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Abstract

The discrete variational method is a method to derive finite difference schemes on uniform grids that inherit the conservation/dissipation property of the original equations. In this paper we extend this method to multidimensional nonuniform grids.

1 Introduction

As is well-known, for PDEs that enjoy the conservation/dissipation property, numerical schemes that inherit that property are often advantageous in that the schemes are fairly stable and give qualitatively better numerical solutions in practice. For example, the Cahn–Hilliard equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}\left(pu + ru^3 + q\frac{\partial^2 u}{\partial x^2}\right), \quad (t,x) \in (0,\infty)\times(0,L), \tag{1.1}$$

has a dissipation property

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \left(\frac{p}{2}u^2 + \frac{r}{4}u^4 - \frac{q}{2}\left(\frac{\partial u}{\partial x}\right)^2\right)\mathrm{d}x \le 0, \quad t > 0 \tag{1.2}$$

under certain boundary conditions. Here p, q and r are real parameters that satisfy $p < 0$, $q < 0$ and $r > 0$. Although the existence of the term related to the negative dispersion effect in this equation often makes naive numerical schemes unstable, some numerical schemes that are designed so that they inherit the dissipation property (1.2) are proved to be stable and convergent [2, 4].

Lately Furihata and Matsuo [3, 4, 5, 8, 9, 10, 11] have developed the so-called “discrete variational method” that *automatically* constructs conservative/dissipative finite difference schemes for a class of PDEs with the conservation/dissipation property that stems from a certain variational structure. Originally Furihata considered two types of equations in his first paper [3]. The first is the class of equations of the form

$$\frac{\partial u}{\partial t} = (-1)^{s+1}\left(\frac{\partial}{\partial x}\right)^{2s}\frac{\delta G}{\delta u}, \quad s = 0,1,2,3,\ldots, \quad x \in [0,L], \tag{1.3}$$

where $\delta G/\delta u$ is the variational derivative, which is defined by

$$\frac{\delta G}{\delta u} = \frac{\partial G}{\partial u} - \frac{\partial}{\partial x}\frac{\partial G}{\partial u_x}. \tag{1.4}$$

This class of equations includes the heat equation and the Cahn–Hilliard equation. The second is the class of equations of the form

$$\frac{\partial u}{\partial t} = \left(\frac{\partial}{\partial x}\right)^{2s+1} \frac{\delta G}{\delta u}, \quad s = 0, 1, 2, 3, \ldots, \quad x \in [0, L], \tag{1.5}$$

which includes the advection equation and the KdV equation. $G(u, u_x)$ denotes a certain energy functional, such as the Hamiltonian or the free energy. The total energy of these equations is defined by

$$H(t) := \int_0^L G(u, u_x)\mathrm{d}x. \tag{1.6}$$

As is widely known, under certain boundary conditions, (1.3) has the dissipation property

$$\frac{\mathrm{d}H}{\mathrm{d}t} \leq 0 \tag{1.7}$$

and (1.5) has the conservation property

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0. \tag{1.8}$$

Furihata proposed a method to derive finite difference schemes for (1.3) and (1.5) that inherit these properties after the discretizations, and his method has been extended to many other equations [5, 8, 10, 11].

Until recently the discrete variational method has been developed on uniform meshes only. However, especially in multidimensional problems, the use of nonuniform meshes is of importance, since the restriction to uniform meshes forces the domains to be rectangles. Furthermore, even in one dimensional cases, nonuniform meshes are often useful when solutions exhibit complicated behaviors locally.

In this paper, we extend the discrete variational method to logically rectangular meshes. Our extension is based on the "mapping method," where change of coordinates plays an important role. For this reason, in the process of the extension we also show that it is retained after the change of coordinates that the conservation/dissipation property is obtained from the variational structure of the original equation.

This paper is organized as follows.

In Section 2 we consider simple one dimensional cases to clarify the idea of the extension. As mentioned above, we use the mapping method for the extension. Therefore, first in Section 2.1, we briefly review the idea of the mapping method and derive the conservation/dissipation property from the variational structure after the change of coordinates. In Section 2.2 we introduce a summation by parts formula on one dimensional nonuniform grids, since it plays a very important role in the discrete variational method, as is similar to that the integration by parts is of importance in the usual variational calculus. By using that formula, we define the discrete variational derivatives in Section 2.3. The dissipative/conservative schemes are defined in Section 2.4 and 2.5 respectively. In Section 3, as an example, we show a conservative scheme for the KdV equation and a numerical example.

In Section 4 we extend the discrete variational method to multidimensional nonuniform meshes. Although we consider two dimensional cases only for convenience of notation, the same procedure can be applied to more than two dimensional cases. Since the integration by parts is replaced by the Gauss theorem in multidimensional cases, we show the discrete analogue of the Gauss theorem and derive the dissipative/conservative schemes by using that theorem. As an example, a dissipative scheme for the Cahn–Hilliard equation is provided in Section 5, which is accompanied by a numerical example.

The discrete variational method has been extended to equations other than those of the form (1.3) or (1.5), which include complex valued equations and nonlinear wave equations [5, 8, 10]. Our extension is also applicable to such equations. As an example, in Section 6, an application to a class of one dimensional complex valued equations is described.

2 Extension to One Dimensional Nonuniform Grids

In this section, we extend the discrete variational method to one dimensional nonuniform grids. We consider two classes of equations that are shown below.

The first class is equations of the form (1.3). Equations in this class are dissipative in the following sense.

Theorem 2.1 (e.g. [3])**.** *Suppose that the boundary condition satisfies*

$$\left[\frac{\partial u}{\partial t}\frac{\partial G}{\partial u_x}\right]_0^L = 0. \tag{2.1}$$

Suppose also that

$$\left[\left(\frac{\partial^{p-1}}{\partial x^{p-1}}\frac{\delta G}{\delta u}\right)\left(\frac{\partial^{2s-p}}{\partial x^{2s-p}}\frac{\delta G}{\delta u}\right)\right]_0^L = 0, \quad p = 1, \ldots, s \tag{2.2}$$

if $s \geq 1$. Then solutions of (1.3) *have the dissipation property:*

$$\frac{\mathrm{d}H}{\mathrm{d}t} \leq 0, \quad H(t) = \int_0^L G(u, u_x)\mathrm{d}x.$$

The second class is equations of the form (1.5). Equations in this class are conservative.

Theorem 2.2 (e.g. [3])**.** *Suppose that the boundary condition satisfies*

$$\left[\frac{\partial u}{\partial t}\frac{\partial G}{\partial u_x}\right]_0^L = 0, \quad \left[\left(\frac{\partial^s}{\partial x^s}\frac{\delta G}{\delta u}\right)^2\right]_0^L = 0.$$

Suppose also that

$$\left[\left(\frac{\partial^{p-1}}{\partial x^{p-1}}\frac{\delta G}{\delta u}\right)\left(\frac{\partial^{2s+1-p}}{\partial x^{2s+1-p}}\frac{\delta G}{\delta u}\right)\right]_0^L = 0, \quad p = 1, \ldots, s \tag{2.3}$$

if $s \geq 1$. Then solutions of (1.5) *have the conservation property:*

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0, \quad H(t) = \int_0^L G(u, u_x)\mathrm{d}x = 0.$$

These theorems are proved by the following lemma:

Lemma 2.3 (e.g. [3])**.** *Suppose that a solution of* (1.3) *or* (1.5) *satisfies the condition*

$$\left[\frac{\partial u}{\partial t}\frac{\partial G}{\partial u_x}\right]_0^L = 0.$$

Then

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \int_0^L \frac{\partial u}{\partial t}\frac{\delta G}{\delta u}\mathrm{d}x. \tag{2.4}$$

Proof of Theorem 2.1. From Lemma 2.3 it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L G(u, u_x)\mathrm{d}x = \int_0^L \frac{\partial u}{\partial t}\frac{\delta G}{\delta u}\mathrm{d}x.$$

Substituting the equation (1.3) into the right-hand side and repeating applications of the integration by parts give

$$
\begin{aligned}
&= \int_0^L \left((-1)^{s+1} \left(\frac{\partial}{\partial x}\right)^{2s} \frac{\delta G}{\delta u} \right) \frac{\delta G}{\delta u} \mathrm{d}x \\
&= \int_0^L \left((-1)^{s+2} \left(\frac{\partial}{\partial x}\right)^{2s-1} \frac{\delta G}{\delta u} \right) \left(\frac{\partial}{\partial x} \frac{\delta G}{\delta u} \right) \mathrm{d}x + (-1)^{s+1} \left[\left(\left(\frac{\partial}{\partial x}\right)^{2s-1} \frac{\delta G}{\delta u} \right) \frac{\delta G}{\delta u} \right]_0^{x=L} \\
&\quad \vdots \\
&= (-1)^{2s+1} \int_0^L \left(\left(\frac{\partial}{\partial x}\right)^{s} \frac{\delta G}{\delta u} \right)^2 \mathrm{d}x \\
&\leq 0.
\end{aligned}
$$

□

Proof of Theorem 2.2. From Lemma 2.3 it follows that

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int_0^L G(u, u_x) \mathrm{d}x = \int_0^L \frac{\partial u}{\partial t} \frac{\delta G}{\delta u} \mathrm{d}x.
$$

Substituting of the equation (1.3) into the right-hand side and repeating applications of the integration by parts give

$$
\begin{aligned}
&= \int_0^L \left(\left(\frac{\partial}{\partial x}\right)^{2s+1} \frac{\delta G}{\delta u} \right) \frac{\delta G}{\delta u} \mathrm{d}x \\
&= - \int_0^L \left(\left(\frac{\partial}{\partial x}\right)^{2s} \frac{\delta G}{\delta u} \right) \left(\frac{\partial}{\partial x} \frac{\delta G}{\delta u} \right) \mathrm{d}x + \left[\left(\left(\frac{\partial}{\partial x}\right)^{2s} \frac{\delta G}{\delta u} \right) \frac{\delta G}{\delta u} \right]_0^{x=L} \\
&\quad \vdots \\
&= (-1)^s \int_0^L \left(\left(\frac{\partial}{\partial x}\right)^{s+1} \frac{\delta G}{\delta u} \right) \left(\left(\frac{\partial}{\partial x}\right)^{s} \frac{\delta G}{\delta u} \right) \mathrm{d}x \\
&= (-1)^{s+1} \int_0^L \left(\left(\frac{\partial}{\partial x}\right)^{s} \frac{\delta G}{\delta u} \right) \left(\left(\frac{\partial}{\partial x}\right)^{s+1} \frac{\delta G}{\delta u} \right) \mathrm{d}x + (-1)^s \left[\left(\frac{\partial^s}{\partial x^s} \frac{\delta G}{\delta u} \right)^2 \right]_0^L \\
&= (-1)^{s+1} \int_0^L \left(\left(\frac{\partial}{\partial x}\right)^{s} \frac{\delta G}{\delta u} \right) \left(\left(\frac{\partial}{\partial x}\right)^{s+1} \frac{\delta G}{\delta u} \right) \mathrm{d}x.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_0^L G(u, u_x) \mathrm{d}x &= (-1)^s \int_0^L \left(\left(\frac{\partial}{\partial x}\right)^{s+1} \frac{\delta G}{\delta u} \right) \left(\left(\frac{\partial}{\partial x}\right)^{s} \frac{\delta G}{\delta u} \right) \mathrm{d}x \\
&= (-1)^{s+1} \int_0^L \left(\left(\frac{\partial}{\partial x}\right)^{s} \frac{\delta G}{\delta u} \right) \left(\left(\frac{\partial}{\partial x}\right)^{s+1} \frac{\delta G}{\delta u} \right) \mathrm{d}x = 0.
\end{aligned}
$$

□

Lemma 2.3 is proved by a kind of calculus of variations. In fact, by the integration by parts, it is shown that

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int_0^L G(u, u_x) \mathrm{d}x = \int_0^L \left(\frac{\partial u}{\partial t} \frac{\partial G}{\partial u} + \frac{\partial u_x}{\partial t} \frac{\partial G}{\partial u_x} \right) \mathrm{d}x
$$

$$= \int_0^L \left(\frac{\partial u}{\partial t} \frac{\partial G}{\partial u} - \frac{\partial u}{\partial t} \frac{\partial}{\partial x} \frac{\partial G}{\partial u_x} \right) \mathrm{d}x + \left[\frac{\partial u}{\partial t} \frac{\partial G}{\partial u_x} \right]_0^{x=L}$$
$$= \int_0^L \frac{\partial u}{\partial t} \frac{\delta G}{\delta u} \mathrm{d}x.$$

In this sense, we call Lemma 2.3 "the variational structure" of equations of the form (1.3) or (1.5). By discretizing this structure, the discrete variational method derives the schemes that preserve Theorem 2.1 or Theorem 2.2. It is notable that the proofs of Theorem 2.1 and 2.2 are based on the following three:

- the integration by parts,
- the variational structure, that is, the variational derivative that satisfies Lemma 2.3,
- the fact that the equations are written in the form (1.3) or (1.5).

The idea of the discrete variational method is to discretize these three. Indeed, in the discrete variational method,

- the summation by parts is introduced,
- the variational structure is preserved, that is, the discrete variational derivative that satisfies a discrete analogue of Lemma 2.3 is introduced,
- the schemes are defined by using the discrete variational derivative so that they have a similar form to (1.3) or (1.5).

In the remainder of this section, we extend the discrete variational method to one dimensional nonuniform grids, by showing that these three can be retained after the discretization even on such grids. Our idea is use of the mapping method, in which the spatial coordinate is transformed to so-called "computational space" that is a domain whose axis is the index of the grid. With this idea in mind,

- first, in Section 2.1, we show that the conservation/dissipation properties are obtained from the variational structure even in the computational space,

and then,

- in Section 2.2, we give a summation by parts formula on nonuniform grids,
- in Section 2.3, we introduce the discrete variational derivative on nonuniform grids and provide an analogue of Lemma 2.3,
- in Section 2.4 and 2.5 we derive dissipative/conservative finite difference schemes respectively, by using the discrete variational derivative that is defined in 2.3.

2.1 The Mapping Method and the Dissipation/Conservation Properties in the Computational Space

We set the $N+1$ points $0 = x_0 < x_1 < x_2 < \cdots < x_N = L$ on the target domain $X = \{x \mid x \in [0, L]\}$. The approximated value of $u(n\Delta t, x_j)$ is denoted by $U_j^{(n)}$. Because we wish to use the mapping method, first the target domain $X = \{x \mid x \in [0, L]\}$ is mapped to the computational space $\Xi = \{\xi \mid \xi \in [0, N]\}$. We denote this map from Ξ to X by $x(\xi)$ and assume that $x(\xi)$ is a sufficiently smooth function that satisfies

$$x(j) = x_j, \quad J = \frac{\mathrm{d}x}{\mathrm{d}\xi} > 0,$$

where J is the Jacobian. In the mapping method, the differential operator $\partial/\partial x$ is discretized by approximating the right-hand side of

$$\frac{\partial}{\partial x} = \left(\frac{\mathrm{d}x}{\mathrm{d}\xi} \right)^{-1} \frac{\partial}{\partial \xi}$$

by some finite difference operators. For example, if we choose

$$\frac{\mathrm{d}x}{\mathrm{d}\xi} \simeq x_{j+1} - x_j, \quad \frac{\partial u}{\partial \xi} \simeq U_{j+1}^{(n)} - U_j^{(n)},$$

for the approximation of $\mathrm{d}x/\mathrm{d}\xi$ and $\partial u/\partial \xi$, $\partial u/\partial x$ is discretized by

$$\frac{\partial u}{\partial x} \simeq \frac{U_{j+1}^{(n)} - U_j^{(n)}}{x_{j+1} - x_j}.$$

We are to apply this method to the discrete variational method; however, it is not obvious whether the conservation/dissipation property stems from the variational structure after the change of coordinates. So first we must confirm it.

The transformation of (1.3) to the computational space results in

$$\frac{\partial u}{\partial t} = (-1)^{s+1} \left(\frac{\mathrm{d}\xi}{\mathrm{d}x} \frac{\mathrm{d}}{\mathrm{d}\xi} \right)^{2s} \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}}, \quad s = 0, 1, 2, 3, \ldots,$$

where $\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}$ is the transformed variational derivative

$$\left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} = \frac{\partial G}{\partial u} - J^{-1} \frac{\partial}{\partial \xi} \left(J \frac{\mathrm{d}\xi}{\mathrm{d}x} \frac{\partial G}{\partial u_x} \right).$$

This is a natural form as the variational derivative in the computational space, as is shown in Lemma 2.7 later. Since $J = \mathrm{d}x/\mathrm{d}\xi$, we can write this equation as

$$\frac{\partial u}{\partial t} = - \left(-J^{-1} \frac{\mathrm{d}}{\mathrm{d}\xi} \right) \left(-\frac{\mathrm{d}\xi}{\mathrm{d}x} \frac{\mathrm{d}}{\mathrm{d}\xi} \right)^{s-1} \left(\frac{\mathrm{d}\xi}{\mathrm{d}x} J \frac{\mathrm{d}\xi}{\mathrm{d}x} \frac{\mathrm{d}}{\mathrm{d}\xi} \right) \left(\frac{\mathrm{d}\xi}{\mathrm{d}x} \frac{\mathrm{d}}{\mathrm{d}\xi} \right)^{s-1} \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}}, \quad s = 0, 1, 2, 3, \ldots \tag{2.5}$$

Similarly (1.5) is transformed to

$$\frac{\partial u}{\partial t} = \left(\frac{\mathrm{d}\xi}{\mathrm{d}x} \frac{\mathrm{d}}{\mathrm{d}\xi} \right)^{2s+1} \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}}, \quad s = 0, 1, 2, 3, \ldots$$

and this becomes

$$\frac{\partial u}{\partial t} = \left(J^{-1} \frac{\mathrm{d}}{\mathrm{d}\xi} \right) \left(\frac{\mathrm{d}\xi}{\mathrm{d}x} \frac{\mathrm{d}}{\mathrm{d}\xi} \right)^{2s} \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}}, \quad s = 0, 1, 2, 3, \ldots \tag{2.6}$$

Remark 2.1. *In the above, we denote* $\mathrm{d}x/\mathrm{d}\xi$ *in two ways,* $\mathrm{d}x/\mathrm{d}\xi$ *and* $J = \mathrm{d}x/\mathrm{d}\xi$*. Although these are the same operators of course, we distinguish these two because they are discretized in different manners in the later sections. Indeed,* $\mathrm{d}x/\mathrm{d}\xi$ *is discretized to approximate* $\mathrm{d}x/\mathrm{d}\xi$ *in the transformed differential operator* $\mathrm{d}/\mathrm{d}x = \mathrm{d}\xi/\mathrm{d}x \cdot \mathrm{d}/\mathrm{d}\xi$*, and* J *is to approximate the Jacobian in the transformed integral* $\int \cdot \, \mathrm{d}x = \int \cdot \, J \mathrm{d}\xi$*. Similarly, since* $J \cdot \mathrm{d}\xi/\mathrm{d}x = 1$*, it is verbose to write* $J \cdot \mathrm{d}\xi/\mathrm{d}x$ *and other similar terms, but we do not omit them for the same reason.*

Theorem 2.1 and 2.2 in the computational space are as shown below.

Theorem 2.4 (Theorem 2.1 in the computational space)**.** *Suppose that the boundary condition satisfies*

$$\left[\frac{\partial u}{\partial t} \frac{\partial G}{\partial u_x} \right]_{\xi=0}^{\xi=N} = 0. \tag{2.7}$$

Suppose also that

$$\left[\left(\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\partial}{\partial\xi}\right)^{s-p}\frac{\mathrm{d}\xi}{\mathrm{d}x}J\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\partial}{\partial\xi}\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\partial}{\partial\xi}\right)^{s-1}\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}\right)\left(\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\partial}{\partial\xi}\right)^{p-1}\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}\right)\right]_{\xi=0}^{\xi=N}=0,\quad p=1,\ldots,s \tag{2.8}$$

if $s\geq 1$*. Then the solutions of* (2.5) *have the dissipation property:*

$$\frac{\mathrm{d}H_{\mathrm{cs}}}{\mathrm{d}t}\leq 0,\quad H_{\mathrm{cs}}(t)=\int_0^N G(u,u_x)J\mathrm{d}\xi.$$

Theorem 2.5 (Thorem 2.2 in the computational space)**.** *Suppose that the boundary condition satisfies*

$$\left[\frac{\partial u}{\partial t}\frac{\partial G}{\partial u_x}\right]_{\xi=0}^{\xi=N}=0,\quad \left[\left(\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\partial}{\partial\xi}\right)^{s}\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}\right)^2\right]_{\xi=0}^{\xi=N}=0.$$

Suppose also that

$$\left[\left(\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\partial}{\partial\xi}\right)^{p-1}\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}\right)\left(\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\partial}{\partial\xi}\right)^{2s+1-p}\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}\right)\right]_{\xi=0}^{\xi=N}=0,\quad p=1,\ldots,s \tag{2.9}$$

if $s\geq 1$*. Then the solutions of* (2.6) *have the conservation property:*

$$\frac{\mathrm{d}H_{\mathrm{cs}}}{\mathrm{d}t}=0,\quad H_{\mathrm{cs}}(t)=\int_0^N G(u,u_x)J\mathrm{d}\xi.$$

To prove these theorems, we use the integration by parts that is transformed to the computational space.

Lemma 2.6 (The integration by parts in the computational space)**.** *Let* $u(\xi)$ *and* $v(\xi)$ *be functions on* $[0,N]$ *that satisfy*

$$\left[J\frac{\mathrm{d}\xi}{\mathrm{d}x}uv\right]_{\xi=0}^{\xi=N}=0. \tag{2.10}$$

Then

$$\int_0^N Ju\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}\xi}\right)\mathrm{d}\xi=-\int_0^N Jv\left(J^{-1}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}Ju\right)\right)\mathrm{d}\xi. \tag{2.11}$$

Proof. Lemma 2.6 is immediately obtained, because this is just a transformed form of the integration by parts. However, since we discretize this lemma by the mapping method later, we prove it by using calculations on the computational space only.

By applying the integration by parts with respect to ξ, we have

$$\begin{aligned}\int_0^N Ju\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}\xi}\right)\mathrm{d}\xi&=-\int_0^N v\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}Ju\right)\mathrm{d}\xi+\left[J\frac{\mathrm{d}\xi}{\mathrm{d}x}uv\right]_{\xi=0}^{\xi=N}\\&=-\int_0^N v\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}Ju\right)\mathrm{d}\xi\\&=-\int_0^N Jv\left(J^{-1}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}Ju\right)\right)\mathrm{d}\xi.\end{aligned}$$

□

Now we show Lemma 2.3, the variational structure, in the computational space:

Lemma 2.7. *Suppose that a solution of* (1.3) *or* (1.5) *satisfies the condition*

$$\left[\frac{\partial u}{\partial t}J\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\partial G}{\partial u_x}\right]_{\xi=0}^{\xi=N}=0. \tag{2.12}$$

Then

$$\frac{\mathrm{d}H_{\mathrm{cs}}}{\mathrm{d}t}=\int_0^N\frac{\partial u}{\partial t}\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}J\mathrm{d}\xi,\quad \left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}:=\frac{\partial G}{\partial u}-J^{-1}\frac{\partial}{\partial\xi}\left(J\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\partial G}{\partial u_x}\right), \tag{2.13}$$

and $\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}$ *satisfies*

$$\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}=\frac{\delta G}{\delta u}.$$

Proof. By the chain rule, we get

$$\begin{aligned}\frac{\mathrm{d}H_{\mathrm{cs}}}{\mathrm{d}t}&=\frac{\mathrm{d}}{\mathrm{d}t}\int_0^N GJ\mathrm{d}\xi\\&=\int_0^N\left(\frac{\partial G}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial G}{\partial u_x}\frac{\partial u_x}{\partial t}\right)J\mathrm{d}\xi\\&=\int_0^N\frac{\partial G}{\partial u}\frac{\partial u}{\partial t}J\mathrm{d}\xi+\int_0^N J\left(\frac{\partial G}{\partial u_x}\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\partial u_t}{\partial\xi}\right)\mathrm{d}\xi.\end{aligned}$$

By applying the integration by parts of the form in Lemma 2.6, we have

$$=\int_0^N\frac{\partial G}{\partial u}\frac{\partial u}{\partial t}J\mathrm{d}\xi-\int_0^N\frac{\partial u}{\partial t}\left(J^{-1}\frac{\partial}{\partial\xi}\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}J\frac{\partial G}{\partial u_x}\right)\right)J\mathrm{d}\xi=\int_0^N\frac{\partial u}{\partial t}\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}J\mathrm{d}\xi. \tag{2.14}$$

For the latter part, we have

$$\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}=\frac{\partial G}{\partial u}-J^{-1}\frac{\partial}{\partial\xi}\left(J\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\partial G}{\partial u_x}\right)=\frac{\partial G}{\partial u}-\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\partial}{\partial\xi}\frac{\partial G}{\partial u_x}=\frac{\partial G}{\partial u}-\frac{\partial}{\partial x}\frac{\partial G}{\partial u_x}=\frac{\delta G}{\delta u},$$

since $J=\mathrm{d}x/\mathrm{d}\xi$. ☐

Thus we have confirmed that the variational structure is retained in the computational space, so now we can proceed to prove Theorem 2.4 and 2.5.

