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Abstract

A mixed matrix pencil is a matrix pencil which has two kinds of nonzero coefficients: fixed constants that account for conservation laws and independent parameters that represent physical characteristics. In this paper, we characterize the indices of nilpotency of the Kronecker canonical form for a mixed matrix pencil in terms of matroids. As a byproduct, we provide an algorithm for computing the rank of a power product of a square mixed matrix.

1 Introduction

A *matrix pencil* is a polynomial matrix in which the degree of each entry is at most one. We express a matrix pencil as $D(s) = sX + Y$ by a pair of constant matrices X and Y. A matrix pencil $D(s)$ can be brought into the *Kronecker canonical form* by equivalence transformations with constant nonsingular matrices. The Kronecker canonical form plays an important role in many applications such as control theory [3, 24] and differential-algebraic equations [13, 22]. Several numerical algorithms for computing it are available [1, 4, 5, 11, 25].

Matrix pencils arising in practice are often very sparse, and it is tempting to exploit the combinatorial structures. The Kronecker canonical form is a block diagonal matrix which consists of nilpotent blocks, rectangular blocks, and a residual square block. Among them, nilpotent blocks admit two combinatorial characterizations. The first one utilizes the highest degree of subdeterminants, which can be computed by combinatorial relaxation algorithms [9, 16]. The second characterization is based on the ranks of larger constant matrices, called *expanded matrices*. Under the genericity assumption that the set of nonzero coefficients is algebraically independent, it is shown in [10] that the rank of the expanded matrix coincides with the maximum weight of a matching in a bipartite graph, which can be computed efficiently by combinatorial algorithms.

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The genericity assumption is justified by the fact that physical characteristics in engineering systems are not precise in values because of noises. However, it is not always valid in practical situations. In fact, exact numbers do arise in dynamical systems such as in Kirchhoff's conservation laws in electric circuits, or in the law of conservation of mass, energy, or momentum and the principle of action and reaction in mechanical systems. This observation led Murota and Iri [21] to introduce the notion of a *mixed matrix*.

A mixed matrix is a constant matrix that consists of two kinds of numbers as follows.

Accurate Numbers (Fixed Constants) Numbers that account for conservation laws are precise in values. These numbers should be treated numerically.

Inaccurate Numbers (Independent Parameters) Numbers that represent physical characteristics are not precise in values. These numbers should be treated combinatorially as nonzero parameters without reference to their nominal values. Since each such nonzero entry often comes from a single physical device, the parameters are assumed to be independent.

In order to deal with dynamical systems, it is natural to consider the polynomial matrix version, which is called a *mixed polynomial matrix* [20]. In particular, the matrix pencil version is called a *mixed matrix pencil*.

For mixed polynomial matrices, Murota [19] showed that the computation of the highest degree of subdeterminants reduces to solving a *valuated independent assignment problem* [17, 18]. This enables us to determine nilpotent blocks for a mixed matrix pencil. Murota also investigated the Smith normal form [14, 15] and the Smith-McMillan form at infinity [19] of a mixed polynomial matrix in terms of the degree of subdeterminants. However, this approach based on the valuated matroid intersection has a drawback that it requires to deal with rational function matrices.

In this paper, we analyze the Kronecker canonical form of a mixed matrix pencil in terms of the ranks of expanded matrices. Extending the results in [10], we prove that the computation of the ranks of expanded matrices for mixed matrix pencils reduces to solving *independent matching problems*. This leads to an algorithm for determining nilpotent blocks of a mixed matrix pencil. An independent matching problem is equivalent to a matroid intersection problem, and in particular, a linear matroid intersection problem in this case, which has been studied in [2, 7, 8].

As a byproduct, we provide an algorithm for computing the rank of a power product A^k of a square mixed matrix A. In general, A^k is not a mixed matrix, because A^k has an independent parameter appearing multiple times. Therefore, we can not apply directly an algorithm for the rank of a mixed matrix [21]. Instead, we reduce the computation of the rank of A^k to solving an independent matching problem via the expanded matrix.

The preceding paper [10] provided combinatorial characterizations on the sizes of rectangular blocks under the genericity assumption. It remains open to extend this result to mixed matrix pencils.

The organization of this paper is as follows. In Section 2, we recapitulate the Kronecker canonical form and its relation to the ranks of expanded matrices. We provide some key lemmas concerning the rank of an expanded matrix in Section 3. Section 4 explains mixed matrix pencils. Section 5 describes independent matching problems and valuated independent assignment problems, which are useful in the proof of our main theorem. In Section 6, we prove that the rank of an expanded matrix for a mixed matrix pencil can be computed by solving an independent matching problem. In Section 7, we apply our approach to the computation of the rank of a power product of a square mixed matrix.

2 Kronecker Canonical Form of Matrix Pencils

Let $D(s) = sX + Y$ be an $m \times n$ matrix pencil with row set R and column set C. A matrix pencil $D(s)$ is said to be *regular* if $D(s)$ is square and $\det D(s) \neq 0$ as a polynomial in s. The rank of $D(s)$ is the maximum size of its submatrix that is a regular matrix pencil. A matrix pencil $\bar{D}(s)$ is said to be *strictly equivalent* to $D(s)$ if there exists a pair of nonsingular constant matrices F and H such that $\bar{D}(s) = FD(s)H$.

For a positive integer μ, we consider a $\mu \times \mu$ matrix pencil N_μ defined by

$$N_\mu = \begin{pmatrix} 1 & s & 0 & \cdots & 0 \\ 0 & 1 & s & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & s \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

For a positive integer ϵ, we further denote by L_ϵ an $\epsilon \times (\epsilon + 1)$ matrix pencil

$$L_\epsilon = \begin{pmatrix} s & 1 & 0 & \cdots & 0 \\ 0 & s & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & s & 1 \end{pmatrix}.$$

We also denote by $L_\eta^\top$ the transpose matrix of L_η.

Let us denote by block-diag$(D_1, \ldots, D_b)$ the block-diagonal matrix pencil with diagonal blocks $D_1, \ldots, D_b$. A matrix pencil is known to be strictly equivalent to its Kronecker canonical form as follows.

Theorem 2.1. By a strict equivalence transformation, a matrix pencil $D(s)$ can be brought into its Kronecker canonical form $\bar{D}(s)$ with

$$\bar{D}(s) = \text{block-diag}(sI_\nu + J_\nu, N_{\mu_1}, \ldots, N_{\mu_d}, L_{\epsilon_1}, \ldots, L_{\epsilon_p}, L_{\eta_1}^\top, \ldots, L_{\eta_q}^\top, O),$$

where

$$\mu_1 \geq \cdots \geq \mu_d > 0, \quad \epsilon_1 \geq \cdots \geq \epsilon_p > 0, \quad \eta_1 \geq \cdots \geq \eta_q > 0,$$

I_ν is a $\nu \times \nu$ identity matrix, and J_ν is a $\nu \times \nu$ constant matrix. The numbers ν, d, p, q, $\mu_1, \ldots, \mu_d$, $\epsilon_1, \ldots, \epsilon_p$, $\eta_1, \ldots, \eta_q$ are uniquely determined.

The matrices $N_{\mu_1}, \ldots, N_{\mu_d}$ are called the *nilpotent blocks*, and the numbers $\mu_1, \ldots, \mu_d$ are called the *indices of nilpotency*. The numbers $\epsilon_1, \ldots, \epsilon_p$ and $\eta_1, \ldots, \eta_q$ are the *minimal column indices* and *minimal row indices*, respectively. The numbers $(\nu, \mu_1, \ldots, \mu_d, \epsilon_1, \ldots, \epsilon_p, \eta_1, \ldots, \eta_q)$ are called the *structural indices* of $D(s)$. For the rank r of $D(s)$, it holds that

$$r = \nu + \sum_{i=1}^{d} \mu_i + \sum_{i=1}^{p} \epsilon_i + \sum_{i=1}^{q} \eta_i. \tag{1}$$

For an $m \times n$ matrix pencil $D(s) = sX + Y$, we consider a $km \times kn$ matrix $\Theta_k(D)$ defined by

$$\Theta_k(D) = \begin{pmatrix} X & O & \cdots & \cdots & O \\ Y & X & \ddots & & \vdots \\ O & Y & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & X & O \\ O & \cdots & O & Y & X \end{pmatrix}.$$

We also construct a $(k+1)m \times kn$ matrix $\Psi_k(D)$ and a $km \times (k+1)n$ matrix $\Phi_k(D)$ defined by

$$\Psi_k(D) = \begin{pmatrix} X & O & \cdots & O \\ Y & X & \ddots & \vdots \\ O & Y & \ddots & O \\ \vdots & \ddots & \ddots & X \\ O & \cdots & O & Y \end{pmatrix} \quad \text{and} \quad \Phi_k(D) = \begin{pmatrix} X & Y & O & \cdots & O \\ O & X & Y & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & O \\ O & \cdots & O & X & Y \end{pmatrix}.$$

The rank of each expanded matrix is denoted by

$$\theta_k(D) = \operatorname{rank} \Theta_k(D), \quad \psi_k(D) = \operatorname{rank} \Psi_k(D), \quad \varphi_k(D) = \operatorname{rank} \Phi_k(D).$$

The following theorem shows a close relationship between the ranks of the expanded matrices and the structural indices.

Theorem 2.2 ([10, Theorem 2.3])**.** Let $D(s)$ be a matrix pencil of rank r with the structural indices $(\nu, \mu_1, \ldots, \mu_d, \epsilon_1, \ldots, \epsilon_p, \eta_1, \ldots, \eta_q)$. Then we have

$$\begin{aligned} \theta_k(D) &= rk - \sum_{i=1}^{d} \min\{k, \mu_i\}, \\ \psi_k(D) &= rk + \sum_{i=1}^{p} \min\{k, \epsilon_i\}, \\ \varphi_k(D) &= rk + \sum_{i=1}^{q} \min\{k, \eta_i\}. \end{aligned}$$

By Theorem 2.2, the ranks of the expanded matrices determine the indices. In this paper, we analyze $\theta_k(D)$ for a mixed matrix pencil $D(s)$ in order to obtain the indices of nilpotency $\mu_1, \ldots, \mu_d$.

3 Maximum Degree of Subdeterminants

In this section, we provide two key lemmas which are useful for the proof of our results. For a polynomial matrix $Z(s)$, let $\delta(Z)$ denote the highest degree of subdeterminants, i.e.,

$$\delta(Z) = \max\{\deg\det Z[I,J] \mid |I| = |J|\}.$$

The first key lemma is as follows.

Lemma 3.1. Let $D(s) = sX + Y$ be a matrix pencil. Then $\theta_k(D) = \delta(s^{k-1}D)$ holds.

Proof. Since $\theta_k(D)$ and $\delta(s^{k-1}D)$ are invariant under strict equivalence transformations for $D(s)$, we may assume that $D(s)$ is in a Kronecker canonical form with the structural indices $(\nu, \mu_1, \dots, \mu_d, \epsilon_1, \dots, \epsilon_p, \eta_1, \dots, \eta_q)$. Then it holds that

$$\delta(s^{k-1}D) = \delta(s^k I_\nu + s^{k-1} J_\nu) + \sum_{i=1}^{d} \delta(s^{k-1} N_{\mu_i}) + \sum_{i=1}^{p} \delta(s^{k-1} L_{\epsilon_i}) + \sum_{i=1}^{q} \delta(s^{k-1} L_{\eta_i}^\top).$$

Since $\delta(s^k I_\nu + s^{k-1} J_\nu) = k\nu$, $\delta(s^{k-1} N_\mu) = k\mu - \min\{k, \mu\}$, $\delta(s^{k-1} L_\epsilon) = k\epsilon$, and $\delta(s^{k-1} L_\eta^\top) = k\eta$, we obtain

$$\delta(s^{k-1}D) = k\nu + k\sum_{i=1}^{d} \mu_i - \sum_{i=1}^{d} \min\{k, \mu_i\} + k\sum_{i=1}^{p} \epsilon_i + k\sum_{i=1}^{q} \eta_i = kr - \sum_{i=1}^{d} \min\{k, \mu_i\},$$

where r denotes the rank of $D(s)$ and the last step is due to (1). This coincides with $\theta_k(D)$ by Theorem 2.2. $\square$

Let $Z(s)$ be a polynomial matrix with column set C and $p : C \to \mathbf{Z}$ a nonnegative function. For any subset J, we denote $p(J) = \sum_{j \in J} p(j)$. We now define $\delta(Z; p)$ by

$$\delta(Z; p) = \max\{\deg\det Z[I,J] - p(J) \mid |I| = |J|\}.$$

Then, the second key lemma is as follows.

