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$\mathcal{H}_2$ and $\mathcal{H}_\infty$ Norm Computations for LTI Systems with Generalized Frequency Variables

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Abstract

A class of large-scale systems with decentralized information structures such as multi-agent dynamical systems can be represented by a linear system with a generalized frequency variable. In this paper, we propose efficient $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm computations based on the generalized frequency variable. Specifically, we first derive a way of $\mathcal{H}_2$ norm computation from state-space realizations of subsystems. We then discuss a region on the complex plane specified by the generalized frequency variable for achieving the $\mathcal{H}_\infty$ norm bound for a simple feedback case, and a graphical test and three different numerical methods for computing the $\mathcal{H}_\infty$ norm are derived. The last part is devoted to the loop shaping type $\mathcal{H}_\infty$ norm, where a graphical test for the condition and three different ways of computing the $\mathcal{H}_\infty$ norm are provided.

1 Introduction

Due to the insatiable growth of computing power and the increasing demand of complex networking, modern engineering systems have become more and more complex and subject to multitude of system dimensions. To cope with these challenges, many studies of different approaches in a variety of areas have been made in the last decade. One of the bulk flows in these studies is the decentralized autonomous control of the multi-agent dynamical systems (See e.g., [9] and references therein.). There have been many researches in the form of proposing a specific approach within an individual problem formulation, but very few results are available so far to provide a unifying theoretical framework.

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This situation motivates us to establish a unified approach for the analysis and synthesis of multi-agent systems in which agents with dynamics exchange information each other and autonomously cooperate. To this end our research group recently proposed a linear time-invariant system with a generalized frequency variable as one of the unifying expressions for multi-agent dynamical systems [3, 4]. Specifically, the transfer function $\mathcal{G}(s)$ representing the overall dynamics of a multi-agent system is described by simply replacing s by a rational function $\phi(s)$ in a transfer function $G(s)$, i.e., $\mathcal{G}(s) := G(\phi(s))$. We call $\phi(s)$ the generalized frequency variable, because s in a continuous-time transfer function represents the frequency variable.

The system description has a potential to provide a theoretical foundation for analyzing and designing homogeneous large-scale networked dynamical systems in a variety of areas. For example, the framework of the generalized frequency variable can be applied to the analysis and synthesis of central pattern generators (CPGs) [6] and gene-protein regulatory networks [1, 11] as well as consensus and formation problems as surveyed in [9].

The very fundamental properties including controllability/observability have been discussed in [3, 4]. Reference [12] investigated the stability and provided two systematic ways of stability check, namely an algebraic condition and LMI condition, which are different from graphical tests in [10, 9]. A Hurwitz type stability criterion for characteristic polynomials with complex coefficients in [2] was used for the derivation of the former condition, and it can be reduced to a set of LMIs by applying a generalized Lyapunov theorem in [5]. However, there are only a few results for the control performances, and the target classes of systems are very restricted as in [8, 7] for the $\mathcal{H}_2$ norm computations and in [10] for robust stability analysis.

This paper is concerned with control performances rather than stability for LTI systems with generalized frequency variables to derive systematic ways of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm computations for fairly general class of multi-agent dynamical systems. The $\mathcal{H}_2$ norm for example can evaluate a variety of control performances including rapidness of consensus, and the $\mathcal{H}_\infty$ norm relates conditions for robust stability and robust performances, which bring us a systematic treatment of heterogeneous multi-agent dynamical systems.

In Section 2, we define the class of linear systems with generalized frequency variables and show the existing results including several stability conditions as a preliminary part. Section 3 provides a way of $\mathcal{H}_2$ norm computation from state-space realizations of subsystems. It is a fairly general results, and it includes existing results in [8, 7] as special cases. Sections 4 and 5 are devoted to $\mathcal{H}_\infty$ norm computations. In Section 4, we discuss a region on the complex plane specified by the generalized frequency variable for achieving the $\mathcal{H}_\infty$ norm bound for a simple feedback case. A graphical test and three different numerical methods for computing the $\mathcal{H}_\infty$ norm are derived. The result can be applied to robust stability analysis for feedback type perturbations, and we compare it with a robust stability result in [10]. Section 5 is concerned with the loop-shaping type H_∞ norm computations for the case of normal interconnected matrix. It is one of typical setting in robust control, since it relates the normalized coprime factor perturbations and it includes both the sensitivity and complementarity sensitivity functions. A graphical test for the condition and three different ways of computing the $\mathcal{H}_\infty$ norm are provided. Finally, we make concluding remarks in Section 6.

We use the following notation. The sets of real, complex and natural numbers, are denoted by $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{N}$, respectively. The complex conjugate of $z \in \mathbb{C}$ is denoted by $\bar{z}$. For a matrix A, its transpose and complex conjugate transpose are denoted by A^T and A^*, respectively. For a square matrix A, the set of eigenvalues is denoted by $\sigma(A)$. The symbols $\mathbf{S}_n$ and $\mathbf{S}_n^+$ stand for the sets of $n \times n$ real symmetric matrices and its positive definite subsets. For matrices A and B, $A \otimes B$ means their Kronecker product. The open left-half complex plane and the closed right-half complex plane are denoted by $\mathbb{C}_-$ and $\mathbb{C}_+$, respectively.