Proof of Theorem 2.4. By Lemma 2.7, we have

$$\frac{\mathrm{d}H_{\mathrm{cs}}}{\mathrm{d}t}=\int_0^N\frac{\partial u}{\partial t}\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}J\mathrm{d}\xi.$$

Substituting the equation (2.5) and an application of the integration by parts in Lemma 2.6 give

$$\begin{aligned}&=-\int_0^N\left\{\left(-J^{-1}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\left(-\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^{s-1}\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}J\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^{s-1}\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}\right\}\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}J\mathrm{d}\xi\\&=-\int_0^N\left\{\left(-\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^{s-1}\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}J\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^{s-1}\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}\right\}\left\{\left(J^{-1}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}\right\}J\mathrm{d}\xi\\&=-\int_0^N\left\{\left(-J^{-1}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\left(-\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^{s-2}\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}J\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^{s-1}\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}\right\}\left\{\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}\right\}J\mathrm{d}\xi.\end{aligned}$$

We continue in this fashion to obtain

$$= -\int_0^N \left\{ \left(-J^{-1}\frac{\mathrm{d}}{\mathrm{d}\xi} \right) \left(-\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi} \right)^{s-3} \left(\frac{\mathrm{d}\xi}{\mathrm{d}x} J \frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi} \right) \left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi} \right)^{s-1} \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} \right\} \left\{ \left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi} \right)^{2} \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} \right\} J \mathrm{d}\xi$$

$$\vdots$$

$$= -\int_0^N \left\{ \left(-J^{-1}\frac{\mathrm{d}}{\mathrm{d}\xi} \right) \left(\frac{\mathrm{d}\xi}{\mathrm{d}x} J \frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi} \right) \left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi} \right)^{s-1} \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} \right\} \left\{ \left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi} \right)^{s-1} \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} \right\} J \mathrm{d}\xi$$

$$= -\int_0^N \left\{ \left(\frac{\mathrm{d}\xi}{\mathrm{d}x} J \frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi} \right) \left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi} \right)^{s-1} \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} \right\} \left\{ J^{-1}\frac{\mathrm{d}}{\mathrm{d}\xi} \left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi} \right)^{s-1} \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} \right\} J \mathrm{d}\xi$$

$$= -\int_0^N J \left\{ \left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi} \right)^{s} \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} \right\}^2 \leq 0.$$

□

Proof of Theorem 2.5. By Lemma 2.7, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^N G(u, u_x) J \mathrm{d}\xi = \int_0^N \frac{\partial u}{\partial t} \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} J \mathrm{d}\xi.$$

Substituting the equation (2.5) and an application of the integration by parts in Lemma 2.6 give

$$= \int_0^N \left\{ \left(J^{-1}\frac{\mathrm{d}}{\mathrm{d}\xi} \right) \left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi} \right)^{2s} \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} \right\} \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} J \mathrm{d}\xi$$

$$= -\int_0^N \left\{ \left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi} \right)^{2s} \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} \right\} \left\{ \left(J^{-1}\frac{\mathrm{d}}{\mathrm{d}\xi} \right) \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} \right\} J \mathrm{d}\xi$$

$$= -\int_0^N \left\{ \left(J^{-1}\frac{\mathrm{d}}{\mathrm{d}\xi} \right) \left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi} \right)^{2s-1} \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} \right\} \left\{ \left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi} \right) \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} \right\} J \mathrm{d}\xi.$$

We continue in this fashion to obtain

$$= (-1)^2 \int_0^N \left\{ \left(J^{-1}\frac{\mathrm{d}}{\mathrm{d}\xi} \right) \left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi} \right)^{2s-2} \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} \right\} \left\{ \left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi} \right)^{2} \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} \right\} J \mathrm{d}\xi$$

$$\vdots$$

$$= (-1)^s \int_0^N \left\{ \left(J^{-1}\frac{\mathrm{d}}{\mathrm{d}\xi} \right) \left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi} \right)^{s} \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} \right\} \left\{ \left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi} \right)^{s} \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} \right\} J \mathrm{d}\xi$$

$$= (-1)^{s+1} \int_0^N \left\{ \left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi} \right)^{s} \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} \right\} \left\{ \left(J^{-1}\frac{\mathrm{d}}{\mathrm{d}\xi} \right) \left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi} \right)^{s} \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} \right\} J \mathrm{d}\xi.$$

The last equality shows that this value equals 0. □

2.2 Discrete Symbols and a Summation by Parts Formula on Nonuniform Grids

In this section we introduce symbols which are useful for notation and show a summation by parts formula on nonuniform grids. As in the previous section, we divide the interval $[0, L]$ into nonuniform meshes with grids $0 = x_0 < x_1 < x_2 < \cdots < x_N = L$. The approximated value of $u(n\Delta t, x_j)$ is denoted by $U_j^{(n)}$. In what follows, δ's with suffixes denote difference operators in the computational space, that is, approximations of $\partial/\partial\xi$. For example, we write the forward difference operator in the computational space as $\delta_+ U_j^{(n)} = U_{j+1}^{(n)} - U_j^{(n)}$, the backward difference operator as $\delta_- U_j^{(n)} = U_j^{(n)} - U_{j-1}^{(n)}$ and the central difference operator as $\delta_{\mathrm{c}} U_j^{(n)} = (U_{j+1}^{(n)} - U_{j-1}^{(n)})/2$. We denote by $(x_\xi)_j$ or similar notations the approximated value of $\mathrm{d}x/\mathrm{d}\xi$, which may be set, for example, to $(x_\xi)_j = x_{j+1} - x_j$. $(x_\xi)_j$'s are used to approximate the values that are denoted by $\mathrm{d}x/\mathrm{d}\xi$ in the previous section. We also denote by w_j's the positive weights that are defined so that $\sum_{j=0}^{N} w_j$ approximates the integral operator. w_j's are used to approximate the values that are denoted by J in the previous section. δ, $(x_\xi)_j$ and w_j are chosen arbitrarily, unless specified otherwise.

To discretize Lemma 2.6, we introduce useful notations:

Definition 2.1. *For a finite difference operator δ with the α-point stencil*

$$\delta U_j = \sum_{k=-\alpha}^{\alpha} a_k U_{j+k},$$

we define δ^* by

$$\delta^* U_j = -\sum_{k=-\alpha}^{\alpha} a_{-k} U_{j+k}.$$

Definition 2.2. *Let δ be a finite difference operator with the α-point stencil that are represented as*

$$\delta U_j = \sum_{k=-\alpha}^{\alpha} a_k U_{j+k}.$$

Let $(x_\xi)_j$ be an arbitrarily chosen approximation of $\mathrm{d}x/\mathrm{d}\xi$. *For any sequences U_j and V_j, we define averaging operator $\mu_{(\pm,\delta,(x_\xi)_j)}\left(\{U_j\},\{V_j\}\right)$ by*

$$\mu_{(+,\delta,(x_\xi)_j)}\left(\{U_j\},\{V_j\}\right) := \sum_{j=N-\alpha+1}^{N} \sum_{k=N+1}^{N+\alpha} \tilde{a}_{j,k}\left(w_j (x_\xi)_j^{-1} U_j V_k\right) + \sum_{j=N-\alpha+1}^{N} \sum_{k=N+1}^{N+\alpha} \tilde{a}_{j,k}^*\left(w_k (x_\xi)_k^{-1} U_k V_j\right),$$

$$\mu_{(-,\delta,(x_\xi)_j)}\left(\{U_j\},\{V_j\}\right) := \sum_{j=0}^{\alpha-1} \sum_{k=-\alpha}^{-1} \tilde{a}_{j,k}\left(w_j (x_\xi)_j^{-1} U_j V_k\right) + \sum_{j=0}^{\alpha-1} \sum_{k=-\alpha}^{-1} \tilde{a}_{j,k}^*\left(w_k (x_\xi)_k^{-1} U_k V_j\right),$$

where $\tilde{a}_{j,k}$ is defined by

$$\tilde{a}_{j,k} = \begin{cases} a_{-j+k} & (k = j-\alpha, j-\alpha+1, \ldots, j+\alpha-1, j+\alpha), \\ 0 & (\text{otherwise}) \end{cases}$$

and $\tilde{a}_{j,k}^*$ is defined corresponding to δ^* in a similar way:

$$\tilde{a}_{j,k}^* = \begin{cases} a_{-j+k}^* & (k = j-\alpha, j-\alpha+1, \ldots, j+\alpha-1, j+\alpha), \\ 0 & (\text{otherwise}), \end{cases}$$
$$a_k^* = -a_{-k}.$$

$\mu_{(+,\delta,(x_\xi)_j)}\left(\{U_j\},\{V_j\}\right)$ and $\mu_{(-,\delta,(x_\xi)_j)}\left(\{U_j\},\{V_j\}\right)$ approximate $U_N V_N$ and $-U_0 V_0$ respectively. An example is provided in Remark 2.2 below. We now give the summation by parts formula:

Lemma 2.8. *Let δ be a finite difference operator that is represented as*

$$\delta U_j = \sum_{k=-\alpha}^{\alpha} a_k U_{j+k}$$

and $(x_\xi)_j$*'s be approximated values of* $\mathrm{d}x/\mathrm{d}\xi$. *For any sequences* U_j *and* V_j *that satisfy*

$$\mu_{(+,\delta,(x_\xi)_j)}\left(\{U_j\},\{V_j\}\right)+\mu_{(-,\delta,(x_\xi)_j)}\left(\{U_j\},\{V_j\}\right)=0, \tag{2.15}$$

a summation by parts formula

$$\sum_{j=0}^{N} w_j U_j((x_\xi)_j^{-1}\delta V_j) = -\sum_{j=0}^{N} w_j V_j w_j^{-1}\delta^*\left((x_\xi)_j^{-1} w_j U_j\right) \tag{2.16}$$

holds.

Remark 2.2. *The condition* (2.15) *corresponds to the condition* (2.10) *in Lemma 2.6. To clarify this, let us consider the simplest case, where uniform grids*

$$(x_\xi)_j = w_j = \Delta x$$

and the central difference operator $\delta = \delta_{\mathrm{c}}$ *are employed. In this case,*

$$\delta U_j = \frac{1}{2}\left(U_{j+1}-U_{j-1}\right) = \sum_{k=-\alpha}^{\alpha} a_k U_{j+k}, \quad \alpha = 1, \quad a_k = \begin{cases} -1/2 & (k=-1) \\ 0 & (k=0) \\ 1/2 & (k=1) \end{cases}$$

and hence $\tilde{a}_{j,k}$ *is*

$$\tilde{a}_{j,k} = (\underset{\cdots}{\cdots} \quad \underset{k=j-2}{0,} \quad \underset{k=j-1}{-1/2,} \quad \underset{k=j}{0,} \quad \underset{k=j+1}{1/2,} \quad \underset{k=j+2}{0,} \quad \underset{\cdots}{\cdots}).$$

The central difference operator is self-adjoint in the sense that $\delta^* = \delta$ *and* $\tilde{a}^*_{j,k} = \tilde{a}_{j,k}$. *The averaging operators become*

$$\begin{aligned}
\mu_{(+,\delta,(x_\xi)_j)}\left(\{U_j\},\{V_j\}\right) &= \sum_{j=N}^{N}\sum_{k=N+1}^{N+1} \tilde{a}_{j,k}\left(w_j (x_\xi)_j^{-1} U_j V_k\right) + \sum_{j=N}^{N}\sum_{k=N+1}^{N+1} \tilde{a}^*_{j,k}\left(w_k (x_\xi)_k^{-1} U_k V_j\right) \\
&= \tilde{a}_{N,N+1} w_N (x_\xi)_N^{-1} U_N V_{N+1} + \tilde{a}^*_{N,N+1} w_{N+1} (x_\xi)_{N+1}^{-1} U_{N+1} V_N \\
&= \frac{1}{2}\left(U_N V_{N+1} + U_{N+1} V_N\right)
\end{aligned}$$

and

$$\begin{aligned}
\mu_{(-,\delta,(x_\xi)_j)}\left(\{U_j\},\{V_j\}\right) &= \sum_{j=0}^{0}\sum_{k=-1}^{-1} \tilde{a}_{j,k}\left(w_j (x_\xi)_j^{-1} U_j V_k\right) + \sum_{j=0}^{0}\sum_{k=-1}^{-1} \tilde{a}^*_{j,k}\left(w_k (x_\xi)_k^{-1} U_k V_j\right) \\
&= \tilde{a}_{0,-1} w_0 (x_\xi)_0^{-1} U_0 V_{-1} + \tilde{a}^*_{0,-1} w_{-1} (x_\xi)_{-1}^{-1} U_{-1} V_0 \\
&= -\frac{1}{2}\left(U_0 V_{-1} + U_{-1} V_0\right).
\end{aligned}$$

Thus the left-hand side of (2.15) *is*

$$\frac{1}{2}\left(U_N V_{N+1} + U_{N+1} V_N\right) - \frac{1}{2}\left(U_0 V_{-1} + U_{-1} V_0\right),$$

which is an approximation of (2.10).

Remark 2.3. *Examples for the boundary conditions that enjoy the condition* (2.15) *includes the Dirichlet boundary condition*

$$U_j = V_j = 0 \qquad \text{for all } j \text{ such that } j > N \text{ or } j < 0, \tag{2.17}$$

and the periodic boundary condition

$$U_{j+N+1} = U_j, \quad V_{j+N+1} = V_j, \quad (x_\xi)^{-1}_{j+N+1} = (x_\xi)^{-1}_j, \quad w_{j+N+1} = w_j \qquad \text{for all } j. \tag{2.18}$$

These are confirmed in the following way. Under the Dirichlet boundary condition we have

$$\begin{aligned}
\mu_{(+,\delta,(x_\xi)_j)}\left(\{U_j\},\{V_j\}\right) &= \sum_{j=N-\alpha+1}^{N}\sum_{k=N+1}^{N+\alpha}\tilde{a}_{j,k}\left(w_j(x_\xi)^{-1}_j U_j V_k\right) + \sum_{j=N-\alpha+1}^{N}\sum_{k=N+1}^{N+\alpha}\tilde{a}^*_{j,k}\left(w_k(x_\xi)^{-1}_k U_k V_j\right) \\
&= \sum_{j=N-\alpha+1}^{N}\sum_{k=N+1}^{N+\alpha}\tilde{a}_{j,k}\left(w_j(x_\xi)^{-1}_j U_j \cdot 0\right) + \sum_{j=N-\alpha+1}^{N}\sum_{k=N+1}^{N+\alpha}\tilde{a}^*_{j,k}\left(w_k(x_\xi)^{-1}_k \cdot 0 \cdot V_j\right) \\
&= 0.
\end{aligned}$$

A similar calculation yields $\mu_{(-,\delta,(x_\xi)_j)}\left(\{U_j\},\{V_j\}\right) = 0$ *and combining these gives* (2.15)*. In the case of the periodic boundary condition we have*

$$\begin{aligned}
&\mu_{(+,\delta,(x_\xi)_j)}\left(\{U_j\},\{V_j\}\right) + \mu_{(-,\delta,(x_\xi)_j)}\left(\{U_j\},\{V_j\}\right) \\
&= \sum_{j=N-\alpha+1}^{N}\sum_{k=N+1}^{N+\alpha}\tilde{a}_{j,k}\left(w_j(x_\xi)^{-1}_j U_j V_k\right) + \sum_{j=N-\alpha+1}^{N}\sum_{k=N+1}^{N+\alpha}\tilde{a}^*_{j,k}\left(w_k(x_\xi)^{-1}_k U_k V_j\right) \\
&\quad + \sum_{j=0}^{\alpha-1}\sum_{k=-\alpha}^{-1}\tilde{a}_{j,k}\left(w_j(x_\xi)^{-1}_j U_j V_k\right) + \sum_{j=0}^{\alpha-1}\sum_{k=-\alpha}^{-1}\tilde{a}^*_{j,k}\left(w_k(x_\xi)^{-1}_k U_k V_j\right) \\
&= \sum_{j=N-\alpha+1}^{N}\sum_{k=0}^{\alpha-1}\tilde{a}_{j,k}\left(w_j(x_\xi)^{-1}_j U_j V_k\right) + \sum_{j=N-\alpha+1}^{N}\sum_{k=0}^{\alpha-1}\tilde{a}^*_{j,k}\left(w_k(x_\xi)^{-1}_k U_k V_j\right) \\
&\quad + \sum_{j=0}^{\alpha-1}\sum_{k=N-\alpha+1}^{N}\tilde{a}_{j,k}\left(w_j(x_\xi)^{-1}_j U_j V_k\right) + \sum_{j=0}^{\alpha-1}\sum_{k=N-\alpha+1}^{N}\tilde{a}^*_{j,k}\left(w_k(x_\xi)^{-1}_k U_k V_j\right) \\
&= \sum_{j=N-\alpha+1}^{N}\sum_{k=0}^{\alpha-1}\tilde{a}_{j,k}\left(w_j(x_\xi)^{-1}_j U_j V_k\right) - \sum_{j=N-\alpha+1}^{N}\sum_{k=0}^{\alpha-1}\tilde{a}_{k,j}\left(w_k(x_\xi)^{-1}_k U_k V_j\right) \\
&\quad + \sum_{j=0}^{\alpha-1}\sum_{k=N-\alpha+1}^{N}\tilde{a}_{j,k}\left(w_j(x_\xi)^{-1}_j U_j V_k\right) - \sum_{j=0}^{\alpha-1}\sum_{k=N-\alpha+1}^{N}\tilde{a}_{k,j}\left(w_k(x_\xi)^{-1}_k U_k V_j\right) \\
&= 0.
\end{aligned}$$

Remark 2.4. *It is common to use the inner product in order to derive summation-by-parts-type formulas [1, 6, 7, 12]. The summation by parts formula* (2.16) *in Lemma 2.8 is comprehensible, if it is represented by using the inner product as well. We see it by an example where the difference operator* δ *is the forward difference operator* $\delta = \delta_+$ *and the boundary condition is given by the Dirichlet condition* (2.17) *or the periodic boundary condition* (2.18)*.*

First we compute the difference matrix D *that represents* δ*. When the Dirichlet condition is imposed, we have for* $j \neq N$

$$\delta U_j = U_{j+1} - U_j$$

and for $j = N$

$$\delta U_N = U_{N+1} - U_N = -U_N.$$

Therefore

$$\begin{pmatrix} \delta U_0 \\ \delta U_1 \\ \delta U_2 \\ \vdots \\ \delta U_{N-2} \\ \delta U_{N-1} \\ \delta U_N \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & -1 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & -1 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} U_0 \\ U_1 \\ U_2 \\ \vdots \\ U_{N-2} \\ U_{N-1} \\ U_N \end{pmatrix},$$

and hence

$$D = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & -1 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & -1 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & -1 \end{pmatrix}.$$

Similarly we have for $j \neq 0$

$$\delta^* U_j = U_j - U_{j-1}$$

and for $j = 0$

$$\delta^* U_0 = U_0 - U_{-1} = U_0,$$

and hence the difference matrix for δ^**, which is denoted as* D^**, is*

$$D^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 1 & 0 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & 1 \end{pmatrix}.$$

In a similar way, when the periodic boundary condition is imposed, the difference matrices become

$$D = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & -1 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & -1 & 1 \\ 1 & \cdots & \cdots & \cdots & 0 & 0 & -1 \end{pmatrix}, \quad D^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & -1 \\ -1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 1 & 0 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & 1 \end{pmatrix}.$$

Note that these matrices satisfy $D^* = -D^\top$.

Let matrices W *and* X *be* $W = \mathrm{diag}(w_j)$ *and* $X = \mathrm{diag}((x_\xi)_j)$*. Let vectors* $\vec{U}$ *and* $\vec{V}$ *be* $\vec{U} = (U_0, \ldots, U_N)$ *and* $\vec{V} = (V_0, \ldots, V_N)$*. By using these notations we can rewrite*

$$\sum_{j=0}^{N} w_j U_j ((x_\xi)_j^{-1} \delta V_j) = \langle \vec{U}, X^{-1} D \vec{V} \rangle_W,$$

where $\langle \vec{U}, \vec{V}\rangle_W := \vec{U}^\top W \vec{V}$ is the inner product with the wight W. Since the adjoint operator of $X^{-1}D$ with respect to this inner product is $W^{-1}D^\top X^{-1}W$, we have

$$\langle \vec{U}, X^{-1}D\vec{V}\rangle_W = \langle W^{-1}D^\top X^{-1}W\vec{U}, \vec{V}\rangle_W$$

and

$$= -\langle W^{-1}(-D)^\top X^{-1}W\vec{U}, \vec{V}\rangle_W.$$

Rewriting this to the form with operators, we have

$$= -\sum_{j=0}^{N} w_j V_j w_j^{-1}\delta^*\left((x_\xi)_j^{-1} w_j U_j\right),$$

because $-D^\top$ corresponds to δ^*. This coincides with the summation by parts (2.16).

Furthermore the above argument is a discrete counterpart of the fact that the transformed integration by parts (2.11) is written as

$$\langle u, \frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\partial v}{\partial \xi}\rangle_J = -\langle J^{-1}\frac{\partial}{\partial \xi}\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}Ju\right), v\rangle_J$$

by using the adjoint operator of the differential operator $\mathrm{d}\xi/\mathrm{d}x \cdot \partial/\partial\xi$ with respect to the inner product $\langle\cdot,\cdot\rangle_J$ whose weight is the Jacobian.

Proof of Lemma 2.8. In a straightforward way we obtain

$$\begin{aligned}
\sum_{j=0}^{N} w_j U_j\left((x_\xi)_j^{-1}\delta V_j\right) &= \sum_{j=0}^{N}\sum_{k=-\alpha}^{\alpha} w_j U_j (x_\xi)_j^{-1} a_k V_{j+k} \\
&= \sum_{j=0}^{N}\sum_{k=0}^{N} w_j U_j (x_\xi)_j^{-1}\tilde{a}_{j,k} V_k \\
&\quad + \sum_{j=N-\alpha+1}^{N}\sum_{k=N+1}^{N+\alpha} \tilde{a}_{j,k}\left(w_j (x_\xi)_j^{-1} U_j V_k\right) + \sum_{j=0}^{\alpha-1}\sum_{k=-\alpha}^{-1} \tilde{a}_{j,k}\left(w_j (x_\xi)_j^{-1} U_j V_k\right) \\
&= \sum_{j=0}^{N}\sum_{k=0}^{N} w_k U_k (x_\xi)_k^{-1}\tilde{a}_{k,j} V_j \\
&\quad + \sum_{j=N-\alpha+1}^{N}\sum_{k=N+1}^{N+\alpha} \tilde{a}_{j,k}\left(w_j (x_\xi)_j^{-1} U_j V_k\right) + \sum_{j=0}^{\alpha-1}\sum_{k=-\alpha}^{-1} \tilde{a}_{j,k}\left(w_j (x_\xi)_j^{-1} U_j V_k\right).
\end{aligned}$$

Using $\tilde{a}_{j,k} = -\tilde{a}^*_{k,j}$, we get

$$\begin{aligned}
&= -\sum_{j=0}^{N}\sum_{k=0}^{N} w_k U_k (x_\xi)_k^{-1}\tilde{a}^*_{j,k} V_j \\
&\quad + \sum_{j=N-\alpha+1}^{N}\sum_{k=N+1}^{N+\alpha} \tilde{a}_{j,k}\left(w_j (x_\xi)_j^{-1} U_j V_k\right) + \sum_{j=0}^{\alpha-1}\sum_{k=-\alpha}^{-1} \tilde{a}_{j,k}\left(w_j (x_\xi)_j^{-1} U_j V_k\right) \\
&= -\sum_{j=0}^{N}\sum_{k=-\alpha}^{\alpha} w_k U_k (x_\xi)_k^{-1} a^*_k V_j \\
&\quad + \sum_{j=N-\alpha+1}^{N}\sum_{k=N+1}^{N+\alpha} \tilde{a}^*_{j,k}\left(w_k (x_\xi)_k^{-1} U_k V_j\right) + \sum_{j=0}^{\alpha-1}\sum_{k=-\alpha}^{-1} \tilde{a}^*_{j,k}\left(w_k (x_\xi)_k^{-1} U_k V_j\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=N-\alpha+1}^{N}\sum_{k=N+1}^{N+\alpha}\tilde{a}_{j,k}\left(w_j(x_\xi)_j^{-1}U_jV_k\right)+\sum_{j=0}^{\alpha-1}\sum_{k=-\alpha}^{-1}\tilde{a}_{j,k}\left(w_j(x_\xi)_j^{-1}U_jV_k\right)\\
&=-\sum_{j=0}^{N}\sum_{k=-\alpha}^{\alpha}w_kU_k(x_\xi)_k^{-1}a_k^*V_j+\mu_{(+,\delta,(x_\xi)_j)}\left(\{U_j\},\{V_j\}\right)+\mu_{(-,\delta,(x_\xi)_j)}\left(\{U_j\},\{V_j\}\right)\\
&=-\sum_{j=0}^{N}\sum_{k=-\alpha}^{\alpha}w_kU_k(x_\xi)_k^{-1}a_k^*V_j.
\end{aligned}
$$

The last equality is due to (2.15). Rewriting this to the form with difference operators, we obtain

$$
=-\sum_{j=0}^{N}w_jV_jw_j^{-1}\delta^*\left((x_\xi)_j^{-1}w_jU_j\right).
$$

□

2.3 Definition of the Discrete Variational Derivative

In this section, we introduce the discrete variational derivative on one dimensional nonuniform grids. The discrete variational derivative will be defined by an approximation of the variational derivative in the computational space

$$
\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}=\frac{\partial G}{\partial u}-J^{-1}\frac{\partial}{\partial\xi}\left(J\frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\partial G}{\partial u_x}\right).
$$

Suppose that the energy functional G is given in the next form:

$$
G(u,u_x)=\sum_{l=1}^{K}f_l(u)g_l(u_x). \tag{2.19}
$$

We define the discrete energy functional by

$$
G_{\mathrm{d}}(\vec{U}^{(n)})_j:=\sum_{l=1}^{K}f_l(U_j^{(n)})\left(\frac{1}{M_l}\sum_{m=1}^{M_l}g_l((x_\xi)_{l,m,j}^{-1}\delta_{l,m}U_j^{(n)})\right), \tag{2.20}
$$

where each $(x_\xi)_{l,m,j}^{-1}$ is an approximation of $(x_\xi)^{-1}$ and $\delta_{l,m}$ is a difference operator. They can be chosen arbitrarily. The summation with respect to m is introduced in consideration of the situation where each g_l is approximated by using more than one difference operators. An example for the KdV equation that is shown in Section 3 helps understanding of the meaning of this summation. We define the discrete total energy $H^{(n)}$ by

$$
H^{(n)}:=\sum_{j=0}^{N}w_jG_{\mathrm{d}}(\vec{U}^{(n)})_j\simeq\int_0^L G(u,u_x)\mathrm{d}x, \tag{2.21}
$$

where w_j's are the weights that are defined so that $\sum_{j=0}^{N}w_j$ becomes an approximation of the integral.