Lemma 3.2. Let $D(s) = sX + Y$ be a matrix pencil with column set C and $p : C \to \mathbf{Z}$ a nonnegative function. We denote by $\bar{R}$ and $\bar{C}$ the row set and the column set of $\Theta_k(D)$. For the subset W of $\bar{C}$ obtained by deleting the first $p(j)$ columns corresponding to j for each $j \in C$, it holds that

$$\operatorname{rank} \Theta_k(D)[\bar{R}, W] \le \delta(s^{k-1}D; p).$$

If $p(j) = 0$ for all $j \in C$, Lemma 3.2 implies that $\theta_k(D) \le \delta(s^{k-1}D)$, which follows from Lemma 3.1.

Before entering the proof of Lemma 3.2, we recall some terminologies. For a rational function $f(s) = g(s)/h(s)$ with polynomials $g(s)$ and $h(s)$, its degree is defined by $\deg f(s) = \deg g(s) - \deg h(s)$. A rational function $f(s)$ is called *proper* if $\deg f(s) \le 0$, and *strictly proper* if $\deg f(s) < 0$. We call a rational function matrix (*strictly*) *proper* if its entries are (strictly) proper rational functions. A square proper rational function matrix is called *biproper* if it is invertible and its inverse is a proper rational function matrix. A proper rational function matrix is biproper if and only if its determinant is a nonzero constant.

A rational function matrix $Z(s)$ is called a *Laurent polynomial matrix* if $s^N Z(s)$ is a polynomial matrix for some integer N. For a Laurent polynomial matrix $Z(s)$, we denote by Z_l the coefficient matrix of s^l. If $Z(s)$ is a proper Laurent polynomial matrix, then it is expressed as

$$Z(s) = Z_0 + s^{-1} Z_{-1} + s^{-2} Z_{-2} + s^{-3} Z_{-3} + \cdots .$$

For a Laurent polynomial matrix $Z(s)$, we define $\delta(Z)$ and $\delta(Z;p)$ in a similar way to the definitions for a polynomial matrix. It is known that $\delta(Z)$ is invariant under biproper equivalence transformations.

For a Laurent polynomial matrix $Z(s) = \sum_i s^i Z_i$, we define an expanded matrix $\Xi_k(Z)$ by

$$\Xi_k(Z) = \begin{pmatrix} Z_k & O & \cdots & O \\ Z_{k-1} & Z_k & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ Z_1 & \cdots & Z_{k-1} & Z_k \end{pmatrix}.$$

Note that $\Xi_k(s^k X + s^{k-1} Y)$ coincides with $\Theta_k(sX + Y)$.

For a Laurent polynomial matrix $Z(s) = (Z_{ij}(s))$ with row set R and column set C, we construct a bipartite graph $G(Z) = (R, C; E(Z))$ with

$$E(Z) = \{(i,j) \mid i \in R, j \in C, Z_{ij}(s) \neq 0\}.$$

The maximum size of a matching in $G(Z)$ is called the *term-rank* of $Z(s)$.

The weight $c(e)$ of an edge $e = (i,j)$ is given by

$$c(e) = c_{ij} = \deg Z_{ij}(s).$$

We remark that $c(e)$ is integer for any $e \in E(Z)$ if $Z(s)$ is a Laurent polynomial matrix. The maximum weight of a matching in $G(Z)$ is denoted by $\hat{\delta}(Z)$.

Consider the following linear program ($\mathbf{PLP}(Z)$):

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{e \in E} c(e)\xi(e) \\ \text{subject to} & \displaystyle\sum_{\partial e \ni i} \xi(e) \leq 1 \quad (\forall i \in R \cup C), \\ & \xi(e) \geq 0 \quad (\forall e \in E(Z)). \end{array}$$

Then $\mathbf{PLP}(Z)$ has an integral optimal solution with $\xi(e) \in \{0,1\}$ for any $e \in E(Z)$. This optimal solution corresponds to the maximum weight matching in $G(Z)$, and its optimal value is equal to $\hat{\delta}(Z)$. The dual program ($\mathbf{DLP}(Z)$) is expressed as follows:

$$\begin{array}{ll} \text{minimize} & p(R \cup C) \\ \text{subject to} & p(i) + p(j) \geq c(e) \quad (\forall e = (i,j) \in E(Z)), \\ & p(i) \geq 0 \quad (\forall i \in R \cup C). \end{array}$$

Since $c(e)$ is integer for any $e \in E(Z)$, $\mathbf{DLP}(Z)$ has an integral optimal solution by the total unimodularity of the coefficient matrix.

In order to prove Lemma 3.2, we first provide the following lemma, which can be derived in a similar way to [16]. A proof is given in Appendix A.

Lemma 3.3. Let $Z(s)$ be a Laurent polynomial matrix. Then, there exists a biproper Laurent polynomial matrix $F(s) = F_0 + F_{-1}s^{-1} + \cdots$ such that $\hat{\delta}(FZ) = \delta(FZ) = \delta(Z)$ and F_0 is nonsingular.

For $H(s) = (H_{ij}(s))$ with $H_{ij}(s) = s^{-p(j)}Z_{ij}(s)$, it follows from the definition of $\delta(Z;p)$ that $\delta(H) = \delta(Z;p)$. By Lemma 3.3, there exists a biproper Laurent polynomial matrix $F(s) = F_0 + F_{-1}s^{-1} + \cdots$ such that $\hat{\delta}(FH) = \delta(FH) = \delta(H)$ and F_0 is nonsingular. Consider the linear program $\mathbf{PLP}(FH)$ and its dual program $\mathbf{DLP}(FH)$. Let p^* be an integral optimal dual solution. Since $\hat{\delta}(FH) = p^*(R \cup C)$ holds, we obtain

$$\delta(Z;p) = \delta(H) = \hat{\delta}(FH) = p^*(R \cup C). \tag{2}$$

Consider the expanded matrix $\Xi_k(FZ)$ with row set $\bar{R}$ and column set $\bar{C}$. For $Z(s) = s^k X + s^{k-1} Y$, the rank of $\Xi_k(FZ)$ has the following property.

Lemma 3.4. Let $D(s) = sX + Y$ be a matrix pencil and $F(s) = F_0 + F_{-1}s^{-1} + \cdots$ a biproper Laurent polynomial matrix such that F_0 is nonsingular. Then, the polynomial matrix $Z(s) = s^{k-1}D$ satisfies $\operatorname{rank}\Xi_k(FZ) = \operatorname{rank}\Theta_k(D)$. Moreover, $\operatorname{rank}\Xi_k(FZ)[\bar{R}, W] = \operatorname{rank}\Theta_k(D)[\bar{R}, W]$ holds for any subset W of $\bar{C}$, where $\bar{C}$ denotes the column set of $\Xi_k(FZ)$.

Proof. For a constant matrix $\tilde{F}$ defined by

$$\tilde{F} = \begin{pmatrix} F_0 & O & \cdots & \cdots & O \\ F_{-1} & F_0 & \ddots & & \vdots \\ F_{-2} & F_{-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & F_0 & O \\ F_{-k+1} & \cdots & F_{-2} & F_{-1} & F_0 \end{pmatrix},$$

we have $\Xi_k(FZ) = \tilde{F}\Xi_k(Z) = \tilde{F}\Theta_k(D)$. Since F_0 is nonsingular, $\tilde{F}$ is also nonsingular. Thus, we obtain $\operatorname{rank}\Xi_k(FZ) = \operatorname{rank}\Theta_k(D)$ and $\operatorname{rank}\Xi_k(FZ)[\bar{R}, W] = \operatorname{rank}\Theta_k(D)[\bar{R}, W]$. □

The rest of this section is devoted to the proof of Lemma 3.2.

Proof of Lemma 3.2. We now prove

$$\operatorname{rank}\Xi_k(FZ)[\bar{R}, W] \leq p^*(R \cup C),$$

which completes the proof of Lemma 3.2 by (2) and Lemma 3.4.

For the expanded matrix $\Xi_k(FZ)$, let us define the ith row set of $\Xi_k(FZ)$ by R_i and the jth column set by C_j. This means that $\bar{R} = R_1 \cup \cdots \cup R_k$ and $\bar{C} = C_1 \cup \cdots \cup C_k$. For

$$S_l = \{i \mid i \in R_l, p^*(i) \geq k - l + 1\} \quad \text{and} \quad T_h = \{j \mid j \in C_h, p^*(j) \geq h - p(j) > 0\},$$

we show that $(S_1 \cup \cdots \cup S_k, T_1 \cup \cdots \cup T_k)$ is a cover of the bipartite graph $G(\Xi_k(FZ)[\bar{R}, W])$, namely, $i \in S_1 \cup \cdots \cup S_k$ or $j \in T_1 \cup \cdots \cup T_k$ holds for any edge (i, j) in $G(\Xi_k(FZ)[\bar{R}, W])$. Let (i, j) be an edge with $i \in R_l$ and $j \in C_h$. Then, the degree of the (i, j)-entry in $F(s)Z(s)$ is equal to $k - (l - h)$. Since p^* is feasible for $\mathbf{DLP}(FH)$, we have

$$p^*(i) + p^*(j) \geq k - (l - h) - p(j) = (k - l) + (h - p(j)).$$

This implies that $p^*(j) \geq h - p(j)$ holds if $p^*(i) \leq k - l$. Hence, $i \in S_l$ or $j \in T_h$ holds, which means that $(S_1 \cup \cdots \cup S_k, T_1 \cup \cdots \cup T_k)$ is a cover of $G(\Xi_k(FZ)[\bar{R}, W])$.

Now it holds that

$$\left| \bigcup_{l=1}^{k} S_l \right| = \sum_{l=1}^{k} |\{i \mid i \in R_l, p^*(i) \geq k - l + 1\}| = p^*(R),$$

because $0 \leq p^*(i) \leq k$ for any $i \in R$. Similarly, we obtain $\left|\bigcup_{h=1}^{k} T_h\right| = p^*(C)$. Thus, the size of the cover $(S_1 \cup \cdots \cup S_k, T_1 \cup \cdots \cup T_k)$ is equal to $p^*(R \cup C)$, which implies that $\operatorname{rank} \Xi_k(FZ)[\bar{R}, W] \leq \operatorname{term-rank} \Xi_k(FZ)[\bar{R}, W] \leq p^*(R \cup C)$. □

4 Mixed Matrix Pencil

In this section, we first describe the definition of mixed matrix pencils. Secondly, for a mixed matrix pencil $D_{\mathrm{M}}(s)$, we reduce the computation of $\theta_k(D_{\mathrm{M}})$ to the computation of $\theta_k(D)$ for an associated special mixed matrix pencil $D(s)$.

A *generic matrix* is a matrix in which each nonzero entry is an independent parameter. A matrix D is called a *mixed matrix* if D is given by $D = Q + T$ with a constant matrix Q and a generic matrix T. A *layered mixed matrix* (or an *LM-matrix* for short) is defined to be a mixed matrix such that Q and T have disjoint nonzero rows. An LM-matrix D is expressed by $D = \binom{Q}{T}$.

A mixed matrix pencil is a matrix pencil version of mixed matrices. A matrix pencil $D(s)$ is called a *mixed matrix pencil* if $D(s)$ is given by $D(s) = Q(s) + T(s)$ with a pair of matrix pencils $Q(s) = sX_Q + Y_Q$ and $T(s) = sX_T + Y_T$ that satisfy the following two conditions.

(MP-Q) X_Q and Y_Q are constant matrices.

(MP-T) X_T and Y_T are generic matrices.

Note that each independent parameter in X_T and Y_T appears only once. A *layered mixed matrix pencil* (or an *LM-matrix pencil* for short) is defined to be a mixed matrix pencil such that $Q(s)$ and $T(s)$ satisfying (MP-Q) and (MP-T) have disjoint nonzero rows. An LM-matrix pencil $D(s)$ is expressed by $D(s) = \binom{Q(s)}{T(s)}$. The polynomial matrix version of a mixed matrix is called a *mixed polynomial matrix*. We define a *layered mixed polynomial matrix* (or an *LM-polynomial matrix* for short) in a similar way.