2 Linear Systems with Generalized Frequency Variable

2.1 System representations

In this section, we define linear systems with generalized frequency variables and provide their dynamical equations in the frequency and time domains. Specifically, consider the linear time-invariant system described by the transfer function

$$\mathcal{G}(s) = C\left(\frac{1}{h(s)}I_n - A\right)^{-1} B + D = \mathcal{F}_u\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, h(s)I_n\right), \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, $h(s)$ is a single-input single-output, νth-order, strictly proper transfer function, and $\mathcal{F}_u$ denotes the upper linear fractional transformation. The system $\mathcal{G}(s)$ can be viewed as an interconnection of n identical agents, each of which has the internal dynamics $h(s)$. As shown in Figure 1, the interconnection structure is specified by A, and the input-output structure for the whole system is specified by B, C, and D. Defining the transfer function

$$G(s) = C(sI_n - A)^{-1}B + D, \tag{2}$$

the system can be described as

$$\mathcal{G}(s) = G(\phi(s)), \quad \phi(s) := 1/h(s). \tag{3}$$

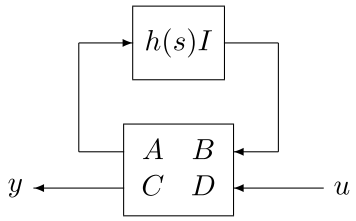

Figure 1: LFT representation of $\mathcal{G}(s)$

Note that the variable s in (2) characterizes frequency properties of the transfer function $G(s)$ and that $\mathcal{G}(s)$ is generated by simply replacing s by $\phi(s)$ in G. Hence, we say that the system (1) is described by the transfer function G with the *generalized frequency variable* $\phi(s)$ [3, 4].

Let $h(s)$ have a minimal realization $h(s) \sim (A_h, b_h, c_h, 0)$ where $A_h \in \mathbb{R}^{\nu\times\nu}$, $b_h \in \mathbb{R}^{\nu}$, $c_h \in \mathbb{R}^{1\times\nu}$. It can be shown [3, 4] that a realization of $\mathcal{G}(s)$ is given by $\mathcal{G}(s) \sim (\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ where

$$\begin{aligned} &\mathcal{A} = I_n \otimes A_h + A \otimes (b_h c_h) \in \mathbb{R}^{n\nu\times n\nu}, \\ &\mathcal{B} = B \otimes b_h \in \mathbb{R}^{n\nu\times m},\ \mathcal{C} = C \otimes c_h \in \mathbb{R}^{p\times n\nu}, \\ &\mathcal{D} = D \in \mathbb{R}^{p\times m}. \end{aligned} \tag{4}$$

2.2 Stability conditions

It should be first noticed that $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ is a minimal realization if (A, B, C, D) and $(A_h, b_h, c_h, 0)$ are both minimal realizations [3, 4]. Hence, the linear time-invariant system with the generalized frequency variable $\mathcal{G}(s)$ given by (3) is stable (all the poles of $\mathcal{G}(s)$ are in $\mathbb{C}_-$), if and only if the feedback system $\Sigma(h(s), A)$ shown Fig 2 is internally stable. This condition is in turn equivalent to stability of

$$\mathcal{H}_A(s) := \left(\frac{1}{h(s)}I - A\right)^{-1} = (\phi(s)I - A)^{-1}, \tag{5}$$

or $\mathcal{H}_A(s)$ is proper and analytic in the closed right half complex plane.

In other words, we can check the stability of an LTI system with generalized frequency variable $\phi(s) = 1/h(s)$ from the pair $(A, h(s))$, and we have the following theorem [12].

Theorem 1. *Let a matrix $A \in \mathbb{R}^{n\times n}$, and a strictly proper rational function $h(s) = n(s)/d(s)$ be given and define $\mathcal{H}_A(s)$ by* (5) *and $p(\lambda, s)$ by*

$$p(\lambda, s) := d(s) - \lambda n(s), \tag{6}$$

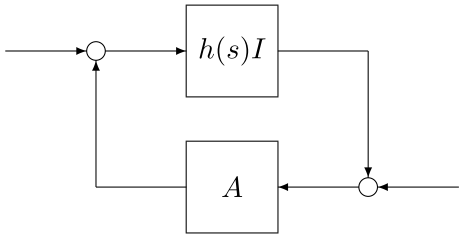

Figure 2: Feedback system $\Sigma(h(s), A)$

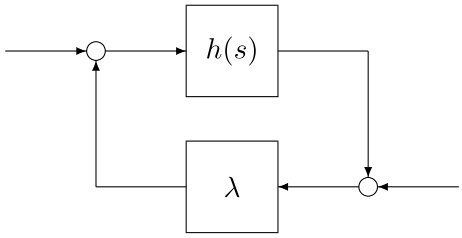

Figure 3: Feedback system $\Sigma(h(s), \lambda)$

respectively. Suppose that $n(s)$ and $d(s)$ are coprime. The following five statements are equivalent, where the positive integer $\ell_k \in \mathbb{N}$ and $\Phi_k \in \mathbf{S}_{\ell_k}$ for $k = 1, 2, \ldots, \nu$ are specified by applying a Hurwitz-type stability test for polynomials with complex coefficients in [5] to the corresponding closed-loop characteristic polynomial $p(\lambda, s)$:

(i) $\mathcal{H}_A(s)$ *is stable.*

(ii) $\sigma(A) \subset \Lambda := \{ \lambda \in \mathbb{C} \mid p(\lambda, s)$ *is Hurwitz* $\}$.

(iii) For all $\lambda \in \sigma(A)$, all the eigenvalues of $A_h + \lambda b_h c_h$ belong to the open left-half complex plane.

(iv) $\sigma(A) \subset \bigcap_{k=1}^{\nu} \Sigma_k$, *where* $\Sigma_k := \{ \lambda \in \mathbb{C} \mid l_{\ell_k}(\lambda)^* \Phi_k l_{\ell_k}(\lambda) > 0 \}$.

(v) For each $k = 1, 2, \ldots, \nu$, there exists $X_k \in \mathbf{S}_n^+$ such that

$$L_{\ell_k}(A)^T (\Phi_k \otimes X_k) L_{\ell_k}(A) > 0. \tag{7}$$

The equivalence among (i), (ii), (iv), and (v) was proved in [12], Condition (ii) gives us an algebraic condition so that all the eigenvalues of A should belong to guarantee the stability of the total system. Condition (v) provides an LMI feasibility problem, where we need no prior computation of the set of all eigenvalues of A. Condition (iii) which is clearly equivalent to that of (ii) will be used for the the $\mathcal{H}_2$-norm computation in the next section.

