MATHEMATICAL ENGINEERING TECHNICAL REPORTS

Ranking Patterns of Unfolding Models of Codimension One

Hidehiko KAMIYA, Akimichi TAKEMURA and Hiroaki TERAO

METR 2010–05 February 2010

DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

WWW page: http://www.keisu.t.u-tokyo.ac.jp/research/techrep/index.html

Ranking patterns of unfolding models of codimension one

Hidehiko Kamiya [*]
Akimichi Takemura [†]
Hiroaki Terao [‡]

February 2010

Abstract

We consider the problem of counting the number of possible sets of rankings (called ranking patterns) generated by unfolding models of codimension one. We express the ranking patterns as slices of the braid arrangement and show that all braid slices, including those not associated with unfolding models, are in one-to-one correspondence with the chambers of an arrangement. By identifying those which are associated with unfolding models, we find the number of ranking patterns. We also give an upper bound for the number of ranking patterns when the difference by a permutation of objects is ignored.

Keywords: all-subset arrangement, braid arrangement, chamber, characteristic polynomial, finite field method, hyperplane arrangement, ideal point, mid-hyperplane arrangement, ranking pattern, unfolding model.

1 Introduction

The unfolding model, also known as the ideal point model, is a model for preference rankings, and was introduced by Coombs [3] in psychometrics. Since then, this model has been widely used not only in psychometrics (De Soete, Feger and Klauer [6]) but also in other fields such as marketing science (DeSarbo and Hoffman [5], MacKay, Easley and Zinnes [17]). The same mathematical structure can also be found in voting theory (Hinich and Munger [10]).

In this paper, we consider the problem of counting the number of possible sets of rankings (called ranking patterns) generated by the unfolding model. We deal with the

[*]This work was partially supported by JSPS KAKENHI (19540131). *Faculty of Economics, Okayama University*

[†]This research was supported by JST CREST. *Graduate School of Information Science and Technology, University of Tokyo*

[‡]This work was partially supported by JSPS KAKENHI (21340001). *Department of Mathematics, Hokkaido University*

case where the restriction by dimension is weakest, and give the answer in terms of the number of chambers of a hyperplane arrangement.

Suppose we have a set of m objects labeled $1, 2, \ldots, m$ and an individual who ranks these m objects according to his/her preference. In the unfolding model, it is assumed that the m objects $1, 2, \ldots, m$ are represented by points $\mu_1, \mu_2, \ldots, \mu_m$ in the Euclidean space $\mathbb{R}^n$. Moreover, the individual is also represented by a point y in the same $\mathbb{R}^n$. This y is called the ideal point of the individual, and is identified with the individual. Then $\mathbb{R}^n$ containing both the objects and the individual is called the joint space in the psychometric literature. Now, according to the unfolding model, individual y prefers object i to object j if and only if y is closer to μ_i than to μ_j in the usual Euclidean distance, i.e., $\|y - \mu_i\| < \|y - \mu_j\|$. So individual y gives ranking $(i_1 i_2 \cdots i_m)$, meaning that i_1 is the individual's best object, i_2 is his/her second best object, and so on, if and only if y is closest to μ_{i_1}, second closest to μ_{i_2}, and so on.

In general, of course, we can think of $m!$ rankings among m objects. But in the unfolding model, not all the $m!$ rankings are generated; there are admissible rankings and inadmissible rankings. That is, if there is a point y in the joint space $\mathbb{R}^n$ which is closest to μ_{i_1}, second closest to μ_{i_2}, and so on, then the ranking $(i_1 i_2 \cdots i_m)$ is admissible. On the contrary, if there is no such point y, then $(i_1 i_2 \cdots i_m)$ is inadmissible. So a natural question is: What is the number of admissible rankings for a given set of m objects? This problem has been solved, and the number is expressed in terms of the signless Stirling numbers of the first kind (Good and Tideman [9], Kamiya and Takemura [12, 13], Zaslavsky [24]).

Now, as we explained, for a given set of objects $\mu_1, \mu_2, \ldots, \mu_m$, we have admissible rankings. Let us call the set of all admissible rankings the ranking pattern of the unfolding model with $\mu_1, \mu_2, \ldots, \mu_m$. Then, if we change $\mu_1, \mu_2, \ldots, \mu_m$, we obtain a different ranking pattern. Our question is: How many ranking patterns are possible by taking different choices of $\mu_1, \mu_2, \ldots, \mu_m$?

In the unidimensional case $n = 1$, determining the ranking pattern corresponds to determining the order of $m(m-1)/2$ midpoints of the objects on the real line $\mathbb{R}^1$. Thrall [21] gave an upper bound for the number of ranking patterns in this unidimensional case. He obtained his upper bound by considering a problem similar to that of counting the number of standard Young tableaux. Recently, Kamiya, Orlik, Takemura and Terao [11] found the exact number of ranking patterns of the unidimensional unfolding model. They showed that the exact number can be obtained by counting the number of chambers of an arrangement called the mid-hyperplane arrangement. (See also Stanley [19].) However, the problem of counting the number of ranking patterns is not easy for general dimension.

In the present paper, we consider the problem of counting the number of ranking patterns when the unfolding model is "of codimension one," i.e., when $n = m - 2$ so that the restriction by dimension is weakest. In this case, we show that there is a one-to-one correspondence between the set of ranking patterns and a subset of the set of chambers of an arrangement (a restriction of the "all-subset arrangement"). By this one-to-one correspondence, we can obtain the number of ranking patterns.

In some cases, specific labelings of the objects matter; e.g., the ranking pattern consisting of all rankings except those with the status quo at the bottom does not mean the same thing that is meant by the ranking pattern consisting of all rankings except those with an object other than the status quo at the bottom. In that case, the abovementioned one-to-one correspondence between the set of ranking patterns and a subset of chambers of a restriction of the all-subset arrangement gives the answer, i.e., the number of ranking patterns. But in other cases, the objects are neutral, and we do not care about the difference between two equivalent ranking patterns in the sense that one is obtained from the other by a permutation of the objects. In the latter case, what we actually want to know is the number of inequivalent ranking patterns. We give an upper bound for this number.

The organization of this paper is as follows. In Section 2, we see that the ranking pattern of the unfolding model of codimension one can be obtained by slicing the braid arrangement by an affine hyperplane, although not all these slices can be realized by unfolding models. In Section 3, the set of braid slices is shown to be in one-to-one correspondence with the set of chambers of a restriction of the all-subset arrangement. Of these chambers, some correspond to braid slices realizable by unfolding models, and others correspond to unrealizable ones. This distinction is made in Section 4. Based on these results, we give the number of ranking patterns of unfolding models of codimension one in Section 5. In the final section (Section 6), we provide an upper bound for the number of inequivalent ranking patterns.

2 The unfolding model as a braid slice

In this section, we show that the ranking pattern of the unfolding model of codimension one can be obtained by slicing the braid arrangement by an affine hyperplane.

Let m be an integer with $m \geq 3$. Denote by $\mathbb{P}_m$ the set of all permutations of $[m] := \{1, \dots, m\}$: $\mathbb{P}_m := \{(i_1 \cdots i_m) : (i_1 \cdots i_m) \text{ is a permutation of } [m]\}$.

Define

$$H_0 := \{x = (x_1, \dots, x_m)^T \in \mathbb{R}^m : x_1 + \cdots + x_m = 0\},$$

and put

$$C_{i_1 \cdots i_m} := \{x = (x_1, \dots, x_m)^T \in H_0 : x_{i_1} > \cdots > x_{i_m}\}, \quad (i_1 \cdots i_m) \in \mathbb{P}_m.$$

Note that $C_{i_1 \cdots i_m}$ is a chamber of the arrangement $\mathcal{B}_m^{H_0} := \{H \cap H_0 : H \in \mathcal{B}_m\}$ in H_0, where

$$\mathcal{B}_m := \{H_{ij} : 1 \leq i < j \leq m\}$$

with $H_{ij} := \{x = (x_1, \dots, x_m)^T \in \mathbb{R}^m : x_i = x_j\}$ is the braid arrangement.

Now, for any $v \in \mathbb{S}^{m-2} := \{x \in H_0 : \|x\| = 1\}$, let us define a hyperplane K_v in H_0 by

$$K_v := \{x \in H_0 : v^T x = 1\}.$$

We call

$$\mathrm{RP}(v) := \{(i_1 \cdots i_m) \in \mathbb{P}_m : K_v \cap C_{i_1 \cdots i_m} \neq \emptyset\}, \quad v \in \mathbb{S}^{m-2}, \tag{1}$$

the *ranking pattern of the braid slice* by K_v.

In general, for m distinct points $\nu_1, \dots, \nu_m \in \mathbb{R}^N$ ($m \geq N+1$), let $\overline{\nu_i \nu_j}$ denote the one-simplex connecting two points ν_i and ν_j ($i < j$). Consider the following condition:

(A) The union of N distinct one-simplices $\overline{\nu_{i_k} \nu_{j_k}}$ ($i_k < j_k,\ k = 1, \dots, N$) contains no loop if and only if the corresponding vectors $\nu_{i_k} - \nu_{j_k}$ ($k = 1, \dots, N$) are linearly independent.