Definition 2.3. *We define the discrete variational derivative of* $G_{\mathrm{d}}(\vec{U}^{(n)})_j$ *by*

$$
\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j:=\sum_{l=1}^{K}\frac{1}{M_l}\sum_{m=1}^{M_l}\left(\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{l,m,j}\right.
$$

$$-w_j^{-1}\delta^*_{l,m}\left((x_\xi)^{-1}_{l,m,j}w_j\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)},\vec{U}_x^{(n)})}\right)_{l,m,j}\right)\Bigg), \tag{2.22}$$

where

$$\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{l,m,j} := \left(\frac{f_l(U_j^{(n+1)})-f_l(U_j^{(n)})}{U_j^{(n+1)}-U_j^{(n)}}\right)\left(\frac{g_l((x_\xi)^{-1}_{l,m,j}\delta_{l,m}U_j^{(n+1)})+g_l((x_\xi)^{-1}_{l,m,j}\delta_{l,m}U_j^{(n)})}{2}\right), \tag{2.23}$$

$$\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)},\vec{U}_x^{(n)})}\right)_{l,m,j} := \left(\frac{f_l(U_j^{(n+1)})+f_l(U_j^{(n)})}{2}\right)\left(\frac{g_l((x_\xi)^{-1}_{l,m,j}\delta_{l,m}U_j^{(n+1)})-g_l((x_\xi)^{-1}_{l,m,j}\delta_{l,m}U_j^{(n)})}{(x_\xi)^{-1}_{l,m,j}\delta_{l,m}U_j^{(n+1)}-(x_\xi)^{-1}_{l,m,j}\delta_{l,m}U_j^{(n)}}\right). \tag{2.24}$$

The definition is chosen carefully so that the variational structure is retained after the discretization in the sense that a discrete counterpart of Lemma 2.7 holds.

Lemma 2.9. *Suppose the condition*

$$\sum_{l=1}^{K}\frac{1}{M_l}\sum_{m=1}^{M_l}\Bigg(\mu_{(+,\delta_{l,m},(x_\xi)^{-1}_{l,m,j})}\left(\left\{\frac{U_j^{(n+1)}-U_j^{(n)}}{\Delta t}\right\},\left\{\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)},\vec{U}_x^{(n)})}\right)_{l,m,j}\right\}\right)$$
$$+\mu_{(-,\delta_{l,m},(x_\xi)^{-1}_{l,m,j})}\left(\left\{\frac{U_j^{(n+1)}-U_j^{(n)}}{\Delta t}\right\},\left\{\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)},\vec{U}_x^{(n)})}\right)_{l,m,j}\right\}\right)\Bigg)=0$$

is satisfied. Then

$$\frac{1}{\Delta t}\left(H^{(n+1)}-H^{(n)}\right)=\sum_{j=0}^{N}w_j\left(\frac{U_j^{(n+1)}-U_j^{(n)}}{\Delta t}\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j. \tag{2.25}$$

Proof. By (2.23) and (2.24) we get

$$\frac{1}{\Delta t}\left(H^{(n+1)}-H^{(n)}\right)=\frac{1}{\Delta t}\sum_{j=0}^{N}\sum_{l=1}^{K}w_j\Bigg(f_l(U_j^{(n+1)})\left(\frac{1}{M_l}\sum_{m=1}^{M_l}g_l((x_\xi)^{-1}_{l,m,j}\delta_{l,m}U_j^{(n+1)})\right)$$
$$-f_l(U_j^{(n)})\left(\frac{1}{M_l}\sum_{m=1}^{M_l}g_l((x_\xi)^{-1}_{l,m,j}\delta_{l,m}U_j^{(n)})\right)\Bigg)$$
$$=\sum_{j=0}^{N}\sum_{l=1}^{K}\frac{w_j}{M_l}\sum_{m=1}^{M_l}\Bigg(\left(\frac{U_j^{(n+1)}-U_j^{(n)}}{\Delta t}\right)\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{l,m,j}$$
$$+\left((x_\xi)^{-1}_{l,m,j}\delta_{l,m}\left(\frac{U_j^{(n+1)}-U_j^{(n)}}{\Delta t}\right)\right)\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)},\vec{U}^{(n)})_x}\right)_{l,m,j}\Bigg).$$

Applying Lemma 2.8, we obtain

$$=\sum_{j=0}^{N}\sum_{l=1}^{K}\frac{w_j}{M_l}\sum_{m=1}^{M_l}\left(\frac{U_j^{(n+1)}-U_j^{(n)}}{\Delta t}\right)\Bigg(\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{l,m,j}$$
$$-w_j^{-1}\delta^*_{l,m}\left((x_\xi)^{-1}_{l,m,j}w_j\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)},\vec{U}^{(n)})_x}\right)_{l,m,j}\right)\Bigg)$$

$$= \sum_{j=0}^{N} w_j \left(\frac{U_j^{(n+1)} - U_j^{(n)}}{\Delta t} \right) \left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)}, \vec{U}^{(n)})} \right)_j .$$

□

2.4 Design of Schemes for the Dissipative Equations

As is usual in the discrete variational method, we design schemes so that they corresponds to (2.5) for the dissipative equations and to (2.6) for the conservative equations. Defining the schemes in this way allows us to obtain the dissipative/conservative property in almost the same way as Section 2.1.

We define the scheme for the dissipative equation (2.5) by

$$\frac{U_j^{(n+1)} - U_j^{(n)}}{\Delta t} = -\left(-w_j^{-1}\delta_s^*\right)\left(-(x_\xi)_{s,j}^{-1}\delta_{s-1}^*\right)\cdots\left(-(x_\xi)_{2,j}^{-1}\delta_1^*\right)$$
$$\left((x_\xi)_{1,j}^{-1} w_j (x_\xi)_{1,j}^{-1}\delta_1\right)\left((x_\xi)_{2,j}^{-1}\delta_2\right)\cdots\left((x_\xi)_{s-1,j}^{-1}\delta_{s-1}\right)\left((x_\xi)_{s,j}^{-1}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)}, \vec{U}^{(n)})}\right)_j . \tag{2.26}$$

$(x_\xi)_{m,j}$ and δ_m $(m = 1, \ldots, s)$ are arbitrarily chosen dependently on, for example, the accuracy of the scheme. For this scheme, we claim a discrete counterpart of Theorem 2.4.

Theorem 2.10. *Let w_j's be positive. Let $U_j^{(n)}$ be a numerical solution of the scheme* (2.26) *under the boundary condition that satisfies the assumption of Lemma 2.9, and*

$$\mu_{(+,\delta_{s-p+1},w_j)}\Big(\left(\left((x_\xi)_{s-p+1,j}^{-1}\delta_{s-p}^*\right)\cdots\left((x_\xi)_{2,j}^{-1}\delta_1^*\right)\right.$$
$$\left((x_\xi)_{1,j}^{-1} w_j (x_\xi)_{1,j}^{-1}\delta_1\right)\left((x_\xi)_{2,j}^{-1}\delta_2\right)\cdots\left((x_\xi)_{s-1,j}^{-1}\delta_{s-1}\right)\left((x_\xi)_{s,j}^{-1}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)}, \vec{U}^{(n)})}\right)_j ,$$
$$\left.(x_\xi)_{s-p+2,j}^{-1}\delta_{s-p+2}\cdots(x_\xi)_{s,j}^{-1}\delta_s\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)}, \vec{U}^{(n)})}\right)_j\right)$$
$$+\,\mu_{(-,\delta_{s-p+1},w_j)}\Big(\left(\left((x_\xi)_{s-p+1,j}^{-1}\delta_{s-p}^*\right)\cdots\left((x_\xi)_{2,j}^{-1}\delta_1^*\right)\right.$$
$$\left((x_\xi)_{1,j}^{-1} w_j (x_\xi)_{1,j}^{-1}\delta_1\right)\left((x_\xi)_{2,j}^{-1}\delta_2\right)\cdots\left((x_\xi)_{s-1,j}^{-1}\delta_{s-1}\right)\left((x_\xi)_{s,j}^{-1}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)}, \vec{U}^{(n)})}\right)_j ,$$
$$\left.(x_\xi)_{s-p+2,j}^{-1}\delta_{s-p+2}\cdots(x_\xi)_{s,j}^{-1}\delta_s\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)}, \vec{U}^{(n)})}\right)_j\right) = 0 \qquad (p = 1, \ldots, s),$$

if $s \geq 1$. Then

$$\frac{1}{\Delta t}\left(H^{(n+1)} - H^{(n)}\right) \leq 0. \tag{2.27}$$

Proof. This theorem is proved in almost the same way as Theorem 2.4. By Lemma 2.9, we have

$$\frac{1}{\Delta t}\left(H^{(n+1)} - H^{(n)}\right) = \sum_{j=0}^{N} w_j \left(\frac{U_j^{(n+1)} - U_j^{(n)}}{\Delta t}\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)}, \vec{U}^{(n)})}\right)_j .$$

Substituting the scheme (2.26) and an application of the summation by parts in Lemma 2.8 give

$$= -\sum_{j=0}^{N} w_j \Big\{ \left(-w_j^{-1}\delta_s^*\right)\left(-(x_\xi)_{s,j}^{-1}\delta_{s-1}^*\right)\cdots\left(-(x_\xi)_{2,j}^{-1}\delta_1^*\right)$$

$$\left((x_\xi)^{-1}_{1,j} w_j (x_\xi)^{-1}_{1,j}\delta_1\right)\cdots\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\Bigg\}\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j$$

$$= -\sum_{j=0}^{N} w_j\left\{\left(-(x_\xi)^{-1}_{s,j}\delta^*_{s-1}\right)\cdots\left(-(x_\xi)^{-1}_{2,j}\delta^*_1\right)\left((x_\xi)^{-1}_{1,j} w_j (x_\xi)^{-1}_{1,j}\delta_1\right)\cdots\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\right\}$$

$$\left\{w_j^{-1}\delta_s\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\right\}$$

$$= -\sum_{j=0}^{N} w_j\left\{\left(-w_j^{-1}\delta^*_{s-1}\right)\cdots\left(-(x_\xi)^{-1}_{2,j}\delta^*_1\right)\left((x_\xi)^{-1}_{1,j} w_j (x_\xi)^{-1}_{1,j}\delta_1\right)\cdots\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\right\}$$

$$\left\{(x_\xi)^{-1}_{s,j}\delta_s\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\right\}.$$

We continue in this fashion to obtain

$$= -\sum_{j=0}^{N} w_j\left\{\left(-w_j^{-1}\delta^*_{s-2}\right)\cdots\left(-(x_\xi)^{-1}_{2,j}\delta^*_1\right)\left((x_\xi)^{-1}_{1,j} w_j (x_\xi)^{-1}_{1,j}\delta_1\right)\cdots\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\right\}$$

$$\left\{\left((x_\xi)^{-1}_{s-1,j}\delta_{s-1}\right)\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\right\}$$

$$\vdots$$

$$= -\sum_{j=0}^{N} w_j\left\{\left(-w_j^{-1}\delta^*_1\right)\left((x_\xi)^{-1}_{1,j} w_j (x_\xi)^{-1}_{1,j}\delta_1\right)\cdots\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\right\}$$

$$\left\{\left((x_\xi)^{-1}_{2,j}\delta_2\right)\cdots\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\right\}$$

$$= -\sum_{j=0}^{N} w_j\left\{\left((x_\xi)^{-1}_{1,j} w_j (x_\xi)^{-1}_{1,j}\delta_1\right)\cdots\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\right\}$$

$$\left\{\left(w_j^{-1}\delta^*_1\right)\left((x_\xi)^{-1}_{2,j}\delta_2\right)\cdots\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\right\}$$

$$= -\sum_{j=0}^{N} w_j\left\{\left((x_\xi)^{-1}_{1,j}\delta_1\right)\cdots\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\right\}^2 \le 0.$$

□

2.5 Design of Schemes for the Conservative Equations

We define the scheme for the conservative equation (2.6) by

$$\frac{U_j^{(n+1)} - U_j^{(n)}}{\Delta t} = \left(w_j^{-1}\delta^*_s\right)\left((x_\xi)^{-1}_{s,j}\delta^*_{s-1}\right)\cdots\left((x_\xi)^{-1}_{2,j}\delta^*_1\right)$$

$$\left((x_\xi)^{-1}_{1,j}\delta_{\mathrm{c}}\right)\left((x_\xi)^{-1}_{1,j}\delta_1\right)\cdots\left((x_\xi)^{-1}_{s-1,j}\delta_{s-1}\right)\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j. \tag{2.28}$$

δ_{c} is the central difference operator. $(x_\xi)_{m,j}$ and δ_m $(m=1,\dots,s)$ can be chosen arbitrarily. We claim a discrete counterpart of Theorem 2.5 for this scheme.

Theorem 2.11. *Let $U_j^{(n)}$ be a numerical solution of the scheme (2.28) under the boundary condition that satisfies the assumption of Lemma 2.9 and*

$$\begin{aligned}
&\mu_{(+,\delta_{\mathrm{c}},w_j)}\left(\left\{\left((x_\xi)^{-1}_{1,j}\delta_1\right)\cdots\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\right\},\right.\\
&\qquad\left.\left\{\left((x_\xi)^{-1}_{1,j}\delta_1\right)\cdots\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\right\}\right)\\
&+\mu_{(-,\delta_{\mathrm{c}},w_j)}\left(\left\{\left((x_\xi)^{-1}_{1,j}\delta_1\right)\cdots\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\right\},\right.\\
&\qquad\left.\left\{\left((x_\xi)^{-1}_{1,j}\delta_1\right)\cdots\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\right\}\right)=0.
\end{aligned}\tag{2.29}$$

We assume also that

$$\begin{aligned}
&\mu_{(+,\delta_{s-p+1},w_j)}\left(\left\{\left((x_\xi)^{-1}_{s-p+1,k}\delta^*_{s-p}\right)\left((x_\xi)^{-1}_{s-p,k}\delta^*_{s-p-1}\right)\cdots\left((x_\xi)^{-1}_{2,k}\delta^*_1\right)\right.\right.\\
&\quad\left.\left((x_\xi)^{-1}_{1,k}\delta_{\mathrm{c}}\right)\left((x_\xi)^{-1}_{1,k}\delta_1\right)\cdots\left((x_\xi)^{-1}_{s,k}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_k\right\},\\
&\quad\left.\left\{\left((x_\xi)^{-1}_{s-p+2,j}\delta_{s-p+2}\right)\cdots\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\right\}\right)\\
&+\mu_{(-,\delta_{s-p+1},w_j)}\left(\left\{\left((x_\xi)^{-1}_{s-p+1,k}\delta^*_{s-p}\right)\left((x_\xi)^{-1}_{s-p,k}\delta^*_{s-p-1}\right)\cdots\left((x_\xi)^{-1}_{2,k}\delta^*_1\right)\right.\right.\\
&\quad\left.\left((x_\xi)^{-1}_{1,k}\delta_{\mathrm{c}}\right)\left((x_\xi)^{-1}_{1,k}\delta_1\right)\cdots\left((x_\xi)^{-1}_{s,k}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_k\right\},\\
&\quad\left.\left\{(x_\xi)^{-1}_{s-p+2,j}\delta_{s-p+2}\cdots(x_\xi)^{-1}_{s,j}\delta_s\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\right\}\right)=0\qquad(p=1,\dots,s),
\end{aligned}$$

if $s\ge 1$. Then

$$\frac{1}{\Delta t}\left(H^{(n+1)}-H^{(n)}\right)=0. \tag{2.30}$$

Proof. This theorem is proved in almost the same way as Theorem 2.5. By Lemma 2.9, we have

$$\frac{1}{\Delta t}\left(H^{(n+1)}-H^{(n)}\right)=\sum_{j=0}^{N}w_j\left(\frac{U_j^{(n+1)}-U_j^{(n)}}{\Delta t}\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j$$

Substituting the scheme (2.28) and an application of the summation by parts in Lemma 2.8 give

$$=\sum_{j=0}^{N}w_j\left\{\left(w_j^{-1}\delta^*_s\right)\left((x_\xi)^{-1}_{s,j}\delta^*_{s-1}\right)\cdots\left((x_\xi)^{-1}_{2,j}\delta^*_1\right)\right.$$

$$
\begin{aligned}
&\left((x_\xi)^{-1}_{1,j}\delta_{\mathrm{c}}\right)\left((x_\xi)^{-1}_{1,j}\delta_1\right)\cdots\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\Bigg\}\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\\
&=-\sum_{j=0}^{N}w_j\Bigg\{\left((x_\xi)^{-1}_{s,j}\delta^*_{s-1}\right)\cdots\left((x_\xi)^{-1}_{2,j}\delta^*_1\right)\\
&\left((x_\xi)^{-1}_{1,j}\delta_{\mathrm{c}}\right)\left((x_\xi)^{-1}_{1,j}\delta_1\right)\cdots\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\Bigg\}\Bigg\{\left(w_j^{-1}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\Bigg\}\\
&=-\sum_{j=0}^{N}w_j\Bigg\{\left(w_j^{-1}\delta^*_{s-1}\right)\cdots\left((x_\xi)^{-1}_{2,j}\delta^*_1\right)\\
&\left((x_\xi)^{-1}_{1,j}\delta_{\mathrm{c}}\right)\left((x_\xi)^{-1}_{1,j}\delta_1\right)\cdots\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\Bigg\}\Bigg\{\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\Bigg\}.
\end{aligned}
$$

We continue in this fashion to obtain

$$
\begin{aligned}
&=(-1)^2\sum_{j=0}^{N}w_j\left\{\left(w_j^{-1}\delta^*_{s-2}\right)\cdots\left((x_\xi)^{-1}_{2,j}\delta^*_1\right)\left((x_\xi)^{-1}_{1,j}\delta_{\mathrm{c}}\right)\left((x_\xi)^{-1}_{1,j}\delta_1\right)\cdots\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\right\}\\
&\left\{\left((x_\xi)^{-1}_{s-1,j}\delta_{s-1}\right)\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\right\}\\
&\quad\vdots\\
&=(-1)^s\sum_{j=0}^{N}w_j\left\{\left(w_j^{-1}\delta_{\mathrm{c}}\right)\left((x_\xi)^{-1}_{1,j}\delta_1\right)\cdots\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\right\}\\
&\left\{\left((x_\xi)^{-1}_{1,j}\delta_1\right)\cdots\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\right\}\\
&=(-1)^{s+1}\sum_{j=0}^{N}w_j\left\{\left((x_\xi)^{-1}_{1,j}\delta_1\right)\cdots\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\right\}\\
&\left\{\left(w_j^{-1}\delta^*_{\mathrm{c}}\right)\left((x_\xi)^{-1}_{1,j}\delta_1\right)\cdots\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\right\}.
\end{aligned}
$$

Since $\delta^*_{\mathrm{c}}=\delta_{\mathrm{c}}$, we have

$$
\begin{aligned}
&=(-1)^{s+1}\sum_{j=0}^{N}w_j\left\{\left((x_\xi)^{-1}_{1,j}\delta_1\right)\cdots\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\right\}\\
&\left\{\left(w_j^{-1}\delta_{\mathrm{c}}\right)\left((x_\xi)^{-1}_{1,j}\delta_1\right)\cdots\left((x_\xi)^{-1}_{s,j}\delta_s\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j\right\}\\
&=0.
\end{aligned}
$$

□

3 An Example in the One Dimensional Case

In this section, we show an example for the KdV equation

$$u_t + uu_x + \gamma^2 u_{xxx} = 0. \tag{3.1}$$

Famously this equation can be written in the following form

$$u_t = \frac{\partial}{\partial x}\frac{\delta G}{\delta u}, \quad G(u, u_x) = -\frac{1}{6}u^3 - \frac{\gamma^2}{2}u_x^2. \tag{3.2}$$

3.1 An Energy Conservative Scheme for the KdV Equation

The energy functional $G(u, u_x)$ is written in the form of (2.19) with $K = 2$ and

$$f_1(u) = -\frac{1}{6}u^3, \quad f_2(u) = 1, \quad g_1(u_x) = 1, \quad g_2(u_x) = -\frac{\gamma^2}{2}u_x^2.$$

The discrete energy functional is introduced so that it corresponds to these. First, on the given nonuniform mesh, we introduce

$$(x_{\xi,+})_j := x(j+1) - x(j), \quad (x_{\xi,-})_j := x(j) - x(j-1), \quad w_j := \frac{x(j+1) - x(j-1)}{2}.$$

For $l = 1$ we set $M_1 = 1$ and approximate the term $f_1(u)g_1(u_x)$ by

$$f_1(u)g_1(u_x) = -\frac{1}{6}u^3 \simeq -\frac{1}{6}\frac{1}{M_1}\left(U_j^{(n)}\right)^3.$$

With $g_1(u_x) = 1$, the differential operator is not included in this term. Therefore the definitions of $(x_\xi)_{1,1,j}$ and $\delta_{1,1}$ do not affect the definition of the scheme, and so we formally define them by $(x_\xi)_{1,1,j} = (x_{\xi,+})_j$, $\delta_{1,1} = \delta_+$.