3 $\mathcal{H}_2$-norm Computation

This section is devoted to the $\mathcal{H}_2$ norm computation for $\mathcal{G}(s)$, where we assume that $D = O$ to assure the boundedness of the norm and that A is diagonalizable for the notational simplicity.

3.1 General case

The following theorem can be derived for the $\mathcal{H}_2$ norm computation.

Theorem 2. *For a given stable $\mathcal{G}(s)$ with $D = O$ in (1). We assume that A is diagonalizable and that A is represented by $A = T\Lambda T^{-1}$ with a non-singular matrix T and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Then, we have*

$$\|\mathcal{G}\|_2^2 = \mathrm{tr}\Big[\big(\Theta \otimes (c_h^* c_h)\big)P\Big] = \mathrm{tr}\Big[\big(\Pi \otimes (b_h b_h^*)\big)Q\Big], \tag{8}$$

where

$$\Pi := T^{-1}BB^*(T^{-1})^*, \quad \Theta := T^* C^* C T,$$

and the (i,j) block of P, Q for $i = 1, 2, \ldots, n,\ j = i, i+1, \ldots, n$ denoted by $P_{ij}, Q_{ij} \in \mathbb{C}^{\nu\times\nu}, (i = 1, 2, \ldots, n, j = 1, 2, \ldots, n)$ are the unique solutions of the following Sylvester equations:

$$\hat{A}_i P_{ij} + P_{ij}\hat{A}_j^* = -\pi_{ij} b_h b_h^* \tag{9}$$

$$\hat{A}_i^* Q_{ij} + Q_{ij}\hat{A}_j = -\theta_{ij} c_h^* c_h, \tag{10}$$

where $\hat{A}_i := A_h + \lambda_i b_h c_h$, and π_{ij}, θ_{ij} are the (i,j) elements of Π and Θ, respectively.

Proof. We can readily see that (9) can be rewritten as

$$(I_n \otimes A_h + \Lambda \otimes (b_h c_h))P + P(I_n \otimes A_h + \Lambda \otimes (b_h c_h))^* = -\Pi \otimes (b_h b_h^*).$$

Multiplications of $(T \otimes I_\nu)$ from left and $(T \otimes I_\nu)^*$ from right to the above equation yield

$$\begin{aligned}
&(T \otimes I_\nu)(I_n \otimes A_h + \Lambda \otimes (b_h c_h))P(T \otimes I_\nu)^* \\
&+ (T \otimes I_\nu)P(I_n \otimes A_h + \Lambda \otimes (b_h c_h))^*(T \otimes I_\nu)^* \\
&= -(T \otimes I_\nu)\Pi \otimes (b_h b_h^*)(T \otimes I_\nu)^*.
\end{aligned}$$

Using $A = T\Lambda T^{-1}$ and $\Pi = T^{-1}BB^*(T^{-1})^*$, we have

$$\begin{aligned}
&(I_n \otimes A_h + A \otimes (b_h c_h))(T \otimes I_\nu)P(T \otimes I_\nu)^* \\
&+ (T \otimes I_\nu)P(T \otimes I_\nu)^*(I_n \otimes A_h + A \otimes (b_h c_h))^* = -(B \otimes b_h)(B \otimes b_h)^*,
\end{aligned}$$

or equivalently

$$\mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A}^* = -\mathcal{B}\mathcal{B}^*, \tag{11}$$

where $\mathcal{P} := (T \otimes I_\nu)P(T \otimes I_\nu)^*$ is the controllability gramian of the system given by (4). Since the system $\mathcal{G}(s)$ is stable, we have

$$\begin{aligned}
\|\mathcal{G}\|_2^2 &= \mathrm{tr}(\mathcal{C}\mathcal{P}\mathcal{C}^*) \\
&= \mathrm{tr}\Big[(C \otimes c_h)(T \otimes I_\nu)P(T \otimes I_\nu)^*(C \otimes c_h)^*\Big] \\
&= \mathrm{tr}\Big[\big((CT) \otimes c_h\big)P\big((T^*C^*) \otimes c_h^*\big)\Big] \\
&= \mathrm{tr}\Big[\big((T^*C^*CT) \otimes (c_h^*c_h)\big)P\Big] \\
&= \mathrm{tr}\Big[\big(\Theta \otimes (c_h^*c_h)\big)P\Big].
\end{aligned}$$

This completes the proof, since the dual version can be proved in a completely similar manner. □

3.2 Special cases

If we restrict the class of systems, we have the more compact results.

Corollary 1. *For a given stable $\mathcal{G}(s)$ with $D = O$ in (1). We assume that A is a normal matrix* [1] *which is represented by $A = T\Lambda T^{-1}$ with a unitary matrix T and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ and that $B = I$ or $C = I$. Then, we have*

$$\begin{aligned}
\|\mathcal{G}\|_2^2 &= \mathrm{tr}\Big[\big((T^*C^*CT) \otimes (c_h^*c_h)\big)P\Big] \quad \textit{(when } B = I\textit{)} &(12)\\
&= \mathrm{tr}\Big[\big((T^*BB^*T) \otimes (b_h b_h^*)\big)Q\Big], \quad \textit{(when } C = I\textit{)} &(13)
\end{aligned}$$

where P and Q are the block diagonal matrices defined by $P = \mathrm{diag}(P_1,\ P_2,\ \dots,\ P_n)$, $P_i \in \mathbb{C}^{\nu\times\nu}, (i = 1, 2, \dots, n)$ and $Q = \mathrm{diag}(Q_1, Q_2, \dots, Q_n), Q_i \in \mathbb{C}^{\nu\times\nu}, (i = 1, 2, \dots, n)$), with P_i and Q_i $(i = 1, 2, \dots, n)$ being the unique solutions of Lyapunov equations

$$\hat{A}_i P_i + P_i \hat{A}_i^* = -b_h b_h^* \tag{14}$$

$$\hat{A}_i^* Q_i + Q_i \hat{A}_i = -c_h^* c_h, \tag{15}$$

where $\hat{A}_i := A_h + \lambda_i b_h c_h$.