Recall, in general, that $N+1$ points $\tilde{\nu}_1, \ldots, \tilde{\nu}_{N+1} \in \mathbb{R}^N$ are said to be in general position if they are the vertices of an N-simplex, in other words, the N vectors $\tilde{\nu}_1 - \tilde{\nu}_2, \tilde{\nu}_2 - \tilde{\nu}_3, \ldots, \tilde{\nu}_N - \tilde{\nu}_{N+1}$ are linearly independent. It is not hard to see that condition (A) implies that any $N+1$ points out of the m points $\nu_1, \ldots, \nu_m$ are in general position. The converse, however, is not true. For example, $\nu_1 = (0,0)^T,\ \nu_2 = (2,0)^T,\ \nu_3 = (0,1)^T$ and $\nu_4 = (1,1)^T$ do not satisfy condition (A) because $\nu_1 - \nu_2$ and $\nu_3 - \nu_4$ are linearly dependent, although any three of these $\nu_1, \ldots, \nu_4$ are in general position.

Next, we move on to the ranking pattern of the unfolding model. Let $n \geq 1$ be a positive integer. By definition, $(i_1 \cdots i_m) \in \mathbb{P}_m$ appears in the unfolding model with objects $\mu_1, \ldots, \mu_m \in \mathbb{R}^n$ iff there exists $y \in \mathbb{R}^n$ such that $\|y - \mu_{i_1}\| < \cdots < \|y - \mu_{i_m}\|$. Let us call

$$\mathrm{RP}^{\mathrm{UF}}(\mu_1, \ldots, \mu_m) := \{(i_1 \cdots i_m) \in \mathbb{P}_m : \|y - \mu_{i_1}\| < \cdots < \|y - \mu_{i_m}\| \text{ for some } y \in \mathbb{R}^n\} \tag{2}$$

the *ranking pattern of the unfolding model* with $\mu_1, \ldots, \mu_m \in \mathbb{R}^n$. Note that for every $c \in \mathbb{R}^n$, $\|y-\mu_{i_1}\| < \cdots < \|y-\mu_{i_m}\|$ for some $y \in \mathbb{R}^n$ iff $\|y-\mu_{i_1}-c\| < \cdots < \|y-\mu_{i_m}-c\|$ for some $y \in \mathbb{R}^n$. Hence the ranking pattern of the unfolding model is invariant with respect to translations of $\mu_1, \ldots, \mu_m$:

$$\mathrm{RP}^{\mathrm{UF}}(\mu_1, \ldots, \mu_m) = \mathrm{RP}^{\mathrm{UF}}(\mu_1 + c, \ldots, \mu_m + c), \quad c \in \mathbb{R}^n.$$

Thus we can assume $\sum_{j=1}^m \mu_j = 0_n$ without loss of generality, where $0_n \in \mathbb{R}^n$ is the vector of zeros. As long as not all $\mu_1, \ldots, \mu_m$ are zero, we can also assume $\sum_{j=1}^m ||\mu_j||^2/m = 1$ without loss of generality, because the ranking pattern of the unfolding model is invariant with respect to nonzero multiplications of $\mu_1, \ldots, \mu_m$:

$$\mathrm{RP}^{\mathrm{UF}}(\mu_1, \ldots, \mu_m) = \mathrm{RP}^{\mathrm{UF}}(a\mu_1, \ldots, a\mu_m), \quad a \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}.$$

Therefore, we assume from now on that $\mu_1, \ldots, \mu_m \in \mathbb{R}^n$ satisfy $\sum_{j=1}^m \mu_j = 0_n$ and $\sum_{j=1}^m ||\mu_j||^2/m = 1$.

Define an $m \times n$-matrix W and an m-dimensional column vector u by

$$W = \mathrm{W}(\mu_1, \ldots, \mu_m) = (w_1, \ldots, w_n) := \begin{pmatrix} \mu_1^T \\ \vdots \\ \mu_m^T \end{pmatrix} \in \mathrm{Mat}_{m\times n}(\mathbb{R}), \tag{3}$$

$$u = \mathrm{u}(\mu_1, \ldots, \mu_m) := -\frac{1}{2} \begin{pmatrix} \|\mu_1\|^2 - 1 \\ \vdots \\ \|\mu_m\|^2 - 1 \end{pmatrix} \in \mathbb{R}^m, \tag{4}$$

where $\mathrm{Mat}_{m\times n}(\mathbb{R})$ stands for the set of $m\times n$-matrices with real entries. Consider an affine map $\kappa : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined by $\kappa(y) := Wy + u$ for $y \in \mathbb{R}^n$. Let $K := \operatorname{im} \kappa = u + \operatorname{col} W$, where $\operatorname{im} \kappa := \{\kappa(y) : y \in \mathbb{R}^n\}$ is the image of κ, and $\operatorname{col} W$ is the column space of W. Note $1_m^T W = 0_n^T$ and $1_m^T u = 0$ (or $w_1, \ldots, w_n, u \in H_0$), where $1_m \in \mathbb{R}^m$ is the vector of ones. Thus K is a subspace of H_0. The condition defining $\mathrm{RP}^{\mathrm{UF}}(\mu_1, \ldots, \mu_m)$ in (2) can be expressed as follows:

$$
\begin{aligned}
&\|y-\mu_{i_1}\| < \cdots < \|y-\mu_{i_m}\| \text{ for some } y \in \mathbb{R}^n && (5)\\
&\iff \mu_{i_1}^T y - \frac{1}{2}(\|\mu_{i_1}\|^2 - 1) > \cdots > \mu_{i_m}^T y - \frac{1}{2}(\|\mu_{i_m}\|^2 - 1) \text{ for some } y \in \mathbb{R}^n\\
&\iff \kappa(y) \in C_{i_1\cdots i_m} \text{ for some } y \in \mathbb{R}^n\\
&\iff K \cap C_{i_1\cdots i_m} \neq \emptyset. && (6)
\end{aligned}
$$

Condition (6) means that $\mathrm{RP}^{\mathrm{UF}}(\mu_1, \dots, \mu_m)$ can be obtained by slicing the braid arrangement by an affine subspace.

Consider the following two conditions on $\mu_1, \dots, \mu_m \in \mathbb{R}^n$ $(n \le m-2)$:

(A1) The m points $\mu_1, \dots, \mu_m \in \mathbb{R}^n$ satisfy condition (A).

(A2) The m points $\begin{pmatrix} \mu_1 \\ \|\mu_1\|^2 \end{pmatrix}, \dots, \begin{pmatrix} \mu_m \\ \|\mu_m\|^2 \end{pmatrix} \in \mathbb{R}^{n+1}$ satisfy condition (A).

When $\mu_1, \dots, \mu_m \in \mathbb{R}^n$ with $n \le m-2$ satisfy (A1) and (A2), we will say the unfolding model with $\mu_1, \dots, \mu_m$ is (or $\mu_1, \dots, \mu_m$ themselves are) *generic*. Note that (A1) and (A2) are translation invariant and nonzero multiplication invariant, i.e., $\mu_1, \dots, \mu_m$ are generic iff $\mu_1 + c, \dots, \mu_m + c$ are generic for any $c \in \mathbb{R}^n$ (or $a\mu_1, \dots, a\mu_m$ are generic for any $a \in \mathbb{R}^*$).

Remark 2.1. *When $n \ge m-1$, condition (A1) with the $N = n$ in (A) replaced by $m-1$ implies $\dim K = \operatorname{rank} W = m-1$ and thus $K = H_0$. In this case, $K \cap C_{i_1\cdots i_m} = C_{i_1\cdots i_m} \neq \emptyset$ for all $(i_1 \cdots i_m) \in \mathbb{P}_m$, and hence $\mathrm{RP}^{\mathrm{UF}}(\mu_1, \dots, \mu_m)$ is the whole $\mathbb{P}_m$: $\mathrm{RP}^{\mathrm{UF}}(\mu_1, \dots, \mu_m) = \mathbb{P}_m$.*

In the present paper, we will treat exclusively the case $n = m-2$.

Suppose $\mu_1, \dots, \mu_m \in \mathbb{R}^{m-2}$ are generic. Let us define

$$\tilde{v} = \tilde{\mathrm{v}}(\mu_1, \dots, \mu_m) := u - \mathrm{proj}_{\mathrm{col}\, W}(u), \tag{7}$$

where $\mathrm{proj}_{\mathrm{col}\, W}$ stands for the orthogonal projection on $\mathrm{col}\, W$. Thanks to (A1), we have $\operatorname{rank} W = m-2$, so we can write $\tilde{v}$ as

$$\tilde{v} = (I_m - W(W^T W)^{-1} W^T)u,$$

where I_m denotes the identity matrix. Since the vector u does not lie on $\mathrm{col}\, W$ because of (A2), we have $\tilde{v} \neq 0_m$. Besides, we have $\dim K = m-2 = \dim H_0 - 1$. These two facts imply that we can write $K = u + \mathrm{col}\, W$ in terms of $\tilde{v}$ as

$$K = K_{\tilde{v}} := \{x \in H_0 : \tilde{v}^T x = \|\tilde{v}\|^2\}.$$

Defining

$$\mathrm{v}(\mu_1, \dots, \mu_m) := \frac{1}{\|\tilde{v}\|}\tilde{v}, \tag{8}$$

we obtain the following equivalence: For $(i_1 \cdots i_m) \in \mathbb{P}_m$,

$$\|y-\mu_{i_1}\| < \cdots < \|y-\mu_{i_m}\| \text{ for some } y \in \mathbb{R}^{m-2} \iff K_{\mathrm{v}(\mu_1,\dots,\mu_m)} \cap C_{i_1\cdots i_m} \neq \emptyset, \tag{9}$$

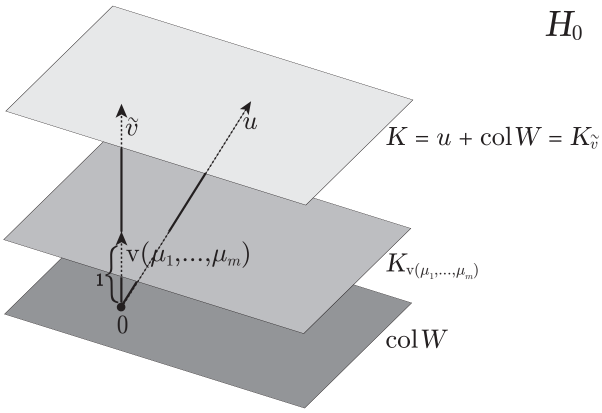

Figure 1: $K_{\mathrm{v}(\mu_1,\ldots,\mu_m)}$.

where

$$K_{\mathrm{v}(\mu_1,\ldots,\mu_m)} = \{x \in H_0 : \mathrm{v}(\mu_1, \ldots, \mu_m)^T x = 1\}.$$

(See Figure 1.)