The term that corresponds to $l = 2$ includes the differential operator. We approximate it by the average of the value by the forward difference δ_+ and that by the backward difference δ_-. For this reason, we set $M_2 = 2$ and

$$f_2(u)g_2(u_x) = -\frac{\gamma^2}{2}u_x^2 \simeq -\frac{\gamma^2}{2}\frac{1}{M_2}\left(\left(\frac{1}{(x_{\xi,+})_j}\delta_+ U_j^{(n)}\right)^2 + \left(\frac{1}{(x_{\xi,-})_j}\delta_- U_j^{(n)}\right)^2\right),$$

which gives

$$(x_\xi)_{2,1,j} = (x_{\xi,+})_j, \quad \delta_{2,1} = \delta_+, \quad (x_\xi)_{2,2,j} = (x_{\xi,-})_j, \quad \delta_{2,2} = \delta_-.$$

The discrete variational derivative is defined by (2.22)–(2.24):

$$\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)}, \vec{U}^{(n)})}\right)_j = \sum_{l=1}^{2}\frac{1}{M_l}\sum_{m=1}^{M_l}\left(\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}^{(n+1)}, \vec{U}^{(n)})}\right)_{l,m,j} - w_j^{-1}\delta_{l,m}^*\left((x_\xi)_{l,m,j}^{-1} w_j\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)}, \vec{U}_x^{(n)})}\right)_{l,m,j}\right)\right),$$

$$\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}^{(n+1)}, \vec{U}^{(n)})}\right)_{1,1,j} = -\frac{1}{6}\frac{(U_j^{(n+1)})^3 - (U_j^{(n)})^3}{U_j^{(n+1)} - U_j^{(n)}} = -\frac{(U_j^{(n+1)})^2 + U_j^{(n+1)}U_j^{(n)} + (U_j^{(n)})^2}{6},$$

$$\left(\frac{\partial G_{\rm d}}{\partial(\vec{U}_x^{(n+1)},\vec{U}_x^{(n)})}\right)_{1,1,j} = -\frac{1}{6}\left(\frac{(U_j^{(n+1)})^3+(U_j^{(n)})^3}{2}\right)\left(\frac{1-1}{(x_\xi)_{1,1,j}^{-1}\delta_{1,1}U_j^{(n+1)}-(x_\xi)_{1,1,j}^{-1}\delta_{1,1}U_j^{(n)}}\right)=0,$$

$$\left(\frac{\partial G_{\rm d}}{\partial(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{2,1,j} = -\frac{\gamma^2}{2}\left(\frac{1-1}{U_j^{(n+1)}-U_j^{(n)}}\right)\left(\frac{((x_\xi)_{2,1,j}^{-1}\delta_{2,1}U_j^{(n+1)})^2+((x_\xi)_{2,1,j}^{-1}\delta_{2,1}U_j^{(n)})^2}{2}\right)=0,$$

$$\begin{aligned}\left(\frac{\partial G_{\rm d}}{\partial(\vec{U}_x^{(n+1)},\vec{U}_x^{(n)})}\right)_{2,1,j} &= -\frac{\gamma^2}{2}\left(\frac{1+1}{2}\right)\left(\frac{((x_\xi)_{2,1,j}^{-1}\delta_{2,1}U_j^{(n+1)})^2-((x_\xi)_{2,1,j}^{-1}\delta_{2,1}U_j^{(n)})^2}{(x_\xi)_{2,1,j}^{-1}\delta_{2,1}U_j^{(n+1)}-(x_\xi)_{2,1,j}^{-1}\delta_{2,1}U_j^{(n)}}\right)\\ &= -\frac{\gamma^2}{2}(x_{\xi,+})_j^{-1}\delta_+\left(U_j^{(n+1)}+U_j^{(n)}\right),\end{aligned}$$

$$\left(\frac{\partial G_{\rm d}}{\partial(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{2,2,j} = -\frac{\gamma^2}{2}\left(\frac{1-1}{U_j^{(n+1)}-U_j^{(n)}}\right)\left(\frac{((x_\xi)_{2,2,j}^{-1}\delta_{2,2}U_j^{(n+1)})^2+((x_\xi)_{2,2,j}^{-1}\delta_{2,2}U_j^{(n)})^2}{2}\right)=0,$$

$$\begin{aligned}\left(\frac{\partial G_{\rm d}}{\partial(\vec{U}_x^{(n+1)},\vec{U}_x^{(n)})}\right)_{2,2,j} &= -\frac{\gamma^2}{2}\left(\frac{1+1}{2}\right)\left(\frac{((x_\xi)_{2,2,j}^{-1}\delta_{2,2}U_j^{(n+1)})^2-((x_\xi)_{2,2,j}^{-1}\delta_{2,2}U_j^{(n)})^2}{(x_\xi)_{2,2,j}^{-1}\delta_{2,2}U_j^{(n+1)}-(x_\xi)_{2,2,j}^{-1}\delta_{2,2}U_j^{(n)}}\right)\\ &= -\frac{\gamma^2}{2}(x_{\xi,-})_j^{-1}\delta_-\left(U_j^{(n+1)}+U_j^{(n)}\right).\end{aligned}$$

The scheme is defined (2.28) with $s=0$

$$\frac{U_j^{(n+1)}-U_j^{(n)}}{\Delta t} = w_j^{-1}\delta_{\rm c}\left(\frac{\delta G_{\rm d}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j.$$

By substituting the discrete variational derivative, we get the scheme

$$\begin{aligned}\frac{U_j^{(n+1)}-U_j^{(n)}}{\Delta t} &= w_j^{-1}\delta_{\rm c}\Bigg\{\frac{(U_j^{(n+1)})^2+U_j^{(n+1)}U_j^{(n)}+(U_j^{(n)})^2}{6}\\ &\quad+\frac{\gamma^2}{4}\Bigg(\left(w_j^{-1}\delta_-\right)\left((x_{\xi,+})_j^{-1}w_j(x_{\xi,+})_j^{-1}\delta_+\right)\left(U_j^{(n+1)}+U_j^{(n)}\right)\\ &\quad+\left(w_j^{-1}\delta_+\right)\left((x_{\xi,-})_j^{-1}w_j(x_{\xi,-})_j^{-1}\delta_-\right)\left(U_j^{(n+1)}+U_j^{(n)}\right)\Bigg)\Bigg\}.\end{aligned} \tag{3.3}$$

This scheme is the same as that by Furihata [3] if the grids are placed uniformly.

3.2 Numerical Example

We solved the KdV equation numerically by the scheme (3.3). The problem is set in the same way as the famous experiment by Zabusky and Kruskal [13], where the domain is set to $[0,2]$ and the initial condition is given by

$$u(0,x)=\cos(\pi x).$$

The boundary condition is periodic. We set $\gamma=0.022$. Zabusky and Kruskal reported that the solution exhibits a sharp slope near $x=0.5$ at $t=1/\pi$. We compare the numerical solutions that are obtained by using a usual uniform mesh and a nonuniform mesh as shown in Fig. 1 in which the grids are concentrated near $x=0.5$. Both meshes consist of 55 grids. The spatial interval of the nonuniform mesh is about 0.005 at the finest area and about 0.06 at the coarsest. The time interval is $\Delta t=0.0001$.

The numerical solutions at $t=0.44$ by the uniform mesh and the nonuniform mesh are shown in Fig. 2 and Fig. 3 respectively. The solid line in each figure is a numerical result that is obtained by using the finer uniform mesh that consists of the 400 grids. Comparing these two figures, we deduce that the use

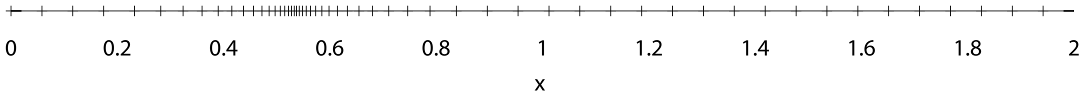

Fig. 1 The nonuniform grids that is used in Section 3.2.

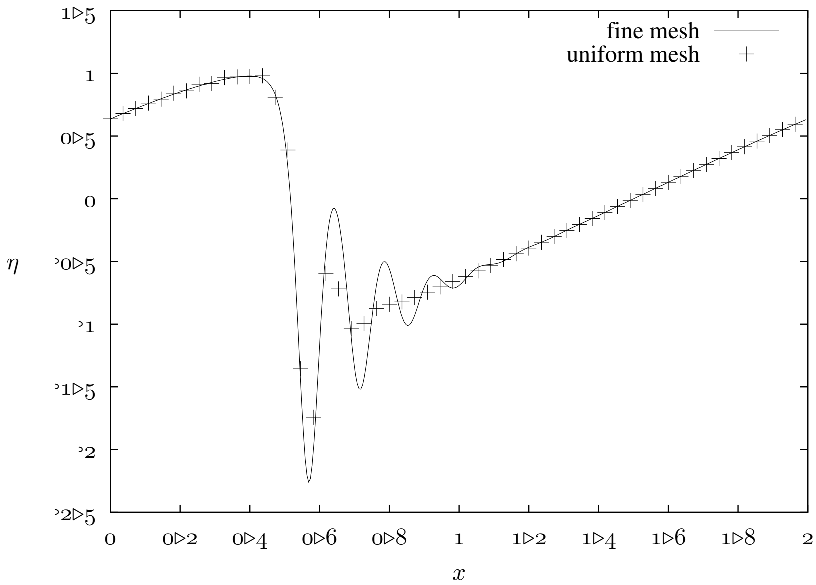

Fig. 2 Numerical solution at $t = 0.44$ by the uniform grid. The solid line is that by the finer uniform mesh that consists of the 400 grids.

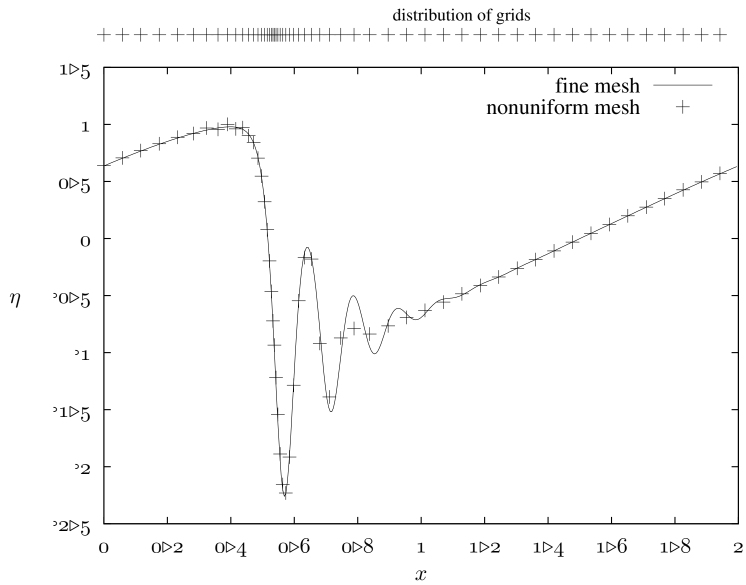

Fig. 3 Numerical solution at $t = 0.44$ by the nonuniform grid. The solid line is that by the finer uniform mesh that consists of the 400 grids.

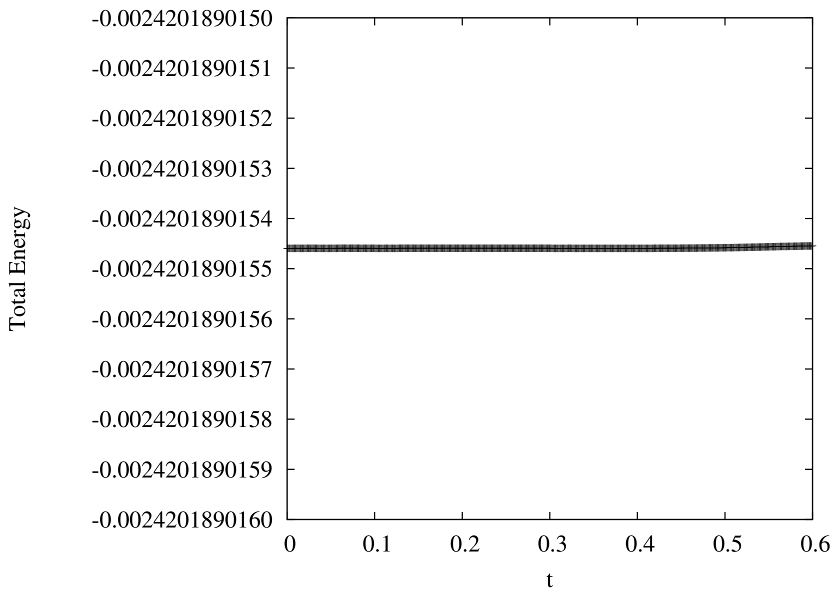

Fig. 4 The time evolution of the total energy in the case where the nonuniform mesh is employed.

of the nonuniform mesh can improve the numerical solution in the sense that the oscillation near $x = 0.6$ can be captured on the nonuniform mesh, unlike on the uniform mesh.

The time evolution of the energy in the case where the nonuniform mesh is employed is shown in Fig. 4, which confirms the energy conservation property that is stated in Theorem 2.11.

4 Extension to Multidimensional Nonuniform Grids

In this section, we extend the discrete variational method to multidimensional nonuniform grids. In multidimensional cases, notation becomes extremely complicated. For this reason we consider the two dimensional case only, but the extensions to more than two dimensional problems are obtained in the same manner.

4.1 Target Equations

Here we consider the following equations on a two dimensional domain Ω, which are two dimensional analogues of the equations considered in the previous sections. We assume that there exists a sufficiently smooth homeomorphism between Ω and the computational space.

In two dimensional cases, the variational derivative is defined by

$$\frac{\delta G}{\delta u} := \frac{\partial G}{\partial u} - \frac{\partial}{\partial x}\frac{\partial G}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial G}{\partial u_y}.$$

The dissipative equations that correspond to (1.3) become

$$\frac{\partial u}{\partial t} = (-1)^{s+1}\left(\frac{\partial^{2s}}{\partial x^{2s}} + \frac{\partial^{2s}}{\partial y^{2s}}\right)\frac{\delta G}{\delta u}, \quad (t, x, y) \in (0, \infty) \times \Omega, \quad s = 0, 1. \tag{4.1}$$

We consider just $s = 0, 1$, since there exists little equation with $s > 1$ in multidimensional cases. Equations of this form enjoy the dissipation property.

Theorem 4.1. *Suppose that the boundary condition satisfies*

$$\int_{\partial\Omega} \frac{\partial u}{\partial t} \left(\frac{\partial G}{\partial u_x}, \frac{\partial G}{\partial u_y} \right)^{\top} \cdot n \mathrm{d}s = 0, \tag{4.2}$$

where n *is the unit normal vector to the boundary and* $\mathrm{d}s$ *is the area element. Suppose also that*

$$\int_{\partial\Omega} \frac{\delta G}{\delta u} \left(\nabla \frac{\delta G}{\delta u} \cdot n \right) \mathrm{d}s = 0 \tag{4.3}$$

if $s = 1$*. Then the solutions of* (4.1) *have the dissipation property:*

$$\frac{\mathrm{d}H}{\mathrm{d}t} \leq 0, \quad H(t) = \int_{\Omega} G(u, u_x, u_y) \mathrm{d}x\mathrm{d}y.$$

The conservative equations that correspond to (1.5) become

$$\frac{\partial u}{\partial t} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \frac{\delta G}{\delta u}, \quad (t, x, y) \in (0, \infty) \times \Omega. \tag{4.4}$$

Theorem 4.2. *Suppose that the boundary condition satisfies*

$$\int_{\partial\Omega} \frac{\partial u}{\partial t} \left(\frac{\partial G}{\partial u_x}, \frac{\partial G}{\partial u_y} \right)^{\top} \cdot n \mathrm{d}s = 0, \quad \int_{\partial\Omega} \left(\frac{\delta G}{\delta u} \right)^2 \mathrm{d}s = 0 \tag{4.5}$$

Then the solutions of (4.4) *have the conservation property:*

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0, \quad H(t) = \int_{\Omega} G(u, u_x, u_y) \mathrm{d}x\mathrm{d}y. \tag{4.6}$$

We show "the variational structure" of these equations:

Lemma 4.3. *Suppose that a solution of* (4.1) *or* (4.4) *satisfies the condition*

$$\int_{\partial\Omega} \frac{\partial u}{\partial t} \left(\frac{\partial G}{\partial u_x}, \frac{\partial G}{\partial u_y} \right)^{\top} \cdot n \mathrm{d}s = 0.$$

Then

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \int_{\Omega} \frac{\partial u}{\partial t} \frac{\delta G}{\delta u} \mathrm{d}x\mathrm{d}y. \tag{4.7}$$

Proof. By the Gauss theorem, we have

$$\begin{aligned}
\frac{\mathrm{d}H}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} G(u, u_x, u_y) \mathrm{d}x\mathrm{d}y \\
&= \int_{\Omega} \left(\frac{\partial G}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial G}{\partial u_x} \frac{\partial u_x}{\partial t} + \frac{\partial G}{\partial u_y} \frac{\partial u_y}{\partial t} \right) \mathrm{d}x\mathrm{d}y \\
&= \int_{\Omega} \left(\frac{\partial G}{\partial u} - \frac{\partial}{\partial x} \frac{\partial G}{\partial u_x} - \frac{\partial}{\partial y} \frac{\partial G}{\partial u_y} \right) \frac{\partial u}{\partial t} \mathrm{d}x\mathrm{d}y + \int_{\partial\Omega} \frac{\partial u}{\partial t} \left(\frac{\partial G}{\partial u_x}, \frac{\partial G}{\partial u_y} \right)^{\top} \cdot n \mathrm{d}s \\
&= \int_{\Omega} \frac{\delta G}{\delta u} \frac{\partial u}{\partial t} \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

□

Proof of Theorem 4.1. Applying Lemma 4.3 gives

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \int_\Omega \frac{\delta G}{\delta u}\frac{\partial u}{\partial t}\mathrm{d}x\mathrm{d}y.$$

In the case where $s = 0$, substituting the equation yields

$$\int_\Omega \frac{\delta G}{\delta u}\frac{\partial u}{\partial t}\mathrm{d}x\mathrm{d}y = -\int_\Omega \left(\frac{\delta G}{\delta u}\right)^2 \mathrm{d}x\mathrm{d}y \leq 0.$$

In the case where $s = 1$, applying the Gauss theorem one more time gives

$$\begin{aligned}
\int_\Omega \frac{\delta G}{\delta u}\frac{\partial u}{\partial t}\mathrm{d}x\mathrm{d}y &= \int_\Omega \left(\frac{\delta G}{\delta u}\right)\left(\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{\delta G}{\delta u}\right)\mathrm{d}x\mathrm{d}y \\
&= -\int_\Omega \left(\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\frac{\delta G}{\delta u}\right)^2 \mathrm{d}x\mathrm{d}y + \int_{\partial\Omega} \frac{\delta G}{\delta u}\left(\nabla\frac{\delta G}{\delta u}\cdot n\right)\mathrm{d}s \\
&= -\int_\Omega \left(\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\frac{\delta G}{\delta u}\right)^2 \mathrm{d}x\mathrm{d}y \\
&\leq 0.
\end{aligned}$$

□

Proof of Theorem 4.2. Applying Lemma 4.3 gives

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \int_\Omega \frac{\delta G}{\delta u}\frac{\partial u}{\partial t}\mathrm{d}x\mathrm{d}y$$

Substituting the equation and applying the Gauss theorem yield

$$\begin{aligned}
&= \int_\Omega \left(\frac{\delta G}{\delta u}\right)\left(\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\frac{\delta G}{\delta u}\right)\mathrm{d}x\mathrm{d}y \\
&= -\int_\Omega \left(\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\frac{\delta G}{\delta u}\right)\left(\frac{\delta G}{\delta u}\right)\mathrm{d}x\mathrm{d}y + \int_{\partial\Omega} \left(\frac{\delta G}{\delta u}\right)^2 \mathrm{d}s \\
&= -\int_\Omega \left(\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\frac{\delta G}{\delta u}\right)\left(\frac{\delta G}{\delta u}\right)\mathrm{d}x\mathrm{d}y,
\end{aligned}$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega G(u, u_x, u_y)\mathrm{d}x\mathrm{d}y = 0.$$

□

4.2 The Dissipation/Conservation Properties in the Computational Space

In this section, we extend the discrete variational method to multidimensional nonuniform grids such as the one shown in Fig. 5. We assume that the number of grids is $(N+1)\times(M+1)$. As in the one dimensional case, we use the mapping method. First we describe the proofs of the conservation/dissipation properties in the computational space with the coordinates (ξ, η), and then, we define the discrete variational derivative and the schemes.

In the computational space, the partial derivatives are transformed to

$$\frac{\partial v}{\partial x} = J^{-1}\left(\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta}\right)v, \quad \frac{\partial v}{\partial y} = J^{-1}\left(\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi}\right)v, \tag{4.8}$$

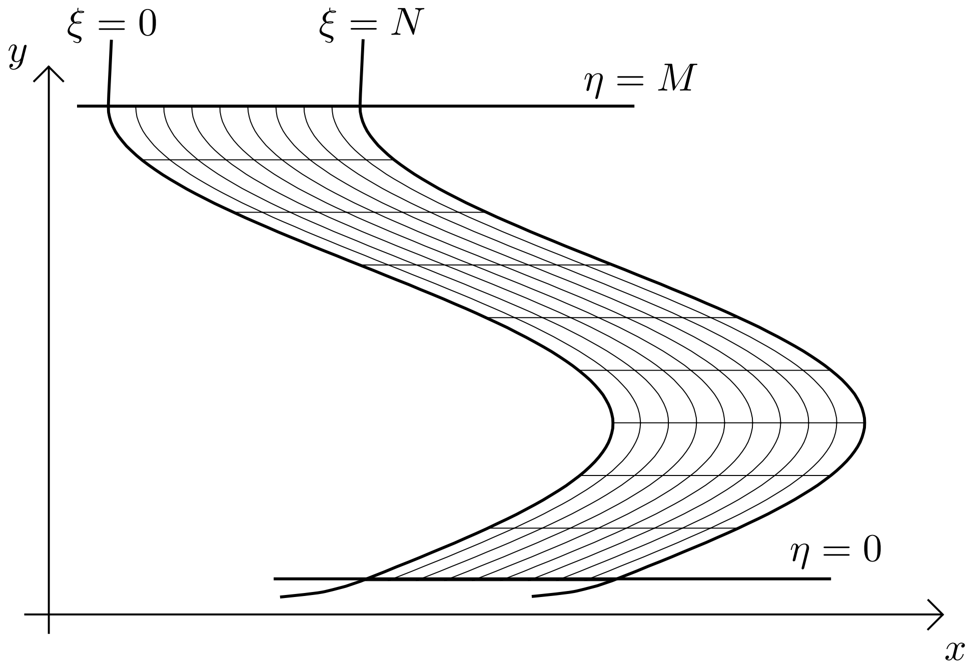

Fig. 5 An example of nonuniform grids in $\mathbb{R}^2$.

where v is a smooth function and J is the Jacobian

$$J = \det \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix}.$$

Note that the partial derivatives are also written as

$$\frac{\partial v}{\partial x} = J^{-1}\left(\frac{\partial}{\partial \xi}\left(\frac{\partial y}{\partial \eta}v\right) - \frac{\partial}{\partial \eta}\left(\frac{\partial y}{\partial \xi}v\right)\right), \quad \frac{\partial v}{\partial y} = J^{-1}\left(\frac{\partial}{\partial \eta}\left(\frac{\partial x}{\partial \xi}v\right) - \frac{\partial}{\partial \xi}\left(\frac{\partial x}{\partial \eta}v\right)\right), \tag{4.9}$$

since $x_{\xi\eta} = x_{\eta\xi}$, $y_{\xi\eta} = y_{\eta\xi}$.

By using these formulas, we rewrite the equations in the following forms that are suitable for the discretizations. We transform the dissipative equations (4.1) to

$$\frac{\partial u}{\partial t} = -\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}, \tag{4.10}$$

if $s = 0$, and to

$$\begin{aligned} \frac{\partial u}{\partial t} &= \\ & J^{-1}\left\{\frac{\partial}{\partial \xi}\left(\frac{\partial y}{\partial \eta}\left(J^{-1}\left(\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta}\right)\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}\right)\right) - \frac{\partial}{\partial \eta}\left(\frac{\partial y}{\partial \xi}\left(J^{-1}\left(\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta}\right)\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}\right)\right)\right\} \\ & + J^{-1}\left\{\frac{\partial}{\partial \eta}\left(\frac{\partial x}{\partial \xi}\left(J^{-1}\left(\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi}\right)\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}\right)\right) - \frac{\partial}{\partial \xi}\left(\frac{\partial x}{\partial \eta}\left(J^{-1}\left(\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi}\right)\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}\right)\right)\right\}, \end{aligned} \tag{4.11}$$

if $s = 1$. We transform the conservative equations (4.4) to

$$\frac{\partial u}{\partial t} = \frac{1}{2}\left[\left\{J^{-1}\left(\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta}\right)\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}} + J^{-1}\left(\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi}\right)\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}\right\}\right.$$

$$+\left\{J^{-1}\left(\frac{\partial}{\partial\xi}\left(\frac{\partial y}{\partial\eta}\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}\right)-\frac{\partial}{\partial\eta}\left(\frac{\partial y}{\partial\xi}\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}\right)\right)+J^{-1}\left(\frac{\partial}{\partial\eta}\left(\frac{\partial x}{\partial\xi}\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}\right)-\frac{\partial}{\partial\xi}\left(\frac{\partial x}{\partial\eta}\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}\right)\right)\right\}\Bigg]. \tag{4.12}$$

$(\delta G/\delta u)_{\mathrm{cs}}$ is the transformed variational derivative

$$\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}:=\frac{\partial G}{\partial u}-\left(J^{-1}\left(\frac{\partial}{\partial\xi}\left(\frac{\partial y}{\partial\eta}\frac{\partial G}{\partial u_x}\right)-\frac{\partial}{\partial\eta}\left(\frac{\partial y}{\partial\xi}\frac{\partial G}{\partial u_x}\right)\right)+J^{-1}\left(\frac{\partial}{\partial\eta}\left(\frac{\partial x}{\partial\xi}\frac{\partial G}{\partial u_y}\right)-\frac{\partial}{\partial\xi}\left(\frac{\partial x}{\partial\eta}\frac{\partial G}{\partial u_y}\right)\right)\right). \tag{4.13}$$

Theorem 4.1 and 4.2 are transformed to the followings.

