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An Improved Approximation Algorithm for the Traveling Tournament Problem

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Abstract. This paper describes the traveling tournament problem, a well-known benchmark problem in the field of tournament timetabling. We propose an approximation algorithm for the traveling tournament problem with the constraints such that both the number of consecutive away games and that of consecutive home games are at most k. When $k \leq 5$, the approximation ratio of the proposed algorithm is bounded by $(2k-1)/k + \mathrm{O}(k/n)$ where n denotes the number of teams; when $k > 5$, the ratio is bounded by $(5k-7)/(2k) + \mathrm{O}(k/n)$. For $k = 3$, the most investigated case of the traveling tournament problem to date, the approximation ratio of the proposed algorithm is $5/3 + \mathrm{O}(1/n)$; this is better than the previous approximation algorithm proposed for $k = 3$, whose approximation ratio is $2 + \mathrm{O}(1/n)$.
Key words: traveling tournament problem, approximation algorithm, lower bound, timetabling, scheduling

1 Traveling Tournament Problem

In the field of tournament timetabling, the traveling tournament problem (TTP) is a well-known benchmark problem established by Easton, Nemhauser and Trick [2]. The objective of TTP is to make a round-robin tournament that minimizes the total traveling distance of participating teams. The problem TTP includes optimization aspects similar to those of the traveling salesman problem (TSP) and vehicle routing problems. However, TTP is surprisingly harder than TSP: there is a 10-team TTP instance that has not yet been solved exactly [6]. This contrasts starkly to TSP, for which a 10-city instance of TSP is easy. For further discussions related to TTP and its variations, see [4, 5].

In the following, we introduce some terminology and then define TTP. We are given a set T of n teams, where $n \geq 4$ and even. Each team in T has its home venue. A game is specified by an ordered pair of teams. A double round-robin tournament is a set of games in which every team plays every other team once at its home venue and once at away (i.e., at the venue of the opponent); consequently, exactly $2(n-1)$ slots are necessary to complete a double round-robin tournament.

Each team stays at its home venue before a tournament; then it travels to play its games at the chosen venues. After a tournament, each team returns to its home venue if the last game is played at away. When a team plays two consecutive away games, the team goes directly from the venue of the first opponent to the other without returning to its home venue.

Let V be the set of venues satisfying $|V| = n$. For any pair of venues $i, j \in V$, $d_{ij} \geq 0$ denotes the distance between the venues i and j. We denote the distance matrix (d_{ij}) by D, whose rows and columns are indexed by V. Throughout this paper we assume that triangle inequality ($d_{ij} + d_{jk} \geq d_{ik}$), symmetry ($d_{ij} = d_{ji}$), and $d_{ii} = 0$ hold for any $i, j, k \in V$.

Given a constant (positive integer) $k \geq 3$, the traveling tournament problem [2] is defined as follows.

Traveling Tournament Problem (TTP(k))
Input: a set of teams T and a distance matrix $D = (d_{ij})$, indexed by V.
Output: a double round-robin tournament S of n teams such that

C1. no team plays more than k consecutive away games;

C2. no team plays more than k consecutive home games;

C3. game i at j immediately followed by game j at i is prohibited;

C4. the total distance traveled by the teams is minimized.

In this paper, we assume that n is sufficiently larger than a fixed parameter k. Constraints C1 and C2 are called the *atmost* constraints, and Constraint C3 is called the *no-repeater* constraint. In the remainder of this paper, a double round-robin tournament satisfying the above conditions C1–C3/C1–C4 are called a *feasible*/*optimal* tournaments.

Various studies on TTP have been appeared in recent years, and most of them considered TTP(3) [6]. Most of the best upper bounds of TTP instances are obtained using metaheuristic algorithms; on the other hand, few researches have been done to explore lower bounds and exact methods for TTP (see [5] for example). Recently, three of the authors of this paper proposed $(2+(9/4)/(n-1))$-approximation algorithm for TTP(3), which is the first approximation algorithm with a constant ratio [3].

In this paper, we propose an approximation algorithm for TTP(k). When $k \leq 5$, the approximation ratio of our algorithm is bounded by $(2k-1)/k + \mathrm{O}(k/n)$; when $k > 5$, the approximation ratio is bounded by $(5k-7)/(2k) + \mathrm{O}(k/n)$. For $k = 3$, the approximation ratio of our algorithm is $5/3 + \mathrm{O}(1/n)$; that improves the approximation ratio of the previous algorithm for TTP(3), whose ratio is $2 + (9/4)/(n-1) = 2 + \mathrm{O}(1/n)$.

2 Algorithm

A key idea of our algorithm is the use of the Kirkman schedule and a shortest Hamilton cycle. The classical Kirkman schedule satisfies the property that the orders of opponents in almost all teams are very similar to a mutual cyclic order of teams. Roughly speaking, our algorithm constructs an almost shortest Hamilton cycle passing all the venues and finds a permutation of teams such that the above cyclic order corresponds to the obtained Hamilton cycle.

In Section 2.1, we introduce how to make a single round-robin schedule with a specific structure. In Section 2.2, we construct double round-robin schedules based on the single round-robin schedule proposed in Section 2.1. In Section 2.3, we consider an assignment of venues to teams in the schedules of Section 2.2.

In the following, "schedule without HA-assignment" means that "round-robin schedule without concepts of home game, away game and venue." In other words, in a schedule without HA-assignment, only a sequence of opponents of each team is decided, but the venues of these games are not specified.