In the generic case with $n = m - 2$, we have that K is an affine hyperplane in H_0: $\dim K = \dim H_0 - 1,\ 0_m \notin K$. We will say the unfolding model is *of codimension one* when $\mu_1, \ldots, \mu_m$ are generic with $n = m - 2$.

By (1), (2) and (9), we obtain the following proposition.

Proposition 2.2. *The ranking pattern of the unfolding model of codimension one with* $\mu_1, \ldots, \mu_m \in \mathbb{R}^{m-2}$ *is given by the ranking pattern of the braid slice by* $K_{\mathrm{v}(\mu_1,\ldots,\mu_m)}$*:*

$$\mathrm{RP}^{\mathrm{UF}}(\mu_1, \ldots, \mu_m) = \mathrm{RP}(\mathrm{v}(\mu_1, \ldots, \mu_m)) \text{ for generic } \mu_1, \ldots, \mu_m \in \mathbb{R}^{m-2}.$$

3 Ranking patterns of braid slices

In this section, we show that the set of ranking patterns of braid slices by K_v ($v \in \mathbb{S}^{m-2}$) for "generic" v's is in one-to-one correspondence with the set of chambers of an arrangement of hyperplanes in H_0. The discussions in this section are about braid slices, and the unfolding model does not concern us (except in a few places) in this section.

We begin by defining an arrangement $\mathcal{A}$ of hyperplanes in $\mathbb{R}^m$ by

$$\mathcal{A} = \mathcal{A}_m := \{H_I : I \subseteq [m],\ |I| \geq 1\}, \quad H_I := \{x = (x_1, \ldots, x_m)^T \in \mathbb{R}^m : \sum_{i \in I} x_i = 0\}.$$

Note that $H_0 = H_{[m]}$. We will call $\mathcal{A}$ the *all-subset arrangement*. Next we consider the restriction of $\mathcal{A}$ to H_0:

$$\mathcal{A}^0 = \mathcal{A}_m^0 := \mathcal{A}_m^{H_0} = \{H_I^0 : I \subset [m],\ 1 \leq |I| \leq m - 1\}, \quad H_I^0 := H_I \cap H_0.$$

We notice that $H^0_{[m]\setminus I} = H^0_I$.

Now define

$$\mathcal{V} := (H_0 \setminus \bigcup \mathcal{A}^0) \cap \mathbb{S}^{m-2},$$

where $\bigcup \mathcal{A}^0 := \bigcup_{H \in \mathcal{A}^0} H$. Then we have the following basic lemma.

Lemma 3.1. *Take an arbitrary* $v = (v_1, \dots, v_m)^T \in \mathcal{V}$. *Then for* $(i_1 \cdots i_m) \in \mathbb{P}_m$, *we have the equivalences below:*

$$\begin{aligned}
&K_v \cap C_{i_1 \cdots i_m} = \emptyset \\
&\qquad \Longleftrightarrow\ v_{i_1} < 0,\ v_{i_1} + v_{i_2} < 0, \dots, v_{i_1} + \cdots + v_{i_{m-1}} < 0, \\
&K_v \cap C_{i_1 \cdots i_m} \neq \emptyset \text{ is bounded} \\
&\qquad \Longleftrightarrow\ v_{i_1} > 0,\ v_{i_1} + v_{i_2} > 0, \dots, v_{i_1} + \cdots + v_{i_{m-1}} > 0, \\
&K_v \cap C_{i_1 \cdots i_m} \text{ is unbounded} \\
&\qquad \Longleftrightarrow\ \text{there exist } k, l \in [m-1]\ (k \neq l) \text{ such that } (v_{i_1} + \cdots + v_{i_k})(v_{i_1} + \cdots + v_{i_l}) < 0.
\end{aligned}$$

Proof. Without loss of generality, we may consider the case $(i_1 \cdots i_m) = (1 \cdots m)$. Let $c_1, \dots, c_{m-1} \in H_0$ be defined by

$$\begin{aligned}
c_1 &:= (1, 0, 0, \dots, 0)^T - \frac{1}{m} 1_m, \\
c_2 &:= (1, 1, 0, \dots, 0)^T - \frac{2}{m} 1_m, \\
&\ \vdots \\
c_{m-1} &:= (1, 1, \dots, 1, 0)^T - \frac{m-1}{m} 1_m.
\end{aligned}$$

Then $c_1, \dots, c_{m-1}$ are linearly independent. Consider the pointed cone with apex 0_m and generators $c_1, \dots, c_{m-1}$:

$$\mathrm{cone}\{c_1, \dots, c_{m-1}\} := \{d_1 c_1 + \cdots + d_{m-1} c_{m-1} : d_1, \dots, d_{m-1} \geq 0\},$$

which is a simplicial cone in H_0. Then $C_{1 \cdots m} = \{(x_1, \dots, x_m)^T \in H_0 : x_1 > \cdots > x_m\}$ is the relative interior of this cone:

$$C_{1 \cdots m} = \mathrm{relint}(\mathrm{cone}\{c_1, \dots, c_{m-1}\}) = \{d_1 c_1 + \cdots + d_{m-1} c_{m-1} : d_1, \dots, d_{m-1} > 0\}.$$

Suppose $v_1 < 0,\ v_1 + v_2 < 0, \dots, v_1 + \cdots + v_{m-1} < 0$. This is equivalent to saying that $c_j^T v < 0$ for all $j \in [m-1]$, which in turn is equivalent to $K_v \cap \mathrm{relint}(\mathrm{cone}\{c_1, \dots, c_{m-1}\}) = \emptyset$.

Suppose on the contrary that $v_1 > 0,\ v_1 + v_2 > 0, \dots, v_1 + \cdots + v_{m-1} > 0$. Then $c_j^T v > 0,\ j \in [m-1]$, and hence we have

$$K_v \cap \mathrm{relint}(\mathrm{cone}\{c_1, \dots, c_{m-1}\}) = \mathrm{relint}\left(\mathrm{conv}\left\{\frac{1}{c_1^T v} c_1, \dots, \frac{1}{c_{m-1}^T v} c_{m-1}\right\}\right), \tag{10}$$

where conv{ } denotes the convex hull of the points in the braces. Noting that $c_1, \dots, c_{m-1}$ are linearly independent, we can see that the right-hand side of (10) is nonempty. Also, it is clearly bounded.

Suppose instead that $v_1 + \cdots + v_k$ and $v_1 + \cdots + v_l$ have different signs for some k and l. Then $c_k^T v$ and $c_l^T v$ have different signs. Hence, there exists $c \in \mathrm{relint}(\mathrm{cone}\{c_1, \dots, c_{m-1}\}) = C_{1 \cdots m}$ such that $c^T v = 0$. We have $v + dc \in K_v$ for any $d \in \mathbb{R}$; moreover, we can see $v + dc \in C_{1 \cdots m}$ for all sufficiently large $d > 0$. Therefore, $K_v \cap C_{1 \cdots m}$ is an unbounded set.

Since there are no other cases than the three above for the signs of $\sum_{j=1}^{s} v_j$ ($s \in [m-1]$) for $v = (v_1, \dots, v_m)^T \in \mathcal{V}$, the preceding arguments are enough to prove the three equivalences in the lemma. □

By (1) and Lemma 3.1, it is easily seen that $|\mathbb{P}_m \setminus \mathrm{RP}(v)| = (m-1)!$ for any $v \in \mathcal{V}$. When $\mathrm{RP}(v)$ can be realized by the unfolding model, this follows also from the general result on the cardinality of a ranking pattern of the unfolding model (Good and Tideman [9], Kamiya and Takemura [12, 13], Zaslavsky [24]).

Let $\mathbf{Ch}(\mathcal{A}^0)$ stand for the set of chambers of $\mathcal{A}^0$. Then we can write $\mathcal{V}$ as

$$\mathcal{V} = \bigsqcup_{\tilde{D} \in \mathbf{Ch}(\mathcal{A}^0)} (\tilde{D} \cap \mathbb{S}^{m-2}) = \bigsqcup_{D \in \mathbf{D}(\mathcal{A}^0)} D \quad \text{(disjoint union)},$$

where

$$\mathbf{D}(\mathcal{A}^0) := \{D = \tilde{D} \cap \mathbb{S}^{m-2} : \tilde{D} \in \mathbf{Ch}(\mathcal{A}^0)\}$$

is in one-to-one correspondence with $\mathbf{Ch}(\mathcal{A}^0)$. Using Lemma 3.1, we can prove the following proposition.