Theorem 4.4. *Suppose that the boundary condition satisfies*

$$\left[\int_0^M \frac{\partial u}{\partial t}\left(\frac{\delta G}{\delta u_x}\frac{\partial y}{\partial\eta}-\frac{\delta G}{\delta u_y}\frac{\partial x}{\partial\eta}\right)\mathrm{d}\eta\right]_{\xi=0}^{\xi=N}+\left[\int_0^N \frac{\partial u}{\partial t}\left(\frac{\delta G}{\delta u_x}\frac{\partial y}{\partial\xi}-\frac{\delta G}{\delta u_y}\frac{\partial x}{\partial\xi}\right)\mathrm{d}\xi\right]_{\eta=M}^{\eta=0}=0. \tag{4.14}$$

Suppose also that

$$\begin{aligned}&\left[\int_0^M\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}\left(J^{-1}\left(\frac{\partial y}{\partial\eta}\frac{\partial}{\partial\xi}-\frac{\partial y}{\partial\xi}\frac{\partial}{\partial\eta}\right)\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}\frac{\partial y}{\partial\eta}-J^{-1}\left(\frac{\partial x}{\partial\xi}\frac{\partial}{\partial\eta}-\frac{\partial x}{\partial\eta}\frac{\partial}{\partial\xi}\right)\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}\frac{\partial x}{\partial\eta}\right)\mathrm{d}\eta\right]_{\xi=0}^{\xi=N}\\&+\left[\int_0^N\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}\left(J^{-1}\left(\frac{\partial y}{\partial\eta}\frac{\partial}{\partial\xi}-\frac{\partial y}{\partial\xi}\frac{\partial}{\partial\eta}\right)\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}\frac{\partial y}{\partial\xi}-J^{-1}\left(\frac{\partial x}{\partial\xi}\frac{\partial}{\partial\eta}-\frac{\partial x}{\partial\eta}\frac{\partial}{\partial\xi}\right)\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}\frac{\partial x}{\partial\xi}\right)\mathrm{d}\xi\right]_{\eta=M}^{\eta=0}\\&=0,\end{aligned} \tag{4.15}$$

if $s=1$. Then the solutions of (4.10) *or* (4.11) *have the dissipation property:*

$$\frac{\mathrm{d}H_{\mathrm{cs}}}{\mathrm{d}t}\le 0,\quad H_{\mathrm{cs}}(t)=\int_{\eta=0}^{\eta=M}\int_{\xi=0}^{\xi=N}G(u,u_x,u_y)J\mathrm{d}\xi\mathrm{d}\eta.$$

Theorem 4.5. *Suppose that the boundary condition satisfies*

$$\left[\int_0^M \frac{\partial u}{\partial t}\left(\frac{\delta G}{\delta u_x}\frac{\partial y}{\partial\eta}-\frac{\delta G}{\delta u_y}\frac{\partial x}{\partial\eta}\right)\mathrm{d}\eta\right]_{\xi=0}^{\xi=N}+\left[\int_0^N \frac{\partial u}{\partial t}\left(\frac{\delta G}{\delta u_x}\frac{\partial y}{\partial\xi}-\frac{\delta G}{\delta u_y}\frac{\partial x}{\partial\xi}\right)\mathrm{d}\xi\right]_{\eta=M}^{\eta=0}=0, \tag{4.16}$$

$$\left[\int_0^M\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}^2\left(\frac{\partial y}{\partial\eta}-\frac{\partial x}{\partial\eta}\right)\mathrm{d}\eta\right]_{\xi=0}^{\xi=N}+\left[\int_0^N\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}^2\left(\frac{\partial y}{\partial\xi}-\frac{\partial x}{\partial\xi}\right)\mathrm{d}\xi\right]_{\eta=M}^{\eta=0}=0. \tag{4.17}$$

Then the solutions of (4.12) *have the conservation property:*

$$\frac{\mathrm{d}H_{\mathrm{cs}}}{\mathrm{d}t}=0,\quad H_{\mathrm{cs}}(t)=\int_{\eta=0}^{\eta=M}\int_{\xi=0}^{\xi=N}G(u,u_x,u_y)J\mathrm{d}\xi\mathrm{d}\eta.$$

For the proofs of these theorems, we prepare the Gauss theorem that is transformed to the computational space.

Lemma 4.6. *Let u, v_1, v_2 be smooth functions that satisfy*

$$\left[\int_0^M u\left(v_1\frac{\partial y}{\partial\eta}-v_2\frac{\partial x}{\partial\eta}\right)\mathrm{d}\eta\right]_{\xi=0}^{\xi=N}+\left[\int_0^N u\left(v_1\frac{\partial y}{\partial\xi}-v_2\frac{\partial x}{\partial\xi}\right)\mathrm{d}\xi\right]_{\eta=M}^{\eta=0}=0. \tag{4.18}$$

Then

$$\int_{\eta=0}^{\eta=M}\int_{\xi=0}^{\xi=N}\left(v_1J^{-1}\left(\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi}-\frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta}\right)u+v_2J^{-1}\left(\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta}-\frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi}\right)u\right)J\mathrm{d}\xi\mathrm{d}\eta$$
$$=-\int_{\eta=0}^{\eta=M}\int_{\xi=0}^{\xi=N}u\left(J^{-1}\left(\frac{\partial}{\partial \xi}\left(\frac{\partial y}{\partial \eta}v_1\right)-\frac{\partial}{\partial \eta}\left(\frac{\partial y}{\partial \xi}v_1\right)\right)+J^{-1}\left(\frac{\partial}{\partial \eta}\left(\frac{\partial x}{\partial \xi}v_2\right)-\frac{\partial}{\partial \xi}\left(\frac{\partial x}{\partial \eta}v_2\right)\right)\right)J\mathrm{d}\xi\mathrm{d}\eta.$$

Proof. We can obtain this lemma by applying the integration by parts in the ξ and the η directions. In fact,

$$\int_{\eta=0}^{\eta=M}\int_{\xi=0}^{\xi=N}\left(v_1J^{-1}\left(\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi}-\frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta}\right)u+v_2J^{-1}\left(\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta}-\frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi}\right)u\right)J\mathrm{d}\xi\mathrm{d}\eta$$
$$=\int_{\eta=0}^{\eta=M}\int_{\xi=0}^{\xi=N}\left(v_1\left(\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi}-\frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta}\right)u+v_2\left(\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta}-\frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi}\right)u\right)\mathrm{d}\xi\mathrm{d}\eta$$
$$=-\int_{\eta=0}^{\eta=M}\int_{\xi=0}^{\xi=N}u\frac{\partial}{\partial \xi}\left(\frac{\partial y}{\partial \eta}v_1\right)\mathrm{d}\xi\mathrm{d}\eta+\left[\int_0^M\frac{\partial y}{\partial \eta}uv_1\mathrm{d}\eta\right]_{\xi=0}^{\xi=N}$$
$$+\int_{\eta=0}^{\eta=M}\int_{\xi=0}^{\xi=N}u\frac{\partial}{\partial \eta}\left(\frac{\partial y}{\partial \xi}v_1\right)\mathrm{d}\xi\mathrm{d}\eta-\left[\int_0^N\frac{\partial y}{\partial \xi}uv_1\mathrm{d}\xi\right]_{\eta=0}^{\eta=M}$$
$$-\int_{\eta=0}^{\eta=M}\int_{\xi=0}^{\xi=N}u\frac{\partial}{\partial \eta}\left(\frac{\partial x}{\partial \xi}v_2\right)\mathrm{d}\xi\mathrm{d}\eta+\left[\int_0^N\frac{\partial x}{\partial \xi}uv_2\mathrm{d}\xi\right]_{\eta=0}^{\eta=M}$$
$$+\int_{\eta=0}^{\eta=M}\int_{\xi=0}^{\xi=N}u\frac{\partial}{\partial \xi}\left(\frac{\partial x}{\partial \eta}v_2\right)\mathrm{d}\xi\mathrm{d}\eta-\left[\int_0^M\frac{\partial x}{\partial \eta}uv_2\mathrm{d}\eta\right]_{\xi=0}^{\xi=N}$$
$$=-\int_{\eta=0}^{\eta=M}\int_{\xi=0}^{\xi=N}u\left(J^{-1}\left(\frac{\partial}{\partial \xi}\left(\frac{\partial y}{\partial \eta}v_1\right)-\frac{\partial}{\partial \eta}\left(\frac{\partial y}{\partial \xi}v_1\right)\right)+J^{-1}\left(\frac{\partial}{\partial \eta}\left(\frac{\partial x}{\partial \xi}v_2\right)-\frac{\partial}{\partial \xi}\left(\frac{\partial x}{\partial \eta}v_2\right)\right)\right)J\mathrm{d}\xi\mathrm{d}\eta$$
$$+\left[\int_0^M u\left(v_1\frac{\partial y}{\partial \eta}-v_2\frac{\partial x}{\partial \eta}\right)\mathrm{d}\eta\right]_{\xi=0}^{\xi=N}+\left[\int_0^N u\left(v_1\frac{\partial y}{\partial \xi}-v_2\frac{\partial x}{\partial \xi}\right)\mathrm{d}\xi\right]_{\eta=M}^{\eta=0}$$
$$=-\int_{\eta=0}^{\eta=M}\int_{\xi=0}^{\xi=N}u\left(J^{-1}\left(\frac{\partial}{\partial \xi}\left(\frac{\partial y}{\partial \eta}v_1\right)-\frac{\partial}{\partial \eta}\left(\frac{\partial y}{\partial \xi}v_1\right)\right)+J^{-1}\left(\frac{\partial}{\partial \eta}\left(\frac{\partial x}{\partial \xi}v_2\right)-\frac{\partial}{\partial \xi}\left(\frac{\partial x}{\partial \eta}v_2\right)\right)\right)J\mathrm{d}\xi\mathrm{d}\eta.$$

□

Next, we transform Lemma 4.3.

Lemma 4.7. *Suppose that a solution of* (4.1) *or* (4.4) *satisfies the condition*

$$\left[\int_0^M u_t\left(\frac{\delta G}{\delta u_x}\frac{\partial y}{\partial \eta}-\frac{\delta G}{\delta u_y}\frac{\partial x}{\partial \eta}\right)\mathrm{d}\eta\right]_{\xi=0}^{\xi=N}+\left[\int_0^N u_t\left(\frac{\delta G}{\delta u_x}\frac{\partial y}{\partial \xi}-\frac{\delta G}{\delta u_y}\frac{\partial x}{\partial \xi}\right)\mathrm{d}\xi\right]_{\eta=M}^{\eta=0}=0.$$

Then

$$\frac{\mathrm{d}H}{\mathrm{d}t}=\int_{\eta=0}^{\eta=M}\int_{\xi=0}^{\xi=N}\frac{\partial u}{\partial t}\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}}J\mathrm{d}\xi\mathrm{d}\eta. \tag{4.19}$$

Proof. By the chain rule, we have

$$\frac{\mathrm{d}H}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\eta=0}^{\eta=M}\int_{\xi=0}^{\xi=N}G\mathrm{d}x\mathrm{d}y \tag{4.20}$$

$$
\begin{aligned}
&= \int_{\eta=0}^{\eta=M} \int_{\xi=0}^{\xi=N} \left(\frac{\partial G}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial G}{\partial u_x}\frac{\partial u_x}{\partial t} + \frac{\partial G}{\partial u_y}\frac{\partial u_y}{\partial t} \right) \mathrm{d}x\mathrm{d}y \\
&= \int_{\eta=0}^{\eta=M} \int_{\xi=0}^{\xi=N} \frac{\partial G}{\partial u}\frac{\partial u}{\partial t} J\mathrm{d}\xi\mathrm{d}\eta + \int_{\eta=0}^{\eta=M} \int_{\xi=0}^{\xi=N} \left(\frac{\partial G}{\partial u_x} J^{-1} \left(\frac{\partial y}{\partial \eta}\frac{\partial u_t}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial u_t}{\partial \eta} \right) \right. \\
&\quad \left. + \frac{\partial G}{\partial u_y} J^{-1} \left(\frac{\partial x}{\partial \xi}\frac{\partial u_t}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial u_t}{\partial \xi} \right) \right) J\mathrm{d}\xi\mathrm{d}\eta.
\end{aligned}
$$

By applying the transformed Gauss theorem shown in Lemma 4.6, we get

$$
\begin{aligned}
&= \int_{\eta=0}^{\eta=M} \int_{\xi=0}^{\xi=N} \frac{\partial G}{\partial u}\frac{\partial u}{\partial t} J\mathrm{d}\xi\mathrm{d}\eta - \int_{\eta=0}^{\eta=M} \int_{\xi=0}^{\xi=N} \frac{\partial u}{\partial t} \left\{ J^{-1} \left(\frac{\partial}{\partial \xi}\left(\frac{\partial y}{\partial \eta}\frac{\partial G}{\partial u_x} \right) - \frac{\partial}{\partial \eta}\left(\frac{\partial y}{\partial \xi}\frac{\partial G}{\partial u_x} \right) \right) \right. \\
&\qquad\qquad \left. + J^{-1} \left(\frac{\partial}{\partial \eta}\left(\frac{\partial x}{\partial \xi}\frac{\partial G}{\partial u_y} \right) - \frac{\partial}{\partial \xi}\left(\frac{\partial x}{\partial \eta}\frac{\partial G}{\partial u_y} \right) \right) \right\} J\mathrm{d}\xi\mathrm{d}\eta \\
&= \int_{\eta=0}^{\eta=M} \int_{\xi=0}^{\xi=N} \frac{\partial u}{\partial t} \\
&\quad \left\{ \frac{\partial G}{\partial u} - \left(J^{-1} \left(\frac{\partial}{\partial \xi}\left(\frac{\partial y}{\partial \eta}\frac{\partial G}{\partial u_x} \right) - \frac{\partial}{\partial \eta}\left(\frac{\partial y}{\partial \xi}\frac{\partial G}{\partial u_x} \right) \right) + J^{-1} \left(\frac{\partial}{\partial \eta}\left(\frac{\partial x}{\partial \xi}\frac{\partial G}{\partial u_y} \right) - \frac{\partial}{\partial \xi}\left(\frac{\partial x}{\partial \eta}\frac{\partial G}{\partial u_y} \right) \right) \right) \right\} J\mathrm{d}\xi\mathrm{d}\eta \\
&= \frac{\partial u}{\partial t} \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} J\mathrm{d}\xi\mathrm{d}\eta.
\end{aligned}
$$

□

Now we can prove Theorem 4.4 and 4.5.

Proof of Theorem 4.4. By Lemma 4.7, we have

$$
\frac{\mathrm{d}H}{\mathrm{d}t} = \int_{\eta=0}^{\eta=M} \int_{\xi=0}^{\xi=N} \frac{\partial u}{\partial t} \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} J\mathrm{d}\xi\mathrm{d}\eta. \tag{4.21}
$$

In the case of $s = 0$, substituting the equation (4.10) yields

$$
(4.21) = -\int_{\eta=0}^{\eta=M} \int_{\xi=0}^{\xi=N} \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}}^2 J\mathrm{d}\xi\mathrm{d}\eta \le 0.
$$

In the case of $s = 1$, substituting the equation (4.11) and applying the Gauss theorem in the form of Lemma 4.6 yield

$$
\begin{aligned}
&(4.21) = \\
&\int_{\eta=0}^{\eta=M} \int_{\xi=0}^{\xi=N} \left[J^{-1} \left\{ \frac{\partial}{\partial \xi}\left(\frac{\partial y}{\partial \eta}\left(J^{-1}\left(\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta} \right)\left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} \right) \right) \right. \right. \\
&\quad \left. - \frac{\partial}{\partial \eta}\left(\frac{\partial y}{\partial \xi}\left(J^{-1}\left(\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta} \right)\left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} \right) \right) \right\} + J^{-1} \left\{ \frac{\partial}{\partial \eta}\left(\frac{\partial x}{\partial \xi}\left(J^{-1}\left(\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi} \right)\left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} \right) \right) \right. \\
&\quad \left. \left. - \frac{\partial}{\partial \xi}\left(\frac{\partial x}{\partial \eta}\left(J^{-1}\left(\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi} \right)\left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} \right) \right) \right\} \right] \left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} J\mathrm{d}\xi\mathrm{d}\eta \\
&= -\int_{\eta=0}^{\eta=M} \int_{\xi=0}^{\xi=N} \left\{ \left(J^{-1}\left(\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta} \right)\left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} \right) + \left(J^{-1}\left(\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi} \right)\left(\frac{\delta G}{\delta u} \right)_{\mathrm{cs}} \right) \right\}^2 J\mathrm{d}\xi\mathrm{d}\eta \\
&\le 0.
\end{aligned}
$$

□

Proof of Theorem 4.5. Applying Lemma 4.7 we have

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \int_{\eta=0}^{\eta=M} \int_{\xi=0}^{\xi=N} \frac{\partial u}{\partial t} \left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}} J \mathrm{d}\xi \mathrm{d}\eta.$$

Substituting the equation (4.12) yields

$$= \int_{\eta=0}^{\eta=M} \int_{\xi=0}^{\xi=N} \frac{1}{2} \left[\left\{ J^{-1} \left(\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta} \right) \left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}} + J^{-1} \left(\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi} \right) \left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}} \right\} \right.$$
$$\left. \left\{ J^{-1} \left(\frac{\partial}{\partial \xi} \left(\frac{\partial y}{\partial \eta} \left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}} \right) - \frac{\partial}{\partial \eta} \left(\frac{\partial y}{\partial \xi} \left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}} \right) \right) + J^{-1} \left(\frac{\partial}{\partial \eta} \left(\frac{\partial x}{\partial \xi} \left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}} \right) - \frac{\partial}{\partial \xi} \left(\frac{\partial x}{\partial \eta} \left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}} \right) \right) \right\} \right]$$
$$\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}} J \mathrm{d}\xi \mathrm{d}\eta.$$

Applying Lemma 4.6, we have

$$= - \int_{\eta=0}^{\eta=M} \int_{\xi=0}^{\xi=N} \frac{1}{2}$$
$$\left[\left\{ J^{-1} \left(\frac{\partial}{\partial \xi} \left(\frac{\partial y}{\partial \eta} \left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}} \right) - \frac{\partial}{\partial \eta} \left(\frac{\partial y}{\partial \xi} \left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}} \right) \right) + J^{-1} \left(\frac{\partial}{\partial \eta} \left(\frac{\partial x}{\partial \xi} \left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}} \right) - \frac{\partial}{\partial \xi} \left(\frac{\partial x}{\partial \eta} \left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}} \right) \right) \right\} \right.$$
$$\left. + \left\{ J^{-1} \left(\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta} \right) \left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}} + J^{-1} \left(\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi} \right) \left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}} \right\} \right] \left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}} J \mathrm{d}\xi \mathrm{d}\eta,$$

and hence, this equals 0. □

4.3 The Discrete Gauss Theorem on Multidimensional Nonuniform Grids

In this section, we provide a discrete counterpart of the Gauss theorem. As is seen in the proof of Lemma 4.6, the Gauss theorem in the computational space is obtained simply by applying the integration by parts in the ξ and the η directions. Therefore the discrete Gauss theorem is obtained by applying the summation by parts in each direction.

First, similarly to the one dimensional case, we introduce an averaging operator to simplify notation.

Definition 4.1. *Let δ_ξ and δ_η be finite difference operators in the ξ and the η directions respectively:*

$$\delta_\xi U_{i,j} = \sum_{k=-\alpha}^{\alpha} a_k U_{i+k,j}, \quad \delta_\eta U_{i,j} = \sum_{k=-\beta}^{\beta} b_k U_{i,j+k}.$$

We define $\tilde{a}_{i,k}$, $\tilde{a}^*_{i,k}$, $\tilde{b}_{j,k}$ and $\tilde{b}^*_{j,k}$ by

$$\tilde{a}_{i,k} = \begin{cases} a_{-i+k} & (k = i-\alpha, i-\alpha+1, \ldots, i+\alpha-1, i+\alpha), \\ 0 & (\text{otherwise}), \end{cases}$$
$$a^*_{i,k} = -a_{k,i},$$
$$\tilde{b}_{j,k} = \begin{cases} b_{-j+k} & (k = j-\beta, j-\beta+1, \ldots, j+\beta-1, j+\beta), \\ 0 & (\text{otherwise}), \end{cases}$$
$$b^*_{j,k} = -b_{k,j}.$$

For $U_{i,j}$, $V_{1,i,j}$ and $V_{2,i,j}$ we define an averaging operator $\mu_{(\partial\Omega,\delta_\xi,\delta_\eta,x_\xi,x_\eta,y_\xi,y_\eta)}(\{U_{i,j}\},\{V_{1,i,j}\},\{V_{2,i,j}\})$ by

$$\mu_{(\partial\Omega,\delta_\xi,\delta_\eta,x_\xi,x_\eta,y_\xi,y_\eta)}(\{U_{i,j}\},\{V_{1,i,j}\},\{V_{2,i,j}\})$$

$$
\begin{aligned}
:= &\sum_{j=0}^{M}\sum_{i=N-\alpha+1}^{N}\sum_{k=N+1}^{N+\alpha} w_{i,j}U_{i,j}\left(w_{i,j}^{-1}(y_\eta)_{i,j}\tilde{a}_{i,k}V_{1,k,j} - w_{i,j}^{-1}(x_\eta)_{i,j}\tilde{a}_{i,k}V_{2,k,j}\right)\\
&+\sum_{j=0}^{M}\sum_{i=0}^{\alpha-1}\sum_{k=-\alpha}^{-1} w_{i,j}U_{i,j}\left(w_{i,j}^{-1}(y_\eta)_{i,j}\tilde{a}_{i,k}V_{1,k,j} - w_{i,j}^{-1}(x_\eta)_{i,j}\tilde{a}_{i,k}V_{2,k,j}\right)\\
&+\sum_{i=0}^{N}\sum_{j=M-\beta+1}^{M}\sum_{k=M+1}^{M+\beta} w_{i,j}U_{i,j}\left(w_{i,j}^{-1}(x_\xi)_{i,j}\tilde{b}_{j,k}V_{2,i,k} - w_{i,j}^{-1}(y_\xi)_{i,j}\tilde{b}_{j,k}V_{1,i,k}\right)\\
&+\sum_{i=0}^{N}\sum_{j=0}^{\beta-1}\sum_{k=-\beta}^{-1} w_{i,j}U_{i,j}\left(w_{i,j}^{-1}(x_\xi)_{i,j}\tilde{b}_{j,k}V_{2,i,k} - w_{i,j}^{-1}(y_\xi)_{i,j}\tilde{b}_{j,k}V_{1,i,k}\right)\\
+&\sum_{j=0}^{M}\sum_{i=N-\alpha+1}^{N}\sum_{k=N+1}^{N+\alpha} w_{i,j}U_{k,j}\left(w_{i,j}^{-1}(y_\eta)_{k,j}\tilde{a}^*_{i,k}V_{1,i,j} - w_{i,j}^{-1}(x_\eta)_{k,j}\tilde{a}^*_{i,k}V_{2,i,j}\right)\\
&+\sum_{j=0}^{M}\sum_{i=0}^{\alpha-1}\sum_{k=-\alpha}^{-1} w_{i,j}U_{k,j}\left(w_{i,j}^{-1}(y_\eta)_{k,j}\tilde{a}^*_{i,k}V_{1,i,j} - w_{i,j}^{-1}(x_\eta)_{k,j}\tilde{a}^*_{i,k}V_{2,i,j}\right)\\
&+\sum_{i=0}^{N}\sum_{j=M-\beta+1}^{M}\sum_{k=M+1}^{M+\beta} w_{i,j}U_{i,k}\left(w_{i,j}^{-1}(x_\xi)_{i,k}\tilde{b}^*_{j,k}V_{2,i,j} - w_{i,j}^{-1}(y_\xi)_{i,k}\tilde{b}^*_{j,k}V_{1,i,j}\right)\\
&+\sum_{i=0}^{N}\sum_{j=0}^{\beta-1}\sum_{k=-\beta}^{-1} w_{i,j}U_{i,k}\left(w_{i,j}^{-1}(x_\xi)_{i,k}\tilde{b}^*_{j,k}V_{2,i,j} - w_{i,j}^{-1}(y_\xi)_{i,k}\tilde{b}^*_{j,k}V_{1,i,j}\right).
\end{aligned}
$$

When $U_{i,j}$, $V_{1,i,j}$ and $V_{2,i,j}$ approximate functions u, v_1 and v_2 respectively, this value approximates the boundary term (4.18) in the Gauss theorem.

We can now state the discrete Gauss theorem.

