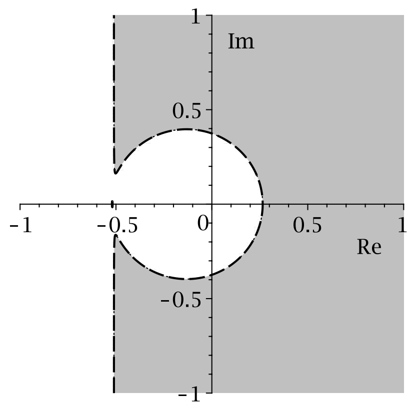

Figure 4: Region satisfying $\Delta_2 > 0$

Applying the stability condition provided in [10], the stability region in which all the eigenvalues of F should lie is illustrated in the shaded region of Fig. 7. We can see from the figure that the stability region does not include the real axis, and hence we can conclude that $h(s)$ is not solely stabilizable. However, the region is not empty, which implies that $h(s)$ is cooperatively stabilizable. Actually, any multi-agent system with even number of agents can be stabilized if we set the matrix F so that all the eigenvalues lie in the shaded region.

For instance, if we set

$$F = -\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}, \tag{30}$$

or the two eigenvalues of F are set at $(-1 \pm j)/2$ marked as white circles in Fig. 7 . Then, we can stabilize the total system. Note that the control law

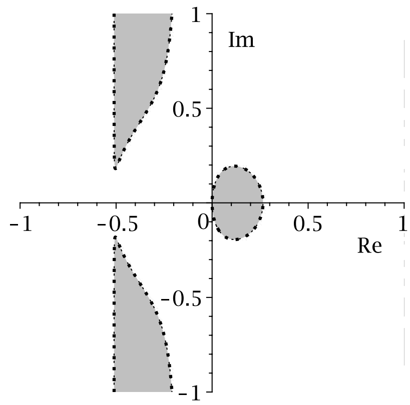

Figure 5: Region satisfying $\Delta_3 > 0$

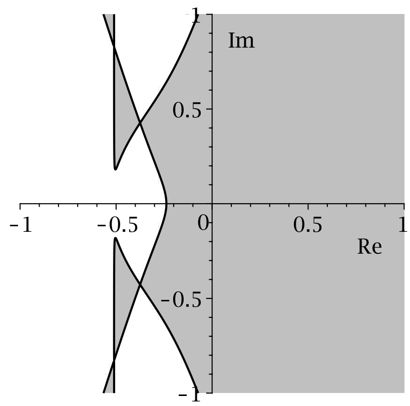

Figure 6: Region satisfying $\Delta_4 > 0$

in this case is given by

$$\left\{ \begin{array}{lcl} u_1 & = & -(y_1 - y_2)/2 \\ u_2 & = & -(y_1 + y_2)/2 \end{array} \right. , \tag{31}$$

which has the following interesting features:

1. The aim of the second agent's control is to make the average of the outputs of two agents.

2. The first agent is trying to follow the second agent, or to reduce the difference of the two agents.

That is, the second agent plays a role of the leader and the first one is just a follower. It should be emphasized that the different roles of two agents only achieves the stability requirement, since any symmetric matrix F or A has only real eigenvalues which cannot stabilize the system.

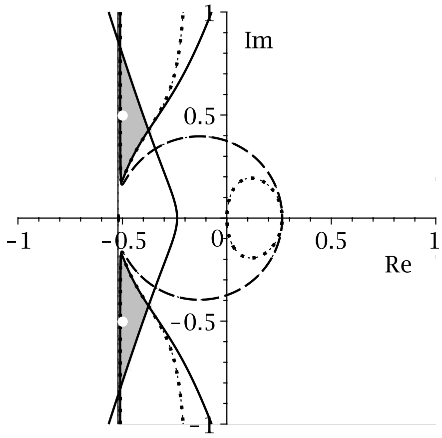

Figure 7: Stability region for $h(s)$ given by (27)

In order to confirm the cooperative stabilization by the control law (31), we check the behaviours of the outputs of two agents with an initial condition given by

$$x_0 = [0.3919, 0.3998, 0.2771, -0.9328, -0.8624, -0.3608, 0.0617, 0.3089]^T .$$

The simulation result depicted in Fig. 8 confirms the achievement of stabilization, although the responses are oscillatory.

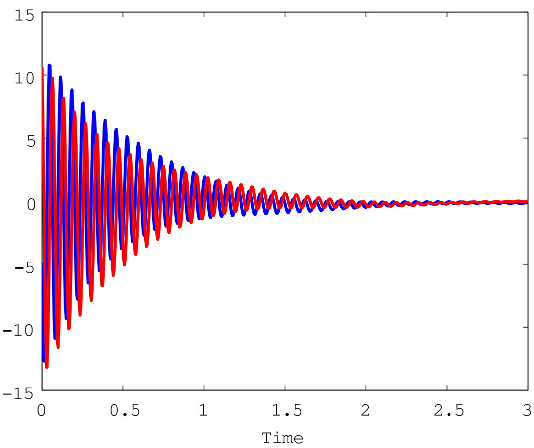

Figure 8: Simulation for $h(s)$ given by (27)

8 Concluding Remarks

This paper has investigated cooperative stabilization for LTI homogeneous multi-agent dynamical systems. We first defined the cooperative stabilization problem by constant output feedback and showed that it can be reduced to a stabilization problem with complex gain feedback. We then presented several classes of systems in which the system is cooperatively stabilizable if and only if it can be stabilized alone. We have also shown a multi-agent system with even number of agents whose dynamics is represented by a 4th order transfer function, which can be stabilized by cooperation even if any single agent alone is not stabilizable.

Acknowledgment: This work was supported in part by The Ministry of Education, Science, Sport, and Culture, Japan under Grant 19656104 and 21246067.

References

[1] H. Banks and J. Mahaffy, "Stability of cyclic gene models for systems involving repression," *Journal of Theoretical Biology*, vol. 74, pp.323–334, 1978.

[2] E. Frank, "On the zeros of polynomials with complex coefficients," *Bulletin of the American Mathematical Society*, vol. 52, pp. 144–157, 1946.

[3] S. Hara, T. Hayakawa, and H. Sugata, "Stability analysis of linear systems with generalized frequency variables and its applications to formation control," in *Proceedings of the 46th IEEE Conference on Decision and Control*, pp. 1459–1466, 2007.

[4] S. Hara, T. Hayakawa, and H. Sugata, "LTI Systems with Generalized Frequency Variables: A Unified Framework for Homogeneous Multi-agent Dynamical Systems," to be published in *the SICE Journal of Control, Measurement, and System Integration*, 2009.

[5] D. Hinrichsen and A.J. Pritchard, *Mathematical Systems Theory I: Modelling, State Space Analysis, Stabilty And Robustness (Texts in Applied Mathematics)*, Springer, 2004.

[6] T. Iwasaki, "Multivariable harmonic balance for central pattern generators," *Automatica*, vol. 44, pp. 3061–3069, 2008.

[7] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp.215–233, 2007.

[8] B. T. Polyak and Y. Z. Tsypkin, "Stability and robust stability of uniform system," *Automation and remote control*, vol. 57, no. 11, pp.1606–1617, 1996.

[9] H. E. Samad, D. D. Vecchio, and M. Khammash, "Repressilators and promotilators: loop dynamics in synthetic gene networks," in *Proceedings of the 2005 American Control Conference*, 2005.

[10] H. Tanaka, S. Hara, and T .Iwasaki, "Stability Analysis of Linear Systems with Generalized Frequency Variables, " in *Proceedings of the 7th Asian Control Conference*, pp.136–141, 2009.