Proposition 3.2. *There is a one-to-one correspondence between* $\mathbf{D}(\mathcal{A}^0)$ *and* $\{\mathrm{RP}(v) : v \in \mathcal{V}\}$ *given by the bijection*

$$\mathbf{D}(\mathcal{A}^0) \ni D \longmapsto \mathrm{RP}(v), \quad v \in D. \tag{11}$$

Proof. It is clear that the map (11) is well-defined and surjective. We will show that it is injective. Suppose D and D' ($D, D' \in \mathbf{D}(\mathcal{A}^0)$) are different. Take arbitrary $v \in D$ and $v' \in D'$. Then there exists $I \subset [m]$, $1 \le |I| \le m-1$, such that $\sum_{i \in I} v_i$ and $\sum_{i \in I} v'_i$ have different signs. Without loss of generality, we may assume $\sum_{i \in I} v_i < 0$ and $\sum_{i \in I} v'_i > 0$. Define $I_- = \{i \in I : v_i < 0\} \neq \emptyset$, $I_+ = \{i \in I : v_i > 0\}$, $\bar{I}_- = \{i \in [m] \setminus I : v_i < 0\}$, $\bar{I}_+ = \{i \in [m] \setminus I : v_i > 0\} \neq \emptyset$. Take an arbitrary $(i_1 \cdots i_m) \in \mathbb{P}_m$ such that $\{i_1, \dots, i_{|I_-|}\} = I_-$, $\{i_{|I_-|+1}, \dots, i_{|I|}\} = I_+$, $\{i_{|I|+1}, \dots, i_{|I|+|\bar{I}_-|}\} = \bar{I}_-$, $\{i_{|I|+|\bar{I}_-|+1}, \dots, i_{[m]}\} = \bar{I}_+$. Then $v = (v_i, \dots, v_m)^T$ satisfies $v_{i_1} < 0$, $v_{i_1} + v_{i_2} < 0, \dots, v_{i_1} + \dots + v_{i_{m-1}} < 0$. Thus we have $(i_1 \cdots i_m) \notin \mathrm{RP}(v)$ by Lemma 3.1. On the other hand, this is not the case with $v' = (v'_1, \dots, v'_m)^T$ because $v'_{i_1} + \dots + v'_{i_{|I|}} = \sum_{i \in I} v'_i > 0$, and we have $(i_1 \cdots i_m) \in \mathrm{RP}(v')$ by Lemma 3.1. Therefore, we obtain $\mathrm{RP}(v) \neq \mathrm{RP}(v')$. □

Proposition 3.2 implies that the ranking patterns $\mathrm{RP}(v)$, $v \in \mathcal{V} = \bigsqcup_{D \in \mathbf{D}(\mathcal{A}^0)} D$, are the same on a common D and different on different D's. So we can write $\mathrm{RP}(v)$ with $v \in D$ as RP_D:

$$\mathrm{RP}_D := \mathrm{RP}(v), \quad v \in D \in \mathbf{D}(\mathcal{A}^0),$$

and we have $\mathrm{RP}_D \neq \mathrm{RP}_{D'}$ for $D \neq D'$.

We will say the braid slice by K_v ($v \in \mathbb{S}^{m-2}$) is *generic* when $v \in \mathcal{V}$. It can be checked that if $\mu_1, \dots, \mu_m \in \mathbb{R}^{m-2}$ are generic, $\mathrm{v}(\mu_1, \dots, \mu_m)$ defined in (8) satisfies $\mathrm{v}(\mu_1, \dots, \mu_m) \in \mathcal{V}$, i.e., the braid slice by $K_{\mathrm{v}(\mu_1, \dots, \mu_m)}$ is generic.

4 Realizable braid slices

By Proposition 2.2, we know that the ranking pattern of any unfolding model of codimension one can be obtained as the ranking pattern of a generic braid slice. However, not all the ranking patterns of generic braid slices, $\mathrm{RP}_D,\ D \in \mathbf{D}(\mathcal{A}^0)$, can be realized as the ranking patterns of unfolding models of codimension one. In this section, we establish conditions on $D \in \mathbf{D}(\mathcal{A}^0)$ which guarantee that RP_D can be realized by an unfolding model of codimension one.

Let $\mathcal{V}_2$ be the set of all $v = (v_1, \ldots, v_m)^T \in \mathcal{V}$ having at least two positive entries $v_i, v_j > 0$ and at the same time at least two negative entries $v_k, v_l < 0$ $(i, j, k, l$: all distinct). Then put $\mathcal{V}_1 := \mathcal{V} \setminus \mathcal{V}_2$. We see that $\mathcal{V}_1$ is the set of all $v = (v_1, \ldots, v_m)^T \in \mathcal{V}$ having exactly one positive entry or exactly one negative entry. Note that for any $D \in \mathbf{D}(\mathcal{A}^0)$, we have either $D \subset \mathcal{V}_2$ or $D \subset \mathcal{V}_1$.

It is helpful to consider $D \in \mathbf{D}(\mathcal{A}^0)$ and $-D = \{-v : v \in D\} \in \mathbf{D}(\mathcal{A}^0)$ in a pair. Obviously, $D \subset \mathcal{V}_i$ implies $-D \subset \mathcal{V}_i$ for each $i = 1, 2$.

Theorem 4.1. *For any $D \in \mathbf{D}(\mathcal{A}^0)$, we have the following.*

1. *Suppose $D \subset \mathcal{V}_2$. Then each of RP_D and RP_{-D} can be realized as the ranking pattern of an unfolding model of codimension one, i.e., there exist generic $\mu_1, \ldots, \mu_m \in \mathbb{R}^{m-2}$ and $\mu'_1, \ldots, \mu'_m \in \mathbb{R}^{m-2}$ such that*

$$\mathrm{RP}_D = \mathrm{RP}^{\mathrm{UF}}(\mu_1, \ldots, \mu_m), \quad \mathrm{RP}_{-D} = \mathrm{RP}^{\mathrm{UF}}(\mu'_1, \ldots, \mu'_m).$$

2. *Suppose $D \subset \mathcal{V}_1$. Then exactly one of RP_D and RP_{-D} can be realized as the ranking pattern of an unfolding model of codimension one. In fact, $\mathrm{RP}_{\varepsilon D}$ can be realized and $\mathrm{RP}_{-\varepsilon D}$ cannot be realized, where $\varepsilon = \pm 1$ is such that εv for any $v \in D$ has exactly one positive entry.*

The proof of Theorem 4.1 is based on the following two lemmas. For $v = (v_1, \ldots, v_m)^T \in \mathbb{R}^m$, let $\mathrm{diag}(v) := \mathrm{diag}(v_1, \ldots, v_m) \in \mathrm{Mat}_{m\times m}(\mathbb{R})$ stand for the diagonal matrix with diagonal elements $v_1, \ldots, v_m$.

Lemma 4.2. *Suppose $v \in \mathbb{R}^m$ and $W \in \mathrm{Mat}_{m\times(m-2)}(\mathbb{R})$ satisfy*

$$v \neq 0_m, \quad 1_m^T v = 0, \quad 1_m^T W = v^T W = 0_{m-2}^T.$$

1. *If v has at least two positive entries as well as at least two negative entries, then $W^T \mathrm{diag}(v) W$ is indefinite (i.e., has at least one positive eigenvalue and at least one negative eigenvalue).*

2. *If v has exactly one positive (resp. negative) entry, then $W^T \mathrm{diag}(v) W$ is non-positive (resp. non-negative) definite. If in addition v has at least two negative (resp. positive) entries, then $W^T \mathrm{diag}(v) W$ has at least one negative (resp. positive) eigenvalue, and hence $\mathrm{tr}\{W^T \mathrm{diag}(v) W\}$ is negative (resp. positive).*

Proof. We can assume without loss of generality that $v = (v_1, \ldots, v_m)^T$ is of unit length: $\|v\|^2 = \sum_{i=1}^m v_i^2 = 1$. Define $C := (1_m, v - 2^{-1} \sum_{i=1}^m v_i^3 1_m, W - 1_m(v_1^2, \ldots, v_m^2) W) \in$

$\mathrm{GL}(m,\mathbb{R})$. Then, by direct calculations, we can see that

$$C^T \operatorname{diag}(v) C = \begin{pmatrix} 0 & 1 & 0_{m-2}^T \\ 1 & 0 & 0_{m-2}^T \\ 0_{m-2} & 0_{m-2} & W^T \operatorname{diag}(v) W \end{pmatrix}. \tag{12}$$

Equation (12) implies that the number of positive (resp. negative) eigenvalues of $W^T \operatorname{diag}(v) W$ plus one is equal to the number of positive (resp. negative) eigenvalues of $C^T \operatorname{diag}(v) C$, which in turn is equal to the number of positive (resp. negative) entries of v by Sylvester's law of inertia. □

Lemma 4.3. *Suppose an $m' \times m'$ real symmetric matrix A is indefinite. Then we have* $\{\mathrm{tr}(BAB^T) : B \in \mathrm{GL}(m',\mathbb{R})\} = \mathbb{R}$.