2.1 Single Round-Robin Schedule

Denote the set of n teams by $T = \{0, 1, \ldots, n-1\}$ and the set of $n-1$ slots $S = \{0, 1, \ldots, n-2\}$. A single round-robin schedule (without HA-assignment) is a matrix K whose (t, s) element $K(t, s)$ denotes the opponent of team t at slot s. More precisely, K is a matrix such that

(1) rows and columns are indexed by T and S, respectively,

(2) every element of K is a team,

(3) a row of K indexed by t consists of teams $T \setminus \{t\}$, and

(4) for any pair $(t, s) \in T \times S$, a team $t' = K(t, s)$ satisfies $K(t', s) = t$.

The *Kirkman schedule* K^* is a matrix defined by

$$K^*(t,s) = \begin{cases} s - t \mod n-1 & (t \neq n-1 \text{ and } [s - t \neq t \mod n-1]), \\ n-1 & (t \neq n-1 \text{ and } [s - t = t \mod n-1]), \\ s/2 & (t = n-1 \text{ and } s \text{ is even}), \\ (s+n-1)/2 & (t = n-1 \text{ and } s \text{ is odd}). \end{cases}$$

Lemma 1. *The Kirkman schedule K^* is a single round-robin schedule.*

Proof. The following table shows the Kirkman schedule with 10 teams.

t \ s	0	1	2	3	4	5	6	7	8
0	9	1	2	3	4	5	6	7	8
1	8	0	9	2	3	4	5	6	7
2	7	8	0	1	9	3	4	5	6
3	6	7	8	0	1	2	9	4	5
4	5	6	7	8	0	1	2	3	9
5	4	9	6	7	8	0	1	2	3
6	3	4	5	9	7	8	0	1	2
7	2	3	4	5	6	9	8	0	1
8	1	2	3	4	5	6	7	9	0
9	0	5	1	6	2	7	3	8	4

In the following, we prove that the Kirkman schedule satisfies Conditions:
(1) rows and columns are indexed by T and S, respectively,
(2) every element of K is a team,
(3) a row of K indexed by t consists of teams $T \setminus \{t\}$, and
(4) for any pair $(t, s) \in T \times S$, a team $t' = K(t, s)$ satisfies $K(t', s) = t$.

Conditions (1) and (2) are obvious.

First, we discuss condition (3). For any team $t \in T \setminus \{n-1\}$, it is clear from the definition that the row of K^* indexed by t consists of teams $T \setminus \{t\}$. Consider the row of K^* indexed by $n-1$. For any team $t \in T \setminus \{n-1\}$, if $t \leq n/2 - 1$, t appears at slot $2t$, and if $n/2 - 1 < t \leq n-2$, t appears at slot $2t - n + 1$. Thus, $(n-1)$th row contains $T \setminus \{n-1\}$.

Next, we consider condition (4).
Case 1: Consider the case that $t \in T \setminus \{n-1\}$ and $s - t \neq t \mod n-1$. The opponent of team t at slot s, denoted by t', satisfies $t' = s - t \mod n-1$. Clearly, $t' \neq n-1$. Since $s - t' = s - (s - t) = t \mod n-1$, the assumption $s - t \neq t \mod n-1$ implies that $s - t' \neq t' \mod n-1$. Thus, the opponent of team t' at slot s satisfies $K^*(t', s) = s - (s - t) = t \mod n-1$.
Case 2: Consider the case that $t \in T \setminus \{n-1\}$ and $s - t = t \mod n-1$. The opponent of team t at slot s is team $n-1$. When s is even, the opponent of team $n-1$ at slot s is team $t' = s/2$. Since $s = 2t \mod n-1$ and s is even, $t' = s/2 = t \mod n-1$. If s is odd, the opponent of team $n-1$ at slot s is team $t' = (s + n - 1)/2$. Since $s + n - 1 = s = 2t \mod n-1$ and s is odd, $s + n - 1$ is even and thus $t' = (s + n - 1)/2 = t \mod n-1$.
Case 3: It is obvious for the case that $t = n-1$. □

Next, we define HA-assignments of the Kirkman schedule. For constructing variations of HA-assignments, we introduce a function $f : U \to \{\mathrm{H}, \mathrm{A}\}$ where $U = \{i \in \mathbb{Z} \mid i \neq 0 \mod n-1\}$. Given a function f, we define the negated function $\neg f : U \to \{\mathrm{H}, \mathrm{A}\}$ by

$$\neg f(i) = \begin{cases} \mathrm{H} \ (f(i) = \mathrm{A}), \\ \mathrm{A} \ (f(i) = \mathrm{H}). \end{cases}$$

Similarly, we define that $\neg$H is A and $\neg$A is H. We say that the function f is *HA-feasible* if f satisfies

(F1) $\forall i, \forall j \in U$, $[i = j \mod n-1$ implies $f(i) = f(j)]$, and

(F2) $\forall i \in U$, $f(i) = \neg f(-i)$.

From the above definition, an HA-feasible function f is uniquely defined by the sequence $(f(1), f(2), \ldots, f(n/2-1))$. The second condition (F2) implies that the sequence $(f(-n/2+1), \ldots, f(-2), f(-1))$ is the reverse of the sequence $(\neg f(1), \neg f(2), \ldots, \neg f(n/2-1))$. Given an HA-feasible function f, we define that for any pair of teams $t, t' \in T\backslash\{n-1\}$, the game between t and t' is $f(t'-t)$-game of team t and $\neg f(t'-t) = f(t-t')$-game of team t'. (Here we note that H-game means a home game, and A-game means an away game.) For constructing a complete HA-assignment, we need to define an HA-pattern of team $n-1$. We introduce a *root sequence* $(r_0, r_1, \ldots, r_{n-2}) \in \{\mathrm{H}, \mathrm{A}\}^{n-1}$ defined by

$$r_i = \begin{cases} \mathrm{H} & (i \neq n-3 \text{ and } [i \in \{0,1,\ldots,k-1\} \mod 2k]), \\ \mathrm{A} & (i \neq n-3 \text{ and } [i \in \{k,k+1,\ldots,2k-1\} \mod 2k]), \\ r_{n-2} & (i = n-3). \end{cases}$$

For example, when $n = 32$ and $k = 5$ the root sequence is

$$(\underbrace{\mathrm{HHHHH}}_{k}\underbrace{\mathrm{AAAAA}}_{k}\underbrace{\mathrm{HHHHH}}_{k}\underbrace{\mathrm{AAAAA}}_{k}\underbrace{\mathrm{HHHHH}}_{k}\underbrace{\mathrm{AAAA}\overset{r_{n-3}}{\mathrm{H}}}_{k}\overset{r_{n-2}}{\mathrm{H}}).$$

We define that team $n-1$ plays r_s-game at slot s and the opponent of $n-1$ at slot s plays $\neg r_s$-game (at slot s).