[1] A matrix A can be diagonalized by a unitary matrix if and only if it is normal. The class of normal matrices include Hermitian, skew-Hermitian, unitary, and circulant matrices.

The proof is straightforward from Theorem 2 using the facts T is unitary and $\Pi = I$ or $\Theta = I$.

Corollary 1 exploits the normal structure of the interconnection matrix A to reduce the computational complexity. We only need to solve n independent Lyapunov equations with size $\nu \times \nu$ for the $\mathcal{H}_2$ norm computation.

Example 1. We consider the system (1) with

$$h(s) = \frac{1}{s^2 + s + 1}, \; A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix}, \; B = I_4, \; C = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}^{\mathsf{T}}.$$

Note that A is a normal matrix and that $\mathcal{G}(s)$ is stable. Since $B = I_4$, we can apply Corollary 1 to compute the $\mathcal{H}_2$ norm. For this example, we have

$$\mathrm{tr}\,[(T^* C^* C T \otimes c_h^* c_h) P] = 2.0000,$$

which can be confirmed by the MATLAB computation for $\mathcal{G}(s)$.

Another corollary which corresponds to the results in [8, 7] is given as follows, where we assume the following.

Assumption 1. *A is a normal matrix, i.e., $AA^* = A^*A$, $B = I_n$, $C = I_n$, and $D = O_n$.*

The proof is omitted, since it is straightforward.

Corollary 2. *For a given stable $\mathcal{G}(s)$ in (1) which satisfies Assumption 1. Then, we have*

$$\|\mathcal{G}\|_2^2 = \sum_{i=1}^{n} (c_h P_i c_h^*) = \sum_{i=1}^{n} (b_h^* Q_i b_h), \tag{16}$$

where $P_i, Q_i \in \mathbb{C}^{\nu \times \nu}, (i = 1, 2, \ldots, n)$ are the unique solutions of Lyapunov equations (14).

4 $\mathcal{H}_\infty$-norm for a simple feedback system

Consider a class of LTI systems with generalized frequency variables $\mathcal{G}(s)$ represented by (1) which satisfy Assumption 1. In other words, this section focuses on the condition for the H_∞ norm of

$$\mathcal{G}(s) = \left(\frac{1}{h(s)} I_n - A \right)^{-1} = (I_n - h(s)A)^{-1} h(s), \tag{17}$$

or the condition of $\|\mathcal{G}\|_\infty < \gamma$ under Assumption 1.

4.1 $\mathcal{H}_\infty$ norm conditions

The following theorem provides two exact conditions for the $\mathcal{H}_\infty$ norm under Assumption 1.

Theorem 3. *For a given positive number $\gamma > 0$ and an LTI system with generalized frequency variable $\mathcal{G}(s)$ represented by (17), which satisfies Assumption 1, the following statements are equivalent.*

(i) $\|\mathcal{G}\|_\infty < \gamma$

(ii) For all $\lambda \in \sigma(A)$,

$$\left\| \frac{h}{1 - \lambda h} \right\|_\infty < \gamma. \tag{18}$$

(iii) For all $\lambda \in \sigma(A)$ and

$$\phi \in \mathbf{\Phi} := \{1/h(j\omega) : \omega \in \mathbb{R}\}, \tag{19}$$

$$\left| \frac{1}{\phi - \lambda} \right| < \gamma. \tag{20}$$

Proof. Suppose A admits a decomposition of the form $A = T^{-1}\Lambda T$ where T is nonsingular and Λ is diagonal. Then

$$\mathcal{G}(s) = T^{-1}H(s)T, \quad H(s) := (I - h(s)\Lambda)^{-1}h(s).$$

Note that

$$\|G_\omega\| < \gamma \quad \Leftrightarrow \quad H_\omega X H_\omega^* < \gamma^2 X, \quad X := TT^*,$$

where $G_\omega := \mathcal{G}(j\omega)$ and H_ω is similarly defined. If A is normal, then T is unitary, or $X = I$, and we have

$$\begin{aligned} \|G_\omega\| < \gamma \quad &\Leftrightarrow \quad H_\omega H_\omega^* < \gamma^2 I, \\ &\Leftrightarrow \quad \left| \frac{1}{\phi_\omega - \lambda_i} \right| < \gamma, \quad \forall\, i = 1, \dots, n \end{aligned}$$

where λ_i is the i^{th} diagonal entry of Λ and $\phi_\omega := 1/h(j\omega)$. Hence, $\|\mathcal{G}\|_\infty < \gamma$ holds if and only if condition (ii) holds, or equivalently condition (iii) holds. □

Thus, the H_∞ norm calculation of the system (1) can be decomposed into that of n subsystems with a complex coefficient if A is normal.

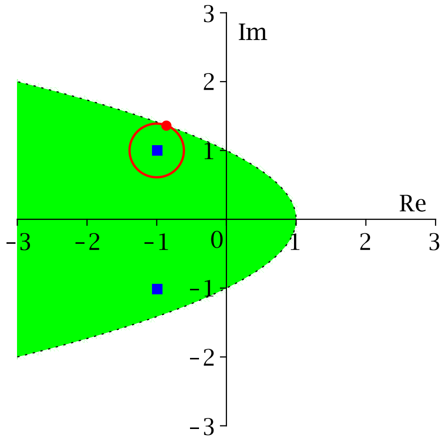

Figure 4: Distance between eigenvalue of A and Locus of $\phi(j\omega)$

Example 2. Consider a system $\mathcal{G}(s)$ with

$$h(s) = \frac{1}{s^2+s+1}, \quad A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

Note that A above is a normal matrix. The blue points in Fig. 4 represent the eigenvalues of A which are $-1 \pm j$. We see that the distance to the locus $\phi(j\omega) = 1/h(j\omega)$, or the radius of red circle, is $r = \sqrt{11 - 6\sqrt{3}}/2$. Hence, from condition (iii) in Theorem 3, we have $\|\mathcal{G}(s)\|_\infty = 1/r \approx 2.5656$, which can be confirmed by a direct application of MATLAB computation for $\mathcal{G}(s)$.