Lemma 4.8. *Let δ_ξ and δ_η be finite difference operators in the ξ and the η directions respectively. Let $U_{i,j}$, $V_{1,i,j}$ and $V_{2,i,j}$ satisfy*

$$
\mu_{(\partial\Omega,\delta_\xi,\delta_\eta,x_\xi,x_\eta,y_\xi,y_\eta)}\left(\{U_{i,j}\},\{V_{1,i,j}\},\{V_{2,i,j}\}\right)=0.
$$

Then

$$
\begin{aligned}
&\sum_{i=0}^{N}\sum_{j=0}^{M} w_{i,j}U_{i,j}\left(w_{i,j}^{-1}\left((y_\eta)_{i,j}\delta_\xi-(y_\xi)_{i,j}\delta_\eta\right)V_{1,i,j}+w_{i,j}^{-1}\left((x_\xi)_{i,j}\delta_\eta-(x_\eta)_{i,j}\delta_\xi\right)V_{2,i,j}\right)\\
&=-\sum_{i=0}^{N}\sum_{j=0}^{M} w_{i,j}\Bigg(V_{1,i,j}w_{i,j}^{-1}\left(\delta_\xi^*\left((y_\eta)_{i,j}U_{i,j}\right)-\delta_\eta^*\left((y_\xi)_{i,j}U_{i,j}\right)\right)\\
&\qquad+V_{2,i,j}w_{i,j}^{-1}\left(\delta_\eta^*\left((x_\xi)_{i,j}U_{i,j}\right)-\delta_\xi^*\left((x_\eta)_{i,j}U_{i,j}\right)\right)\Bigg).
\end{aligned}
\tag{4.22}
$$

Proof. The proof is the same as Lemma 4.6. By applying the summation by parts in the ξ and the η directions, we obtain

$$
\begin{aligned}
&\sum_{i=0}^{N}\sum_{j=0}^{M} w_{i,j}U_{i,j}\left(w_{i,j}^{-1}\left((y_\eta)_{i,j}\delta_\xi-(y_\xi)_{i,j}\delta_\eta\right)V_{1,i,j}+w_{i,j}^{-1}\left((x_\xi)_{i,j}\delta_\eta-(x_\eta)_{i,j}\delta_\xi\right)V_{i,j}\right)\\
&=\sum_{j=0}^{M}\sum_{i=0}^{N}\sum_{k=0}^{N} w_{i,j}U_{i,j}\left(w_{i,j}^{-1}(y_\eta)_{i,j}\tilde{a}_{i,k}V_{1,k,j}-w_{i,j}^{-1}(x_\eta)_{i,j}\tilde{a}_{i,k}V_{2,k,j}\right)+(\text{boundary term corresponding to }\delta_\xi)
\end{aligned}
$$

$$+\sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{M} w_{i,j}U_{i,j}\left(w_{i,j}^{-1}(x_\xi)_{i,j}\tilde{b}_{j,k}V_{2,i,k} - w_{i,j}^{-1}(y_\xi)_{i,j}\tilde{b}_{j,k}V_{1,i,k}\right) + (\text{boundary term corresponding to } \delta_\eta)$$

$$= -\sum_{j=0}^{M}\sum_{k=0}^{N}\sum_{i=0}^{N} w_{k,j}U_{k,j}\left(w_{k,j}^{-1}(y_\eta)_{k,j}\tilde{a}^*_{i,k}V_{1,i,j} - w_{k,j}^{-1}(x_\eta)_{k,j}\tilde{a}^*_{i,k}V_{2,i,j}\right) + (\text{boundary term corresponding to } \delta_\xi)$$

$$-\sum_{i=0}^{N}\sum_{k=0}^{M}\sum_{j=0}^{M} w_{i,k}U_{i,k}\left(w_{i,k}^{-1}(x_\xi)_{i,k}\tilde{b}^*_{j,k}V_{2,i,j} - w_{i,k}^{-1}(y_\xi)_{i,k}\tilde{b}^*_{j,k}V_{1,i,j}\right) + (\text{boundary term corresponding to } \delta_\eta)$$

$$= -\sum_{i=0}^{N}\sum_{j=0}^{M} w_{i,j}\Bigg(V_{1,i,j}w_{i,j}^{-1}\left(\delta^*_\xi\left((y_\eta)_{i,j}U_{i,j}\right) - \delta^*_\eta\left((y_\xi)_{i,j}U_{i,j}\right)\right)$$
$$- V_{2,i,j}w_{i,j}^{-1}\left(\delta^*_\eta\left((x_\xi)_{i,j}U_{i,j}\right) - \delta^*_\xi\left((x_\eta)_{i,j}U_{i,j}\right)\right)\Bigg)$$
$$+ (\text{boundary term corresponding to } \delta_\xi,\ \delta^*_\xi,\ \delta_\eta,\ \delta^*_\eta).$$

The straightforward calculation shows that the boundary term equals $\mu_{(\partial\Omega,\delta_\xi,\delta_\eta,x_\xi,x_\eta,y_\xi,y_\eta)}\left(\{U_{i,j}\},\{V_{1,i,j}\},\{V_{2,i,j}\}\right)$. Therefore

$$= -\sum_{i=0}^{N}\sum_{j=0}^{M} w_{i,j}\Bigg(V_{1,i,j}w_{i,j}^{-1}\left(\delta^*_\xi\left((y_\eta)_{i,j}U_{i,j}\right) - \delta^*_\eta\left((y_\xi)_{i,j}U_{i,j}\right)\right)$$
$$- V_{2,i,j}w_{i,j}^{-1}\left(\delta^*_\eta\left((x_\xi)_{i,j}U_{i,j}\right) - \delta^*_\xi\left((x_\eta)_{i,j}U_{i,j}\right)\right)\Bigg).$$

□

4.4 The Definition of the Discrete Variational Derivative

In this section we define the discrete variational derivative on two dimensional nonuniform meshes. Similarly to one dimensional case, we define the discrete variational derivative so that they approximate the transformed variational derivative

$$\left(\frac{\delta G}{\delta u}\right)_{\mathrm{cs}} = \frac{\partial G}{\partial u} - \left(J^{-1}\left(\frac{\partial}{\partial \xi}\left(\frac{\partial y}{\partial \eta}\frac{\partial G}{\partial u_x}\right) - \frac{\partial}{\partial \eta}\left(\frac{\partial y}{\partial \xi}\frac{\partial G}{\partial u_x}\right)\right) + J^{-1}\left(\frac{\partial}{\partial \eta}\left(\frac{\partial x}{\partial \xi}\frac{\partial G}{\partial u_y}\right) - \frac{\partial}{\partial \xi}\left(\frac{\partial x}{\partial \eta}\frac{\partial G}{\partial u_y}\right)\right)\right).$$

We assume that the energy functional $G(u, u_x, u_y)$ is given in the next form:

$$G(u, u_x, u_y) = \sum_{l=1}^{K} f_l(u)g_l(u_x)h_l(u_y). \tag{4.23}$$

We define the discrete energy functional $G_{\mathrm{d}}(\vec{U}^{(n)})_{i,j}$ by

$$G_{\mathrm{d}}(\vec{U}^{(n)})_{i,j} = \sum_{l=1}^{K} f_l(U_{i,j}^{(n)})\Bigg(\frac{1}{M_l}\sum_{m_l=1}^{M_l} g_l\left(\left(w_{i,j}^{-1}(y_\eta)_{l,m,i,j}\delta_{\xi,l,m} - w_{i,j}^{-1}(y_\xi)_{l,m,i,j}\delta_{\eta,l,m}\right)U_{i,j}^{(n)}\right)$$
$$h_l\left(\left(w_{i,j}^{-1}(x_\xi)_{l,m,i,j}\delta_{\eta,l,m} - w_{i,j}^{-1}(x_\eta)_{l,m,i,j}\delta_{\xi,l,m}\right)U_{i,j}^{(n)}\right)\Bigg) \tag{4.24}$$

and the discrete total energy $H^{(n)}$ by

$$H^{(n)} = \sum_{i=0}^{N}\sum_{j=0}^{M} w_{i,j}G_{\mathrm{d}}(\vec{U}^{(n)})_{i,j}. \tag{4.25}$$

Definition 4.2. *We define the discrete variational derivative of $G_{\mathrm{d}}(\vec{U}^{(n)})_{i,j}$ by*

$$
\begin{aligned}
&\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\\
=&\sum_{l=1}^{K}\frac{1}{M_l}\sum_{m=1}^{M_l}\Bigg\{\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{l,m,i,j}\\
&-w_{i,j}^{-1}\left(\delta^*_{\xi,l,m}\left((y_\eta)_{l,m,i,j}\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)},\vec{U}_x^{(n)})}\right)_{l,m,i,j}\right)-\delta^*_{\eta,l,m}\left((y_\xi)_{l,m,i,j}\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)},\vec{U}_x^{(n)})}\right)_{l,m,i,j}\right)\right)\\
&-w_{i,j}^{-1}\left(\delta^*_{\eta,l,m}\left((x_\xi)_{l,m,i,j}\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_y^{(n+1)},\vec{U}_y^{(n)})}\right)_{l,m,i,j}\right)-\delta^*_{\xi,l,m}\left((x_\eta)_{l,m,i,j}\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_y^{(n+1)},\vec{U}_y^{(n)})}\right)_{l,m,i,j}\right)\right)\Bigg\},
\end{aligned}
\tag{4.26}
$$

where

$$
\begin{aligned}
&\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{l,m,i,j}=\frac{1}{6}\left(\frac{f_l(U_{i,j}^{(n+1)})-f_l(U_{i,j}^{(n)})}{U_{i,j}^{(n+1)}-U_{i,j}^{(n)}}\right)\\
&\Big(2g_l\left(\left(w_{i,j}^{-1}(y_\eta)_{l,m,i,j}\delta_{\xi,l,m}-w_{i,j}^{-1}(y_\xi)_{l,m,i,j}\delta_{\eta,l,m}\right)U_{i,j}^{(n+1)}\right)h_l\left(\left(w_{i,j}^{-1}(x_\xi)_{l,m,i,j}\delta_{\eta,l,m}-w_{i,j}^{-1}(x_\eta)_{l,m,i,j}\delta_{\xi,l,m}\right)U_{i,j}^{(n+1)}\right)\\
&+g_l\left(\left(w_{i,j}^{-1}(y_\eta)_{l,m,i,j}\delta_{\xi,l,m}-w_{i,j}^{-1}(y_\xi)_{l,m,i,j}\delta_{\eta,l,m}\right)U_{i,j}^{(n+1)}\right)h_l\left(\left(w_{i,j}^{-1}(x_\xi)_{l,m,i,j}\delta_{\eta,l,m}-w_{i,j}^{-1}(x_\eta)_{l,m,i,j}\delta_{\xi,l,m}\right)U_{i,j}^{(n)}\right)\\
&+g_l\left(\left(w_{i,j}^{-1}(y_\eta)_{l,m,i,j}\delta_{\xi,l,m}-w_{i,j}^{-1}(y_\xi)_{l,m,i,j}\delta_{\eta,l,m}\right)U_{i,j}^{(n)}\right)h_l\left(\left(w_{i,j}^{-1}(x_\xi)_{l,m,i,j}\delta_{\eta,l,m}-w_{i,j}^{-1}(x_\eta)_{l,m,i,j}\delta_{\xi,l,m}\right)U_{i,j}^{(n+1)}\right)\\
&+2g_l\left(\left(w_{i,j}^{-1}(y_\eta)_{l,m,i,j}\delta_{\xi,l,m}-w_{i,j}^{-1}(y_\xi)_{l,m,i,j}\delta_{\eta,l,m}\right)U_{i,j}^{(n)}\right)h_l\left(\left(w_{i,j}^{-1}(x_\xi)_{l,m,i,j}\delta_{\eta,l,m}-w_{i,j}^{-1}(x_\eta)_{l,m,i,j}\delta_{\xi,l,m}\right)U_{i,j}^{(n)}\right)\Big),
\end{aligned}
\tag{4.27}
$$

$$
\begin{aligned}
\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)},\vec{U}^{(n)})_x}\right)_{l,m,i,j}=&\frac{1}{6}\Big(2f_l(U_j^{(n+1)})h_l\left(\left(w_{i,j}^{-1}(x_\xi)_{l,m,i,j}\delta_{\eta,l,m}-w_{i,j}^{-1}(x_\eta)_{l,m,i,j}\delta_{\xi,l,m}\right)U_{i,j}^{(n+1)}\right)\\
&+f_l(U_j^{(n+1)})h_l\left(\left(w_{i,j}^{-1}(x_\xi)_{l,m,i,j}\delta_{\eta,l,m}-w_{i,j}^{-1}(x_\eta)_{l,m,i,j}\delta_{\xi,l,m}\right)U_{i,j}^{(n)}\right)\\
&+f_l(U_j^{(n)})h_l\left(\left(w_{i,j}^{-1}(x_\xi)_{l,m,i,j}\delta_{\eta,l,m}-w_{i,j}^{-1}(x_\eta)_{l,m,i,j}\delta_{\xi,l,m}\right)U_{i,j}^{(n+1)}\right)\\
&+2f_l(U_j^{(n)})h_l\left(\left(w_{i,j}^{-1}(x_\xi)_{l,m,i,j}\delta_{\eta,l,m}-w_{i,j}^{-1}(x_\eta)_{l,m,i,j}\delta_{\xi,l,m}\right)U_{i,j}^{(n)}\right)\Big)\\
&\frac{g_l\left(\left(w_{i,j}^{-1}(y_\eta)_{l,m,i,j}\delta_{\xi,l,m}-w_{i,j}^{-1}(y_\xi)_{l,m,i,j}\delta_{\eta,l,m}\right)U_{i,j}^{(n+1)}\right)-g_l\left(\left(w_{i,j}^{-1}(y_\eta)_{l,m,i,j}\delta_{\xi,l,m}-w_{i,j}^{-1}(y_\xi)_{l,m,i,j}\delta_{\eta,l,m}\right)U_{i,j}^{(n)}\right)}{\left(w_{i,j}^{-1}(y_\eta)_{l,m,i,j}\delta_{\xi,l,m}-w_{i,j}^{-1}(y_\xi)_{l,m,i,j}\delta_{\eta,l,m}\right)U_{i,j}^{(n+1)}-\left(w_{i,j}^{-1}(y_\eta)_{l,m,i,j}\delta_{\xi,l,m}-w_{i,j}^{-1}(y_\xi)_{l,m,i,j}\delta_{\eta,l,m}\right)U_{i,j}^{(n)}},
\end{aligned}
\tag{4.28}
$$

$$\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_y^{(n+1)},\vec{U}_y^{(n)})}\right)_{l,m,i,j} = \frac{1}{6}\left(2f_l(U_j^{(n+1)})g_l\left(\left(w_{i,j}^{-1}(y_\eta)_{l,m,i,j}\delta_{\xi,l,m} - w_{i,j}^{-1}(y_\xi)_{l,m,i,j}\delta_{\eta,l,m}\right)U_{i,j}^{(n+1)}\right)\right.$$
$$+f_l(U_j^{(n+1)})g_l\left(\left(w_{i,j}^{-1}(y_\eta)_{l,m,i,j}\delta_{\xi,l,m} - w_{i,j}^{-1}(y_\xi)_{l,m,i,j}\delta_{\eta,l,m}\right)U_{i,j}^{(n)}\right)$$
$$+f_l(U_j^{(n)})g_l\left(\left(w_{i,j}^{-1}(y_\eta)_{l,m,i,j}\delta_{\xi,l,m} - w_{i,j}^{-1}(y_\xi)_{l,m,i,j}\delta_{\eta,l,m}\right)U_{i,j}^{(n+1)}\right)$$
$$\left.+2f_l(U_j^{(n)})g_l\left(\left(w_{i,j}^{-1}(y_\eta)_{l,m,i,j}\delta_{\xi,l,m} - w_{i,j}^{-1}(y_\xi)_{l,m,i,j}\delta_{\eta,l,m}\right)U_{i,j}^{(n)}\right)\right)$$
$$\frac{h_l\left(\left(w_{i,j}^{-1}(x_\xi)_{l,m,i,j}\delta_{\eta,l,m} - w_{i,j}^{-1}(x_\eta)_{l,m,i,j}\delta_{\xi,l,m}\right)U_{i,j}^{(n+1)}\right) - h_l\left(\left(w_{i,j}^{-1}(x_\xi)_{l,m,i,j}\delta_{\eta,l,m} - w_{i,j}^{-1}(x_\eta)_{l,m,i,j}\delta_{\xi,l,m}\right)U_{i,j}^{(n)}\right)}{\left(w_{i,j}^{-1}(x_\xi)_{l,m,i,j}\delta_{\eta,l,m} - w_{i,j}^{-1}(x_\eta)_{l,m,i,j}\delta_{\xi,l,m}\right)U_{i,j}^{(n+1)} - \left(w_{i,j}^{-1}(x_\xi)_{l,m,i,j}\delta_{\eta,l,m} - w_{i,j}^{-1}(x_\eta)_{l,m,i,j}\delta_{\xi,l,m}\right)U_{i,j}^{(n)}}. \tag{4.29}$$

The discrete variational derivative is defined carefully again so that this brings the variational structure:

Lemma 4.9. *Suppose the condition*

$$\mu_{(\partial\Omega,\delta_{\xi,l,m},\delta_{\eta,l,m},(x_\xi)_{l,m,i,j},(x_\eta)_{l,m,i,j},(y_\xi)_{l,m,i,j},(y_\eta)_{l,m,i,j})}\left(\left\{\frac{U_{i,j}^{(n+1)}-U_{i,j}^{(n)}}{\Delta t}\right\},\left\{\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)},\vec{U}_x^{(n)})}\right)_{l,m,i,j}\right\},\right.$$
$$\left.\left\{\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_y^{(n+1)},\vec{U}_y^{(n)})}\right)_{l,m,i,j}\right\}\right) = 0 \tag{4.30}$$

is satisfied for each l and m. Then

$$\frac{1}{\Delta t}\left(H^{(n+1)} - H^{(n)}\right) = \sum_{i=0}^{N}\sum_{j=0}^{M} w_{i,j}\left(\frac{U_{i,j}^{(n+1)}-U_{i,j}^{(n)}}{\Delta t}\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}. \tag{4.31}$$

Proof. From (4.27)–(4.29) we have

$$\frac{1}{\Delta t}\left(H^{(n+1)} - H^{(n)}\right) =$$
$$\frac{1}{\Delta t}\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{l=1}^{K}\frac{w_{i,j}}{M_l}\sum_{m_l=1}^{M_l}\left(f_l(U_{i,j}^{(n+1)})\left(g_l\left(\left(w_{i,j}^{-1}(y_\eta)_{l,m,i,j}\delta_{\xi,l,m} - w_{i,j}^{-1}(y_\xi)_{l,m,i,j}\delta_{\eta,l,m}\right)U_{i,j}^{(n+1)}\right)\right.\right.$$
$$\left.h_l\left(\left(w_{i,j}^{-1}(x_\xi)_{l,m,i,j}\delta_{\eta,l,m} - w_{i,j}^{-1}(x_\eta)_{l,m,i,j}\delta_{\xi,l,m}\right)U_{i,j}^{(n+1)}\right)\right)$$
$$-f_l(U_{i,j}^{(n)})\left(g_l\left(\left(w_{i,j}^{-1}(y_\eta)_{l,m,i,j}\delta_{\xi,l,m} - w_{i,j}^{-1}(y_\xi)_{l,m,i,j}\delta_{\eta,l,m}\right)U_{i,j}^{(n)}\right)\right.$$
$$\left.\left.h_l\left(\left(w_{i,j}^{-1}(x_\xi)_{l,m,i,j}\delta_{\eta,l,m} - w_{i,j}^{-1}(x_\eta)_{l,m,i,j}\delta_{\xi,l,m}\right)U_{i,j}^{(n)}\right)\right)\right)$$
$$=\sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{l=1}^{K}\frac{w_{i,j}}{M_l}\sum_{m=1}^{M_l}\left(\left(\frac{U_{i,j}^{(n+1)}-U_{i,j}^{(n)}}{\Delta t}\right)\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{l,m,i,j}\right.$$
$$+ w_{i,j}^{-1}\left((y_\eta)_{l,m,i,j}\delta_{\xi,l,m} - (y_\xi)_{l,m,i,j}\delta_{\eta,l,m}\right)\left(\frac{U_{i,j}^{(n+1)}-U_{i,j}^{(n)}}{\Delta t}\right)\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)},\vec{U}^{(n)})_x}\right)_{l,m,i,j}$$
$$\left.+w_{i,j}^{-1}\left((x_\xi)_{l,m,i,j}\delta_{\eta,l,m} - (x_\eta)_{l,m,i,j}\delta_{\xi,l,m}\right)\left(\frac{U_{i,j}^{(n+1)}-U_{i,j}^{(n)}}{\Delta t}\right)\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_y^{(n+1)},\vec{U}^{(n)})_y}\right)_{l,m,i,j}\right).$$

By applying Lemma 4.8 we get

$$
\begin{aligned}
&= \sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{l=1}^{K}\frac{w_{i,j}}{M_l}\sum_{m=1}^{M_l}\left[\left(\frac{U_{i,j}^{(n+1)}-U_{i,j}^{(n)}}{\Delta t}\right)\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{l,m,i,j}\right.\\
&\quad - w_{i,j}^{-1}\left\{\delta^*_{\xi,l,m}\left((y_\eta)_{l,m,i,j}\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)},\vec{U}_x^{(n)})}\right)_{l,m,i,j}\right)\right.\\
&\quad \left.-\delta^*_{\eta,l,m}\left((y_\xi)_{l,m,i,j}\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)},\vec{U}_x^{(n)})}\right)_{l,m,i,j}\right)\right\}\left(\frac{U_{i,j}^{(n+1)}-U_{i,j}^{(n)}}{\Delta t}\right)\\
&\quad -w_{i,j}^{-1}\left\{\delta^*_{\eta,l,m}\left((x_\xi)_{l,m,i,j}\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_y^{(n+1)},\vec{U}_y^{(n)})}\right)_{l,m,i,j}\right)\right.\\
&\quad \left.\left.-\delta^*_{\xi,l,m}\left((x_\eta)_{l,m,i,j}\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_y^{(n+1)},\vec{U}_y^{(n)})}\right)_{l,m,i,j}\right)\right\}\left(\frac{U_{i,j}^{(n+1)}-U_{i,j}^{(n)}}{\Delta t}\right)\right]\\
&= \sum_{j=0}^{N} w_{i,j}\left(\frac{U_{i,j}^{(n+1)}-U_{i,j}^{(n)}}{\Delta t}\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}.
\end{aligned}
$$

□

4.5 Design of Schemes for the Dissipative Equations

We define the scheme for the dissipative equation (4.1) by

$$
\frac{U_{i,j}^{(n+1)}-U_{i,j}^{(n)}}{\Delta t} = -\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j} \tag{4.32a}
$$

if $s=0$, and by

$$
\begin{aligned}
\frac{U_{i,j}^{(n+1)}-U_{i,j}^{(n)}}{\Delta t} =& w_{i,j}^{-1}\delta^*_\xi\left((y_\eta)_{i,j}\left(w_{i,j}^{-1}\left((y_\eta)_{i,j}\delta_\xi-(y_\xi)_{i,j}\delta_\eta\right)\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right)\\
&- w_{i,j}^{-1}\delta^*_\eta\left((y_\xi)_{i,j}\left(w_{i,j}^{-1}\left((y_\eta)_{i,j}\delta_\xi-(y_\xi)_{i,j}\delta_\eta\right)\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right)\\
&+ w_{i,j}^{-1}\delta^*_\eta\left((x_\xi)_{i,j}\left(w_{i,j}^{-1}\left((x_\xi)_{i,j}\delta_\eta-(x_\eta)_{i,j}\delta_\xi\right)\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right)\\
&- w_{i,j}^{-1}\delta^*_\xi\left((x_\eta)_{i,j}\left(w_{i,j}^{-1}\left((x_\xi)_{i,j}\delta_\eta-(x_\eta)_{i,j}\delta_\xi\right)\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right)
\end{aligned} \tag{4.32b}
$$

if $s=1$.