Let $D_{\mathrm{M}}(s) = s(X_Q + X_T) + (Y_Q + Y_T)$ be an $m \times n$ mixed matrix pencil. We construct an LM-matrix pencil

$$D(s) = s \begin{pmatrix} I & X_Q \\ -D_T & X_T \end{pmatrix} + \begin{pmatrix} O & Y_Q \\ O & Y_T \end{pmatrix}, \tag{3}$$

where D_T is a diagonal matrix with the (i,i)-entry being a new independent parameter t_i. We transform $D(s)$ into its strictly equivalent matrix

$$\begin{pmatrix} I & O \\ O & D_T^{-1} \end{pmatrix} D(s) = s \begin{pmatrix} I & X_Q \\ -I & D_T^{-1}X_T \end{pmatrix} + \begin{pmatrix} O & Y_Q \\ O & D_T^{-1}Y_T \end{pmatrix}.$$

Since D_T is a diagonal generic matrix, we can regard $D_T^{-1}X_T$ and $D_T^{-1}Y_T$ as new generic matrices $\tilde{X}_T$ and $\tilde{Y}_T$, respectively. Hence, $D(s)$ and $s\begin{pmatrix} I & X_Q \\ -I & \tilde{X}_T \end{pmatrix} + \begin{pmatrix} O & Y_Q \\ O & \tilde{Y}_T \end{pmatrix}$, as well as $\bar{D}(s) = s\begin{pmatrix} I & X_Q \\ -I & X_T \end{pmatrix} + \begin{pmatrix} O & Y_Q \\ O & Y_T \end{pmatrix}$, have the same Kronecker canonical form. This observation leads to the following lemma concerning the relation between $D_{\mathrm{M}}(s)$ and $D(s)$.

Lemma 4.1. Let $D_{\mathrm{M}}(s) = s(X_Q + X_T) + (Y_Q + Y_T)$ be an $m \times n$ mixed matrix pencil and $D(s)$ its associated LM-matrix pencil defined by (3). Then we have

$$\theta_k(D_{\mathrm{M}}) + km = \theta_k(D). \tag{4}$$

Proof. As noted above, $D(s)$ has the same Kronecker canonical form as

$$\bar{D}(s) = s\begin{pmatrix} I & X_Q \\ -I & X_T \end{pmatrix} + \begin{pmatrix} O & Y_Q \\ O & Y_T \end{pmatrix}.$$

Moreover, since $\bar{D}(s)$ is strictly equivalent to

$$\bar{D}_{\mathrm{M}}(s) = \begin{pmatrix} I & O \\ I & I \end{pmatrix} \bar{D}(s) = s\begin{pmatrix} I & X_Q \\ O & X_Q + X_T \end{pmatrix} + \begin{pmatrix} O & Y_Q \\ O & Y_Q + Y_T \end{pmatrix},$$

we have

$$\theta_k(D) = \theta_k(\bar{D}) = \theta_k(\bar{D}_{\mathrm{M}}). \tag{5}$$

Now we can transform $\Theta_k(\bar{D}_{\mathrm{M}})$ as follows:

$$\Theta_k(\bar{D}_{\mathrm{M}}) = \left(\begin{array}{cc|cc|cc|cc} I_m & X_Q & O & O & O & O & O & O \\ O & X_Q + X_T & O & O & O & O & O & O \\ \hline O & Y_Q & I_m & X_Q & O & O & O & O \\ O & Y_Q + Y_T & O & X_Q + X_T & O & O & O & O \\ \hline O & O & \ddots & & \ddots & & O & O \\ O & O & & \ddots & & \ddots & O & O \\ \hline O & O & O & O & O & Y_Q & I_m & X_Q \\ O & O & O & O & O & Y_Q + Y_T & O & X_Q + X_T \end{array}\right)$$

$$\xrightarrow{\text{permutations}} \left(\begin{array}{cccc|cccc} I_m & O & O & O & X_Q & O & O & O \\ O & I_m & O & O & Y_Q & X_Q & O & O \\ O & O & \ddots & O & O & \ddots & \ddots & O \\ O & O & O & I_m & O & O & Y_Q & X_Q \\ \hline O & O & O & O & X_Q+X_T & O & O & O \\ O & O & O & O & Y_Q+Y_T & X_Q+X_T & O & O \\ O & O & O & O & O & \ddots & \ddots & O \\ O & O & O & O & O & O & Y_Q+Y_T & X_Q+X_T \end{array}\right)$$

$$= \left(\begin{array}{c|c} I_{km} & * \\ \hline O & \Theta_k(D_{\mathrm{M}}) \end{array}\right).$$

Hence it holds that

$$\theta_k(\bar{D}_{\mathrm{M}}) = \theta_k(D_{\mathrm{M}}) + km.$$

Thus we obtain (4) by (5). □

By Lemma 4.1, we hereafter focus on an LM-matrix pencil.

Remark 4.2. Let $D_{\mathrm{M}}(s) = s(X_Q + X_T) + (Y_Q + Y_T)$ be a mixed matrix pencil and $D(s)$ its associated LM-matrix pencil defined by (3). Then we can not reduce the computation of $\psi_k(D_{\mathrm{M}})$ to $\psi_k(D)$ and $\varphi_k(D_{\mathrm{M}})$ to $\varphi_k(D)$ in a similar way to the proof of Lemma 4.1. This is one of major differences between $\theta_k(D_{\mathrm{M}})$ and $\psi_k(D_{\mathrm{M}}), \varphi_k(D_{\mathrm{M}})$.

5 Independent Matching and Valuated Independent Assignment

This section is devoted to preliminaries on matroids and valuated matroids, which are combinatorial abstractions of matrices and polynomial matrices. After recapitulating matroids and valuated matroids in Section 5.1, we explain the independent matching problem in Section 5.2, and the valuated independent assignment problem in Section 5.3.

5.1 Matroids and Valuated Matroids

A *matroid* is a pair $\mathbf{M} = (V, \mathcal{I})$ of finite set V and a collection $\mathcal{I}$ of subsets of V such that

(I-1) $\emptyset \in \mathcal{I}$,

(I-2) $I \subseteq J \in \mathcal{I} \Rightarrow I \in \mathcal{I}$,

(I-3) $I, J \in \mathcal{I}, |I| < |J| \Rightarrow I \cup \{v\} \in \mathcal{I}$ for some $v \in J \setminus I$.

The set V is called the *ground set*, $I \in \mathcal{I}$ is an *independent set*, and $\mathcal{I}$ is the *family of independent sets*. The *rank function* ρ of $\mathbf{M} = (V, \mathcal{I})$ is defined by

$$\rho(W) = \max\{|I| \mid I \subseteq W, I \in \mathcal{I}\} \quad (W \subseteq V).$$

Hereafter, we denote a matroid by $\mathbf{M} = (V, \mathcal{I}, \rho)$ together with the rank function ρ. Let $\mathcal{B}$ be the family of inclusion-wise maximal members of $\mathcal{I}$. A member of $\mathcal{B}$ is called a *base*, and $\mathcal{B}$ is the *base family*.

Matroids are a combinatorial abstraction of matrices with respect to linear independence. As a generalization of matroids, Dress and Wenzel [6] introduced *valuated matroids*, which originate from a combinatorial structure of polynomial matrices with respect to the degree of determinants.

A valuated matroid is a triple $\mathbf{M} = (V, \mathcal{B}, \omega)$ of a ground set V, a base family $\mathcal{B} \subseteq 2^V$, and a function $\omega : \mathcal{B} \to \mathbf{R}$ that satisfy the following axiom (VM).

(VM) For any $B, B' \in \mathcal{B}$ and $u \in B \setminus B'$, there exists $v \in B' \setminus B$ such that $B \setminus \{u\} \cup \{v\} \in \mathcal{B}$, $B' \cup \{u\} \setminus \{v\} \in \mathcal{B}$, and $\omega(B) + \omega(B') \leq \omega(B \setminus \{u\} \cup \{v\}) + \omega(B' \cup \{u\} \setminus \{v\})$.

The function ω is called a *valuation*. The local optimality for the valuation implies the global optimality as follows.

Theorem 5.1 ([20, Theorem 5.2.7])**.** A base $B \in \mathcal{B}$ satisfies $\omega(B) \geq \omega(B')$ for any $B' \in \mathcal{B}$ if and only if $\omega(B \setminus \{u\} \cup \{v\}) \leq \omega(B)$ holds for any $u \in B$ and $v \in V \setminus B$.

Note that a valuated matroid $\mathbf{M} = (V, \mathcal{B}, \omega)$ such that $\omega(B) = 0$ for all $B \in \mathcal{B}$ coincides with a matroid. For a polynomial $b(s)$, we denote the degree of $b(s)$ by $\deg b$, where $\deg 0 = -\infty$ by convention. For a matrix pencil $D(s)$, $D[I, J]$ denotes the submatrix of $D(s)$ with row set I and column set J. A typical example of a valuated matroid is as follows.

Example 5.2. For an $m \times n$ matrix pencil $D(s)$ of rank m with row set R and column set C, let us define

$$\mathcal{B} = \{B \subseteq C \mid \det D[R, B] \neq 0\} \quad \text{and} \quad \omega(B) = \deg \det D[R, B].$$

Then $(C, \mathcal{B}, \omega)$ is a valuated matroid. □

5.2 Independent Matching Problem

The following problem is an extension of the matching problem.

> **[Independent Matching Problem (IMP)]**
> Given a bipartite graph $G = (V^+, V^-; E)$ with vertex sets V^+, V^- and edge set E, and a pair of matroids $\mathbf{M}^+ = (V^+, \mathcal{I}^+, \rho^+)$ and $\mathbf{M}^- = (V^-, \mathcal{I}^-, \rho^-)$, find a matching $M \subseteq E$ that maximizes $|M|$ subject to
>
> $$\partial^+ M \in \mathcal{I}^+, \quad \partial^- M \in \mathcal{I}^-, \tag{6}$$
>
> where $\partial^+ M$ and $\partial^- M$ denote the set of vertices in V^+ and V^- incident to M, respectively.

A matching $M \subseteq E$ satisfying (6) is called an *independent matching*. A pair (U^+, U^-) is called a *cover* if $U^+ \subseteq V^+$, $U^- \subseteq V^-$, and $\partial^+ e \in U^+$ or $\partial^- e \in U^-$ for each $e \in E$, where $\partial^+ e$ and $\partial^- e$ denote the vertex in V^+ and V^- incident to e.

[Dual Problem for IMP]
Given a bipartite graph $G = (V^+, V^-; E)$ with vertex sets V^+, V^- and edge set E, and a pair of matroids $\mathbf{M}^+ = (V^+, \mathcal{I}^+, \rho^+)$ and $\mathbf{M}^- = (V^-, \mathcal{I}^-, \rho^-)$, find a cover (U^+, U^-) that minimizes $\rho^+(U^+) + \rho^-(U^-)$.

The following min-max theorem is well-known.

Theorem 5.3 ([26])**.** It holds that

$$\max\{|M| \mid M : \text{independent matching}\} = \min\{\rho^+(U^+) + \rho^-(U^-) \mid (U^+, U^-) : \text{cover}\}.$$

The computation of the rank of an LM-matrix pencil $D(s)$ can be reduced to solving an independent matching problem [21].

5.3 Valuated Independent Assignment Problem

Murota [17, 18] introduced the *valuated independent assignment problem* as a generalization of the independent matching problem. We generalize Theorem 5.3 to valuated independent assignment problem.

[Valuated Independent Assignment Problem (VIAP)]
Given a bipartite graph $G = (V^+, V^-; E)$ with vertex sets V^+, V^- and edge set E, a pair of valuated matroids $\mathbf{M}^+ = (V^+, \mathcal{B}^+, \omega^+)$ and $\mathbf{M}^- = (V^-, \mathcal{B}^-, \omega^-)$, a weight function $w : E \to \mathbf{R}$, and subsets $V_0^+ \subseteq V^+$ and $V_0^- \subseteq V^-$, find a triple (M, B^+, B^-) that maximizes

$$\Omega(M, B^+, B^-) := w(M) + \omega^+(B^+) + \omega^-(B^-),$$

where $w(M) = \sum\{w(e) \mid e \in M\}$, subject to the constraint that $M \subseteq E$ is a matching and

$$\partial^+ M \subseteq B^+ \in \mathcal{B}^+, \quad \partial^+ M \cap V_0^+ = B^+ \cap V_0^+, \tag{7}$$
$$\partial^- M \subseteq B^- \in \mathcal{B}^-, \quad \partial^- M \cap V_0^- = B^- \cap V_0^-. \tag{8}$$

Consider the following dual problem for the VIAP.