A bunch of disks in Figs. 5 and 6 illustrate the regions from which the distance to the locus $\phi(j\omega) = 1/h(j\omega)$ are smaller than $1/\gamma$ with $\gamma = 1, 2$, respectively. Note that the blue shaded areas are the stability regions derived from condition (iv) in Theorem 1. Therefore, we can see from condition (iii) in Theorem 3 that the region for allocatable eigenvalues of A to guarantee the stability of $\mathcal{G}(s)$ and the H_∞ norm constraint $\|\mathcal{G}\|_\infty < \gamma$ is the intersection of the blue shaded region and the outside of all the disks.

4.2 Robust stability condition

Since the H_∞ norm constraint corresponds to the allowable uncertainty bound in the context of robust control, the H_∞ norm condition for $\mathcal{G}(s)$, or $\|\mathcal{G}\|_\infty < \gamma$, investigated in the previous subsection relates a robust stability condition. To this end, we consider a feedback loop system with uncertainty $\Delta(s)$ depicted in Fig. 7. It is easy to see that the transfer function from $\mathbf{z}$

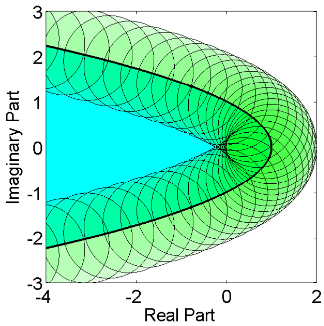

Figure 5: $\gamma = 1$

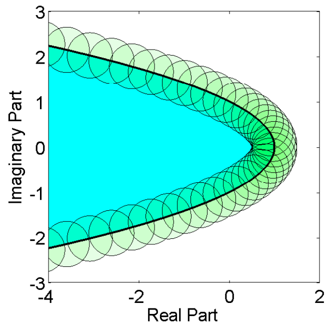

Figure 6: $\gamma = 2$

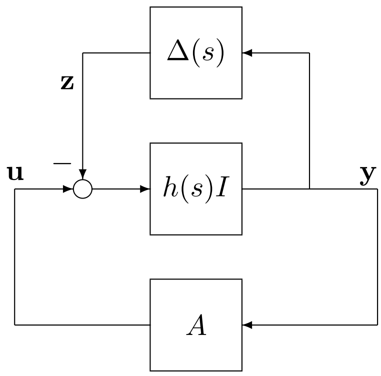

Figure 7: Feedback system with uncertainties of $h(s)I$

to $\mathbf{y}$ in Fig. 7 is $\mathcal{G}(s)$. Hence, if $\Delta(s)$ belongs to

$$\boldsymbol{\Delta}_\gamma := \{\ \Delta(s) \mid \text{proper \& stable},\ \|\Delta\|_\infty \leq \frac{1}{\gamma}\ \}, \tag{21}$$

then $\mathcal{G}(s)$ satisfying $\|\mathcal{G}\|_\infty < \gamma$ is robustly stable under any perturbation $\Delta(s) \in \boldsymbol{\Delta}_\gamma$. This is a natural consequence of the small gain theorem. Note that as seen in Fig. 7 the corresponding class of perturbations to the nominal system $h(s)I$ is the feedback type, and the class of perturbed system without inter connection matrix A is given by

$$\mathbf{H}_\gamma := \{\ (I + h(s)\Delta(s))^{-1} h(s) \mid \Delta(s) \in \boldsymbol{\Delta}_\gamma\ \}. \tag{22}$$

In other words, the result in the previous subsection for the H_∞ norm computation can be applied to the allowable uncertainty bound of feedback type to assure the feedback stability. This is a powerful tool for examination of the stability of heterogeneous multi-agent dynamical systems.

It should be noticed that the block diagram in Fig. 7 can be equivalently translated into another block diagram shown in Fig. 8, where $\phi(s) = 1/h(s)$

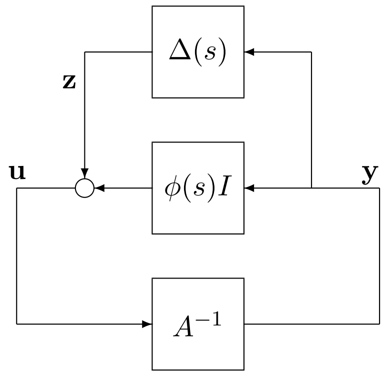

Figure 8: Feedback system with uncertainties of $\phi(s)I$

and A^{-1} are used instead of $h(s)$ and A. In this case, the perturbed system has an additive type perturbation represented by $\phi(s)I+\Delta(s)$. The diagonal perturbation was investigated in [10]. We will show the detailed comparison in the final paper.

4.3 H_∞ norm computation

This subsection provides three different ways of H_∞ norm computation based on condition (ii) in Theorem 3, where we use the γ iteration to get the precise value.

4.3.1 Boundedreal lemma (LMI)

The H_∞ norm condition in (18) is satisfied if and only if

$$\begin{bmatrix} h_\omega \\ 1-\lambda h_\omega \end{bmatrix}^* \begin{bmatrix} 1 & 0 \\ 0 & -\gamma^2 \end{bmatrix} \begin{bmatrix} h_\omega \\ 1-\lambda h_\omega \end{bmatrix} < 0.$$

holds $\forall\ \omega \in \mathbb{R}$. Noting that

$$\begin{bmatrix} h(s) \\ 1-\lambda h(s) \end{bmatrix} = \begin{bmatrix} c_h & 0 \\ -\lambda c_h & 1 \end{bmatrix} \begin{bmatrix} (sI-A_h)^{-1}b_h \\ 1 \end{bmatrix},$$

one can verify that (18) holds if and only if, for each $\lambda \in \sigma(A)$, there exists a Hermitian matrix P such that

$$\begin{bmatrix} A_h & b_h \\ I & 0 \end{bmatrix}^* \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} A_h & b_h \\ I & 0 \end{bmatrix} + \begin{bmatrix} c_h & 0 \\ -\lambda c_h & I \end{bmatrix}^* \begin{bmatrix} 1 & 0 \\ 0 & -\gamma^2 \end{bmatrix} \begin{bmatrix} c_h & 0 \\ -\lambda c_h & I \end{bmatrix} < 0 \tag{23}$$

Hence, (18) holds if and only if the set of $\lambda \in \mathbb{C}$ satisfying this condition for some Hermitian P contains $\sigma(A)$. Note that the dimension of P is $\nu \times \nu$ where ν is the order of $h(s)$.