As a consequence, given an HA-feasible function f (and the root sequence), we can construct an HA-assignment of the Kirkman schedule K^* as follows. For any pair $(t, s) \in T \times S$,

$$\text{at slot } s \text{, team } t \text{ plays } \begin{cases} f(s-2t)\text{-game } (t \neq n-1 \text{ and } [s-t \neq t \mod n-1]), \\ \quad \neg r_s\text{-game } (t \neq n-1 \text{ and } [s-t = t \mod n-1]), \\ \quad\; r_s\text{-game } (t = n-1). \end{cases}$$

When $t \neq n-1$ and $[s-t \neq t \mod n-1]$, the opponent of team t, denoted by $K^*(t,s)$, is defined by $K^*(t,s) = s-t \mod n-1$ and team t plays $f(K^*(t,s)-t)$-game at slot s. Thus, $K^*(t,s)-t = (s-t)-t = s-2t \mod n-1$ and Definition (F1) implies $f(K^*(t,s)-t) = f(s-2t)$. The remaining cases are trivial.

Next, we define variations of HA-assignments by introducing k HA-feasible functions $f_1, f_2, \ldots, f_k$. For each $\alpha \in \{1, 2, \ldots, k\}$, we settle a function f_α by a sequence $(f_\alpha(1), f_\alpha(2), \ldots, f_\alpha(n/2-1))$, defined below. First, we consider an infinite sequence that contains k consecutive 'A's and k consecutive 'H's alternately. Next, we clip a sequence of length $n/2-1$ whose top $k-\alpha+1$ elements are $(\overbrace{\mathrm{A}, \mathrm{A}, \ldots, \mathrm{A}}^{k-\alpha}, \mathrm{H})$. Lastly, we set the first and second elements to 'A' and change the penultimate element to the same element as the last (if it is required). When $k = 3$, we additionally set the third and fourth elements to 'H.' For example, when $n = 32$ and $k = 5$, the sequence

$F_\alpha = (f_\alpha(-15), \ldots, f_\alpha(-2), f_\alpha(-1), *, f_\alpha(1), f_\alpha(2), \ldots, f_\alpha(15))$
becomes
$F_1 = (\mathrm{AAHHHHAAAAAHHHH} * \mathrm{AAAAHHHHAAAAAHH})$
$F_2 = (\mathrm{AAHHHHHAAAAAHHH} * \mathrm{AAAHHHHHAAAAAHH})$
$F_3 = (\mathrm{AAAHHHHHAAAAAHH} * \mathrm{AAHHHHHAAAAAHHH})$
$F_4 = (\mathrm{AAAAHHHHAAAAAHH} * \mathrm{AAHHHHAAAAAHHHH})$
$F_5 = (\mathrm{AAAAAHHHHAAAAHH} * \mathrm{AAHHHAAAAAHHHHH})$.

In the rest of this paper, X_α ($\alpha \in \{1, 2, \ldots, k\}$) denotes the Kirkman schedule with an HA-assignment induced by an HA-feasible function f_α.

2.2 Feasible Double Round-Robin Schedule

In the previous section, we introduced an HA-feasible function f_α, the sequence

$$F_\alpha = (f_\alpha(-n/2+1), \ldots, f_\alpha(-2), f_\alpha(-1), *, f_\alpha(1), f_\alpha(2), \ldots, f_\alpha(n/2-1))$$

and the Kirkman schedule (with an HA-assignment) X_α for any $\alpha \in \{1, 2, \ldots, k\}$. We set the center element (denoted by $*$) of F_α to A or H, and denote the obtained sequence by F_α^{A} or F_α^{H}, respectively. Here we assume that the first element of F_α^{A} is adjacent with the last element of F_α^{A} (and similarly assume for F_α^{H}). Then an HA-pattern of team $t \in T \setminus \{n-1\}$ in schedule X_α is obtained by a cyclic permutation of sequence F_α^{A} or F_α^{H}. In addition, the HA-pattern of team $n-1$ in schedule X_α is obtained by the root sequence.

From the definition of HA-feasible functions $f_1, f_2, \ldots, f_k$ and the root sequence, the following property holds.

Theorem 1. *For any single round-robin schedule $X_\alpha \in \{X_1, X_2, \ldots, X_k\}$, X_α satisfies the atmost constraints.*

Proof. It is clear from the fact that the HA-pattern of team $t \in T \setminus \{n-1\}$ in schedule X_α is obtained by a cyclic permutation of sequence F_α^{A} or F_α^{H}. For team $n-1$, it is obvious. □

Next, we show a property of sequences F_α^{A} and F_α^{H}, which plays an important role in constructing a double round-robin schedule.

Lemma 2. *For any $\alpha \in \{1, 2, \ldots, k\}$, (1) every consecutive three elements of the cyclic sequence F_α^{A} is neither* (HAH) *nor* (AHA)*; (2) every consecutive three elements of the cyclic sequence F_α^{H} is neither* (HAH) *nor* (AHA).