[Dual Problem for VIAP]
Given a bipartite graph $G = (V^+, V^-; E)$ with vertex sets V^+, V^- and edge set E, a pair of valuated matroids $\mathbf{M}^+ = (V^+, \mathcal{B}^+, \omega^+)$ and $\mathbf{M}^- = (V^-, \mathcal{B}^-, \omega^-)$, a weight function $w : E \to \mathbf{R}$, and subsets $V_0^+ \subseteq V^+$ and $V_0^- \subseteq V^-$, find potential functions p^+ and p^- that minimizes

$$\zeta^+(p^+) + \zeta^-(p^-),$$

where

$$\zeta^+(p^+) := \max_{B^+ \in \mathcal{B}^+} \{\omega^+(B^+) + p^+(B^+)\} \quad \text{and} \quad \zeta^-(p^-) := \max_{B^- \in \mathcal{B}^-} \{\omega^-(B^-) + p^-(B^-)\},$$

subject to the constraint that

$$p^+(i) + p^-(j) \geq w(e) \quad (\forall e = (i,j) \in E), \tag{9}$$

$$p^+(i) \geq 0 \quad (\forall i \in V^+ \setminus V_0^+), \tag{10}$$

$$p^-(j) \geq 0 \quad (\forall j \in V^- \setminus V_0^-). \tag{11}$$

These problems are an extension of the problems introduced in [17, 18], where $V_0^+ = V^+$ and $V_0^- = V^-$. The following is a min-max theorem for the VIAP.

Theorem 5.4 (Duality Theorem for **VIAP**)**.** It holds that

$$\begin{aligned}&\max\{\Omega(M, B^+, B^-) \mid (M, B^+, B^-) \text{ satisfies } (7)\text{–}(8)\} \\ &\qquad = \min\{\zeta^+(p^+) + \zeta^-(p^-) \mid (p^+, p^-) \text{ satisfies } (9)\text{–}(11)\}.\end{aligned} \tag{12}$$

A proof is given in Appendix B.

6 Analysis of $\theta_k(D)$

Let $D(s) = \begin{pmatrix} Q(s) \\ T(s) \end{pmatrix}$ be an LM-matrix pencil with $Q(s) = sX_Q + Y_Q$ and $T(s) = sX_T + Y_T$. In this section, we prove that $\theta_k(D)$ coincides with the optimal value of an **IMP**. We give a main theorem in Section 6.1. Sections 6.2 and 6.3 are devoted to the proof. The main theorem leads to an algorithm for computing the indices of nilpotency $\mu_1, \ldots, \mu_d$. In Section 6.4, we discuss the time complexity of our algorithm.

6.1 IMP for $\Theta_k(D)$

For an LM-matrix pencil $D(s) = \begin{pmatrix} Q(s) \\ T(s) \end{pmatrix}$, we denote the row sets of $Q(s)$ and $T(s)$ by R^Q and R^T, respectively. Moreover, we denote the column set of $D(s)$ by C, and its copy by $C^Q = \{j^Q \mid j \in C\}$.

The expanded matrix $\Theta_k(D)$ is expressed as

$$\Theta_k(D) = \left(\begin{array}{ccccc} X_Q & O & \cdots & \cdots & O \\ Y_Q & X_Q & \ddots & & \vdots \\ O & Y_Q & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & X_Q & O \\ O & \cdots & O & Y_Q & X_Q \\ \hline X_T & O & \cdots & \cdots & O \\ Y_T & X_T & \ddots & & \vdots \\ O & Y_T & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & X_T & O \\ O & \cdots & O & Y_T & X_T \end{array}\right)$$

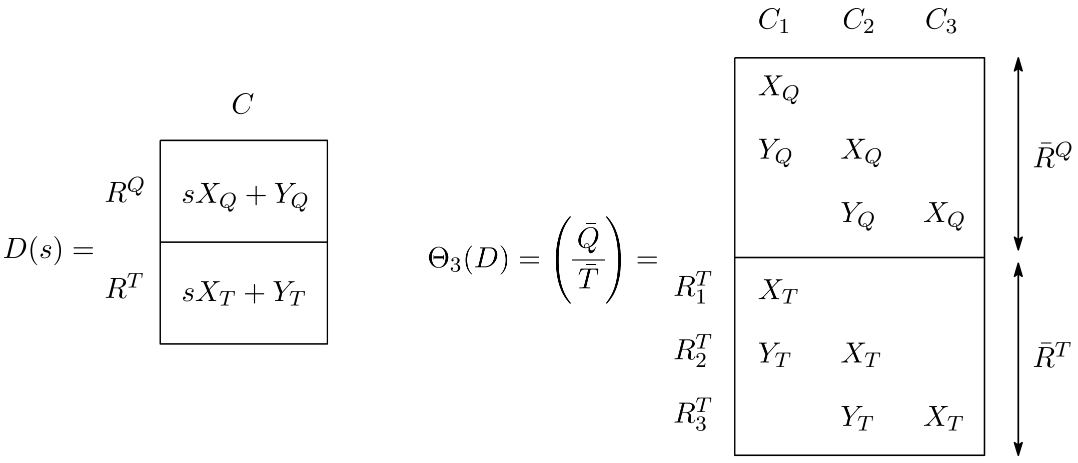

Figure 1: An LM-matrix pencil $D(s)$ and an expanded matrix $\Theta_3(D)$.

by permutations. Although $\Theta_k(D)$ looks like an LM-matrix, it is not. This is because independent parameters in X_T and Y_T appear multiple times. Let us denote the upper half of $\Theta_k(D)$ by $\bar{Q}$, and the lower half by $\bar{T}$. Note that $\bar{Q} = \Theta_k(sX_Q + Y_Q)$ holds and $\bar{T} = \Theta_k(sX_T + Y_T)$ is not a generic matrix.

Let us denote the hth column set of $\Theta_k(D)$ by C_h, and the hth row set of $\bar{T}$ by R_h^T for $h = 1, \dots, k$. Then $C_h = \{j_h \mid j \in C\}$ is the copy of C, and $R_h^T = \{i_h \mid i \in R^T\}$ is the copy of R^T. Moreover, let $C_h^Q = \{i_h^Q \mid i^Q \in C^Q\}$ denote the hth copy of C^Q for $h = 1, \dots, k$. For the sake of simplicity, we use $\bar{R}^T = \bigcup_{h=1}^k R_h^T$ and $\bar{C}^Q = \bigcup_{h=1}^k C_h^Q$. The row set of $\bar{Q}$ is denoted by $\bar{R}^Q$. These notations are summarized in Figure 1.

We define a bipartite graph $G(\Theta_k(D)) = (\bar{V}^+, \bar{V}^-; \bar{E})$ with

$$\bar{V}^+ = \bigcup_{h=1}^k C_h^Q \cup \bigcup_{h=1}^k R_h^T, \quad \bar{V}^- = \bigcup_{h=1}^k C_h, \quad \bar{E} = \bigcup_{h=1}^k E_h^Q \cup \bigcup_{h=1}^k E_h^X \cup \bigcup_{h=1}^{k-1} E_h^Y,$$

where

$$\begin{aligned} E_h^Q &= \{(i_h^Q, i_h) \mid i_h^Q \in C_h^Q, i_h \in C_h\}, \\ E_h^X &= \{(i_h, j_h) \mid i \in R^T, j \in C, \text{the } (i,j)\text{-entry of } X_T \text{ is nonzero}\}, \\ E_h^Y &= \{(i_h, j_{h+1}) \mid j \in R^T, i \in C, \text{the } (j,i)\text{-entry of } Y_T \text{ is nonzero}\}. \end{aligned}$$

The edge set $\bigcup_{h=1}^k E_h^X \cup \bigcup_{h=1}^{k-1} E_h^Y$ corresponds to the set of nonzero entries in $\bar{T}$.

Example 6.1. Consider an LM-matrix pencil

$$D(s) = \left(\begin{array}{cccc} s & 0 & 0 & 1 \\ 0 & 1 & s & s \\ \hline t_1 s & 0 & 0 & t_2 s \\ t_3 & t_4 & 0 & 0 \\ 0 & 0 & t_5 s & t_6 \end{array}\right),$$

where $t_1, \dots, t_6$ are independent parameters. Figure 2 illustrates $G(\Theta_3(D))$. $\square$

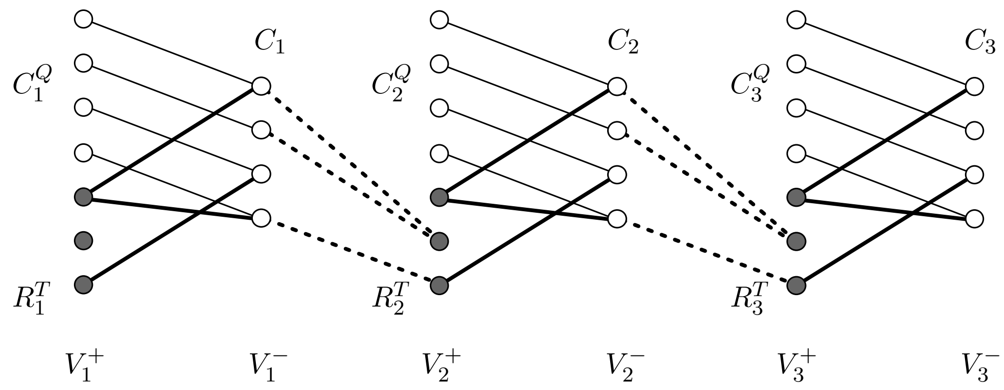

Figure 2: A graph $G(\Theta_3(D))$ with E_h^Q (solid line), E_h^X (heavy line), and E_h^Y (dotted line) of Example 6.1.

We define the following **IMP** on $G(\Theta_k(D))$. The matroids $\mathbf{M}^+ = (\bar{V}^+, \mathcal{I}^+, \rho^+)$ and $\mathbf{M}^- = (\bar{V}^-, \mathcal{I}^-, \rho^-)$ attached to $\bar{V}^+$ and $\bar{V}^-$ are defined by

$$\mathcal{I}^+ = \{I^+ \mid I^+ \subseteq \bar{V}^+, \operatorname{rank} \bar{Q}[\bar{R}^Q, I^+ \cap \bar{C}^Q] = |I^+ \cap \bar{C}^Q|\},$$
$$\rho^+(W^+) = \operatorname{rank} \bar{Q}[\bar{R}^Q, W^+ \cap \bar{C}^Q] + |W^+ \cap \bar{R}^T|,$$

and

$$\mathcal{I}^- = \{I^- \mid I^- \subseteq \bar{V}^-\}, \quad \rho^-(W^-) = |W^-|.$$

The **IMP** is summarized as follows.

[IMP($\Theta_k(D)$)]
Given a bipartite graph $G(\Theta_k(D)) = (\bar{V}^+, \bar{V}^-; \bar{E})$ and a matroid $\mathbf{M}^+ = (\bar{V}^+, \mathcal{I}^+, \rho^+)$, find a matching $M \subseteq E$ that maximizes $|M|$ subject to $\partial^+ M \in \mathcal{I}^+$.

The main theorem of this paper is as follows.

Theorem 6.2. Let $D(s) = \begin{pmatrix} sX_Q + Y_Q \\ sX_T + Y_T \end{pmatrix}$ be an LM-matrix pencil. Then $\theta_k(D)$ coincides with the optimal value of **IMP($\Theta_k(D)$)**.

Theorem 6.2 is known for the special case that $\bar{T}$ is a generic matrix [20]. The key point here is that $\bar{T}$ is not a generic matrix.

The following is the dual problem of **IMP($\Theta_k(D)$)**.