4.3.2 Hamiltonian matrix

The transfer function in (18) has the following state space realization

$$\frac{h(s)}{1-\lambda h(s)} = c_h(sI - A_h - \lambda b_h c_h)^{-1} b_h.$$

The H_∞ norm of this system is less than γ if and only if

$$\det(j\omega I - H_\lambda) \neq 0, \quad \forall\, \omega \in \mathbb{R},$$

$$H_\lambda := \begin{bmatrix} A_h + \lambda b_h c_h & b_h b_h^\mathsf{T}/\gamma^2 \\ -c_h^\mathsf{T} c_h & -(A_h + \lambda b_h c_h)^\mathsf{T} \end{bmatrix}. \tag{24}$$

The frequency variable ω can be removed from this condition via the quantifier elimination (QE), and the resulting equivalent condition is given by polynomial inequalities in terms of λ and $\bar{\lambda}$.

Another way of using the QE technique to check the H_∞ norm condition is as follows. The set of ϕ satisfying $|\phi - \lambda| = r$ can be parametrized as

$$\phi = \lambda + r \cdot \frac{1 - j\omega}{1 + j\omega}, \quad \omega \in \mathbb{R}.$$

Note that the set $\mathbf{\Phi}$ defined in (19) can be characterized by a polynomial equation through the Euclid's algorithm in the following form:

$$\mathbf{\Phi} = \{\phi \in \mathbb{C} \mid l(\phi)^* \Pi l(\phi) = 0\}, \quad l(\phi) := \begin{bmatrix} 1 & \phi & \phi^2 & \ldots & \phi^\nu \end{bmatrix}^\mathsf{T}.$$

Substituting this expression for ϕ, one can define a polynomial of ω with coefficients polynomially dependent on λ and $\bar{\lambda}$:

$$p(\lambda, \omega) := \tau l(\phi)^* \Pi l(\phi)(1 + \omega^2)^\nu$$

Then, in principle, it is possible to find a necessary and sufficient condition for $p(\lambda, \omega)$ to be positive for all $\omega \in \mathbb{R}$, and the resulting condition will be given in terms of multiple polynomial inequalities in λ and $\bar{\lambda}$.

4.3.3 Polynomial KYP lemma

Condition (20) holds if and only if the curve $\mathbf{\Phi}$ does not intersect with the set of points on or inside of the circle of radius $r := 1/\gamma$ centered at λ on the complex plane. This is further equivalent to the sign of $l(\phi)\Pi l(\phi)$ being constant (positive or negative) on the boundary of the circular region:

$$\tau l(\phi)^* \Pi l(\phi) > 0 \quad \text{for all } \phi \text{ such that } |\phi - \lambda| = r,$$

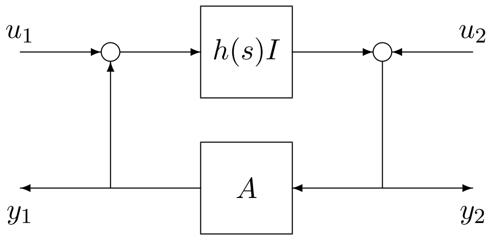

Figure 9: Feedback system for Loop Shaping

where $\tau = 1$ or -1, because $\mathbf{\Phi}$ cannot be contained in the circle due to $h(\infty) = 0$. The points ϕ on the circle $|\phi - \lambda| = r$ can be characterized by

$$\begin{bmatrix} \phi \\ 1 \end{bmatrix}^* \Phi \begin{bmatrix} \phi \\ 1 \end{bmatrix} = 0, \quad \Phi := \begin{bmatrix} 1 & -\lambda \\ -\bar{\lambda} & |\lambda|^2 - r \end{bmatrix}.$$

Then the generalized KYP lemma asserts that condition (20) holds if and only if, for each of $\tau = 1$ and -1, there exists a Hermitian matrix P such that

$$\tau\Pi > \begin{bmatrix} U \\ V \end{bmatrix}^* \begin{bmatrix} P & -\lambda P \\ -\bar{\lambda} P & (|\lambda|^2 - r)P \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix},$$

$$U := \begin{bmatrix} 0 & I_\nu \end{bmatrix}, \quad V := \begin{bmatrix} I_\nu & 0 \end{bmatrix}.$$

Note that the dimension of P is $\nu \times \nu$.

5 Loop Shaping Type H_∞ Norm

Consider the feedback system in Fig. 9, where A is a normal matrix. The closed-loop transfer function $\mathcal{L}(s)$ is given by

$$\mathcal{L}(s) := \begin{bmatrix} A \\ I \end{bmatrix} (I - h(s)A)^{-1} \begin{bmatrix} h(s)I & I \end{bmatrix}. \tag{25}$$

The type of feedback system is used for the H_∞ loop shaping design, It is one of typical setting in robust control, since it relates the normalized coprime factor perturbations and it includes both the sensitivity and complementarity sensitivity functions. Our purpose of this section is to provide several ways of checking the H_∞ norm condition $\|\mathcal{L}\|_\infty < \gamma$.

5.1 Graphical test

Let the spectral decomposition of A be given by $A = T^*\Lambda T$ where T is unitary and Λ is diagonal. Then, we have

$$\mathcal{L}(s) = \begin{bmatrix} T^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Lambda \\ I \end{bmatrix} (I - h(s)\Lambda)^{-1} \begin{bmatrix} h(s)I & I \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix},$$

and it follows that

$$\|\mathcal{L}\|_\infty < \gamma \tag{26}$$

holds if and only if

$$\left\| \begin{bmatrix} \lambda \\ 1 \end{bmatrix} (1 - h\lambda)^{-1} \begin{bmatrix} h & 1 \end{bmatrix} \right\|_\infty < \gamma, \tag{27}$$

holds for all $\lambda \in \sigma(A)$.