Proof. First, consider the cyclic sequence F_α^{A}. In the following, denote the elements of F_α^{A} by $(F_\alpha^{\mathrm{A}}(-n/2+1), F_\alpha^{\mathrm{A}}(-n/2+2), \ldots, F_\alpha^{\mathrm{A}}(n/2-1))$.

When $k \geq 4$, the subsequence $(F_\alpha^{\mathrm{A}}(-4), F_\alpha^{\mathrm{A}}(-3), \ldots, F_\alpha^{\mathrm{A}}(4))$, denoted by F'_α, satisfies that

$$F'_\alpha = \begin{cases} \begin{array}{ccccccccccc} & {\scriptstyle -4} & {\scriptstyle -3} & {\scriptstyle -2} & {\scriptstyle -1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & {\scriptstyle 4} & \\ (& \mathrm{H} & \mathrm{H} & \mathrm{H} & \mathrm{H} & \mathrm{A} & \mathrm{A} & \mathrm{A} & \mathrm{A} & \mathrm{A} &) \end{array} & (\text{if } k-\alpha \geq 4), \\ (\mathrm{A\ H\ H\ H\ A\ A\ A\ A\ H}) & (\text{if } k-\alpha = 3), \\ (\mathrm{A\ A\ H\ H\ A\ A\ A\ H\ H}) & (\text{if } k-\alpha = 2), \\ (\mathrm{A\ A\ H\ H\ A\ A\ A\ H\ H}) & (\text{if } k-\alpha = 1), \\ (\mathrm{A\ A\ H\ H\ A\ A\ A\ H\ H}) & (\text{if } k-\alpha = 0). \end{cases}$$

Thus, every consecutive three elements of F'_α satisfies the required property.

When $k = 3$, the subsequence $(F^{\mathrm{A}}_\alpha(-6), F^{\mathrm{A}}_\alpha(-5), \dots, F^{\mathrm{A}}_\alpha(6))$, denoted by F''_α, satisfies that

$$F''_\alpha = \begin{cases} \overset{-6\,-5\,-4\,-3\,-2\,-1\;0\;\;1\;\;2\;\;3\;\;4\;\;5\;\;6}{(\mathrm{H\ A\ A\ A\ H\ H\ A\ A\ A\ H\ H\ H\ A})} & (\text{if } \alpha = 1), \\ (\mathrm{H\ H\ A\ A\ H\ H\ A\ A\ A\ H\ H\ A\ A}) & (\text{if } \alpha = 2), \\ (\mathrm{H\ H\ A\ A\ H\ H\ A\ A\ A\ H\ H\ A\ A}) & (\text{if } \alpha = 3). \end{cases}$$

Thus, every consecutive three elements of F''_α satisfies the required property.

The subsequence

$$(F^{\mathrm{A}}_\alpha(n/2-4), F^{\mathrm{A}}_\alpha(n/2-3), F^{\mathrm{A}}_\alpha(n/2-2), F^{\mathrm{A}}_\alpha(n/2-1),$$
$$F^{\mathrm{A}}_\alpha(-n/2+1), F^{\mathrm{A}}_\alpha(-n/2+2), F^{\mathrm{A}}_\alpha(-n/2+3), F^{\mathrm{A}}_\alpha(-n/2+4))$$

is contained in a set of sequences defined by

$$\left\{ \begin{array}{cccccccccc} & n/2{-}4 & n/2{-}3 & n/2{-}2 & n/2{-}1 & -n/2{+}1 & -n/2{+}2 & -n/2{+}3 & -n/2{+}4 & \\ (& \mathrm{A} & \mathrm{A} & \mathrm{A} & \mathrm{A} & \mathrm{H} & \mathrm{H} & \mathrm{H} & \mathrm{H} &), \\ (& \mathrm{H} & \mathrm{A} & \mathrm{A} & \mathrm{A} & \mathrm{H} & \mathrm{H} & \mathrm{H} & \mathrm{A} &), \\ (& \mathrm{H} & \mathrm{H} & \mathrm{A} & \mathrm{A} & \mathrm{H} & \mathrm{H} & \mathrm{A} & \mathrm{A} &), \\ (& \mathrm{A} & \mathrm{A} & \mathrm{H} & \mathrm{H} & \mathrm{A} & \mathrm{A} & \mathrm{H} & \mathrm{H} &), \\ (& \mathrm{A} & \mathrm{H} & \mathrm{H} & \mathrm{H} & \mathrm{A} & \mathrm{A} & \mathrm{A} & \mathrm{H} &), \\ (& \mathrm{H} & \mathrm{H} & \mathrm{H} & \mathrm{H} & \mathrm{A} & \mathrm{A} & \mathrm{A} & \mathrm{A} &) \end{array} \right\}.$$

Every sequence in the above set satisfies that every consecutive three elements is neither (HAH) nor (AHA). Since $k \geq 3$, remained subsequences of consecutive three elements satisfy the required property.

For the cyclic sequence F^{H}_α, the result can be proved in a similar way. □

Lemma 2 and the definition of the root sequence imply the following.

Corollary 1. *For any single round-robin schedule $X_\alpha \in \{X_1, X_2, \dots, X_k\}$, X_α satisfies: (1) the HA-pattern of each team at slots $(n-2, 0, 1)$ is neither* (HAH) *nor* (AHA)*; (2) the HA-pattern of each team at slots $(n-3, n-2, 0)$ is neither* (HAH) *nor* (AHA)*.*

For any $\alpha \in \{1, 2, \dots, k\}$, given a single round-robin schedule X_α defined above, we construct a double round-robin schedule as follows. First, we construct a single round-robin schedule, denoted by Y_α, by exchanging the first slot for the last slot of X_α. Next, we construct a double round-robin schedule by the ordinary mirroring as follows. Denote $\overline{Y}_\alpha$ the schedule obtained from Y_α by reversing the home and away. We concatenate two single round-robin schedules Y_α and $\overline{Y}_\alpha$ to obtain a double round-robin schedule, denoted by Z_α.