[DIMP($\Theta_k(D)$)]
Given a bipartite graph $G(\Theta_k(D)) = (\bar{V}^+, \bar{V}^-; \bar{E})$ and a matroid $\mathbf{M}^+ = (\bar{V}^+, \mathcal{I}^+, \rho^+)$, find a cover (U^+, U^-) that minimizes

$$\operatorname{rank} \bar{Q}[\bar{R}^Q, U^+ \cap \bar{C}^Q] + |U^+ \cap \bar{R}^T| + |U^-|.$$

Theorem 5.3 implies that the optimal value of **IMP($\Theta_k(D)$)** coincides with the optimal value of **DIMP($\Theta_k(D)$)**.

6.2 VIAP for $\theta_k(D)$

Here, we introduce a VIAP and its dual problem for an LM-matrix pencil $D(s) = \begin{pmatrix} sX_Q + Y_Q \\ sX_T + Y_T \end{pmatrix}$. These problems are useful for the proof of Theorem 6.2.

With respect to $D(s)$, we consider an LM-polynomial matrix

$$\tilde{Z}(s) = \begin{pmatrix} I & s^k X_Q + s^{k-1} Y_Q \\ O & s^k X_T + s^{k-1} Y_T \end{pmatrix}. \tag{13}$$

By applying Lemma 3.1 to $D(s)$, we obtain the following corollary.

Corollary 6.3. Let $D(s) = \begin{pmatrix} sX_Q + Y_Q \\ sX_T + Y_T \end{pmatrix}$ be an LM-matrix pencil and $\tilde{Z}(s)$ an LM-polynomial matrix defined by (13). Then we have $\theta_k(D) = \delta(\tilde{Z})$.

Proof. It follows from Lemma 3.1 that $\theta_k(D) = \delta(s^{k-1}D)$. Hence we obtain $\theta_k(D) = \delta(\tilde{Z})$ by $\delta(s^{k-1}D) = \delta(\tilde{Z})$. □

By virtue of Corollary 6.3, we focus on $\delta(\tilde{Z})$ instead of $\theta_k(D)$. Note that $\delta(\tilde{Z})$ is the highest degree of a submatrix with row set containing R^Q. For an LM-polynomial matrix, Murota [19] showed that the computation of the highest degree of subdeterminants with row set containing R^Q reduces to a VIAP. We now derive a VIAP for $\delta(\tilde{Z})$ in the same way as [19].

The row set and the column set of $\tilde{Z}(s)$ are denoted by $R^Q \cup R^T$ and $R \cup C$, where we use the same notation R^Q, R^T, and C as the corresponding row/column set of $D(s)$. We denote the upper half of $\tilde{Z}(s)$ by $\tilde{Q}(s)$, and the lower half by $\tilde{T}(s)$. Consider a bipartite graph $G = (V^+, V^-; E)$ with $V^+ = R^Q \cup C^Q \cup R^T$, $V^- = R \cup C$, and $E = E^Q \cup E^X \cup E^Y$, where

$$\begin{aligned} E^Q &= \{(j^Q, j) \mid j \in R \cup C\}, \\ E^X &= \{(i,j) \mid i \in R^T, j \in C, \text{the } (i,j)\text{-entry of } X_T \text{ is nonzero}\}, \\ E^Y &= \{(i,j) \mid i \in R^T, j \in C, \text{the } (i,j)\text{-entry of } Y_T \text{ is nonzero}\}. \end{aligned}$$

The weight $w(e)$ of an edge $e \in E$ is given by

$$w(e) = \begin{cases} 0 & (e \in E^Q), \\ k & (e \in E^X), \\ k-1 & (e \in E^Y). \end{cases}$$

Let us define a valuated matroid $\tilde{\mathbf{M}} = (R^Q \cup C^Q, \tilde{\mathcal{B}}, \tilde{\omega})$ by

$$\tilde{\mathcal{B}} = \{B \subseteq R^Q \cup C^Q \mid \det \tilde{Z}[R^Q, B] \neq 0\} \quad \text{and} \quad \tilde{\omega}(B) = \deg \det \tilde{Z}[R^Q, B] \quad (B \in \tilde{\mathcal{B}}). \tag{14}$$

Figure 3 illustrates G in Example 6.1.

We define the following VIAP on G, where $V_0^+ = R^Q \cup C^Q$ and $V_0^- = \emptyset$. The valuated matroids $\mathbf{M}^+ = (V^+, \mathcal{B}^+, \omega^+)$ and $\mathbf{M}^- = (V^-, \mathcal{B}^-, \omega^-)$ attached to V^+ and V^- are defined by

$$\mathcal{B}^+ = \{R^T \cup B \mid B \in \tilde{\mathcal{B}}\}, \quad \omega^+(B^+) = \tilde{\omega}(B^+ \setminus R^T) \quad (B^+ \in \mathcal{B}^+),$$

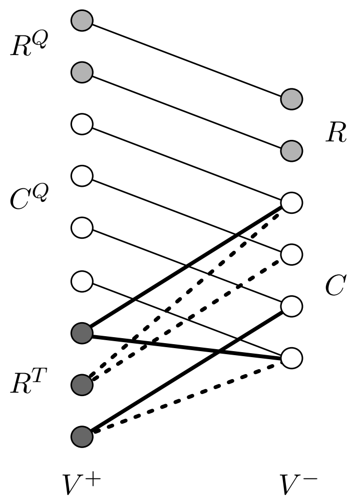

Figure 3: A graph $G = (V^+, V^-; E^Q \cup E^X \cup E^Y)$ with E^Q (solid line), E^X (heavy line), and E^Y (dotted line) of Example 6.1.

and

$$\mathcal{B}^- = \{R \cup C\}, \quad \omega^-(R \cup C) = 0.$$

Consider the dual problem for this VIAP. The constraints are given by

$$p^+(i) + p^-(j) \geq 0 \quad (\forall (i,j) \in E^Q), \tag{15}$$

$$p^+(i) + p^-(j) \geq k \quad (\forall (i,j) \in E^X), \tag{16}$$

$$p^+(i) + p^-(j) \geq k - 1 \quad (\forall (i,j) \in E^Y), \tag{17}$$

$$p^+(i) \geq 0 \quad (\forall i \in R^T), \tag{18}$$

$$p^-(j) \geq 0 \quad (\forall j \in R \cup C). \tag{19}$$

By (15), (19), and the definition of G, we may assume that

$$p^+(i) = 0 \quad (\forall i \in R^Q) \quad \text{and} \quad p^-(j) = 0 \quad (\forall j \in R). \tag{20}$$

Moreover, we may assume that

$$p^+(j^Q) = -p^-(j) \quad (\forall j \in C). \tag{21}$$

Thus, the objective function is expressed by

$$\zeta^+(p^+) + \zeta^-(p^-) = \max_{B \in \tilde{\mathcal{B}}} \{\tilde{\omega}(B) + p^+(B)\} + p^+(R^T) + p^-(C),$$

because it holds that

$$\zeta^+(p^+) = \max_{B \in \tilde{\mathcal{B}}} \{\tilde{\omega}(B) + p^+(B)\} + p^+(R^T),$$

$$\zeta^-(p^-) = \omega^-(R \cup C) + p^-(R \cup C) = p^-(C).$$

Let us summarize the VIAP and its dual problem.

[VIAP(D)]
Given a bipartite graph $G = (V^+, V^-; E)$, a valuated matroid $\tilde{\mathbf{M}} = (R^Q \cup C^Q, \tilde{\mathcal{B}}, \tilde{\omega})$, and a weight function $w : E \to \mathbf{R}$, find a pair (M, B) of a matching $M \subseteq E$ and a base $B \in \tilde{\mathcal{B}}$ that maximizes

$$\Omega(M, B) := w(M) + \tilde{\omega}(B),$$

subject to $\partial^+ M \cap (R^Q \cup C^Q) = B$.

[DVIAP(D)]
Given a bipartite graph $G = (V^+, V^-; E)$, a valuated matroid $\tilde{\mathbf{M}} = (R^Q \cup C^Q, \tilde{\mathcal{B}}, \tilde{\omega})$, and a weight function $w : E \to \mathbf{R}$, find potential functions p^+ and p^- that minimize

$$\max_{B \in \tilde{\mathcal{B}}} \{\tilde{\omega}(B) + p^+(B)\} + p^+(R^T) + p^-(C),$$

subject to the constraints (15)–(19).

By Theorem 5.4, the optimal value of **VIAP(D)** coincides with that of **DVIAP(D)**.

6.3 Proof of Theorem 6.2

In this section, we prove that

$$\theta_k(D) = (\text{optimal value of } \mathbf{IMP}(\Theta_k(D))). \tag{22}$$

The problems introduced in Sections 6.1 and 6.2 have the following relations.

$$\begin{array}{ccccc}
 & \theta_k(D) & & & \\
\boxed{\text{Corollary 6.3}} & \parallel & & & \\
 & \mathbf{VIAP}(D) & \overset{\boxed{\text{Theorem 5.4}}}{=} & \mathbf{DVIAP}(D) & \\
\boxed{\text{Lemma 6.4}} & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\geq$} & \boxed{\text{Lemma 6.5}} \\
 & \mathbf{IMP}(\Theta_k(D)) & \underset{\boxed{\text{Theorem 5.3}}}{=} & \mathbf{DIMP}(\Theta_k(D)) &
\end{array}$$

The equality

$$\theta_k(D) = (\text{optimal value of } \mathbf{VIAP}(D)) \tag{23}$$

is derived from Corollary 6.3, because $\delta(\tilde{Z})$ coincides with the optimal value of **VIAP(D)** by the results in [19].

We now give the proofs of the above two inequalities.

Lemma 6.4. For any LM-matrix pencil $D(s)$, we have

$$(\text{optimal value of } \mathbf{IMP}(\Theta_k(D))) \geq (\text{optimal value of } \mathbf{VIAP}(D)).$$

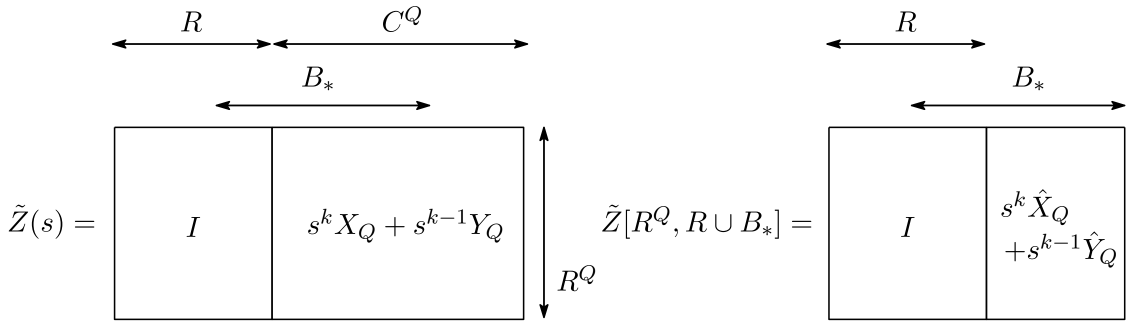

Figure 4: The polynomial matrix $\tilde{Z}(s)$ and its submatrix $\tilde{Z}[R^Q, R \cup B_*]$.