Proof. Let $\lambda_1, \dots, \lambda_{m'}$ be the eigenvalues of A with $\lambda_1 > 0$ and $\lambda_2 < 0$, and write $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_{m'})$. Then

$$\begin{aligned} \{\mathrm{tr}(BAB^T) : B \in \mathrm{GL}(m',\mathbb{R})\} &= \{\mathrm{tr}(B\Lambda B^T) : B \in \mathrm{GL}(m',\mathbb{R})\} \\ &= \{\lambda_1\|b_1\|^2 + \lambda_2\|b_2\|^2 + \lambda_3\|b_3\|^2 + \dots + \lambda_{m'}\|b_{m'}\|^2 : \\ &\qquad (b_1, \dots, b_{m'}) \in \mathrm{GL}(m',\mathbb{R})\}. \end{aligned}$$

For given $b_3, \dots, b_{m'}$, we can take b_1, b_2 with arbitrary positive lengths. □

Proof of Theorem 4.1. Take an arbitrary $v \in D$. Let $\{w_1, \dots, w_{m-2}\}$ be a basis of $H_0 \cap (\mathrm{span}\{v\})^\perp = (\mathrm{span}\{1_m, v\})^\perp$, and take $\mu_1, \dots, \mu_m \in \mathbb{R}^{m-2}$ as

$$\begin{pmatrix} \mu_1^T \\ \vdots \\ \mu_m^T \end{pmatrix} := (w_1, \dots, w_{m-2}).$$

Note that $\sum_{j=1}^m \mu_j = 0_{m-2}$. Moreover, we can take $w_1, \dots, w_{m-2}$ so that $\sum_{j=1}^m \|\mu_j\|^2/m = 1$. For such $\mu_1, \dots, \mu_m$, let us consider $\mathrm{W}(\mu_1, \dots, \mu_m)$ and $\mathrm{u}(\mu_1, \dots, \mu_m)$ defined in (3) and (4):

$$\mathrm{W}(\mu_1, \dots, \mu_m) = \begin{pmatrix} \mu_1^T \\ \vdots \\ \mu_m^T \end{pmatrix} \in \mathrm{Mat}_{m \times (m-2)}(\mathbb{R}), \quad \mathrm{u}(\mu_1, \dots, \mu_m) = -\frac{1}{2} \begin{pmatrix} \|\mu_1\|^2 - 1 \\ \vdots \\ \|\mu_m\|^2 - 1 \end{pmatrix} \in \mathbb{R}^m.$$

We note here that $\mathrm{u}(\mu_1, \dots, \mu_m)^T v$ can be written as

$$\mathrm{u}(\mu_1, \dots, \mu_m)^T v = -\frac{1}{2} \mathrm{tr}\{\mathrm{W}(\mu_1, \dots, \mu_m)^T \operatorname{diag}(v) \mathrm{W}(\mu_1, \dots, \mu_m)\}. \tag{13}$$

Moreover, using the fact that $v \notin \bigcup \mathcal{A}^0$, we can check that $\mu_1, \dots, \mu_m$ satisfy (A1).

We first prove Part 1. Suppose $D \subset \mathcal{V}_2$. Then, since $v \in D \subset \mathcal{V}_2$, we have by Part 1 of Lemma 4.2 that the symmetric matrix $\mathrm{W}(\mu_1, \dots, \mu_m)^T \operatorname{diag}(v) \mathrm{W}(\mu_1, \dots, \mu_m)$

is indefinite. So Lemma 4.3 implies that there exist $B_1 \in \mathrm{GL}(m-2,\mathbb{R})$ and $B_2 \in \mathrm{GL}(m-2,\mathbb{R})$ such that

$$\begin{aligned}\mathrm{tr}\{B_1 \mathrm{W}(\mu_1,\dots,\mu_m)^T \operatorname{diag}(v)\mathrm{W}(\mu_1,\dots,\mu_m)B_1^T\} > 0,\\ \mathrm{tr}\{B_2 \mathrm{W}(\mu_1,\dots,\mu_m)^T \operatorname{diag}(v)\mathrm{W}(\mu_1,\dots,\mu_m)B_2^T\} < 0.\end{aligned}$$

Together with

$$\begin{aligned}\mathrm{u}(B_k\mu_1,\dots,B_k\mu_m)^T v &= -\frac{1}{2}\,\mathrm{tr}\{\mathrm{W}(B_k\mu_1,\dots,B_k\mu_m)^T \operatorname{diag}(v)\mathrm{W}(B_k\mu_1,\dots,B_k\mu_m)\}\\ &= -\frac{1}{2}\,\mathrm{tr}\{B_k\mathrm{W}(\mu_1,\dots,\mu_m)^T \operatorname{diag}(v)\mathrm{W}(\mu_1,\dots,\mu_m)B_k^T\}, \quad k=1,2,\end{aligned}$$

these inequalities imply

$$\mathrm{u}(B_1\mu_1,\dots,B_1\mu_m)^T v < 0, \quad \mathrm{u}(B_2\mu_1,\dots,B_2\mu_m)^T v > 0. \tag{14}$$

We observe that the column space of $\mathrm{W}(B_k\mu_1,\dots,B_k\mu_m) = \mathrm{W}(\mu_1,\dots,\mu_m)B_k^T$ is the same as that of $\mathrm{W}(\mu_1,\dots,\mu_m)$. This fact and $\mathrm{u}(B_k\mu_1,\dots,B_k\mu_m)^T v \neq 0$ yield

$$\mathrm{v}(B_k\mu_1,\dots,B_k\mu_m) = \operatorname{sign}\{\mathrm{u}(B_k\mu_1,\dots,B_k\mu_m)^T v\}v, \quad k=1,2$$

(see (7), (8) and Figure 1). By (14), we obtain

$$\mathrm{v}(B_1\mu_1,\dots,B_1\mu_m) = -v, \quad \mathrm{v}(B_2\mu_1,\dots,B_2\mu_m) = v.$$

Now, since $\mu_1,\dots,\mu_m$ satisfy (A1), clearly so do $B_k\mu_1,\dots,B_k\mu_m$ for $k=1,2$. From this fact and $\mathrm{u}(B_k\mu_1,\dots,B_k\mu_m)^T v \neq 0$, we can check that $B_k\mu_1,\dots,B_k\mu_m$ also satisfy (A2) ($k=1,2$). Now that $B_k\mu_1,\dots,B_k\mu_m$ are generic ($k=1,2$), Proposition 2.2 yields $\mathrm{RP}(-v) = \mathrm{RP}(\mathrm{v}(B_1\mu_1,\dots,B_1\mu_m)) = \mathrm{RP}^{\mathrm{UF}}(B_1\mu_1,\dots,B_1\mu_m)$ and $\mathrm{RP}(v) = \mathrm{RP}(\mathrm{v}(B_2\mu_1,\dots,B_2\mu_m)) = \mathrm{RP}^{\mathrm{UF}}(B_2\mu_1,\dots,B_2\mu_m)$. Thus, we have proved that each of $\mathrm{RP}(v)$ and $\mathrm{RP}(-v)$ is realized by an unfolding model of codimension one, where $v \in D$ and $-v \in -D$. This completes the proof of Part 1.

Next we prove Part 2. Suppose $D \subset \mathcal{V}_1$. Then the fact that $v \in \mathcal{V}_1$ together with Part 2 of Lemma 4.2 and equation (13) implies that $\mathrm{u}(\mu_1,\dots,\mu_m)^T v \neq 0$. Hence we have $\mathrm{v}(\mu_1,\dots,\mu_m) = \varepsilon v$, $\varepsilon = \operatorname{sign}\{\mathrm{u}(\mu_1,\dots,\mu_m)^T v\}$. Also, from $\mathrm{u}(\mu_1,\dots,\mu_m)^T v \neq 0$ and the fact that $\mu_1,\dots,\mu_m$ satisfy (A1), it follows that $\mu_1,\dots,\mu_m$ satisfy (A2) as well. Thus we obtain $\mathrm{RP}(\varepsilon v) = \mathrm{RP}(\mathrm{v}(\mu_1,\dots,\mu_m)) = \mathrm{RP}^{\mathrm{UF}}(\mu_1,\dots,\mu_m)$. This proves that at least one of $\mathrm{RP}(v)$ and $\mathrm{RP}(-v)$ can be realized by an unfolding model of codimension one. It remains to show that not both $\mathrm{RP}(v)$ and $\mathrm{RP}(-v)$ can be realized by unfolding models of codimension one. Suppose on the contrary that both $\mathrm{RP}(v)$ and $\mathrm{RP}(-v)$ were realized. Without loss of generality, assume that $v_{i_0} < 0$, $v_i > 0$ ($i \neq i_0$) for some $i_0 \in [m]$, where v_i ($1 \le i \le m$) are the entries of v. But by taking y in (5) sufficiently close to μ_{i_0}, we see that $\mathrm{RP}(v)$ with such a v cannot be realized by an unfolding model of codimension one, because $\mathrm{RP}(v) = \mathbb{P}_m \setminus \{(i_0 i_1 \cdots i_{m-1}) : (i_1 \cdots i_{m-1}) \text{ is a permutation of } [m]\setminus\{i_0\}\}$. This is a contradiction. □

5 The number of ranking patterns of unfolding models

Based on the results in Sections 2, 3 and 4, we find, in this section, the number of ranking patterns of unfolding models of codimension one.