Theorem 2. *For any $\alpha \in \{1, 2, \dots, k\}$, both of the single round-robin schedules Y_α and $\overline{Y}_\alpha$ satisfy the atmost constraints.*

Proof. First, we consider the schedule Y_α. Assume on the contrary the case that there exist a team $t \in T$ and consecutive $k+1$ slots $(s, s+1, \dots, s+k)$

where team t plays home games at slots in $(s, s+1, \ldots, s+k)$. Theorem 1 states that schedule X_α satisfies the atmost constraints. Thus the set of slots $\{s, s+1, \ldots, s+k\}$ includes slot 0 or slot $n-2$, i.e., either $s = 0$ or $s+k = n-2$ holds.
Case 1: Consider the case that $s = 0$. If team t plays a home game at slot $n-2$ in schedule Y_α, then team t plays a home game at slot 0 in schedule X_α. Thus, team t has consecutive home games with length $k+1$ at slots $(0, 1, \ldots, k)$ in X_α, which contradicts with the fact that X_α satisfies the atmost constraints, shown in Theorem 1. When team t plays an away game at slot $n-2$ in schedule Y_α, team t plays an away game at slot 0 in schedule X_α. Then the HA-pattern of team t at slots $(n-2, 0, 1)$ in schedule X_α is (HAH), which contradicts with Corollary 1.
Case 2: We can deal with the case that $s+k = n-2$ in a similar manner.

In the remained cases, we can derive contradiction similarly. From the above, Y_α satisfies the atmost constraints.

It is obvious that $\overline{Y}_\alpha$ also satisfies the atmost constraints. □

Theorem 3. *For any double round-robin schedule $Z_\alpha \in \{Z_1, Z_2, \ldots, Z_k\}$, Z_α is a feasible schedule.*

Proof. The schedule $Z_\alpha \in \{Z_1, Z_2, \ldots, Z_k\}$ is obtained by concatenating Y_α and $\overline{Y}_\alpha$. It is obvious that schedule Z_α satisfies the no-repeater constraint.

Assume on the contrary the case that there exist a team $t \in T$ and consecutive $k+1$ slots $(s, s+1, \ldots, s+k)$ where team t plays home games at slots in $(s, s+1, \ldots, s+k)$. Theorem 2 states that the single round-robin schedules Y_α and $\overline{Y}_\alpha$ satisfy the atmost constraints. Thus, the set of slots $\{s, s+1, \ldots, s+k\}$ includes both slot $n-2$ and slot $n-1$. Since $k \geq 3$, $\{s, s+1, \ldots, s+k\}$ contains either $\{n-3, n-2, n-1\}$ or $\{n-2, n-1, n\}$.
Case 1: Consider the case that $\{s, s+1, \ldots, s+k\} \supseteq \{n-3, n-2, n-1\}$. The HA-pattern of team t at slots $(n-3, n-2, n-1)$ in schedule Z_α is (HHH). Here we note that
(1) slot $n-3$ in schedule Z_α corresponds to slot $n-3$ in X_α,
(2) slot $n-2$ in schedule Z_α corresponds to slot 0 in X_α, and
(3) slot $n-1$ in schedule Z_α is obtained from slot $n-2$ in schedule X_α by reversing the home and away.
Then the HA-pattern of team t at slots $(n-3, n-2, 0)$ in schedule X_α is (HAH), which contradicts with Corollary 1.
Case 2: We can deal with the case that $\{s, s+1, \ldots, s+k\} \supseteq \{n-2, n-1, n\}$ in a similar manner.

In the remained cases, we can derive contradiction similarly.

The above discussion concludes that Z_α satisfies the atmost constraints. □

2.3 Assignment of Venues

In Section 2.1, we defined that T is a set of teams and described a method for constructing a double round-robin schedule of teams in T. In this section, we say that T is a set of *imaginary* teams (without venues) and each venue in V represents a *real* team. We propose an algorithm for finding a bijection between the set of venues V and the set of imaginary teams T. In this section, 'a team $t \in T$' means 'an imaginary team $t \in T$.'

Herein, we describe our algorithm. First, we choose $\alpha \in \{1, 2, \ldots, k\}$ randomly and construct a double round-robin schedule Z_α with imaginary teams T. Next, we apply Christofides' algorithm for the traveling salesman problem to a complete undirected graph with vertex set (venue set) V and edge length defined by D, and obtain a Hamilton cycle H_{C}. (Here we note that the length of an undirected edge is well-defined by D, since D satisfies symmetry $d_{ij} = d_{ji}$.) We denote a Hamilton cycle H_{C} by a sequence $(v_0, v_1, \ldots, v_{n-1})$ of vertices (venues). Lastly, we choose $\beta \in \{0, 1, \ldots, n-1\}$ randomly and construct a bijection $\pi : T \to V$ defined by $\pi(i) = v_j$ where $T = \{0, 1, \ldots, n-1\}$ and $j = i + \beta \mod n$.

To determine an expected value of total traveling distance obtained by the above algorithm, we introduce an undirected graph G_α defined by a double round-robin schedule Z_α. The graph G_α has a vertex set T, and a (multi) edge set $E(\alpha)$ with partition $\{E_t(\alpha) \mid t \in T\}$ where every edge in a multiset $E_t(\alpha)$ corresponds to a move of team $t \in T$ in Z_α. More precisely, multiset $E_t(\alpha)$ consists of following (at most) four types of edges;

(1) when team t plays two consecutive away games, $E_t(\alpha)$ includes an edge between two opponents,

(2) when team t plays consecutive pair of home and away games, $E_t(\alpha)$ includes an edge between t and opponent of the away game,

(3) if t plays away game at first slot, $E_t(\alpha)$ includes an edge between t and opponent in the away game,

(4) if t plays away game at last slot, $E_t(\alpha)$ includes an edge between t and opponent in the away game.