We can derive the following theorem based on condition (27).

Theorem 4. *Consider a given positive number $\gamma > 0$ and $\mathcal{L}(s)$ represented by (25). We assume that A is a normal matrix. Then, the following statements are equivalent.*

(i) $\|\mathcal{L}\|_\infty < \gamma$

(ii) For all $\lambda \in \sigma(A)$,

$$\frac{(1+|\lambda|^2)(1+|\phi|^2)}{|\phi - \lambda|^2} < \gamma^2, \quad \forall\, \phi \in \mathbf{\Phi}, \tag{28}$$

where $\mathbf{\Phi}$ is defined in (19).

(iii) For all $\lambda \in \sigma(A)$,

$$\begin{cases} (1-\alpha)(|\lambda - \lambda_\phi|^2 - r_\phi^2) > 0, & (\text{if } \alpha \neq 1) \\ 1 + |\lambda|^2 < |\phi - \lambda|^2, & (\text{if } \alpha = 1), \end{cases},$$

where

$$\alpha := \frac{1+|\phi|^2}{\gamma^2},$$

and

$$\lambda_\phi := \frac{\phi}{1-\alpha}, \quad r_\phi := \sqrt{\frac{\alpha}{1-\alpha}\left(\frac{|\phi|^2}{1-\alpha} + 1\right)}.$$

Proof. It is easy to show that condition (27) is equivalent to condition (ii). The equivalence of conditions (ii) and (iii) can be shown as follows. When $\alpha < 1$, the corresponding inequality condition holds if and only if λ is outside of the circle of radius r_ϕ centered at λ_ϕ. When $\alpha > 1$, the inequality holds if and only if λ is inside of the circle. It can be shown using $\gamma > 1$ that the radius r_ϕ is always well defined (i.e., real positive) unless $\alpha = 1$. When $\alpha = 1$, the set of λ satisfying the corresponding inequality is the half plane not containing ϕ with the boundary specified as the straight line, orthogonal to the line connecting the origin and ϕ, passing through the point $\beta\phi$ with $\beta := (1 - 1/|\phi|^2)/2$. □

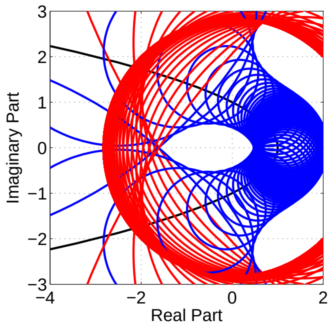

Figure 10: $\gamma = 3$

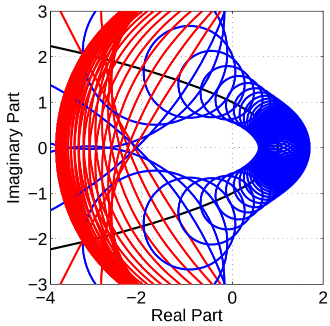

Figure 11: $\gamma = 4$

Condition (iii) in the theorem gives a graphical test for the H_∞ constraints as seen in the following numerical example.

Example 3. Consider the same system as in Example 2, or $h(s) = \frac{1}{s^2+s+1}$. Figs. 10 and 11 respectively illustrate the regions for $\gamma = 3, 4$ based on the condition in Theorem 4. The blue circles and the red circles represent the conditions for $\alpha < 1$ and $\alpha > 1$ respectively in these figures. In other words, $\|\mathcal{L}(s)\|_\infty < \gamma$ holds if and only if all the eigenvalues of A lie outside of the blue circles and inside of the red circles. Thus, $\|\mathcal{G}(s)\|_\infty < \gamma$ holds if and only if all the eigenvalues of A are in the white regions including the origin.

5.2 H_∞ norm computation

Similar to the simple feedback case discussed in the previous section we have three different ways of the norm computation.

5.2.1 Standard KYP lemma

Assuming stability, the H_∞ norm condition in (27) is satisfied if and only if

$$(1 + |\lambda|^2)(1 + |h_\omega|^2) < \gamma^2 |1 - \lambda h_\omega|^2,$$

or equivalently,

$$\begin{bmatrix} h_\omega \\ 1 \end{bmatrix}^* \begin{bmatrix} 1 + (1-\gamma^2)|\lambda|^2 & \gamma^2 \bar{\lambda} \\ \gamma^2 \lambda & 1 + |\lambda|^2 - \gamma^2 \end{bmatrix} \begin{bmatrix} h_\omega \\ 1 \end{bmatrix} < 0 \qquad (29)$$

hold for all $\omega \in \mathbb{R}$, where

$$h_\omega := h(j\omega), \quad h(s) := C(sI - A)^{-1}B.$$

By the standard KYP lemma, this condition holds if and only if there exists a Hermitian matrix P such that

$$\begin{bmatrix} PA + A^*P & PB \\ B^*P & 0 \end{bmatrix} + \begin{bmatrix} (1 + (1-\gamma^2)|\lambda|^2)C^*C & \gamma^2(\lambda C)^* \\ \gamma^2 \lambda C & 1 + |\lambda|^2 - \gamma^2 \end{bmatrix} < 0.$$