For $i \in [m]$, let us define $\mathcal{V}_1(i,+) \subset \mathcal{V}_1$ by

$$\mathcal{V}_1(i,+) := \{v = (v_1,\dots,v_m)^T \in \mathcal{V}_1 : v_i > 0,\ v_j < 0 \text{ for all } j \in [m] \setminus \{i\}\}.$$

Lemma 5.1. *For any $i \in [m]$, we have $\mathcal{V}_1(i,+) = D_i$ for some $D_i \in \mathbf{D}(\mathcal{A}^0)$.*

Proof. Obviously, $\mathcal{V}_1(i,+)$ is a union of some chambers $D \in \mathbf{D}(\mathcal{A}^0)$. So it suffices to show the following: For any $I \subset [m]$ with $1 \le |I| \le m-1$, we have $\mathcal{V}_1(i,+) \subset (H_I^0)^+ \cap \mathbb{S}^{m-2}$ or $\mathcal{V}_1(i,+) \subset (H_I^0)^- \cap \mathbb{S}^{m-2}$, where $(H_I^0)^+ := \{x = (x_1,\dots,x_m)^T \in H_0 : \sum_{j\in I} x_j > 0\}$ and $(H_I^0)^- := H_0 \setminus (H_I^0 \cup (H_I^0)^+)$. If $i \notin I$, any $v = (v_1,\dots,v_m)^T \in \mathcal{V}_1(i,+)$ satisfies $\sum_{j\in I} v_j < 0$, and thus we have $\mathcal{V}_1(i,+) \subset (H_I^0)^- \cap \mathbb{S}^{m-2}$. If $i \in I$, on the other hand, $v = (v_1,\dots,v_m)^T \in \mathcal{V}_1(i,+)$ implies $\sum_{j\in I} v_j = -\sum_{j\in[m]\setminus I} v_j > 0$, so we obtain $\mathcal{V}_1(i,+) \subset (H_I^0)^+ \cap \mathbb{S}^{m-2}$. □

We can write $\mathcal{V}_1$ as

$$\mathcal{V}_1 = D_1 \sqcup (-D_1) \sqcup \cdots \sqcup D_m \sqcup (-D_m),$$

where $-D_i = \{-v : v \in D_i\} \in \mathbf{D}(\mathcal{A}^0)$ for $i \in [m]$. Notice $-D_i = \mathcal{V}_1(i,-)$ with

$$\mathcal{V}_1(i,-) := \{v = (v_1,\dots,v_m)^T \in \mathcal{V}_1 : v_i < 0,\ v_j > 0 \text{ for all } j \in [m] \setminus \{i\}\}$$

for $i \in [m]$.

Now, consider the mapping

$$\begin{aligned} \mathrm{v} : \{(\mu_1,\dots,\mu_m) : \mu_1,\dots,\mu_m \in \mathbb{R}^{m-2} \text{ are generic}\} &\longrightarrow \mathcal{V}, \\ (\mu_1,\dots,\mu_m) &\longmapsto \mathrm{v}(\mu_1,\dots,\mu_m). \end{aligned}$$

From the proof of Theorem 4.1, we can see that the image $\operatorname{im}\mathrm{v} = \{\mathrm{v}(\mu_1,\dots,\mu_m) : \mu_1,\dots,\mu_m \in \mathbb{R}^{m-2} \text{ are generic}\}$ of v is given by

$$\begin{aligned} \operatorname{im}\mathrm{v} &= \bigsqcup_{D\in\mathbf{D}(\mathcal{A}^0),\ D\neq -D_i\ (i\in[m])} D \\ &= \mathcal{V} \setminus ((-D_1) \sqcup \cdots \sqcup (-D_m)) = \mathcal{V}_2 \sqcup D_1 \sqcup \cdots \sqcup D_m. \end{aligned} \tag{15}$$

We are in a position to state the main result of this section. Denote by $q(m)$ the number of ranking patterns of unfolding models of codimension one:

$$q(m) := |\{\mathrm{RP}^{\mathrm{UF}}(\mu_1,\dots,\mu_m) : \text{ generic } \mu_1,\dots,\mu_m \in \mathbb{R}^{m-2}\}|.$$

Theorem 5.2. *It holds that $q(m) = |\mathbf{Ch}(\mathcal{A}^0)| - m,\ \mathcal{A}^0 = \mathcal{A}_m^0$.*

Proof. By Propositions 2.2 and 3.2 and equation (15), we have

$$q(m) = |\{\mathrm{RP}(v) : v \in \mathcal{V} \setminus ((-D_1) \sqcup \cdots \sqcup (-D_m))\}| = |\mathbf{D}(\mathcal{A}^0)| - m.$$

□

We have calculated specific values of $q(m)$ for $m \leq 8$ in the following way.

The number of chambers $|\mathbf{Ch}(\mathcal{A}_m^0)|$ can be obtained by finding the characteristic polynomial $\chi(\mathcal{A}_m^0, t)$ of $\mathcal{A}_m^0$ (Orlik and Terao [18, Definition 2.52]): $|\mathbf{Ch}(\mathcal{A}_m^0)| = (-1)^{m-1}\chi(\mathcal{A}_m^0, -1)$ (Zaslavsky [23, Theorem A], Orlik and Terao [18, Theorem 2.68]). Moreover, when finding $\chi(\mathcal{A}_m^0, t)$, we can use the property $L(\mathcal{A}_m^0) \simeq L(\mathcal{A}_{m-1})$ of the all-subset arrangement, where $L(\,\cdot\,)$ denotes the intersection poset of an arrangement (Orlik and Terao [18, Definition 2.1]). The characteristic polynomials $\chi(\mathcal{A}_m^0, t)$ and the numbers of chambers $|\mathbf{Ch}(\mathcal{A}_m^0)|$ of $\mathcal{A}_m^0$ for $m \leq 8$ are given in the following lemma.

Lemma 5.3. *For $m \leq 8$, $\chi(\mathcal{A}_m^0, t)$ and $|\mathbf{Ch}(\mathcal{A}_m^0)|$ are given by*

$$\begin{aligned}
\chi(\mathcal{A}_3^0, t) &= t^2 - 3t + 2 = (t-1)(t-2), \quad |\mathbf{Ch}(\mathcal{A}_3^0)| = 6; \\
\chi(\mathcal{A}_4^0, t) &= t^3 - 7t^2 + 15t - 9 = (t-1)(t-3)^2, \quad |\mathbf{Ch}(\mathcal{A}_4^0)| = 32; \\
\chi(\mathcal{A}_5^0, t) &= t^4 - 15t^3 + 80t^2 - 170t + 104 = (t-1)(t-4)(t^2 - 10t + 26), \\
&\quad\ |\mathbf{Ch}(\mathcal{A}_5^0)| = 370; \\
\chi(\mathcal{A}_6^0, t) &= t^5 - 31t^4 + 375t^3 - 2130t^2 + 5270t - 3485 \\
&= (t-1)(t^4 - 30t^3 + 345t^2 - 1785t + 3485), \\
&\quad\ |\mathbf{Ch}(\mathcal{A}_6^0)| = 11292; \\
\chi(\mathcal{A}_7^0, t) &= t^6 - 63t^5 + 1652t^4 - 22435t^3 + 159460t^2 - 510524t + 371909 \\
&= (t-1)(t^5 - 62t^4 + 1590t^3 - 20845t^2 + 138615t - 371909), \\
&\quad\ |\mathbf{Ch}(\mathcal{A}_7^0)| = 1066044; \\
\chi(\mathcal{A}_8^0, t) &= t^7 - 127t^6 + 7035t^5 - 215439t^4 + 3831835t^3 \\
&\qquad\qquad -37769977t^2 + 169824305t - 135677633 \\
&= (t-1)(t^6 - 126t^5 + 6909t^4 - 208530t^3 \\
&\qquad\qquad +3623305t^2 - 34146672t + 135677633), \\
&\quad\ |\mathbf{Ch}(\mathcal{A}_8^0)| = 347326352.
\end{aligned}$$

We obtained $\chi(\mathcal{A}_3^0, t)$ and $\chi(\mathcal{A}_4^0, t)$ by direct calculations. For $\chi(\mathcal{A}_5^0, t)$, we used the method of deletion and restriction (Orlik and Terao [18, Theorem 2.56]). Furthermore, we calculated $\chi(\mathcal{A}_6^0, t)$, $\chi(\mathcal{A}_7^0, t)$ and $\chi(\mathcal{A}_8^0, t)$ by the finite field method (Athanasiadis [1, 2], Stanley [20, Lecture 5], Crapo and Rota [4], Kamiya, Takemura and Terao [14, 15, 16]).

Remark 5.4. *We can consider $\mathcal{A}_m^0$ also for $m = 2$, and we have $\chi(\mathcal{A}_2^0, t) = t - 1$ and $|\mathbf{Ch}(\mathcal{A}_2^0)| = 2$. The arrangement $\mathcal{A}_m^0$ $(m \geq 2)$ also appears in thermal field theory (Evans [7, 8], van Eijck [22]). The numbers $|\mathbf{Ch}(\mathcal{A}_m^0)|$ $(m = 2, \ldots, 8)$:*

$$2,\ 6,\ 32,\ 370,\ 11292,\ 1066044,\ 347326352$$

are listed in [8, Table 1] and [22, Table 2.1] as the numbers of regions of the analytic continuations of ITF (imaginary-time formalism) Green functions, although the characteristic polynomials $\chi(\mathbf{Ch}(\mathcal{A}_m^0), t)$ $(m \leq 8)$ are not obtained there.

From Theorem 5.2 and the values of $|\mathbf{Ch}(\mathcal{A}_m^0)|$ $(3 \le m \le 8)$ in Lemma 5.3, we can obtain $q(m)$ $(3 \le m \le 8)$:

Corollary 5.5. *The numbers $r(m)$ of ranking patterns of unfolding models of codimension one for $m \le 8$ are given by*

$$q(3) = 3,\ q(4) = 28,\ q(5) = 365,$$
$$q(6) = 11286,\ q(7) = 1066037,\ q(8) = 347326344.$$

6 Inequivalent ranking patterns

In this section, we define equivalence of ranking patterns, and give an upper bound for the number of inequivalent ranking patterns of unfolding models of codimension one. For $m \le 6$, we will see that this upper bound is actually the exact number.

6.1 The number of inequivalent ranking patterns of unfolding models

Let $\mathfrak{S}_m$ be the symmetric group on m letters, consisting of all bijections $\sigma : [m] \longrightarrow [m]$. Let us say that ranking patterns RP_D and $\mathrm{RP}_{D'}$ $(D, D' \in \mathbf{D}(\mathcal{A}^0))$ of generic braid slices are *equivalent* iff

$$\mathrm{RP}_D = \sigma \mathrm{RP}_{D'} \text{ for some } \sigma \in \mathfrak{S}_m,$$

where

$$\sigma \mathrm{RP}_{D'} := \{(\sigma(i_1) \cdots \sigma(i_m)) : (i_1 \cdots i_m) \in \mathrm{RP}_{D'}\}.$$

We say RP_D and $\mathrm{RP}_{D'}$ are *inequivalent* iff they are not equivalent. We want to know the number of inequivalent ranking patterns of generic braid slices that can be realized by unfolding models of codimension one.