If we have a bijection $\pi : T \to V$, the corresponding total traveling distance becomes $\sum_{\{i,j\} \in E(\alpha)} d_{\pi(i)\pi(j)}$.

Next, we define a partition of $E(\alpha)$ consists of three subsets, called *irregular edges*, *regular Hamilton edges*, and *regular non-Hamilton edges*. For any $t \in T$, an edge e in $E_t(\alpha)$ is called *irregular* if e satisfies at least one of the following conditions;

(I1) e has at least one parallel edge in $E_t(\alpha)$,

(I2) e connects a pair of vertices in $\{t-5, t-4, \ldots, t+5\} \cup \{n-2, n-1, 0\} \cup \{t-n/2+1, t-n/2+2, t-n/2+3\} \cup \{t+n/2-3, t+n/2-2, t+n/2-1\}$, where every integer $t+i$ appearing above corresponds to a vertex (team) $t' \in T$ with $t' = t + i \mod n$,

(I3) e corresponds to a move between a pair of slots in $\{(0,1), (n-3, n-2), (n-2, n-1), (n-1, n), (2n-4, 2n-3)\}$,

(I4) e corresponds to a move caused by an away game at first slot (if it exists),

(I5) e corresponds to a move caused by an away game at last slot (if it exists),

(I6) $e \in E_{n-1}(\alpha)$, i.e., e corresponds to a move of team $n-1$.

Every non-irregular edge in $E(\alpha)$ is called *regular*. If a regular edge in $E(\alpha)$ connects a pair of vertices $\{t, t'\}$ with $t' = t+1 \mod n$, the edge is called *regular Hamilton*. Here we note that every regular Hamilton edge corresponds to an edge in Hamilton cycle H^* defined by cyclic sequence $(0, 1, \ldots, n-1)$ of T. The rests of regular edges in $E(\alpha)$ are called *regular non-Hamilton* edges.

In the following, we determine the number of irregular edges in $E(\alpha)$. For any team $t \in T$, undirected graph $(T, E_t(\alpha))$ is an Eulerian graph such that every vertex $t' \in T \setminus \{t\}$ has two incident edges. Thus, every pair of vertices has at most two parallel edges corresponding to consecutive three games with the HA-pattern (H, A, H). From the definition of Z_α, the number of vertex pairs with parallel edges in $E_t(\alpha)$ is bounded by a constant and thus the number of irregular edges in $E_t(\alpha)$ satisfying Condition I1 is bounded by a constant. Obviously, the number of irregular edges in $E_t(\alpha)$ satisfying Conditions I2–I5 is also bounded by a constant. Consequently, for any team $t \in T \setminus \{n-1\}$, $E_t(\alpha)$ contains a constant number of irregular edges. Since $|E_{n-1}(\alpha)| = \mathrm{O}(n)$, the total number of irregular edges in $E(\alpha)$ is $\mathrm{O}(n)$.

Next, we discuss the number of regular Hamilton edges. Every regular Hamilton edge corresponds to a consecutive pair of away games of a team $t \in T \setminus \{n-1\}$. Thus, every team $t \in T \setminus \{n-1\}$ has at most $(2n-2)(k-1)/(2k) \leq n(k-1)/k$ regular Hamilton edges.

Finally, every regular non-Hamilton edge corresponds to a consecutive pair of home game and away game of a team $t \in T \setminus \{n-1\}$. Consequently, every team $t \in T \setminus \{n-1\}$ has at most $(2n-2)2/(2k) \leq 2n/k$ regular non-Hamilton edges.

The below table shows upper bounds of sizes of three sets.

irregular edges	$\mathrm{O}(n)$
regular Hamilton edges	$n(n-1)(k-1)/k$
regular non-Hamilton edges	$2n(n-1)/k$

3 Lower Bounds

In the rest of this paper, we consider a complete undirected graph $\overline{G}$ with vertex set (venue set) V and edge length defined by the distance matrix D. We can relate every move of a real team to an undirected edge in $\overline{G}$.

Let Ψ^* be an optimal tournament of a given instance and ψ^* the optimal value. In the tournament Ψ^*, each real team $v \in V$ visits every venue of opponent exactly once, and we call the sequence of moves of v among venues a tour of v. For each real team $v \in V$, ψ^*_v denotes a tour distance of team v in Ψ^*. Obviously, $\sum_{v \in V} \psi^*_v = \psi^*$ holds.

Let η^* be the length of shortest Hamilton cycle of a complete undirected graph $\overline{G}$. Since distance matrix D satisfies triangle inequalities, $\psi^*_v \geq \eta^*$ holds for any $v \in V$. Consequently, we have the following lemma.

Lemma 3. *The length η^* of a shortest Hamilton cycle satisfies that $\psi^* \geq n\eta^*$.*

For each real team $v \in V$, ψ_v^{home} denotes the sum of distances corresponding to moves leaving or returning to its home v. We introduce a ratio $a^* = (\sum_{v\in V} \psi_v^{\mathrm{home}})/\psi^*$. Let τ^* be the length of minimum spanning tree of a complete undirected graph $\overline{G}$. Then we have the following.