Proof. Let (M_*, B_*) be an optimal solution of $\mathbf{VIAP}(D)$. We define $M^Q = M_* \cap E^Q$, $M^X = M_* \cap E^X$, and $M^Y = M_* \cap E^Y$. Then we have

$$w(M_*) + \tilde{\omega}(B_*) = k|M^X| + (k-1)|M^Y| + \tilde{\omega}(B_*). \tag{24}$$

It holds that

$$\tilde{\omega}(B_*) = \deg\det \tilde{Z}[R^Q, B_*] \leq \max\{\deg\det \tilde{Z}[R^Q, B] \mid B \subseteq R \cup B_*\}. \tag{25}$$

By applying Corollary 6.3 to $\tilde{Z}[R^Q, R \cup B_*]$, we obtain

$$\max\{\deg\det \tilde{Z}[R^Q, B] \mid B \subseteq R \cup B_*\} = \theta_k(s\hat{X}_Q + \hat{Y}_Q), \tag{26}$$

where $\hat{X}_Q = X_Q[R^Q, B_* \cap C^Q]$ and $\hat{Y}_Q = Y_Q[R^Q, B_* \cap C^Q]$. Figure 4 shows $\tilde{Z}(s)$ and $\tilde{Z}[R^Q, R \cup B_*]$. Thus it follows from (24)–(26) that

$$w(M_*) + \tilde{\omega}(B_*) \leq k|M^X| + (k-1)|M^Y| + \theta_k(s\hat{X}_Q + \hat{Y}_Q).$$

We make a copy M_h^X of M^X on edge set E_h^X in $G(\Theta_k(D))$ for $h = 1, \ldots, k$. Similarly, M_h^Y and M_h^Q denote copies of M^Y and M^Q. In $G(\Theta_k(D))$, consider an independent matching

$$\tilde{M} = (M_1^X \cup \cdots \cup M_k^X) \cup (M_1^Y \cup \cdots \cup M_{k-1}^Y) \cup M',$$

where M' satisfies $M' \subseteq M_1^Q \cup \cdots \cup M_k^Q$ and $|M'| = \theta_k(s\hat{X}_Q + \hat{Y}_Q)$. Then $\tilde{M}$ is an independent matching with $|\tilde{M}| = k|M^X| + (k-1)|M^Y| + \theta_k(s\hat{X}_Q + \hat{Y}_Q)$. Thus we have

$$w(M_*) + \tilde{\omega}(B_*) \leq |\tilde{M}| \leq (\text{optimal value of } \mathbf{IMP}(\Theta_k(D))).$$

$\square$

Lemma 6.5. For any LM-matrix pencil $D(s)$, we have

$$(\text{optimal value of } \mathbf{DVIAP}(D)) \geq (\text{optimal value of } \mathbf{DIMP}(\Theta_k(D))).$$

Proof. Let (p^+, p^-) be an integral optimal solution of $\mathbf{DVIAP}(D)$. We can obtain such (p^+, p^-) by the construction rule given in Appendix B. In $G(\Theta_k(D))$, we construct (U^+, U^-) by taking

- $p^-(j)$ copies of $j \in C$ from left to right,
- $j^Q \in \bar{C}^Q$ if U^- does not contain $j \in \bar{C}$,
- $p^+(i)$ copies of $i \in R^T$ from right to left.

The constraints (16)–(19) ensure that (U^+, U^-) is a cover. Moreover, we have

$$(\text{optimal value of } \mathbf{DIMP}(\Theta_k(D))) \leq \operatorname{rank} \bar{Q}[\bar{R}^Q, U^+ \cap \bar{C}^Q] + |U^+ \cap \bar{R}^T| + |U^-|.$$

The construction rule of (U^+, U^-) implies that $|U^+ \cap \bar{R}^T| = p^+(R^T)$ and $|U^-| = p^-(C)$. In addition, it follows from Lemma 3.2 that $\operatorname{rank} \bar{Q}[\bar{R}^Q, U^+ \cap \bar{C}^Q] \leq \delta(Z_Q; p^-)$, where $Z_Q(s) = s^{k-1}(sX_Q + Y_Q)$. Thus we obtain

$$(\text{optimal value of } \mathbf{DIMP}(\Theta_k(D))) \leq \delta(Z_Q; p^-) + p^+(R^T) + p^-(C).$$

Since we have

$$\deg\det Z_Q[I, J] = \deg\det \tilde{Z}[R^Q, (R \setminus I) \cup J],$$

it holds that

$$\begin{aligned}
\delta(Z_Q; p^-) &= \max\{\deg\det Z_Q[I, J] - p^-(J) \mid |I| = |J|\} \\
&= \max\{\deg\det \tilde{Z}[R^Q, (R \setminus I) \cup J] - p^-(J) \mid |I| = |J|\} \\
&= \max_{B \in \tilde{\mathcal{B}}}\{\tilde{\omega}(B) + p^+(B)\}
\end{aligned}$$

by (20) and (21). Thus we obtain

$$\begin{aligned}
(\text{optimal value of } \mathbf{DIMP}(\Theta_k(D))) &\leq \max_{B \in \tilde{\mathcal{B}}}\{\tilde{\omega}(B) + p^+(B)\} + p^+(R^T) + p^-(C) \\
&= (\text{optimal value of } \mathbf{DVIAP}(D)).
\end{aligned}$$

□

Example 6.6. For an LM-matrix pencil given in Example 6.1, we denote the row set and the column set by $R = \{r_1, r_2, r_3\}$ and $C = \{c_1, c_2, c_3, c_4\}$. Consider the case of $k = 3$. Then there exists a feasible solution of $\mathbf{DVIAP}(D)$ such that

$$\begin{aligned}
&p^-(c_1) = 1, \quad p^-(c_2) = p^-(c_4) = 2, \quad p^-(c_3) = 3, \\
&p^+(r_1) = 2, \quad p^+(r_2) = 1, \quad p^+(r_3) = 0, \quad p^+(j^Q) = -p^-(j) \quad (\forall j \in C), \\
&p^+(i) = 0 \quad (\forall i \in R^Q), \quad p^-(j) = 0 \quad (\forall j \in R).
\end{aligned}$$

With these p^+ and p^-, we construct a cover (U^+, U^-) for $G(\Theta_3(D))$ depicted in Figure 5. □

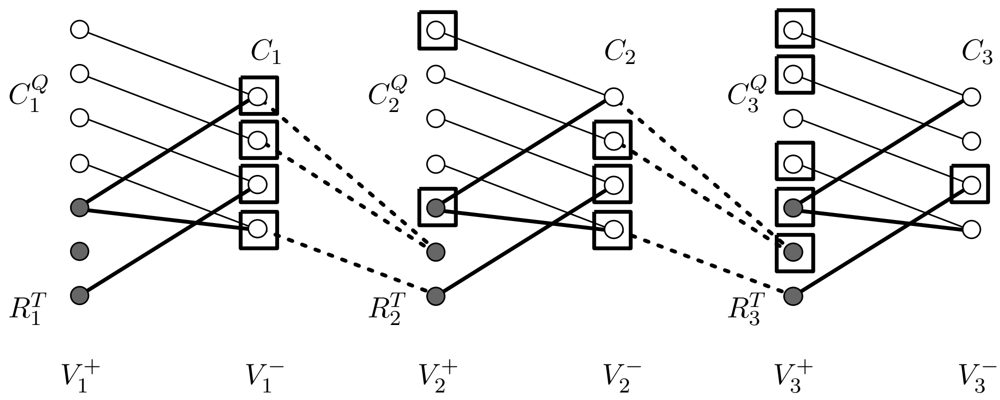

Figure 5: A graph $G(\Theta_3(D))$ and a cover (U^+, U^-) (squares) of Example 6.6.

By Theorems 5.3 and 5.4 and Lemmas 6.4 and 6.5, we obtain

$$\text{(optimal value of } \mathbf{VIAP}(D)) = \text{(optimal value of } \mathbf{IMP}(\Theta_k(D))). \tag{27}$$

Hence (22) follows from (23). This completes the proof of Theorem 6.2.

The equality (27) implies that the independent matching $\tilde{M}$ constructed from an optimal solution (M_*, B_*) of $\mathbf{VIAP}(D)$ in the proof of Lemma 6.4 is in fact an optimal solution of $\mathbf{IMP}(\Theta_k(D))$. Thus, $\mathbf{IMP}(\Theta_k(D))$ has an optimal solution with periodic structure such that each edge in $\bigcup_{h=1}^{k} E_h^X$ has k copies and each edge in $\bigcup_{h=1}^{k-1} E_h^Y$ has $k-1$ copies. We now exploit this optimal solution $\tilde{M}$ to give an alternative proof of Theorem 6.2.

Consider the submatrix of $\Theta_k(D)[\partial^+\tilde{M}, \partial^-\tilde{M}]$, where $\partial^+\tilde{M}$ and $\partial^-\tilde{M}$ denote the set of vertices in $\bar{V}^+$ and $\bar{V}^-$ incident to $\tilde{M}$, respectively. The expansion of $\det \Theta_k(D)[\partial^+\tilde{M}, \partial^-\tilde{M}]$ contains a nonzero term

$$\prod_{(i,j)\in M^X} {(X_T)_{ij}}^k \prod_{(i,j)\in M^Y} {(Y_T)_{ij}}^{k-1},$$

where $(X_T)_{ij}$ and $(Y_T)_{ij}$ denote the (i,j)-entries of X_T and Y_T. Each $(X_T)_{ij}$ appears exactly k times and each $(Y_T)_{ij}$ appears exactly $k-1$ times in $G(\Theta_k(D))$. Hence no other independent matching cancels this term in the expansion of $\det \Theta_k(D)[\partial^+\tilde{M}, \partial^-\tilde{M}]$. Thus $\Theta_k(D)[\partial^+\tilde{M}, \partial^-\tilde{M}]$ is nonsingular, which implies (22). This completes the second proof of Theorem 6.2.

The first proof makes use of (23), which is obtained by the results in [19]. In contrast, the second proof does not rely on (23) but an optimal solution with a periodic structure of $\mathbf{IMP}(\Theta_k(D))$.

6.4 Time Complexity

Let $D_{\mathrm{M}}(s)$ be an $m \times n$ matrix pencil with rank r and $D(s)$ an associated LM-matrix pencil defined by (3). In order to compute the indices of nilpotency $\mu_1, \ldots, \mu_d$, we have to solve r independent matching problems $\mathbf{IMP}(\Theta_1(D))$, ..., $\mathbf{IMP}(\Theta_r(D))$.

Murota's algorithm [19] for computing $\mu_1, \ldots, \mu_d$ solves a valuated independent assignment problem, which needs to deal with rational function matrices. In contrast, our algorithm requires only constant matrix computation, because it solves independent matching problems for linear matroids. Moreover, our algorithm has benefits in that the independent matching problem admits a variety of efficient algorithms in comparison with a valuated independent assignment problem.

In fact, an independent matching problem is known to be equivalent to a matroid intersection problem, and in particular, a linear matroid intersection problem in this case. For a linear matroid intersection problem, several algorithms have been developed in [2, 7, 8]. Let n denote the number of vertices of a linear matroid. Cunningham's algorithm [2] runs in $\mathrm{O}(n^3 \log n)$ time, and Gabow and Xu's algorithm [7] runs in $\mathrm{O}(n^{2+\frac{1}{4-\omega}})$ time, where $\omega < 2.38$ is the matrix multiplication exponent. The current fastest one is Harvey's randomized algorithm [8], which runs in $\mathrm{O}(n^\omega)$ time.

Since $D(s)$ is a $2m \times (m+n)$ LM-matrix pencil, a graph $G(\Theta_k(D))$ has $\mathrm{O}(kn)$ vertices and $\mathrm{O}(kn^2)$ edges under the assumption that $m \leq n$ holds. Therefore, we can solve $\mathbf{IMP}(\Theta_k(D))$ for $k = 1, \ldots, r$ in $\mathrm{O}(r^{3+\frac{1}{4-\omega}} n^{2+\frac{1}{4-\omega}})$ time by Gabow and Xu's algorithm [7], and in $\mathrm{O}(r^{1+\omega} n^\omega)$ time by Harvey's algorithm [8].

7 Application to Power Product of Mixed Matrices

We now consider the problem of computing the rank of A^k for an $n \times n$ mixed matrix A and a positive integer k. An algorithm for computing the rank of mixed matrices was described by Murota and Iri [21]. However, since A^k has an independent parameter appearing multiple times, A^k itself is not a mixed matrix. This prevents us from applying that algorithm directly to A^k.

Instead, we compute $\operatorname{rank} A^k$ via the expanded matrix. For an expanded matrix

$$\Theta_k(sA+I) = \begin{pmatrix} A & O & \cdots & \cdots & O \\ I & A & \ddots & & \vdots \\ O & I & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & A & O \\ O & \cdots & O & I & A \end{pmatrix},$$

we can transform $\Theta_k(sA+I)$ into

$$\begin{pmatrix} O & O & \cdots & O & (-1)^{k+1}A^k \\ I & O & \ddots & \vdots & (-1)^k A^{k-1} \\ O & I & \ddots & O & \vdots \\ \vdots & \ddots & \ddots & O & -A^2 \\ O & \cdots & O & I & A \end{pmatrix}$$

by row operations. Hence we obtain

$$\operatorname{rank} A^k = \theta_k(sA+I) - (k-1)n.$$

Therefore, $\operatorname{rank} A^k$ is determined by $\theta_k(sA+I)$, which can be computed by solving an independent matching problem.