Consider the action of $\mathfrak{S}_m$ on $\mathcal{V}$ defined by

$$\mathfrak{S}_m \times \mathcal{V} \ni (\sigma, v) \longmapsto \sigma v := (v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(m)})^T \in \mathcal{V},$$

where $v = (v_1, \ldots, v_m)^T$. This induces the action of $\mathfrak{S}_m$ on $\mathbf{D}(\mathcal{A}^0)$:

$$\mathfrak{S}_m \times \mathbf{D}(\mathcal{A}^0) \ni (\sigma, D) \longmapsto \sigma D := \{\sigma v : v \in D\} \in \mathbf{D}(\mathcal{A}^0). \tag{16}$$

We can check

$$\mathrm{RP}_{\sigma D} = \sigma \mathrm{RP}_D, \quad D \in \mathbf{D}(\mathcal{A}^0),\ \sigma \in \mathfrak{S}_m.$$

Thus, RP_D and $\mathrm{RP}_{D'}$ are equivalent iff D and D' are on the same orbit under action (16). Therefore, the number of inequivalent ranking patterns of generic braid slices is equal to the number of orbits $\mathfrak{S}_m D := \{\sigma D : \sigma \in \mathfrak{S}_m\}$, $D \in \mathbf{D}(\mathcal{A}^0)$, i.e., the cardinality of the orbit space $\mathbf{D}(\mathcal{A}^0)/\mathfrak{S}_m := \{\mathfrak{S}_m D : D \in \mathbf{D}(\mathcal{A}^0)\}$ under action (16).

For each orbit $\mathfrak{S}_m D \in \mathbf{D}(\mathcal{A}^0)/\mathfrak{S}_m$, either all or none of its elements $D' \in \mathfrak{S}_m D$ correspond to ranking patterns $\mathrm{RP}_{D'}$ realizable by unfolding models of codimension one. Among the orbits in $\mathbf{D}(\mathcal{A}^0)/\mathfrak{S}_m$, exactly one orbit, $\mathfrak{S}_m \mathcal{V}_1(m, -) = \mathfrak{S}_m(-D_m)$, consists of elements (chambers) that correspond to ranking patterns not realizable by unfolding models of codimension one, $\mathrm{RP}_{-D_1}, \ldots, \mathrm{RP}_{-D_m}$:

$$\mathfrak{S}_m \mathcal{V}_1(m,-) = \{-D_1, \ldots, -D_m\}$$

(see (15)). Therefore, the number of inequivalent ranking patterns RP_D $(D \in \mathbf{D}(\mathcal{A}^0))$ realizable by unfolding models of codimension one is $|\mathbf{D}(\mathcal{A}^0)/\mathfrak{S}_m| - 1$.

For ranking patterns of unfolding models of codimension one, we say $\mathrm{RP}^{\mathrm{UF}}(\mu_1, \ldots, \mu_m)$ and $\mathrm{RP}^{\mathrm{UF}}(\mu_1', \ldots, \mu_m')$ are *equivalent* (resp. *inequivalent*) iff they are equivalent (resp. inequivalent) when regarded as ranking patterns of generic braid slices. So $\mathrm{RP}^{\mathrm{UF}}(\mu_1, \ldots, \mu_m) = \mathrm{RP}(\mathrm{v}(\mu_1, \ldots, \mu_m))$ and $\mathrm{RP}^{\mathrm{UF}}(\mu_1', \ldots, \mu_m') = \mathrm{RP}(\mathrm{v}(\mu_1', \ldots, \mu_m'))$ are equivalent iff $D \ni \mathrm{v}(\mu_1, \ldots, \mu_m)$ and $D' \ni \mathrm{v}(\mu_1', \ldots, \mu_m')$ are on the same orbit under action (16). Of course, if $(\mu_1, \ldots, \mu_m) = (\mu'_{\sigma^{-1}(1)}, \ldots, \mu'_{\sigma^{-1}(m)})$ for some $\sigma \in \mathfrak{S}_m$, then $\mathrm{RP}^{\mathrm{UF}}(\mu_1, \ldots, \mu_m)$ and $\mathrm{RP}^{\mathrm{UF}}(\mu_1', \ldots, \mu_m')$ are equivalent (because of $\mathrm{v}(\mu'_{\sigma^{-1}(1)}, \ldots, \mu'_{\sigma^{-1}(m)}) = \sigma \mathrm{v}(\mu_1', \ldots, \mu_m')$ yielding $\mathrm{RP}^{\mathrm{UF}}(\mu_1, \ldots, \mu_m) = \sigma \mathrm{RP}^{\mathrm{UF}}(\mu_1', \ldots, \mu_m')$), but not vice versa.

From the arguments above, we obtain the following proposition.

Proposition 6.1. *The number of inequivalent ranking patterns of unfolding models of codimension one is* $|\mathbf{D}(\mathcal{A}^0)/\mathfrak{S}_m| - 1$.

Finding $|\mathbf{D}(\mathcal{A}^0)/\mathfrak{S}_m|$ is not always easy. We will give an upper bound for the number, $|\mathbf{D}(\mathcal{A}^0)/\mathfrak{S}_m| - 1$, of inequivalent ranking patterns of unfolding models of codimension one.

We have $D \not\subset \bigcup \mathcal{B}_m^{H_0}$ for any $D \in \mathbf{D}(\mathcal{A}^0)$. Thus, to each orbit $\mathfrak{S}_m D \in \mathbf{D}(\mathcal{A}^0)/\mathfrak{S}_m$, there belongs a chamber $\sigma D \in \mathfrak{S}_m D$ $(\sigma \in \mathfrak{S}_m)$ that intersects $C_{1\cdots m}$. Hence, the set

$$\mathbf{D}^{1\cdots m}(\mathcal{A}^0) := \{D \in \mathbf{D}(\mathcal{A}^0) : D \cap C_{1\cdots m} \neq \emptyset\}$$

always includes a cross section (i.e., a complete set of representatives of the orbits) under action (16). Therefore, an upper bound for $|\mathbf{D}(\mathcal{A}^0)/\mathfrak{S}_m|$ is given by the cardinality of $\mathbf{D}^{1\cdots m}(\mathcal{A}^0)$:

$$|\mathbf{D}(\mathcal{A}^0)/\mathfrak{S}_m| \leq |\mathbf{D}^{1\cdots m}(\mathcal{A}^0)| = \frac{|\mathbf{Ch}(\mathcal{A}^0 \cup \mathcal{B}_m^{H_0})|}{m!}. \tag{17}$$

If, in particular, $\mathbf{D}^{1\cdots m}(\mathcal{A}^0)$ is a cross section, then the inequality in (17) is actually an equality.

Note that we can write $\mathbf{D}^{1\cdots m}(\mathcal{A}^0)$ as

$$\mathbf{D}^{1\cdots m}(\mathcal{A}^0) = \{D_1, -D_m\} \cup \mathbf{D}_2^{1\cdots m}(\mathcal{A}^0), \tag{18}$$

where

$$\mathbf{D}_2^{1\cdots m}(\mathcal{A}^0) := \{D \in \mathbf{D}(\mathcal{A}^0) : D \subset \mathcal{V}_2,\ D \cap C_{1\cdots m} \neq \emptyset\}.$$

Note, moreover, that for each $D \in \mathbf{D}_2^{1\cdots m}(\mathcal{A}^0)$, three chambers $D_1, -D_m$ and D are all on different orbits. Thus $\mathbf{D}^{1\cdots m}(\mathcal{A}^0)$ is a cross section if and only if all elements of $\mathbf{D}_2^{1\cdots m}(\mathcal{A}^0)$ are on different orbits. Define $\rho \in \mathfrak{S}_m$ by

$$\rho(i) := m + 1 - i, \quad i \in [m]. \tag{19}$$

Then, for any $D \in \mathbf{D}_2^{1\cdots m}(\mathcal{A}^0)$, we have that D and $-\rho D = \{(-v_m, \dots, -v_1)^T : (v_1, \dots, v_m)^T \in D\} \in \mathbf{D}_2^{1\cdots m}(\mathcal{A}^0)$ are on different orbits. This can be seen as follows. Without loss of generality, suppose $v_1 > \cdots > v_m$ and $v_1 + v_m > 0$. Then $v_1 + v_j > 0$ for all $j = 2, \dots, m$. But then there is no i such that $-v_i - v_j > 0$ for all $j \neq i$. However, the fact that D and $-\rho D$ are on different orbits does not exclude the possibility of some D and D' ($D, D' \in \mathbf{D}_2^{1\cdots m}(\mathcal{A}^0)$, $D \neq D'$) being on the same orbit.

For $\mathbf{D}_2^{1\cdots m}(\mathcal{A}^0)$, we may find $\mathbf{D}_2^{\overline{1\cdots m}}(\mathcal{A}^0)$ instead: $\mathbf{D}_2^{1\cdots m}(\mathcal{A}^0) = \mathbf{D}_2^{\overline{1\cdots m}}(\mathcal{A}^0)$, where $\mathbf{D}_2^{\overline{1\cdots m}}(\mathcal{A}^0) := \{D \in \mathbf{D}(\mathcal{A}^0) : D \subset \mathcal{V}_2,\ D \cap \bar{C}_{1\cdots m} \neq \emptyset\}$ with $\bar{C}_{1\cdots m} := \{(x_1, \dots, x_m)^T \in H_0 : x_1 \geq \cdots \geq x_m\}$.