Lemma 4. *The length τ^* of a minimum spanning tree satisfies $(1-\frac{a^*}{2})\psi^* \geq n\tau^*$.*

Proof. For each real team $v \in V$, the tour of v in Ψ^* consists of subcycles in the complete undirected graph $\overline{G}$, in which each subcycle includes vertex v. Each subcycle includes exactly two edges incident to vertex v. If we delete a longer edge in two edges incident to v for each subcycle, we obtain a spanning tree, whose length is denoted by ψ_v^{tree}. Clearly, an inequality $\psi_v^{\mathrm{tree}} \geq \tau^*$ holds. Since we deleted longer edge alternatively, the sum of distances of deleted edges $\psi_v^* - \psi_v^{\mathrm{tree}}$ is greater than or equal to $(1/2)\psi_v^{\mathrm{home}}$. Thus, we have the following:

$$\begin{aligned}(1-a^*/2)\psi^* &= \psi^* - (1/2)a^*\psi^* = \textstyle\sum_{v\in V} \psi_v^* - (1/2)\sum_{v\in V} \psi_v^{\mathrm{home}} \\ &= \textstyle\sum_{v\in V}(\psi_v^* - (1/2)\psi_v^{\mathrm{home}}) \geq \sum_{v\in V} \psi_v^{\mathrm{tree}} \geq \sum_{v\in V} \tau^* = n\tau^*. \quad \square\end{aligned}$$

We denote the sum total of distances of ordered pairs of venues by Δ, i.e., $\Delta \stackrel{\text{def.}}{=} \sum_{v\in V}\sum_{u\in V} d_{vu}$.

Lemma 5. *The sum of distances Δ satisfies $(a^* + \frac{k-2}{2})\psi^* \geq \Delta$.*

Proof. For each real team $v \in V$, the tour of v in Ψ^* consists of subcycles in the complete undirected graph $\overline{G}$, such that each subcycle includes vertex v. We denote the set of subcycles by Γ_v. For each subcycle $C \in \Gamma_v$, we denote the (weighted) length of C by $\psi_v^*(C)$ and sum of lengths of two edges in C incident to vertex v by $\psi_v^{\mathrm{home}}(C)$. Clearly, $\sum_{C\in\Gamma_v} \psi_v^*(C) = \psi_v^*$ and $\sum_{C\in\Gamma_v} \psi_v^{\mathrm{home}}(C) = \psi_v^{\mathrm{home}}$ hold. For each subcycle $C \in \Gamma_v$, C also denotes the set of vertices in C. We denote the number of vertices in C by $|C|$.

Next, we show that for each cycle $C \in \Gamma_v$, the inequality

$$\textstyle\sum_{u\in C\setminus\{v\}} d_{vu} \leq \psi_v^{\mathrm{home}}(C) + (k-2)(1/2)\psi_v^*(C) \tag{1}$$

holds. When $|C| = 2$, Inequality (1) obviously holds. Next, consider the case $|C| \geq 3$. For each vertex u in $C \setminus \{v\}$, subcycle C consists of two paths between v and u and thus symmetry and triangle inequalities imply the inequality $d_{vu} \leq (1/2)\psi_v^*(C)$ and a (loose) inequality $\sum_{u\in C\setminus\{v\}} d_{vu} \leq (|C|-1)(1/2)\psi_v^*(C)$. Employing the value $\psi_v^{\mathrm{home}}(C)$, we obtain a (tight) inequality as follows

$$\begin{aligned}\textstyle\sum_{u\in C\setminus\{v\}} d_{vu} &\leq \psi_v^{\mathrm{home}}(C) + (|C|-3)(1/2)\psi_v^*(C) \\ &\leq \psi_v^{\mathrm{home}}(C) + (k-2)(1/2)\psi_v^*(C),\end{aligned}$$

where the last relationship comes from the fact that every subcycle corresponds to consecutive away games and thus $|C| \leq k+1$.

The inequality (1) directly implies the following:

$$
\begin{aligned}
\Delta &= \sum_{v\in V}\sum_{u\in V} d_{vu} = \sum_{v\in V}\sum_{u\in V\setminus\{v\}} d_{vu} = \sum_{v\in V}\left(\sum_{C\in\Gamma_v}\sum_{u\in C\setminus\{v\}} d_{vu}\right) \\
&\leq \sum_{v\in V}\sum_{C\in\Gamma_v}\left(\psi_v^{\mathrm{home}}(C) + \frac{k-2}{2}\psi_v^*(C)\right) = \sum_{v\in V}\left(\sum_{C\in\Gamma_v}\psi_v^{\mathrm{home}}(C) + \frac{k-2}{2}\sum_{C\in\Gamma_v}\psi_v^*(C)\right) \\
&= \sum_{v\in V}\left(\psi_v^{\mathrm{home}} + \frac{k-2}{2}\psi_v^*\right) = a^*\psi^* + \frac{k-2}{2}\psi^*. \qquad \square
\end{aligned}
$$

4 Approximation Ratio

Here we discuss the approximation ratio of our algorithm. Let $(v_0, v_1, \ldots, v_{n-1})$ be a sequence of vertices (venues) corresponding to Hamilton cycle H_{C} obtained by Christofides' algorithm. Our algorithm chooses $\beta \in \{0, 1, \ldots, n-1\}$ randomly and construct a bijection $\pi : T \to V$ defined by $\pi(i) = v_j$ where $T = \{0, 1, \ldots, n-1\}$ and $j = i + \beta \mod n$.