Theorem 4.10. *Let $w_{i,j}$'s be positive. Let $U_{i,j}^{(n)}$ be a numerical solution of the scheme* (4.32a) *or* (4.32b) *under the boundary condition that satisfies the assumption of Lemma 4.9. Suppose also that*

$$
\mu_{(\partial\Omega,\delta_\xi,\delta_\eta,(x_\xi)_{i,j},(x_\eta)_{i,j},(y_\xi)_{i,j},(y_\eta)_{i,j})}\left(\left\{\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right\},\right.
$$

$$\left\{w_{k,j}^{-1}\left((y_\eta)_{k,j}\delta_\xi-(y_\xi)_{k,j}\delta_\eta\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right\},$$

$$\left.\left\{w_{k,j}^{-1}\left((x_\xi)_{k,j}\delta_\eta-(x_\eta)_{k,j}\delta_\xi\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right\}\right)=0 \tag{4.33}$$

if $s=1$*. Then*

$$\frac{1}{\Delta t}\left(H^{(n+1)}-H^{(n)}\right)\leq 0. \tag{4.34}$$

Proof. This theorem is obtained in almost the same way as Theorem 4.4. By Lemma 4.9, we have

$$\frac{1}{\Delta t}\left(H^{(n+1)}-H^{(n)}\right)=\sum_{i=0}^{N}\sum_{j=0}^{M}w_{i,j}\left(\frac{U_{i,j}^{(n+1)}-U_{i,j}^{(n)}}{\Delta t}\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}. \tag{4.35}$$

In the case of $s=0$, substituting the scheme (4.32a) yields

$$(4.35)=-\sum_{i=0}^{N}\sum_{j=0}^{M}w_{i,j}\left(\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right)^2\leq 0.$$

In the case of $s=1$, substituting the scheme (4.32b) and applying Lemma 4.8 yield

$$\begin{aligned}
(4.35)=&-\sum_{i=0}^{N}\sum_{j=0}^{M}w_{i,j}\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\\
&\left(w_{i,j}^{-1}\delta_\xi^*\left((y_\eta)_{i,j}\left(w_{i,j}^{-1}\left((y_\eta)_{i,j}\delta_\xi-(y_\xi)_{i,j}\delta_\eta\right)\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right)\right.\\
&-w_{i,j}^{-1}\delta_\eta^*\left((y_\xi)_{i,j}\left(w_{i,j}^{-1}\left((y_\eta)_{i,j}\delta_\xi-(y_\xi)_{i,j}\delta_\eta\right)\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right)\\
&+w_{i,j}^{-1}\delta_\eta^*\left((x_\xi)_{i,j}\left(w_{i,j}^{-1}\left((x_\xi)_{i,j}\delta_\eta-(x_\eta)_{i,j}\delta_\xi\right)\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right)\\
&\left.-w_{i,j}^{-1}\delta_\xi^*\left((x_\eta)_{i,j}\left(w_{i,j}^{-1}\left((x_\xi)_{i,j}\delta_\eta-(x_\eta)_{i,j}\delta_\xi\right)\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right)\right)\\
=&-\sum_{i=0}^{N}\sum_{j=0}^{M}w_{i,j}\left(\left(w_{i,j}^{-1}\left((y_\eta)_{i,j}\delta_\xi-(y_\xi)_{i,j}\delta_\eta\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right)^2\right.\\
&\left.+\left(w_{i,j}^{-1}\left((x_\xi)_{i,j}\delta_\eta-(x_\eta)_{i,j}\delta_\xi\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right)^2\right)\\
\leq&\ 0.
\end{aligned}$$

□

4.6 Design of Schemes for the Conservative Equations

We define the scheme for the conservative equation (4.4) by

$$\frac{U_{i,j}^{(n+1)} - U_{i,j}^{(n)}}{\Delta t} = \frac{1}{2}\left\{ \left(w_{i,j}^{-1}\left((y_{\mathrm{c}})_{i,j}\delta_\xi - (y_{\mathrm{c}}\xi)_{i,j}\delta_\eta \right) + w_{i,j}^{-1}\left((x_\xi)_{i,j}\delta_\eta - (x_\eta)_{i,j}\delta_\xi \right) \right) \left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)}, \vec{U}^{(n)})} \right)_{i,j} \right.$$
$$+ w_{i,j}^{-1}\left(\delta_\xi \left((y_\eta)_{i,j} \left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)}, \vec{U}^{(n)})} \right)_{i,j} \right) - \delta_\eta \left((y_\xi)_{i,j} \left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)}, \vec{U}^{(n)})} \right)_{i,j} \right) \right)$$
$$\left. + w_{i,j}^{-1}\left(\delta_\eta \left((x_\xi)_{i,j} \left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)}, \vec{U}^{(n)})} \right)_{i,j} \right) - \delta_\xi \left((x_\eta)_{i,j} \left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)}, \vec{U}^{(n)})} \right)_{i,j} \right) \right) \right\}, \tag{4.36}$$

where δ_ξ and δ_η are the central difference operators in the ξ and the η directions respectively.

Theorem 4.11. *Let $U_{i,j}^{(n)}$ be a numerical solution of the scheme* (4.36) *under the boundary condition that satisfies the assumption of Lemma 4.9 and*

$$\mu_{(\partial\Omega, \delta_{\mathrm{c}}, \delta_{\mathrm{c}}, (x_\xi)_{i,j}, (x_\eta)_{i,j}, (y_\xi)_{i,j}, (y_\eta)_{i,j})} \left(\left\{ \left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)}, \vec{U}^{(n)})} \right)_{i,j} \right\}, \left\{ \left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)}, \vec{U}^{(n)})} \right)_{i,j} \right\}, \left\{ \left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)}, \vec{U}^{(n)})} \right)_{i,j} \right\} \right) = 0. \tag{4.37}$$

Then

$$\frac{1}{\Delta t}\left(H^{(n+1)} - H^{(n)} \right) = 0. \tag{4.38}$$

Proof. This theorem is obtained in almost the same way as Theorem 4.5. Applying Lemma 4.9 we have

$$\frac{1}{\Delta t}\left(H^{(n+1)} - H^{(n)} \right) = \sum_{i=0}^{N}\sum_{j=0}^{M} w_{i,j} \left(\frac{U_{i,j}^{(n+1)} - U_{i,j}^{(n)}}{\Delta t} \right) \left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)}, \vec{U}^{(n)})} \right)_{i,j}.$$

Substituting the scheme (4.36) yields

$$= \frac{1}{2}\sum_{i=0}^{N}\sum_{j=0}^{M} w_{i,j} \left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)}, \vec{U}^{(n)})} \right)_{i,j}$$
$$\left\{ \left(w_{i,j}^{-1}\left((y_{\mathrm{c}})_{i,j}\delta_\xi - (y_{\mathrm{c}}\xi)_{i,j}\delta_\eta \right) + w_{i,j}^{-1}\left((x_\xi)_{i,j}\delta_\eta - (x_\eta)_{i,j}\delta_\xi \right) \right) \left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)}, \vec{U}^{(n)})} \right)_{i,j} \right.$$
$$+ w_{i,j}^{-1}\left(\delta_\xi \left((y_\eta)_{i,j} \left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)}, \vec{U}^{(n)})} \right)_{i,j} \right) - \delta_\eta \left((y_\xi)_{i,j} \left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)}, \vec{U}^{(n)})} \right)_{i,j} \right) \right)$$
$$\left. + w_{i,j}^{-1}\left(\delta_\eta \left((x_\xi)_{i,j} \left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)}, \vec{U}^{(n)})} \right)_{i,j} \right) - \delta_\xi \left((x_\eta)_{i,j} \left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)}, \vec{U}^{(n)})} \right)_{i,j} \right) \right) \right\}.$$

Since $\delta_\xi = \delta_\eta = \delta_{\mathrm{c}} = \delta_{\mathrm{c}}^*$, applying Lemma 4.8 gives

$$= -\frac{1}{2}\sum_{i=0}^{N}\sum_{j=0}^{M} w_{i,j}\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}$$

$$\Bigg\{ w_{i,j}^{-1}\left(\delta_\eta^*\left((x_\xi)_{i,j}\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right) - \delta_\xi^*\left((x_\eta)_{i,j}\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right)\right)$$

$$+ w_{i,j}^{-1}\left(\delta_\xi^*\left((y_\eta)_{i,j}\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right) - \delta_\eta^*\left((y_\xi)_{i,j}\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right)\right)$$

$$\left(w_{i,j}^{-1}\left((y_{\mathrm{c}})_{i,j}\delta_\xi^* - (y_{\mathrm{c}}\xi)_{i,j}\delta_\eta^*\right) + w_{i,j}^{-1}\left((x_\xi)_{i,j}\delta_\eta^* - (x_\eta)_{i,j}\delta_\xi^*\right)\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\Bigg\}$$

$$= -\frac{1}{2}\sum_{i=0}^{N}\sum_{j=0}^{M} w_{i,j}\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}$$

$$\Bigg\{ w_{i,j}^{-1}\left(\delta_\eta\left((x_\xi)_{i,j}\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right) - \delta_\xi\left((x_\eta)_{i,j}\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right)\right)$$

$$+ w_{i,j}^{-1}\left(\delta_\xi\left((y_\eta)_{i,j}\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right) - \delta_\eta\left((y_\xi)_{i,j}\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right)\right)$$

$$\left(w_{i,j}^{-1}\left((y_{\mathrm{c}})_{i,j}\delta_\xi - (y_{\mathrm{c}}\xi)_{i,j}\delta_\eta\right) + w_{i,j}^{-1}\left((x_\xi)_{i,j}\delta_\eta - (x_\eta)_{i,j}\delta_\xi\right)\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\Bigg\},$$

and hence this equals 0. □

5 An Example in the Two Dimensional Case

As an example in the two dimensional case, we derive a dissipative scheme and show a numerical result for the Cahn–Hilliard equation

$$\frac{\partial u}{\partial t} = \Delta\frac{\delta G}{\delta u},\quad G(u,u_x,u_y) = \frac{p}{2}u^2 + \frac{r}{4}u^4 - \frac{q}{2}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right). \tag{5.1}$$

5.1 An Energy Dissipative Scheme for the Cahn–Hilliard Equation

We first define the discrete energy functional for this equation. The energy functional $G(u,u_x,u_y)$ is written in the form of (4.23) with $K=3$ and

$$f_1(u) = \frac{p}{2}u^2 + \frac{r}{4}u^4,\quad f_2(u) = 1,\quad f_3(u) = 1,$$
$$g_1(u_x) = 1,\quad g_2(u_x) = -\frac{q}{2}u_x^2,\quad g_3(u_x) = 1,$$
$$h_1(u_y) = 1,\quad h_2(u_y) = 1,\quad h_3(u_y) = -\frac{q}{2}u_y^2.$$

We introduce the discrete energy functional so that it corresponds to these. On the given mesh, we introduce the weights by

$$w_{i,j} = (x_{\xi,\mathrm{c}})_{i,j}(y_{\eta,\mathrm{c}})_{i,j} - (x_{\eta,\mathrm{c}})_{i,j}(y_{\xi,\mathrm{c}})_{i,j},$$

$$(x_{\xi,\mathrm{c}})_{i,j} = \frac{x(i+1,j)-x(i-1,j)}{2}, \quad (y_{\xi,\mathrm{c}})_{i,j} = \frac{y(i+1,j)-y(i-1,j)}{2},$$
$$(x_{\eta,\mathrm{c}})_{i,j} = \frac{x(i,j+1)-x(i,j-1)}{2}, \quad (y_{\eta,\mathrm{c}})_{i,j} = \frac{y(i,j+1)-y(i,j-1)}{2}.$$

We use the following difference operators for the discretization:

$$\delta_{\xi,+}U_{i,j} = U_{i+1,j} - U_{i,j},\ \delta_{\xi,-}U_{i,j} = U_{i,j} - U_{i-1,j},\ \delta_{\eta,+}U_{i,j} = U_{i,j+1} - U_{i,j},\ \delta_{\eta,-}U_{i,j} = U_{i,j} - U_{i,j-1}.$$

For $l = 2$ we set $M_2 = 4$ and approximate g_2 by

$$\begin{aligned}
&g_2(u_x) = -\frac{q}{2}u_x^2 \\
&\simeq -\frac{1}{M_2}\frac{q}{2}\left(\left(w_{i,j}^{-1}\left((y_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,+} - (y_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,+}\right)U_{i,j}^{(n)}\right)^2 + \left(w_{i,j}^{-1}\left((y_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,+} - (y_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,-}\right)U_{i,j}^{(n)}\right)^2\right. \\
&\quad \left. + \left(w_{i,j}^{-1}\left((y_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,-} - (y_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,+}\right)U_{i,j}^{(n)}\right)^2 + \left(w_{i,j}^{-1}\left((y_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,-} - (y_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,-}\right)U_{i,j}^{(n)}\right)^2\right),
\end{aligned}$$

which gives

$$\begin{aligned}
&(y_\xi)_{2,1,i,j} = (y_\xi)_{2,2,i,j} = (y_\xi)_{2,3,i,j} = (y_\xi)_{2,4,i,j} = (y_{\xi,\mathrm{c}})_{i,j}, \\
&(y_\eta)_{2,1,i,j} = (y_\eta)_{2,2,i,j} = (y_\eta)_{2,3,i,j} = (y_\eta)_{2,4,i,j} = (y_{\eta,\mathrm{c}})_{i,j}, \\
&\delta_{\xi,2,1} = \delta_{\xi,2,2} = \delta_{\xi,+}, \quad \delta_{\xi,2,3} = \delta_{\xi,2,4} = \delta_{\xi,-}, \quad \delta_{\eta,2,1} = \delta_{\eta,2,3} = \delta_{\eta,+}, \quad \delta_{\eta,2,2} = \delta_{\eta,2,4} = \delta_{\eta,-}.
\end{aligned}$$

Similarly we set $M_3 = 4$ and approximate h_3 by

$$\begin{aligned}
&h_3(u_y) = -\frac{q}{2}u_y^2 \\
&\simeq -\frac{1}{M_3}\frac{q}{2}\left(\left(w_{i,j}^{-1}\left((x_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,+} - (x_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,+}\right)U_{i,j}^{(n)}\right)^2 + \left(w_{i,j}^{-1}\left((x_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,+} - (x_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,-}\right)U_{i,j}^{(n)}\right)^2\right. \\
&\quad \left. + \left(w_{i,j}^{-1}\left((x_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,-} - (x_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,+}\right)U_{i,j}^{(n)}\right)^2 + \left(w_{i,j}^{-1}\left((x_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,-} - (x_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,-}\right)U_{i,j}^{(n)}\right)^2\right),
\end{aligned}$$

which gives

$$\begin{aligned}
&(x_\xi)_{3,1,i,j} = (x_\xi)_{3,2,i,j} = (x_\xi)_{3,3,i,j} = (x_\xi)_{3,4,i,j} = (x_{\xi,\mathrm{c}})_{i,j}, \\
&(x_\eta)_{3,1,i,j} = (x_\eta)_{3,2,i,j} = (x_\eta)_{3,3,i,j} = (x_\eta)_{3,4,i,j} = (x_{\eta,\mathrm{c}})_{i,j}, \\
&\delta_{\xi,3,1} = \delta_{\xi,3,3} = \delta_{\xi,+}, \quad \delta_{\xi,3,2} = \delta_{\xi,3,4} = \delta_{\xi,-}, \quad \delta_{\eta,3,1} = \delta_{\eta,3,2} = \delta_{\eta,+}, \quad \delta_{\eta,3,3} = \delta_{\eta,3,4} = \delta_{\eta,-}.
\end{aligned}$$

From the above we define the discrete energy functional by

$$\begin{aligned}
&G_{\mathrm{d}}(\vec{U}^{(n)})_{i,j} = \frac{p}{2}\left(U_{i,j}^{(n)}\right)^2 + \frac{r}{4}\left(U_{i,j}^{(n)}\right)^4 \\
&- \frac{1}{M_2}\frac{q}{2}\left(\left(w_{i,j}^{-1}\left((y_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,+} - (y_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,+}\right)U_{i,j}^{(n)}\right)^2 + \left(w_{i,j}^{-1}\left((y_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,+} - (y_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,-}\right)U_{i,j}^{(n)}\right)^2\right. \\
&\quad \left. + \left(w_{i,j}^{-1}\left((y_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,-} - (y_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,+}\right)U_{i,j}^{(n)}\right)^2 + \left(w_{i,j}^{-1}\left((y_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,-} - (y_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,-}\right)U_{i,j}^{(n)}\right)^2\right) \\
&- \frac{1}{M_3}\frac{q}{2}\left(\left(w_{i,j}^{-1}\left((x_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,+} - (x_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,+}\right)U_{i,j}^{(n)}\right)^2 + \left(w_{i,j}^{-1}\left((x_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,+} - (x_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,-}\right)U_{i,j}^{(n)}\right)^2\right. \\
&\quad \left. + \left(w_{i,j}^{-1}\left((x_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,-} - (x_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,+}\right)U_{i,j}^{(n)}\right)^2 + \left(w_{i,j}^{-1}\left((x_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,-} - (x_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,-}\right)U_{i,j}^{(n)}\right)^2\right).
\end{aligned}$$

Next we define the discrete variational derivative by (4.26). For $l=1$, we set $M_1=1$ and obtain

$$\begin{aligned}\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{1,1,i,j} &= \frac{p}{2}\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right)\\ &\quad+\frac{r}{4}\left(\left(U_{i,j}^{(n+1)}\right)^3+\left(U_{i,j}^{(n+1)}\right)^2U_{i,j}^{(n)}+U_{i,j}^{(n+1)}\left(U_{i,j}^{(n)}\right)^2+\left(U_{i,j}^{(n)}\right)^3\right),\\ \left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)},\vec{U}_x^{(n)})}\right)_{1,1,i,j} &= \left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_y^{(n+1)},\vec{U}_y^{(n)})}\right)_{1,1,i,j}=0.\end{aligned}$$

For $l=2$, we have already set $M_2=4$ and obtain

$$\begin{aligned}\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)},\vec{U}_x^{(n)})}\right)_{2,1,i,j} &= -\frac{q}{2}w_{i,j}^{-1}\left((y_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,+}-(y_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,+}\right)\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right),\\ \left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)},\vec{U}_x^{(n)})}\right)_{2,2,i,j} &= -\frac{q}{2}w_{i,j}^{-1}\left((y_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,+}-(y_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,-}\right)\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right),\\ \left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)},\vec{U}_x^{(n)})}\right)_{2,3,i,j} &= -\frac{q}{2}w_{i,j}^{-1}\left((y_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,-}-(y_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,+}\right)\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right),\\ \left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)},\vec{U}_x^{(n)})}\right)_{2,4,i,j} &= -\frac{q}{2}w_{i,j}^{-1}\left((y_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,-}-(y_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,-}\right)\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right),\\ \left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{2,m,i,j} &= \left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_y^{(n+1)},\vec{U}_y^{(n)})}\right)_{2,m,i,j}=0 \qquad (m=1,2,3,4).\end{aligned}$$

In a similar manner, we have for $l=3$

$$\begin{aligned}\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_y^{(n+1)},\vec{U}_y^{(n)})}\right)_{3,1,i,j} &= -\frac{q}{2}w_{i,j}^{-1}\left((x_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,+}-(x_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,+}\right)\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right),\\ \left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_y^{(n+1)},\vec{U}_y^{(n)})}\right)_{3,2,i,j} &= -\frac{q}{2}w_{i,j}^{-1}\left((x_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,+}-(x_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,-}\right)\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right),\\ \left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_y^{(n+1)},\vec{U}_y^{(n)})}\right)_{3,3,i,j} &= -\frac{q}{2}w_{i,j}^{-1}\left((x_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,-}-(x_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,+}\right)\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right),\\ \left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_y^{(n+1)},\vec{U}_y^{(n)})}\right)_{3,4,i,j} &= -\frac{q}{2}w_{i,j}^{-1}\left((x_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,-}-(x_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,-}\right)\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right),\\ \left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{3,m,i,j} &= \left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)},\vec{U}_x^{(n)})}\right)_{3,m,i,j}=0 \qquad (m=1,2,3,4).\end{aligned}$$

By using the above symbols, the discrete variational derivative is defined by

$$\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}=$$
$$\sum_{l=1}^{3}\frac{1}{M_l}\sum_{m=1}^{M_l}\left(\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{l,m,i,j}\right.$$

$$
\begin{aligned}
&- w_{i,j}^{-1}\left(\delta^*_{\xi,l,m}\left((y_\eta)_{l,m,i,j}\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)},\vec{U}_x^{(n)})}\right)_{l,m,i,j}\right)-\delta^*_{\eta,l,m}\left((y_\xi)_{l,m,i,j}\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)},\vec{U}_x^{(n)})}\right)_{l,m,i,j}\right)\right)\\
&-w_{i,j}^{-1}\left(\delta^*_{\eta,l,m}\left((x_\xi)_{l,m,i,j}\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_y^{(n+1)},\vec{U}_y^{(n)})}\right)_{l,m,i,j}\right)-\delta^*_{\xi,l,m}\left((x_\eta)_{l,m,i,j}\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_y^{(n+1)},\vec{U}_y^{(n)})}\right)_{l,m,i,j}\right)\right)\Bigg)\\
&=\frac{p}{2}\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right)+\frac{r}{4}\left(\left(U_{i,j}^{(n+1)}\right)^3+\left(U_{i,j}^{(n+1)}\right)^2U_{i,j}^{(n)}+U_{i,j}^{(n+1)}\left(U_{i,j}^{(n)}\right)^2+\left(U_{i,j}^{(n)}\right)^3\right)\\
&\quad-\frac{q}{8}\Bigg(w_{i,j}^{-1}\Big(\delta_{\xi,-}\left((y_{\eta,\mathrm{c}})_{i,j}w_{i,j}^{-1}\left((y_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,+}-(y_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,+}\right)\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right)\right)\\
&\qquad\qquad-\delta_{\eta,-}\left((y_{\xi,\mathrm{c}})_{i,j}w_{i,j}^{-1}\left((y_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,+}-(y_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,+}\right)\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right)\right)\Big)\\
&\qquad+w_{i,j}^{-1}\Big(\delta_{\xi,-}\left((y_{\eta,\mathrm{c}})_{i,j}w_{i,j}^{-1}\left((y_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,+}-(y_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,-}\right)\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right)\right)\\
&\qquad\qquad-\delta_{\eta,+}\left((y_{\xi,\mathrm{c}})_{i,j}w_{i,j}^{-1}\left((y_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,+}-(y_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,-}\right)\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right)\right)\Big)\\
&\qquad+w_{i,j}^{-1}\Big(\delta_{\xi,+}\left((y_{\eta,\mathrm{c}})_{i,j}w_{i,j}^{-1}\left((y_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,-}-(y_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,+}\right)\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right)\right)\\
&\qquad\qquad-\delta_{\eta,-}\left((y_{\xi,\mathrm{c}})_{i,j}w_{i,j}^{-1}\left((y_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,-}-(y_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,+}\right)\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right)\right)\Big)\\
&\qquad+w_{i,j}^{-1}\Big(\delta_{\xi,+}\left((y_{\eta,\mathrm{c}})_{i,j}w_{i,j}^{-1}\left((y_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,-}-(y_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,-}\right)\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right)\right)\\
&\qquad\qquad-\delta_{\eta,+}\left((y_{\xi,\mathrm{c}})_{i,j}w_{i,j}^{-1}\left((y_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,-}-(y_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,-}\right)\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right)\right)\Big)\Bigg)\\
&\quad-\frac{q}{8}\Bigg(w_{i,j}^{-1}\Big(\delta_{\eta,-}\left((x_{\xi,\mathrm{c}})_{i,j}w_{i,j}^{-1}\left((x_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,+}-(x_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,+}\right)\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right)\right)\\
&\qquad\qquad-\delta_{\xi,-}\left((x_{\eta,\mathrm{c}})_{i,j}w_{i,j}^{-1}\left((x_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,+}-(x_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,+}\right)\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right)\right)\Big)\\
&\qquad+w_{i,j}^{-1}\Big(\delta_{\eta,-}\left((x_{\xi,\mathrm{c}})_{i,j}w_{i,j}^{-1}\left((x_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,+}-(x_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,-}\right)\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right)\right)\\
&\qquad\qquad-\delta_{\xi,+}\left((x_{\eta,\mathrm{c}})_{i,j}w_{i,j}^{-1}\left((x_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,+}-(x_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,-}\right)\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right)\right)\Big)\\
&\qquad+w_{i,j}^{-1}\Big(\delta_{\eta,+}\left((x_{\xi,\mathrm{c}})_{i,j}w_{i,j}^{-1}\left((x_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,-}-(x_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,+}\right)\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right)\right)\\
&\qquad\qquad-\delta_{\xi,-}\left((x_{\eta,\mathrm{c}})_{i,j}w_{i,j}^{-1}\left((x_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,-}-(x_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,+}\right)\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right)\right)\Big)\\
&\qquad+w_{i,j}^{-1}\Big(\delta_{\eta,+}\left((x_{\xi,\mathrm{c}})_{i,j}w_{i,j}^{-1}\left((x_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,-}-(x_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,-}\right)\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right)\right)\\
&\qquad\qquad-\delta_{\xi,+}\left((x_{\eta,\mathrm{c}})_{i,j}w_{i,j}^{-1}\left((x_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,-}-(x_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,-}\right)\left(U_{i,j}^{(n+1)}+U_{i,j}^{(n)}\right)\right)\Big)\Bigg).
\end{aligned}
$$

The scheme is defined by (4.32b) with, for example, δ_ξ and δ_η being the backward difference $\delta_{\xi,-}$ and

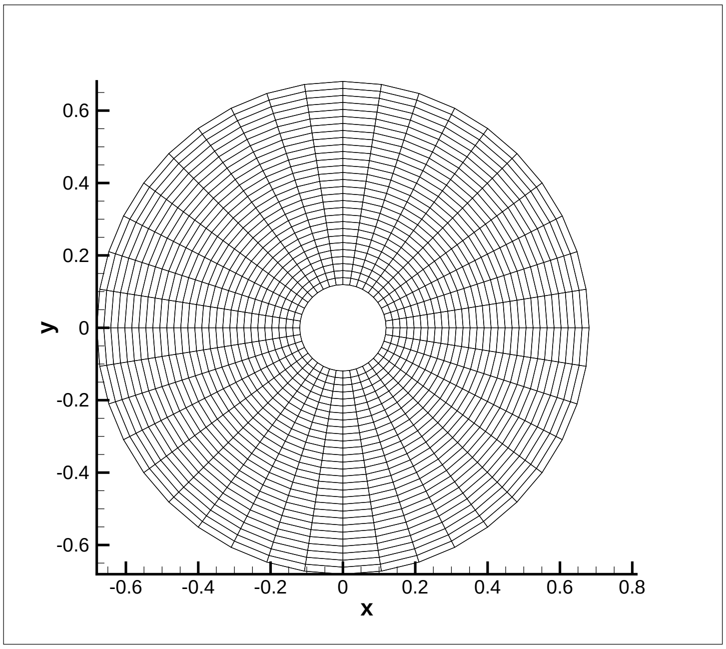

Fig. 6 The grid that is used in the computation.