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A Proof of Lemma 3.3

Consider $\mathbf{PLP}(Z)$ and $\mathbf{DLP}(Z)$ for a Laurent polynomial matrix $Z(s) = (Z_{ij}(s))$ with row set R and column set C. By the complementary slackness condition, we have

$$p(i) + p(j) - c(e) > 0 \quad \Rightarrow \quad \xi(e) = 0, \tag{28}$$

$$\sum_{\partial e \ni i} \xi(e) < 1 \quad \Rightarrow \quad p(i) = 0. \tag{29}$$

For a dual feasible solution p, we define a bipartite graph $G^*(p) = (R, C; E^*(p))$ with the set of tight edges

$$E^*(p) = \{e \in E \mid p(i) + p(j) - c(e) = 0\}.$$

The tight coefficient matrix $Z^* = (Z^*_{ij})$ is defined by

$$Z^*_{ij} = \begin{cases} \text{the coefficient of } s^{p(i)+p(j)} \text{ in } Z_{ij}(s) & \text{if } e = (i,j) \in E^* \\ 0 & \text{otherwise.} \end{cases}$$

The set of nonzero entries of Z^* corresponds to the edge set $E^*(p)$. Let us define $p_{\mathrm{R}} = (p(i) \mid i \in R)$ and $p_{\mathrm{C}} = (p(j) \mid j \in C)$. By the definition of Z^*, we have

$$Z(s) = \mathrm{diag}(s; p_{\mathrm{R}})(Z^* + Z^\infty)\,\mathrm{diag}(s; p_{\mathrm{C}}), \tag{30}$$

where Z^∞ denotes a strictly proper Laurent polynomial matrix, and $\mathrm{diag}(s; r)$ is a diagonal matrix with diagonal entries $s^{r_1}, s^{r_2}, \ldots$ with $r = (r_1, r_2, \ldots)$.

The active rows and columns are defined by

$$I^* = \{i \in R \mid p(i) > 0\} \quad \text{and} \quad J^* = \{j \in C \mid p(j) > 0\}.$$

We now prove the following lemmas in a similar way to [16].

Lemma A.1. Let $Z(s)$ be a Laurent polynomial matrix and p a dual feasible solution of $\mathbf{DLP}(Z)$. Then p is optimal if and only if

$$\text{term-rank}\, Z^*[I^*, C] = |I^*| \quad \text{and} \quad \text{term-rank}\, Z^*[R, J^*] = |J^*|.$$

Lemma A.2. Let $Z(s)$ be a Laurent polynomial matrix and p an optimal dual solution of $\mathbf{DLP}(Z)$. Then $\delta(Z) = \hat{\delta}(Z)$ holds if and only if

$$\mathrm{rank}\, Z^*[I^*, C] = |I^*| \quad \text{and} \quad \mathrm{rank}\, Z^*[R, J^*] = |J^*|.$$

In the proof of these lemmas, we make use of a property of linking functions. A *linking function* is a function $\lambda : 2^R \times 2^C \to \mathbf{Z}$ which satisfies the following conditions [12, 23].

(B-1) $0 \le \lambda(I, J) \le \min\{|I|, |J|\}$ for $I \subseteq R$ and $J \subseteq C$.

(B-2) $\lambda(I', J') \leq \lambda(I, J)$ for $I' \subseteq I \subseteq R$ and $J' \subseteq J \subseteq C$.

(B-3) $\lambda(I, J) + \lambda(I', J') \leq \lambda(I \cup I', J \cap J') + \lambda(I \cap I', J \cup J')$ for $I, I' \subseteq R$ and $J, J' \subseteq C$.

In particular, the condition (B-3) is called the *bisubmodularity*. The rank and term-rank are two principal examples of linking functions [20]. A linking function has the following property.

Lemma A.3. Let $\lambda : 2^R \times 2^C \to \mathbf{Z}$ be a linking function, and let $I^* \subseteq R$ and $J^* \subseteq C$ be given. Then, there exist $I \supseteq I^*$ and $J \supseteq J^*$ such that $\lambda(I, J) = |I| = |J|$ if and only if

$$\lambda(I^*, C) = |I^*| \quad \text{and} \quad \lambda(R, J^*) = |J^*|.$$

Proof. The conditions $\lambda(I^*, C) = |I^*|$ and $\lambda(R, J^*) = |J^*|$ are obviously necessary. We show the sufficiency below. Let us denote $\lambda(I^*, J^*)$ by r^*. Then, there exist $I_1 \subseteq I^*$ and $J_1 \subseteq J^*$ such that $\lambda(I_1, J_1) = |I_1| = |J_1| = r^*$. Hence $\lambda(I^*, J_1) = |J_1|$ holds. Since we have $\lambda(I^*, C) = |I^*|$ by the assumption, there exists $J_2 \subseteq C \setminus J^*$ such that

$$\lambda(I^*, J_1 \cup J_2) = |I^*| = |J_1| + |J_2|. \tag{31}$$

By the bisubmodularity (B-3) of λ, it holds that

$$\begin{aligned} \lambda(R, J^* \cup J_2) + \lambda(I^*, J^*) &\geq \lambda(R, J^*) + \lambda(I^*, J^* \cup J_2) \\ &= |J^*| + |J_1| + |J_2|, \end{aligned}$$

where the last step is due to the assumption and (31). By $\lambda(I^*, J^*) = |J_1|$, we have $\lambda(R, J^* \cup J_2) \geq |J^*| + |J_2|$, and hence

$$\lambda(R, J^* \cup J_2) = |J^*| + |J_2|. \tag{32}$$

On the other hand, it holds that

$$|I^*| \geq \lambda(I^*, J^* \cup J_2) \geq \lambda(I^*, J_1 \cup J_2) = |I^*|,$$

where the last step is due to (31). Hence we obtain $\lambda(I^*, J^* \cup J_2) = |I^*|$. This implies that there exists $I_2 \subseteq R \setminus I^*$ such that

$$|I^*| + |I_2| = \lambda(I^* \cup I_2, J^* \cup J_2) = \lambda(R, J^* \cup J_2) = |J^*| + |J_2|,$$

where the last step is due to (32). This completes the proof by setting $I = I^* \cup I_2$ and $J = J^* \cup J_2$. $\square$

We first prove Lemma A.1 by using Lemma A.3.

Proof of Lemma A.1. By rewriting the conditions (28) and (29) with I^* and J^*, we obtain the following claim.

Claim A.4. Let $G(Z)$ be a bipartite graph defined in Section 3. A matching M in $G(Z)$ and a dual feasible solution p of $\mathbf{DLP}(Z)$ are optimal if and only if $\partial M \cap R \supseteq I^*$, $\partial M \cap C \supseteq J^*$, and $M \subseteq E(p)$.

We now rephrase Claim A.4 in terms of matrices.

Claim A.5. Let p be a dual feasible solution. Then p is optimal if and only if there exist $I \supseteq I^*$, $J \supseteq J^*$, and term-rank $Z^*[I, J] = |I| = |J|$.

This claim together with Lemma A.3 applied to the linking function term-rank $Z^*[\cdot, \cdot]$ completes the proof of Lemma A.1. □

Next, we give a proof of Lemma A.2.

Proof of Lemma A.2. By (30), we have $Z_{ij}(s) = s^{p(i)+p(j)}(Z^*_{ij} + Z^\infty_{ij})$, where $Z^\infty = (Z^\infty_{ij})$ denotes a strictly proper Laurent polynomial matrix. Thus, for $I \subseteq R$ and $J \subseteq C$ with $|I| = |J|$, we obtain

$$\det Z[I, J] = s^{p(I \cup J)} \det(Z^*[I, J] + Z^\infty[I, J]).$$

If p is optimal and if $I \supseteq I^*$ and $J \supseteq J^*$, it holds that $p(I \cup J) = p(R \cup C) = \hat{\delta}(Z)$, which implies that

$$\det Z[I, J] = s^{\hat{\delta}(Z)} \det(Z^*[I, J] + Z^\infty[I, J]).$$

This yields the following claim.

Claim A.6. Let p be an optimal dual solution. Then $\delta(Z) = \hat{\delta}(Z)$ holds if and only if there exist $I \supseteq I^*$ and $J \supseteq J^*$ such that rank $Z^*[I, J] = |I| = |J|$.

This claim together with Lemma A.3 applied to the linking function rank $Z^*[\cdot, \cdot]$ completes the proof of Lemma A.2. □

In order to prove Lemma 3.3, we show the following lemma.

Lemma A.7. Let $Z(s)$ be a Laurent polynomial matrix with $\delta(Z) < \hat{\delta}(Z)$. Then there exists a biproper Laurent polynomial matrix $F(s)$ such that $\hat{\delta}(Z') \leq \hat{\delta}(Z) - 1$ with $Z'(s) = F(s)Z(s)$.

Proof. Let p be an optimal dual solution of $\mathbf{DLP}(Z)$ and Z^* the tight coefficient matrix. It follows from $\delta(Z) < \hat{\delta}(Z)$ that

$$\operatorname{rank} Z^*[I^*, C] < |I^*| \quad \text{or} \quad \operatorname{rank} Z^*[R, J^*] < |J^*| \tag{33}$$

by Lemma A.2.

Consider the former case, where we have rank $Z^*[I^*, C] < |I^*| =$ term-rank $Z^*[I^*, C]$. Then we have the following claim.

Claim A.8. There exists a nonsingular constant matrix $F^* = (F^*_{ih})$ which satisfies

$$\text{term-rank}(F^* Z^*)[I^*, C] \leq |I^*| - 1, \tag{34}$$

$$F^*_{ih} \neq 0 \Rightarrow p(i) \leq p(h) \quad (\forall i, h \in R) \quad \text{and} \quad F^*_{ii} = 1 \quad (\forall i \in R). \tag{35}$$

Proof. For the sake of simplicity, we may assume that $p(i) \geq p(h)$ holds for any $i, h \in I^*$ with $i \leq h$ and that $j > |I^*|$ holds for any $j \notin I^*$. Let $\boldsymbol{z}_i^*$ denote the ith row vector of $Z^*[I^*, C]$. We construct the basis $\{\boldsymbol{z}_i^* \mid i \in B\}$ by picking up the independent vectors from the sequence $\boldsymbol{z}_1^*, \boldsymbol{z}_2^*, \ldots, \boldsymbol{z}_{|I^*|}^*$ in this order. Let $\boldsymbol{z}_l^*$ ($l \leq |I^*|$) be the first row vector in this sequence that does not belong to the basis. We define the lth row vector of the matrix F^* by

$$-\boldsymbol{z}_l^* = \sum_{h<l} F_{lh}^* \boldsymbol{z}_h^*, \quad F_{ll}^* = 1, \quad F_{lh}^* = 0 \ (h > l).$$

For any $i \neq l$, the ith row vector of F^* is defined to be the ith unit vector. Then the lth row vector of $(F^* Z^*)[I^*, C]$ is zero, and hence (34) holds. The construction of F^* indicates that F^* satisfies (35). $\square$

With F^* in Claim A.8, we define $F(s)$ by

$$F(s) = \operatorname{diag}(s; p_{\mathrm{R}}) F^* \operatorname{diag}(s; -p_{\mathrm{R}}), \tag{36}$$

and put $Z'(s) = F(s)Z(s)$. Then $F(s)$ has the following property.

Claim A.9. The matrix $F(s)$ is a biproper Laurent polynomial matrix which satisfies $\hat{\delta}(Z') \leq \hat{\delta}(Z) - 1$.

Proof. First, we show that $F(s)$ is a biproper Laurent polynomial matrix. The matrix $F(s)$ is a Laurent polynomial matrix by (36). Moreover, it follows from (35) that $F(s)$ is proper and $\det F(s)$ is constant. Thus, $F(s)$ is a biproper Laurent polynomial matrix.