5.2.2 Hamiltonian matrix

Recall that the H_∞ norm of a stable transfer function $\mathsf{C}(sI - \mathsf{A})^{-1}\mathsf{B} + \mathsf{D}$ is strictly smaller than γ if and only if

$$\det(j\omega I - H) \neq 0, \quad \forall\, \omega \in \mathbb{R}, \tag{30}$$

where H is the Hamiltonian matrix defined by

$$H := \begin{bmatrix} \mathsf{A} + \mathsf{B}\mathsf{R}^{-1}\mathsf{D}^*\mathsf{C} & \mathsf{B}\mathsf{R}^{-1}\mathsf{B}^* \\ -\mathsf{C}^*(I + \mathsf{D}\mathsf{R}^{-1}\mathsf{D}^*)\mathsf{C} & -(\mathsf{A} + \mathsf{B}\mathsf{R}^{-1}\mathsf{D}^*\mathsf{C})^* \end{bmatrix},$$

$$\mathsf{R} := \gamma^2 I - \mathsf{D}^*\mathsf{D}.$$

Using the determinant formulas

$$\det(MN) = \det(M)\det(N),$$

$$\det \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \det(H)\det(E - FH^{-1}G), \quad (\text{when } \det(H) \neq 0),$$

condition (30) can be equivalently written as

$$\det \begin{bmatrix} j\omega I - \mathsf{A} & 0 & \mathsf{B} & 0 \\ 0 & j\omega I + \mathsf{A}^* & 0 & \mathsf{C}^* \\ 0 & \mathsf{B}^* & \gamma I & \mathsf{D}^* \\ -\mathsf{C} & 0 & \mathsf{D} & \gamma I \end{bmatrix} \neq 0, \quad \forall\, \omega \in \mathbb{R}.$$

Note that the transfer function in (27) has the following state space realization:

$$\begin{bmatrix} \lambda \\ 1 \end{bmatrix} (1 - h\lambda)^{-1} \begin{bmatrix} h & 1 \end{bmatrix} = \left(\begin{array}{c|cc} A_h + \lambda b_h c_h & b_h & \lambda b_h \\ \hline \lambda c_h & 0 & \lambda \\ c_h & 0 & 1 \end{array} \right),$$

Hence, assuming stability, the H_∞ norm condition in (27) is satisfied if and only if

$$\det \begin{bmatrix} j\omega I - A_h - \lambda b_h c_h & 0 & b_h & \lambda b_h & 0 & 0 \\ 0 & j\omega I + (A_h + \lambda b_h c_h)^* & 0 & 0 & (\lambda c_h)^* & c_h^* \\ 0 & b_h^* & \gamma & 0 & 0 & 0 \\ 0 & (\lambda b_h)^* & 0 & \gamma & \bar{\lambda} & 1 \\ -\lambda c_h & 0 & 0 & \lambda & \gamma & 0 \\ -C & 0 & 0 & 1 & 0 & \gamma \end{bmatrix} \neq 0,$$

$$\forall\, \omega \in \mathbb{R}.$$

5.2.3 Polynomial KYP lemma

From (29), condition (27) is satisfied if and only if

$$\eta(\phi) < 0, \quad \forall\, \phi \in \mathbf{\Phi}, \tag{31}$$

where

$$\eta(\phi) := \begin{bmatrix} \phi \\ 1 \end{bmatrix}^* \Phi(\lambda) \begin{bmatrix} \phi \\ 1 \end{bmatrix}, \quad \Phi(\lambda) := \begin{bmatrix} 1 + |\lambda|^2 - \gamma^2 & \gamma^2 \lambda \\ \gamma^2 \bar{\lambda} & 1 + (1-\gamma^2)|\lambda|^2 \end{bmatrix}.$$

Note from (27) that $\|G(\infty)\| < \gamma$ implies $1 + |\lambda|^2 < \gamma^2$ due to $h(\infty) = 0$. Hence, the set

$$\mathbf{\Sigma} := \{\ \phi \in \mathbb{C} \mid \sigma(\phi) \geq 0\ \},$$

is bounded. In fact, $\mathbf{\Sigma}$ is the set of points on or inside of the circle of radius r_λ centered at ϕ_λ where

$$\phi_\lambda := \frac{\lambda}{1-\beta}, \quad r_\lambda := \sqrt{\frac{\beta}{1-\beta}\left(\frac{|\lambda|^2}{1-\beta} + 1\right)}, \quad \beta := \frac{1+|\lambda|^2}{\gamma^2}.$$

Now, condition (31) holds if and only if the curve $\mathbf{\Phi}$ lies outside of $\mathbf{\Sigma}$, which holds if and only if the curve $\mathbf{\Phi}$ does not intersect with the boundary of $\mathbf{\Sigma}$ since $\mathbf{\Phi}$ is unbounded and it cannot be contained inside of $\mathbf{\Sigma}$. Therefore, (31) is equivalent to satisfaction of

$$\tau l(\phi)^* \Pi l(\phi) > 0 \text{ for all } \phi \text{ such that } \sigma(\phi) = 0$$

for $\tau = 1$ or -1. Then the generalized KYP lemma asserts that this condition holds if and only if, for each of $\tau = 1$ and -1, there exists a $\nu \times \nu$ Hermitian matrix P such that

$$\tau \Pi > \begin{bmatrix} U \\ V \end{bmatrix}^* (\Phi(\lambda) \otimes P) \begin{bmatrix} U \\ V \end{bmatrix}.$$

6 Conclusion

In this paper, we have considered linear time-invariant systems with generalized frequency variables $\phi(s)$, described as $C(\phi(s)I - A)^{-1}B + D$. Such systems arise from interconnections of multiple identical subsystems, where $h(s) := 1/\phi(s)$ is the common subsystem dynamics, and A is the connectivity matrix characterizing the information exchange among subsystems.

We have proposed efficient $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm computations based on the generalized frequency variable. We first derived a way of $\mathcal{H}_2$ norm computation from state-space realizations of subsystems. We then discussed a region on the complex plane specified by the generalized frequency variable for achieving the $\mathcal{H}_\infty$ norm bound for a simple feedback case, and a graphical test and three different numerical methods for computing the $\mathcal{H}_\infty$ norm were derived. The $\mathcal{H}_\infty$ norm constraint relates the robust stability condition for feedback perturbation, and the derived result is useful for stability analysis for a class of non-homogeneous multi-agent systems. Finally, we investigated the loop shaping type $\mathcal{H}_\infty$ norm, where we provided a graphical test for the condition and three different ways of computing the $\mathcal{H}_\infty$ norm.

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