$\delta_{\eta,-}$:

$$\begin{aligned}\frac{U_{i,j}^{(n+1)}-U_{i,j}^{(n)}}{\Delta t} =& w_{i,j}^{-1}\delta_{\xi,+}\left((y_{\eta,\mathrm{c}})_{i,j}\left(w_{i,j}^{-1}\left((y_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,-}-(y_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,-}\right)\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right)\\ &-w_{i,j}^{-1}\delta_{\eta,+}\left((y_{\xi,\mathrm{c}})_{i,j}\left(w_{i,j}^{-1}\left((y_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,-}-(y_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,-}\right)\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right)\\ &+w_{i,j}^{-1}\delta_{\eta,+}\left((x_{\xi,\mathrm{c}})_{i,j}\left(w_{i,j}^{-1}\left((x_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,-}-(x_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,-}\right)\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right)\\ &-w_{i,j}^{-1}\delta_{\xi,+}\left((x_{\eta,\mathrm{c}})_{i,j}\left(w_{i,j}^{-1}\left((x_{\xi,\mathrm{c}})_{i,j}\delta_{\eta,-}-(x_{\eta,\mathrm{c}})_{i,j}\delta_{\xi,-}\right)\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{i,j}\right).\end{aligned}$$

5.2 Numerical Example

We solved the Cahn–Hilliard equation on the annular domain Ω as is shown in Fig. 6 by the scheme that is derived in the previous section. We set the parameters of the equation to $p=-1$, $q=-0.001$ and $r=1$. The domain Ω is represented as $\Omega=\{(x,y)\mid x=\rho\cos\theta,\ y=\rho\sin\theta,\ 0.1\le\rho\le 0.7,\ 0\le\theta<2\pi\}$ by the polar coordinate. We used the grid that is shown in Fig. 6, which is written as

$$x(\xi,\eta)=\rho(\xi)\cos(\theta(\eta)),\quad y(\xi,\eta)=\rho(\xi)\sin(\theta(\eta)),\quad \rho(\xi)=\frac{3}{5}\frac{\xi}{N}+\frac{1}{10},\quad \theta(\eta)=\frac{2\pi\eta}{M+1} \tag{5.2}$$

in the computational space. The number of grids are 30 in the ξ direction and 40 in the η direction. We set the periodic boundary condition in the η direction and

$$\frac{\partial u}{\partial\xi}=0,\quad \frac{\partial}{\partial\xi}(\Delta u)=0 \tag{5.3}$$

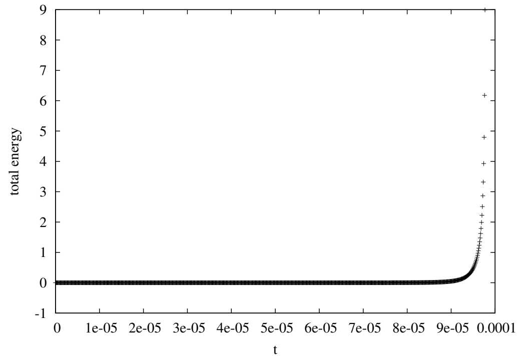

Fig. 7 The time evolution of the total energy of the numerical solution computed by the naive scheme.

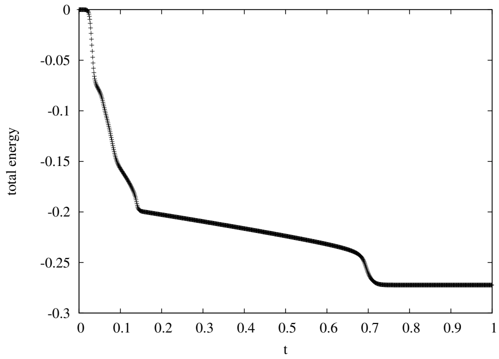

Fig. 8 The time evolution of the total energy of the numerical solution computed by our scheme.

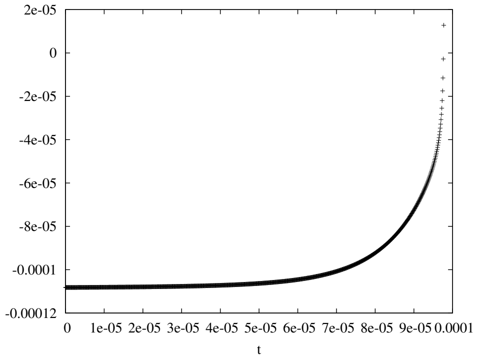

Fig. 9 The time evolution of the total density of the numerical solution computed by the naive scheme.

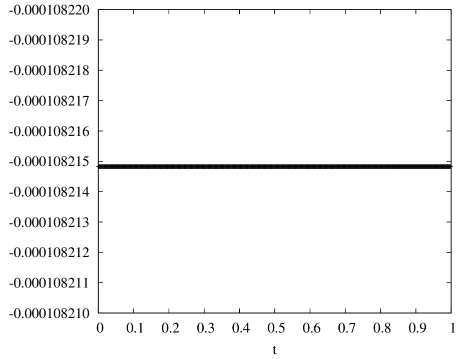

Fig. 10 The time evolution of the total density of the numerical solution computed by our scheme.

in the ξ direction. Under these conditions the solution has the dissipation property. Additionally, the Cahn–Hilliard equation describes the phase separation and the solution u denotes the density of the materials. Under the above conditions, the total density is also conserved:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} u \mathrm{d}x\mathrm{d}y = 0.$$

We set the initial condition by

$$u(0,\xi,\eta) = 0.001\sin(10\pi(r(\xi) - 0.1)) + 0.001\sin(7\theta(\eta)).$$

As a naive method, it is natural to employ the implicit Euler method in time and the central difference that approximates the Laplace operator in the polar coordinate

$$\Delta = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}.$$

So we compared our method with this naive method.

When we used the naive method, the scheme is unstable even with $\Delta t = 10^{-7}$. This result should be due to the fact that the naive method does not retain neither the energy dissipation property nor the conservation of the total density. Indeed both the total energy $H^{(n)}$ and the total density $\sum_{i,j} w_{i,j}U_{i,j}^{(n)}$ diverge as shown in Fig. 7 and 9.

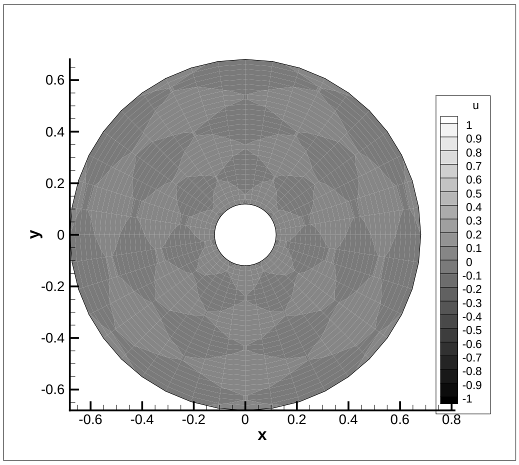

Fig. 11 The initial condition.

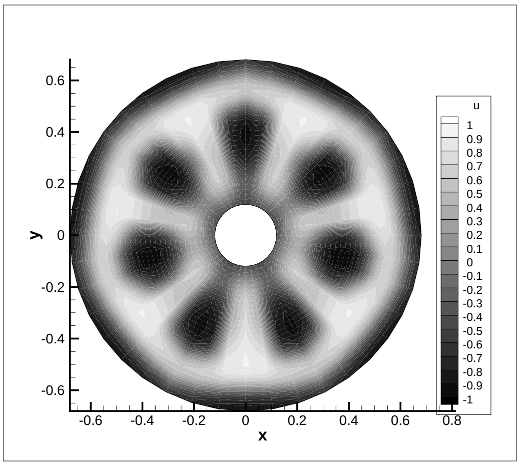

Fig. 12 The numerical solution by our scheme at $t = 0.05$.

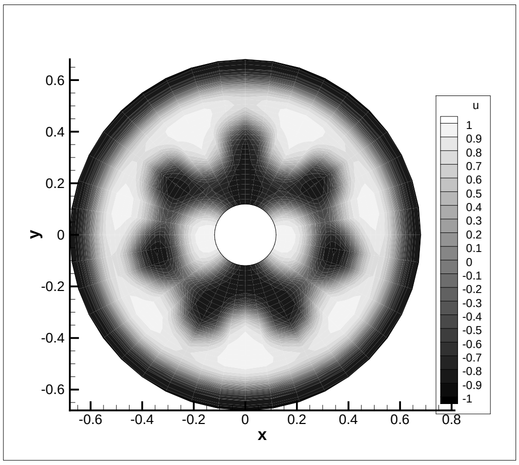

Fig. 13 The numerical solution by our scheme at $t = 0.08$.

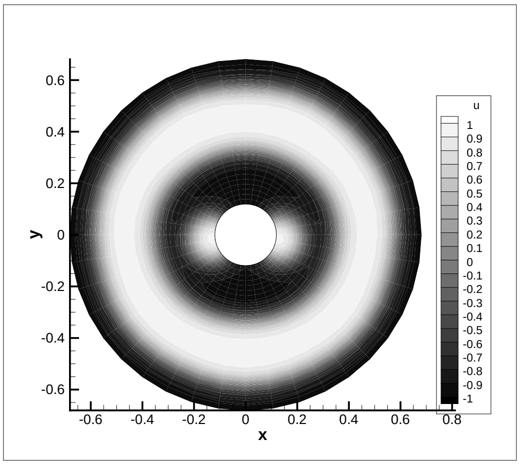

Fig. 14 The numerical solution by our scheme at $t = 0.12$.

On the other hand, our scheme is stable with $\Delta t = 10^{-3}$. The time evolutions of the total energy and the total density in that case are shown in Fig. 8 and 10. The dissipation property of our scheme is confirmed by Fig. 8. Additionally, although we omit the proof, the total density is preserved in our scheme, which is confirmed by Fig. 10.

The numerical solutions by our scheme at $t = 0.0, 0.05, 0.08, 0.12, 0.15, 0.75$ are shown in Fig. 11–16. The black region and the white region in the figures correspond to the regions where the value of u is almost -1 and 1 respectively. They represent the different phases and the total energy decreases when the phases coalesce. In fact, the coalescence of the black regions is observed from $t = 0.05$ to $t = 0.08$, while the decrease of the energy is observed at the same time. We can see similar phenomena also from $t = 0.12$ to $t = 0.15$ and from $t = 0.15$ to $t = 0.75$.

6 Dissipative/Conservative Schemes for Complex Valued Equations

The discrete variational method has been extended to other equations than those of the form (1.3) or (1.5), which include complex valued equations and nonlinear wave equations [5, 8, 10]. Our extension can be also applied to such equations. As an example we show an application to one dimensional complex

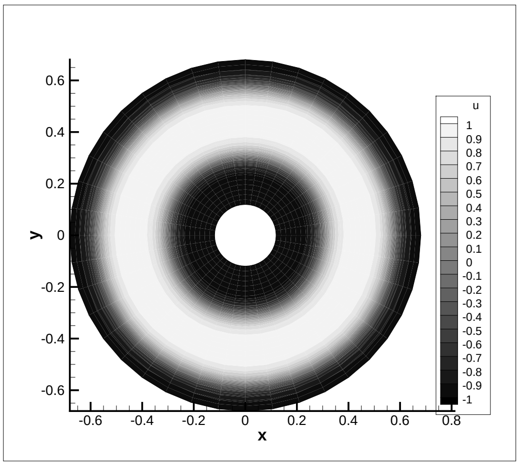

Fig. 15 The numerical solution by our scheme at $t = 0.15$.

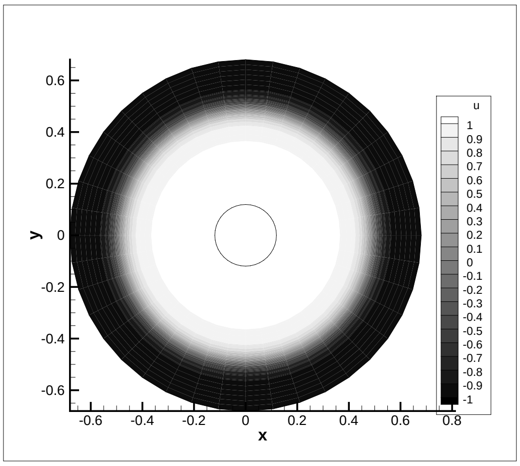

Fig. 16 The numerical solution by our scheme at $t = 0.75$.

valued equations. The extension to two dimensional case is obtained in the same way as Section 4. We omit the proofs of the theorems below, but they are proved in similar ways to those in Section 2. The original discrete variational method for such equations is shown in [10].

We consider the dissipative equations

$$\frac{\partial u}{\partial t} = -\frac{\delta G}{\delta \bar{u}} \tag{6.1}$$

and the conservative equations

$$\mathrm{i}\frac{\partial u}{\partial t} = -\frac{\delta G}{\delta \bar{u}}, \tag{6.2}$$

where u is a complex valued function and $\bar{u}$ denotes the complex conjugate. The dissipative equations include the Ginzburg–Landau equation

$$\frac{\partial u}{\partial t} = p\frac{\partial^2 u}{\partial x^2} + q|u|^2 u + ru,$$

where p, q and r are real parameters, and the conservative equations the nonlinear Schrödinger equation

$$\mathrm{i}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \gamma|u|^{p-1}u,$$

where γ is a real parameter. The variational derivatives are defined by

$$\frac{\delta G}{\delta u} = \frac{\partial G}{\partial u} - \frac{\partial}{\partial x}\frac{\partial G}{\partial u_x}, \quad \frac{\delta G}{\delta \bar{u}} = \frac{\partial G}{\partial \bar{u}} - \frac{\partial}{\partial x}\frac{\partial G}{\partial \bar{u}_x}.$$

The energy functional $G(u, u_x)$ is assumed to be of the next form:

$$G(u, u_x) = \sum_{l=1}^{K} f_l(u) g_l(u_x),$$

where $f_l(u)$ and $g_l(u_x)$ are read valued functions that satisfy

$$f_l(u) = f_l(\bar{u}),\ g_l(u_x) = g_l(\bar{u}_x).$$

As is well-known [10], equations of the form (6.1) are dissipative.

Theorem 6.1. *Suppose that the boundary condition satisfies*

$$\left[\frac{\partial G}{\partial u_x}\frac{\partial u}{\partial t}+\frac{\partial G}{\partial \bar{u}_x}\frac{\partial \bar{u}}{\partial t}\right]_0^L=0.$$

Then the solutions of (6.1) *have the dissipation property:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L G(u,u_x)\mathrm{d}x\le 0.$$

Similarly, equations of the form (6.2) are conservative.

Theorem 6.2. *Suppose that the boundary condition satisfies*

$$\left[\frac{\partial G}{\partial u_x}\frac{\partial u}{\partial t}+\frac{\partial G}{\partial \bar{u}_x}\frac{\partial \bar{u}}{\partial t}\right]_0^L=0.$$

Then the solutions of (6.2) *have the conservation property:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L G(u,u_x)\mathrm{d}x=0.$$

6.1 Definition of the Discrete Variational Derivative

We can introduce the discrete variational derivatives for these equations in a similar way as Section 2. We define the discrete energy functional by

$$G_{\mathrm{d}}(\vec{U})_j^{(n)}=\sum_{l=1}^{K}f_l(U_j^{(n)})\left(\frac{1}{M_l}\sum_{m=1}^{M_l}g_l((x_\xi)_{l,m,j}^{-1}\delta_{l,m}U_j^{(n)})\right).$$

and the discrete total energy $H^{(n)}$ by (2.21).

Definition 6.1. *We define the discrete variational derivatives of* $G_{\mathrm{d}}(\vec{U})_j^{(n)}$ *by*

$$\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j:=\sum_{l=1}^{K}\frac{1}{M_l}\sum_{m=1}^{M_l}\left(\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{l,m,j}\right.$$
$$\left.-w_j^{-1}\delta_{l,m}^{*}\left((x_\xi)_{l,m,j}^{-1}w_j\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)},\vec{U}_x^{(n)})}\right)_{l,m,j}\right)\right),$$

$$\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_{l,m,j}:=\left(\frac{f_l(U_j^{(n+1)})-f_l(U_j^{(n)})}{|U_j^{(n+1)}|^2-|U_j^{(n)}|^2}\right)\left(\frac{\overline{U_j^{(n+1)}+U_j^{(n)}}}{2}\right)$$
$$\left(\frac{g_l((x_\xi)_{l,m,j}^{-1}\delta_{l,m}U_j^{(n+1)})+g_l((x_\xi)_{l,m,j}^{-1}\delta_{l,m}U_j^{(n)})}{2}\right),$$

$$\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)},\vec{U}_x^{(n)})}\right)_{l,m,j}:=\left(\frac{f_l(U_j^{(n+1)})+f_l(U_j^{(n)})}{2}\right)$$
$$\left(\frac{g_l((x_\xi)_{l,m,j}^{-1}\delta_{l,m}U_j^{(n+1)})-g_l((x_\xi)_{l,m,j}^{-1}\delta_{l,m}U_j^{(n)})}{|(x_\xi)_{l,m,j}^{-1}\delta_{l,m}U_j^{(n+1)}|^2-|(x_\xi)_{l,m,j}^{-1}\delta_{l,m}U_j^{(n)}|^2}\right)\left(\frac{\overline{(x_\xi)_{l,m,j}^{-1}\delta_{l,m}U_j^{(n+1)}+(x_\xi)_{l,m,j}^{-1}\delta_{l,m}U_j^{(n)}}}{2}\right).$$

and

$$\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{\vec{U}}^{(n+1)},\vec{\vec{U}}^{(n)})}\right)_j := \sum_{l=1}^{K}\frac{1}{M_l}\sum_{m=1}^{M_l}\left(\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{\vec{U}}^{(n+1)},\vec{\vec{U}}^{(n)})}\right)_{l,m,j} -w_j^{-1}\delta_{l,m}^{*}\left((x_\xi)_{l,m,j}^{-1}w_j\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{\vec{U}}_x^{(n+1)},\vec{\vec{U}}_x^{(n)})}\right)_{l,m,j}\right)\right),$$

$$\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{\vec{U}}^{(n+1)},\vec{\vec{U}}^{(n)})}\right)_{l,m,j} := \left(\frac{f_l(U_j^{(n+1)})-f_l(U_j^{(n)})}{|U_j^{(n+1)}|^2-|U_j^{(n)}|^2}\right)\left(\frac{U_j^{(n+1)}+U_j^{(n)}}{2}\right) \left(\frac{g_l((x_\xi)_{l,m,j}^{-1}\delta_{l,m}U_j^{(n+1)})+g_l((x_\xi)_{l,m,j}^{-1}\delta_{l,m}U_j^{(n)})}{2}\right),$$

$$\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{\vec{U}}_x^{(n+1)},\vec{\vec{U}}_x^{(n)})}\right)_{l,m,j} := \left(\frac{f_l(U_j^{(n+1)})+f_l(U_j^{(n)})}{2}\right) \left(\frac{g_l((x_\xi)_{l,m,j}^{-1}\delta_{l,m}U_j^{(n+1)})-g_l((x_\xi)_{l,m,j}^{-1}\delta_{l,m}U_j^{(n)})}{|(x_\xi)_{l,m,j}^{-1}\delta_{l,m}U_j^{(n+1)}|^2-|(x_\xi)_{l,m,j}^{-1}\delta_{l,m}U_j^{(n)}|^2}\right)\left(\frac{(x_\xi)_{l,m,j}^{-1}\delta_{l,m}U_j^{(n+1)}+(x_\xi)_{l,m,j}^{-1}\delta_{l,m}U_j^{(n)}}{2}\right).$$

Lemma 6.3. *Suppose the boundary condition satisfies*

$$\sum_{l=1}^{K}\frac{1}{M_l}\sum_{m=1}^{M_l}\left(\mu_{(+,\delta_{l,m},(x_\xi)_{l,m,j}^{-1})}\left(\left\{\frac{U_j^{(n+1)}-U_j^{(n)}}{\Delta t}\right\},\left\{\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)},\vec{U}_x^{(n)})}\right)_{l,m,j}\right\}\right) +\mu_{(-,\delta_{l,m},(x_\xi)_{l,m,j}^{-1})}\left(\left\{\frac{U_j^{(n+1)}-U_j^{(n)}}{\Delta t}\right\},\left\{\left(\frac{\partial G_{\mathrm{d}}}{\partial(\vec{U}_x^{(n+1)},\vec{U}_x^{(n)})}\right)_{l,m,j}\right\}\right)\right)+(\mathrm{C.C.})=0$$

where (C.C.) *denotes the complex conjugate of the other terms. Then*

$$\frac{1}{\Delta t}\left(H^{(n+1)}-H^{(n)}\right)=\sum_{j=0}^{N}w_j\left(\frac{U_j^{(n+1)}-U_j^{(n)}}{\Delta t}\right)\left(\frac{\delta G_{\mathrm{d}}}{\delta(\vec{U}^{(n+1)},\vec{U}^{(n)})}\right)_j+(\mathrm{C.C.}).$$

6.2 Design of Schemes

We define the scheme for the dissipative equation of the form (6.1) by

$$\left(\frac{U_j^{(n+1)}-U_j^{(n)}}{\Delta t}\right)=-\left(\frac{\delta\vec{G}_{\mathrm{d}}}{\delta(\vec{\vec{U}}^{(n+1)},\vec{\vec{U}}^{(n)})}\right)_j \tag{6.3}$$

and by

$$\mathrm{i}\left(\frac{U_j^{(n+1)}-U_j^{(n)}}{\Delta t}\right)=-\left(\frac{\delta\vec{G}_{\mathrm{d}}}{\delta(\vec{\vec{U}}^{(n+1)},\vec{\vec{U}}^{(n)})}\right)_j \tag{6.4}$$

for the conservative equation of the form (6.2).

Theorem 6.4. *Let w_j's be positive. Let $U_j^{(n)}$ be a numerical solution of the scheme* (6.3) *under the boundary condition that satisfies the assumption of Lemma 6.3. Then*

$$\frac{1}{\Delta t}\left(H^{(n+1)}-H^{(n)}\right)\le 0.$$

Theorem 6.5. *Let $U_j^{(n)}$ be a numerical solution of the scheme* (6.4) *under the boundary condition that satisfies the assumption of Lemma 6.3. Then*

$$\frac{1}{\Delta t}\left(H^{(n+1)} - H^{(n)}\right) = 0.$$

7 Concluding Remarks

We have extended the discrete variational method to nonuniform meshes. Although we have considered the one and the two dimensional cases for simplicity, we can also derive the conservative/dissipative schemes in three dimensional cases by transforming the differential operators to

$$\frac{\partial}{\partial x} = \frac{1}{J}\left(\left(\frac{\partial y}{\partial \eta}\frac{\partial z}{\partial \zeta} - \frac{\partial y}{\partial \zeta}\frac{\partial z}{\partial \eta}\right)\frac{\partial}{\partial \xi} + \left(\frac{\partial y}{\partial \zeta}\frac{\partial z}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial z}{\partial \zeta}\right)\frac{\partial}{\partial \eta} + \left(\frac{\partial y}{\partial \xi}\frac{\partial z}{\partial \eta} - \frac{\partial y}{\partial \eta}\frac{\partial z}{\partial \xi}\right)\frac{\partial}{\partial \zeta}\right)$$

and so on. In particular in the two dimensional case, it is remarkable that our method is capable of deriving conservative/dissipative, and therefore stable, schemes on various domains such as the annular domain. In fact our scheme for the Cahn–Hilliard equation is far more stable than the naive scheme.

Our extension is based on the mapping method. In order to employ this method, we have shown that the fact that "the conservation/dissipation properties are obtained from the variational structure" is retained after the change of coordinates. If we can get the similar result in the space-time, it brings in a possibility of the extension to moving meshes; this is included in the future works.

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