Next, we prove $\hat{\delta}(Z') \leq \hat{\delta}(Z) - 1$. By the definition of $Z'(s)$, (30), and (36), we have

$$\begin{aligned}\operatorname{diag}(s; -p_{\mathrm{R}}) Z'(s) \operatorname{diag}(s; -p_{\mathrm{C}}) &= \operatorname{diag}(s; -p_{\mathrm{R}}) F(s) Z(s) \operatorname{diag}(s; -p_{\mathrm{C}}) \\ &= F^* \operatorname{diag}(s; -p_{\mathrm{R}}) Z(s) \operatorname{diag}(s; -p_{\mathrm{C}}) \\ &= F^*(Z^* + Z^\infty) = F^* Z^* + F^* Z^\infty,\end{aligned}$$

where Z^∞ designates a strictly proper Laurent polynomial matrix. This implies that $c'(e) - p(i) - p(j) \leq 0$ for $c'(e) = \deg Z'_{ij}(s)$. Hence p is feasible for $\mathbf{DLP}(Z')$, but not optimal by Lemma A.1 and (34). Thus we have

$$\hat{\delta}(Z') < p(R \cup C) = \hat{\delta}(Z),$$

because p is optimal for $\mathbf{DLP}(Z)$. $\square$

By Claims A.8 and A.9, we complete the proof of the former case in (33). We can show the latter case similarly by replacing $Z^*[I^*, C]$ by $Z^*[R, J^*]$. This completes the proof of Lemma A.7. $\square$

We now complete the proof of Lemma 3.3. Let $Z(s)$ be a Laurent polynomial matrix with $\delta(Z) < \hat{\delta}(Z)$. By Lemma A.7, there exists a biproper Laurent polynomial matrix $F_1(s)$ such that $\hat{\delta}(Z_1) \leq \hat{\delta}(Z) - 1$ with $Z_1(s) = F_1(s)Z(s)$. Since $F_1(s)$ is biproper, $\delta(Z_1) = \delta(Z)$ holds.

If $\delta(Z_1) = \hat{\delta}(Z_1)$ holds, then $F_1(s)$ serves as $F(s)$ in Lemma 3.3. Otherwise, there exists a biproper Laurent polynomial matrix $F_2(s)$ such that $\hat{\delta}(Z_2) \leq \hat{\delta}(Z_1) - 1$ with $Z_2(s) = F_2(s)Z_1(s)$ by Lemma A.7.

Thus, by applying Lemma A.7 repeatedly, we obtain biproper Laurent polynomial matrices $F_1(s), F_2(s), \ldots, F_h(s)$ such that $\hat{\delta}(Z_{i+1}) \leq \hat{\delta}(Z_i) - 1$ holds for $i = 1, \ldots, h-1$ and $\hat{\delta}(Z_h) = \delta(Z_h)$ holds, where $Z_{i+1}(s) = F_{i+1}(s)Z_i(s)$. By setting $F(s) = F_h(s)F_{h-1}(s)\cdots F_1(s)$, we have $\delta(Z) = \delta(FZ) = \hat{\delta}(FZ)$, and $F(s) = F_0 + F_{-1}s^{-1} + \cdots$ is a biproper Laurent matrix with nonsingular matrix F_0. This completes the proof of Lemma 3.3.

B Proof of Theorem 5.4

Let (M, B^+, B^-) be a feasible solution for **VIAP** and (p^+, p^-) a feasible solution for **Dual Problem for VIAP**. It follows from (9) that

$$\begin{aligned}\Omega(M, B^+, B^-) &= w(M) + \omega^+(B^+) + \omega^-(B^-) \\ &\leq p^+(\partial^+ M) + p^-(\partial^- M) + \omega^+(B^+) + \omega^-(B^-).\end{aligned}$$

By (7), (8), (10) and (11), it holds that $p^+(\partial^+ M) \leq p^+(B^+)$ and $p^-(\partial^- M) \leq p^-(B^-)$. Hence we have

$$\begin{aligned}p^+(\partial^+ M) + p^-(\partial^- M) + \omega^+(B^+) + \omega^-(B^-) &\leq \{p^+(B^+) + \omega^+(B^+)\} + \{p^-(B^-) + \omega^-(B^-)\} \\ &\leq \zeta^+(p^+) + \zeta^-(p^-).\end{aligned}$$

Thus we obtain

$$\Omega(M, B^+, B^-) \leq \zeta^+(p^+) + \zeta^-(p^-). \tag{37}$$

Let (M_*, B_*^+, B_*^-) be an optimal solution for **VIAP**. We prove that there exists (p^+, p^-) such that

$$\Omega(M_*, B_*^+, B_*^-) = \zeta^+(p^+) + \zeta^-(p^-), \tag{38}$$

which completes the proof by (37).

Let us denote the reorientation of $a \in E$ by a°. In order to find (p^+, p^-) satisfying (38), we construct an auxiliary graph $\tilde{G} = (\tilde{V}, A)$ with

$$\tilde{V} = V^+ \cup V^- \cup \{s\} \quad \text{and} \quad A = \tilde{E} \cup E^+ \cup E^- \cup M^\circ \cup S,$$

where s is a new vertex and

$$\begin{aligned}\tilde{E} &= \{(i,j) \mid (i,j) \in E\} \quad (\text{copy of } E), \\ E^+ &= \{(i,j) \mid i \in B_*^+, j \in V^+ \setminus B_*^+, B_*^+ \setminus \{i\} \cup \{j\} \in \mathcal{B}^+\}, \\ E^- &= \{(j,i) \mid i \in B_*^-, j \in V^- \setminus B_*^-, B_*^- \setminus \{i\} \cup \{j\} \in \mathcal{B}^-\}, \\ M^\circ &= \{a^\circ \mid a \in M_*\}, \\ S &= \{(s,i) \mid i \in (B_*^+ \setminus \partial^+ M_*) \cup (V^- \setminus V_0^-)\}.\end{aligned}$$

We define the arc length $\gamma : A \to \mathbf{Z}$ by

$$\gamma(a) = \begin{cases} -w(a) & (a \in \tilde{E}), \\ -\omega^+(B_*^+ \setminus \{i\} \cup \{j\}) + \omega^+(B_*) & (a = (i,j) \in E^+), \\ -\omega^-(B_*^- \setminus \{i\} \cup \{j\}) + \omega^-(B_*) & (a = (j,i) \in E^-), \\ w(a^\circ) & (a \in M_*^\circ), \\ 0 & (a = (s,i) \in S). \end{cases} \tag{39}$$

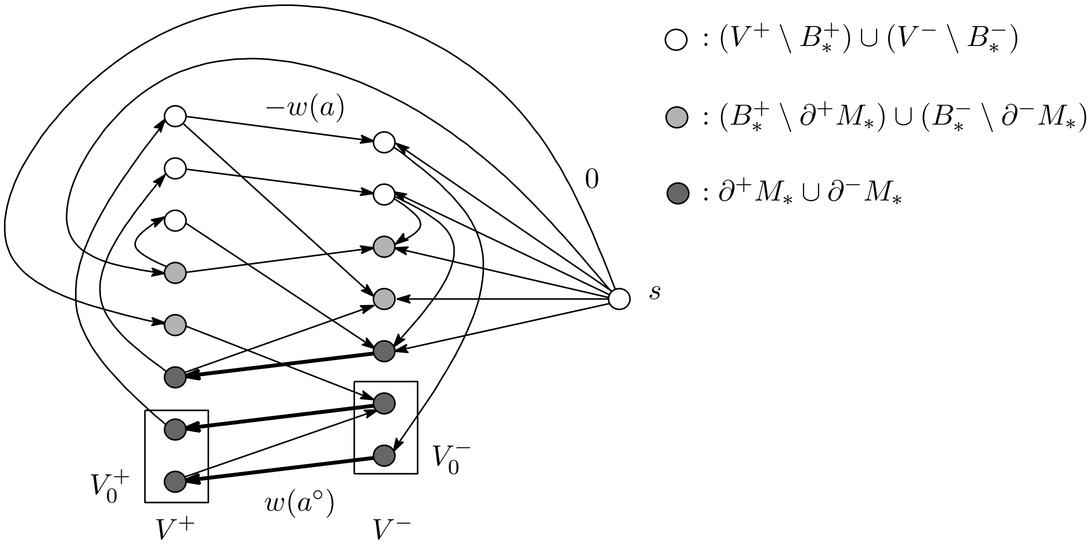

Figure 6: An auxiliary graph $\tilde{G} = (\tilde{V}, A)$, where heavy lines show edges in a matching.

Figure 6 illustrates $\tilde{G} = (\tilde{V}, A)$.

Let $d(i)$ be a shortest distance from s to $i \in V^+ \cup V^-$ with respect to the arc length γ in $\tilde{G}$. If there exists no path from s to i, then we put $d(i) = \infty$. Since all arcs entering $i \in B_*^+ \setminus \partial^+ M_*$ start from s, we have

$$d(i) = 0 \quad (\forall i \in B_*^+ \setminus \partial^+ M_*). \tag{40}$$

Let us assume that there exists a shortest path P from s to $i \in (\partial^+ M_* \setminus V_0^+) \cup (V^+ \setminus B_*^+)$ with negative distance. Then, the triple $(\hat{M}, \hat{B}^+, \hat{B}^-)$ obtained by

$$\begin{aligned}
\hat{M} &= M_* \setminus \{a \in M_* \mid a^\circ \in P \cap M_*^\circ\} \cup (P \cap \tilde{E}), \\
\hat{B}^+ &= B_*^+ \setminus \{i \mid (i,j) \in P \cap E^+\} \cup \{j \mid (i,j) \in P \cap E^+\}, \\
\hat{B}^- &= B_*^- \setminus \{i \mid (j,i) \in P \cap E^-\} \cup \{j \mid (j,i) \in P \cap E^-\},
\end{aligned}$$

satisfies $\Omega(\hat{M}, \hat{B}^+, \hat{B}^-) > \Omega(M_*, B_*^+, B_*^-)$, which contradicts the optimality of (M_*, B_*^+, B_*^-). Hence we have

$$d(i) \geq 0 \quad (\forall i \in (\partial^+ M_* \setminus V_0^+) \cup (V^+ \setminus B_*^+)).$$

Similarly, it holds that $d(j) \geq 0$ for $j \in B_*^- \setminus \partial^- M_*$. Since there exists an arc (s, j) for each $j \in V^- \setminus V_0^-$, we have

$$d(j) = 0 \quad (\forall j \in B_*^- \setminus \partial^- M_*), \tag{41}$$

$$d(j) \leq 0 \quad (\forall j \in (\partial^- M_* \setminus V_0^-) \cup (V^- \setminus B_*^-)). \tag{42}$$

By $p^+(i) = d(i)$ for $i \in V^+$ and $p^-(j) = -d(j)$ for $j \in V^-$, we obtain (p^+, p^-) satisfying (9)–(11).

For $(i,j) \in E^+$, it holds that

$$p^+(i) - \omega^+(B_*^+ \setminus \{i\} \cup \{j\}) + \omega^+(B_*^+) \geq p^+(j),$$

which implies

$$\omega^+(B_*^+) + p^+(B_*^+) \geq \omega^+(B_*^+ \setminus \{i\} \cup \{j\}) + p^+(B_*^+ \setminus \{i\} \cup \{j\}).$$

Hence B_*^+ is a maximizer of $\omega^+ + p^+$ by Theorem 5.1. Similarly B_*^- is a maximizer of $\omega^- + p^-$. Therefore, we have

$$\begin{aligned}\zeta^+(p^+) + \zeta^-(p^-) &= \max_{B^+ \in \mathcal{B}^+} \{\omega^+(B^+) + p^+(B^+)\} + \max_{B^- \in \mathcal{B}^-} \{\omega^-(B^-) + p^-(B^-)\} \\ &= \omega^+(B_*^+) + p^+(B_*^+) + \omega^-(B_*^-) + p^-(B_*^-). \end{aligned} \tag{43}$$

It follows from (40) and (41) that

$$p^+(B_*^+ \setminus \partial^+ M_*) = 0 \quad \text{and} \quad p^-(B_*^- \setminus \partial^- M_*) = 0.$$

Since $p^+(\partial^+ M_*) + p^-(\partial^- M_*) = w(M_*)$ holds, we have

$$\begin{aligned}\omega^+(B_*^+) + p^+(B_*^+) + \omega^-(B_*^-) + p^-(B_*^-) &= \omega^+(B_*^+) + p^+(\partial^+ M_*) + \omega^-(B_*^-) + p^-(\partial^- M_*) \\ &= \omega^+(B_*^+) + \omega^-(B_*^-) + w(M_*) \\ &= \Omega(M_*, B_*^+, B_*^-). \end{aligned} \tag{44}$$

Thus we obtain (p^+, p^-) satisfying (38) by (43) and (44).