First, we consider the sum of weights of irregular edges, denoted by a random variable W_{IR}. For each irregular edge (t, t') connecting $t, t' \in T$, $\pi(t)$ (and $\pi(t')$) is randomly assigned to a vertex in V. For any $v \in V$, the triangle inequalities imply that $d_{\pi(t)\pi(t')} \leq d_{\pi(t)v} + d_{v\pi(t')}$, and thus $d_{\pi(t)\pi(t')} \leq (1/n)\sum_{v\in V}(d_{\pi(t)v} + d_{v\pi(t')})$ holds. The expectation of $d_{\pi(t)\pi(t')}$ with respect to random selection of β satisfies

$$
\begin{aligned}
&\mathrm{E}[d_{\pi(t)\pi(t')}] \leq \left(\frac{1}{n}\right)\sum_{v\in V}\left(\mathrm{E}[d_{\pi(t)v}] + \mathrm{E}[d_{v\pi(t')}]\right) \\
&= \left(\frac{1}{n}\right)\sum_{v\in V}\left(\left(\frac{1}{n}\right)\sum_{u\in V} d_{uv} + \left(\frac{1}{n}\right)\sum_{u\in V} d_{vu}\right) = \left(\frac{2}{n^2}\right)\sum_{v\in V}\sum_{u\in V} d_{uv} = \left(\frac{2}{n^2}\right)\Delta.
\end{aligned}
$$

As discussed in Section 2.3, the number of irregular edges is bounded by $\mathrm{O}(n)$ and consequently $\mathrm{E}[W_{\mathrm{IR}}] \leq \mathrm{O}(n)(2/n^2)\Delta = \mathrm{O}(1/n)\Delta$ holds.

Next, we consider the sum of weights of regular Hamilton edges, denoted by W_{RH}. On the length of H_{C}, the following is a well-known theorem.

Lemma 6. [1] *The length of H_C is less than or equal to $\tau^* + (1/2)\eta^*$, where τ^* and η^* denote the length of minimum spanning tree and shortest Hamilton cycle of a complete undirected graph $\overline{G}$.*

Since regular Hamilton edges in H^* are contained in Hamilton cycle H_{C} for any $\beta \in \{0, 1, \ldots, n-1\}$, the above randomization implies that the expectation satisfies $\mathrm{E}[W_{\mathrm{RH}}] \leq \dfrac{n(n-1)(k-1)}{k}\left(\dfrac{\eta_{\mathrm{C}}}{n}\right) \leq \dfrac{(n-1)(k-1)}{k}(\tau^* + (1/2)\eta^*)$, where η_{C} denotes the length of H_{C}.

Lastly, we consider the sum of weights of regular non-Hamilton edges, denoted by W_{RnH}. Recall that every regular non-Hamilton edge corresponds to a consecutive pair of a home game and an away game of a team $t \in T \setminus \{n-1\}$. We fix team $t \in T \setminus \{n-1\}$, $\beta \in \{0, 1, \ldots, n-1\}$, and permutation $\pi : T \to V$ with respect to β. For any team $t' \in T \setminus \{t, n-1\}$, a regular non-Hamilton edge corresponding to a move of team t (real team $\pi(t)$) between venues $\pi(t)$ and $\pi(t')$ appears with probability at most $2/k$ with respect to random choice of $\alpha \in \{0, 1, \ldots, k-1\}$ (and consequently sequence F_α), because our algorithm constructs a double round-robin tournament by mirroring. Thus, we have that

$$\begin{aligned}
\mathrm{E}[W_{\mathrm{RnH}}] &\leq \sum_{t \in T \setminus \{n-1\}} \sum_{\beta \in \{0,1,\ldots,n-1\}} \left(\frac{1}{n}\right) \left(\sum_{t' \in T \setminus \{n-1,t\}} \left(\frac{2}{k}\right) d_{\pi(t)\pi(t')} \right) \\
&\leq \sum_{t \in T} \sum_{\beta \in \{0,1,\ldots,n-1\}} \left(\frac{1}{n}\right) \left(\sum_{t' \in T} \left(\frac{2}{k}\right) d_{\pi(t)\pi(t')} \right) \\
&= \left(\frac{2}{kn}\right) \sum_{\beta \in \{0,1,\ldots,n-1\}} \left(\sum_{t \in T} \sum_{t' \in T} d_{\pi(t)\pi(t')} \right) = \left(\frac{2}{k}\right) \varDelta.
\end{aligned}$$

Finally, we determine the approximation ratio of our algorithm.

Theorem 4. *When $k \leq 5$, the approximation ratio of our algorithm is bounded by $(2k-1)/k + \mathrm{O}(k/n)$. If $k > 5$, the ratio is bounded by $(5k-7)/(2k) + \mathrm{O}(k/n)$.*

Proof. From the above discussion, the expectation of the objective value of a solution obtained by our algorithm satisfies that

$$\begin{aligned}
&\mathrm{E}[W_{\mathrm{IR}} + W_{\mathrm{RH}} + W_{\mathrm{RnH}}] \leq \mathrm{O}(1/n)\varDelta + \frac{(n-1)(k-1)}{k}(\tau^* + (1/2)\eta^*) + (2/k)\varDelta \\
&\leq \mathrm{O}(1/n)\varDelta + \frac{k-1}{k} n\tau^* + \frac{k-1}{2k} n\eta^* + (2/k)\varDelta \\
&\leq \mathrm{O}(1/n)\left(a^* + \frac{k-2}{2}\right)\psi^* + \frac{k-1}{k}\left(1 - \frac{a^*}{2}\right)\psi^* + \frac{k-1}{2k}\psi^* + \left(\frac{2}{k}\right)\left(a^* + \frac{k-2}{2}\right)\psi^* \\
&\leq \mathrm{O}(k/n)\psi^* + \frac{a^*(5-k) + (5k-7)}{2k}\psi^*.
\end{aligned}$$

When $k \leq 5$, $a^* = 1$ gives an upper bound $(2k-1)/k + \mathrm{O}(k/n)$ of the approximation ratio. For $k > 5$, $a^* = 0$ gives an upper bound $(5k-7)/(2k) + \mathrm{O}(k/n)$. □

We note that, if we run our algorithm for every pair of $\alpha \in \{1, 2, \ldots, k\}$ and $\beta \in \{0, 1, \ldots, n-1\}$, the above approximation ration can be always attainable in polynomial time.

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