By Proposition 6.1, (17) and (18), we obtain an upper bound for the number of inequivalent ranking patterns of unfolding models of codimension one.

Corollary 6.2. *The number of inequivalent ranking patterns of unfolding models of codimension one cannot exceed*

$$1 + |\mathbf{D}_2^{1\cdots m}(\mathcal{A}^0)| = \frac{|\mathbf{Ch}(\mathcal{A}^0 \cup \mathcal{B}_m^{H_0})|}{m!} - 1. \tag{20}$$

Moreover, if all elements of $\mathbf{D}_2^{1\cdots m}(\mathcal{A}^0)$ *are on different orbits under action* (16)*, then* (20) *gives the exact number of inequivalent ranking patterns of unfolding models of codimension one.*

6.2 Inequivalent ranking patterns for $m \leq 6$

In this subsection, we investigate inequivalent ranking patterns of unfolding models of codimension one for $m \leq 6$.

We know

$$\begin{aligned} \mathrm{RP}_{D_1} &= \mathbb{P}_m \setminus \{(i_1 \cdots i_{m-1}\, 1) : (i_1 \cdots i_{m-1}) \text{ is a permutation of } [m] \setminus \{1\}\}, \\ \mathrm{RP}_{-D_m} &= \mathrm{RP}_{-\rho D_1} = \mathbb{P}_m \setminus \{(m\, i_1 \cdots i_{m-1}) : (i_1 \cdots i_{m-1}) \text{ is a permutation of } [m] \setminus \{m\}\} \end{aligned}$$

by Lemma 3.1.

6.2.1 Case $m = 3$

When $m = 3$, we have $\mathcal{V}_2 = \emptyset$ and $\mathbf{D}(\mathcal{A}_3^0) = \{D_1, D_2, D_3, -D_1, -D_2, -D_3\}$. Accordingly, the set of all ranking patterns of unfolding models of codimension one is $\{\mathrm{RP}_{D_1}, \mathrm{RP}_{D_2}, \mathrm{RP}_{D_3}\}$. Since $D_1 = \tau_2 D_2 = \tau_3 D_3$ ($\tau_2 \in \mathfrak{S}_3$ is the transposition of 1 and 2, and $\tau_3 \in \mathfrak{S}_3$ is the transposition of 1 and 3), the number of inequivalent ranking patterns of unfolding models of codimension one is 1. We have $\mathrm{RP}_{D_1} = \mathbb{P}_3 \setminus \{(231), (321)\}$.

Let us consider (17) in this case. We have $\mathbf{D}_2^{123}(\mathcal{A}_3^0) = \emptyset$, and $\mathbf{D}^{123}(\mathcal{A}_3^0) = \{D_1, -D_3\} \subset \mathbf{D}(\mathcal{A}_3^0)$ is a cross section under the action of $\mathfrak{S}_3$ on $\mathbf{D}(\mathcal{A}_3^0)$:

$$\mathbf{D}(\mathcal{A}_3^0) = \mathfrak{S}_3 D_1 \sqcup \mathfrak{S}_3(-D_3) = \{D_1, D_2, D_3\} \sqcup \{-D_1, -D_2, -D_3\}.$$

Thus, the inequality in (17) is actually an equality in this case: $|\mathbf{D}(\mathcal{A}_3^0)/\mathfrak{S}_3| = |\mathbf{D}^{123}(\mathcal{A}_3^0)| = 2$. The number $|\mathbf{D}^{123}(\mathcal{A}_3^0)| = 2$ can also be confirmed by $\chi(\mathcal{A}_3^0 \cup \mathcal{B}_3^{H_0}, t) = t^2 - 6t + 5 = (t-1)(t-5)$ yielding $|\mathbf{Ch}(\mathcal{A}_3^0 \cup \mathcal{B}_3^{H_0})|/(3!) = (-1)^{3-1}\chi(\mathcal{A}_3^0 \cup \mathcal{B}_3^{H_0}, -1)/(3!) = 12/(3!) = 2$.

6.2.2 Case $m = 4$

When $m = 4$, we have

$$\mathcal{V}_2 \cap \bar{C}_{1\cdots 4} = \bigsqcup_{D \subset \mathcal{V}_2,\ D \in \mathbf{D}(\mathcal{A}_4^0)} (D \cap \bar{C}_{1\cdots 4}) = R_4 \sqcup (-\rho R_4), \tag{21}$$

where

$$R_4 := \{(v_1, v_2, v_3, v_4)^T \in \mathbb{S}^2 : v_1 \geq v_2 > 0 > v_3 \geq v_4,\ v_2 > -v_3\},$$

$\rho \in \mathfrak{S}_4$ is defined in (19) and

$$\begin{aligned} -\rho R_4 &:= \{-(v_{\rho^{-1}(1)}, \ldots, v_{\rho^{-1}(4)})^T : (v_1, \ldots, v_4)^T \in R_4\} \\ &= \{(v_1, v_2, v_3, v_4)^T \in \mathbb{S}^2 : v_1 \geq v_2 > 0 > v_3 \geq v_4,\ v_2 < -v_3\}. \end{aligned}$$

Now, there is only one $D \subset \mathcal{V}_2$ ($D \in \mathbf{D}(\mathcal{A}_4^0)$) such that $\emptyset \neq D \cap \bar{C}_{1\cdots 4} \subseteq R_4$. Such a D is the chamber E ($\ni (v_1, \ldots, v_4)^T$) determined by $v_{\{1\}}, v_{\{2\}}, v_{\{1,2\}}, v_{\{1,3\}}, v_{\{2,3\}} > 0$ and $v_{\{3\}}, v_{\{4\}} < 0$, and we have $\emptyset \neq E \cap \bar{C}_{1\cdots 4} = R_4$. Here, we are writing $v_I := \sum_{i \in I} v_i$. As for $-\rho R_4$ in (21), $D = -\rho E$ is the only $D \subset \mathcal{V}_2$ ($D \in \mathbf{D}(\mathcal{A}_4^0)$) such that $\emptyset \neq D \cap \bar{C}_{1\cdots 4} \subseteq -\rho R_4 : \emptyset \neq (-\rho E) \cap \bar{C}_{1\cdots 4} = -\rho R_4$.

From the preceding arguments, we obtain

$$\mathcal{V}_2 \cap \bar{C}_{1\cdots 4} = (E \cap \bar{C}_{1\cdots 4}) \sqcup ((-\rho E) \cap \bar{C}_{1\cdots 4}),$$

and hence $\mathbf{D}_2^{1\cdots 4}(\mathcal{A}^0) = \mathbf{D}_2^{\overline{1\cdots 4}}(\mathcal{A}^0) = \{E, -\rho E\}$. Thus, we get

$$\mathbf{D}^{1\cdots 4}(\mathcal{A}_4^0) = \{D_1, -\rho D_1, E, -\rho E\}$$

by (18). We know that E and $-\rho E$ are on different orbits. Therefore, $\mathbf{D}^{1\cdots 4}(\mathcal{A}_4^0)$ is a cross section under the action of $\mathfrak{S}_4$ on $\mathbf{D}(\mathcal{A}_4^0)$, and we have $|\mathbf{D}(\mathcal{A}_4^0)/\mathfrak{S}_4| = |\mathbf{D}^{1\cdots 4}(\mathcal{A}_4^0)| = 2(1+1) = 4$. So the number of inequivalent ranking patterns of unfolding models of codimension one is $4 - 1 = 3 = 1 + 2 \cdot 1$. (In passing, we can confirm the number $|\mathbf{D}^{1\cdots 4}(\mathcal{A}_4^0)| = 4$ by $\chi(\mathcal{A}_4^0 \cup \mathcal{B}_4^{H_0}, t) = t^3 - 13t^2 + 47t - 35 = (t-1)(t-5)(t-7)$ giving $|\mathbf{Ch}(\mathcal{A}_4^0 \cup \mathcal{B}_4^{H_0})|/(4!) = (-1)^{4-1}\chi(\mathcal{A}_4^0 \cup \mathcal{B}_4^{H_0}, -1)/(4!) = 96/(4!) = 4$.)

The chambers $D_1, E, -\rho E$ correspond to ranking patterns that can be realized by unfolding models of codimension one, $\mathrm{RP}_{D_1}, \mathrm{RP}_E, \mathrm{RP}_{-\rho E}$, while the chamber $-\rho D_1 = -D_4 = \mathcal{V}_1(4, -)$ corresponds to RP_{-D_4}, which cannot be realized. From E, we can take $v = (1/2, 1/2, -1/4, -3/4)^T/(3\sqrt{2}/4) \in E$. Thus by Lemma 3.1, we can see

$$\begin{aligned} \mathrm{RP}_{D_1} &= \mathbb{P}_m \setminus \{(2341), (2431), (3241), (3421), (4231), (4321)\}, \\ \mathrm{RP}_E &= \mathbb{P}_m \setminus \{(3412), (3421), (4312), (4321), (4132), (4231)\}, \\ \mathrm{RP}_{-\rho E} &= \mathbb{P}_m \setminus \{(3412), (3421), (4312), (4321), (4231), (3241)\}. \end{aligned}$$

These three ranking patterns, realized as the ranking patterns of unfolding models of codimension one, are displayed in Figures 2, 3 and 4. (For simplicity, $\mu_1, \mu_2, \mu_3, \mu_4$ are written as $1, 2, 3, 4$ in the figures.)

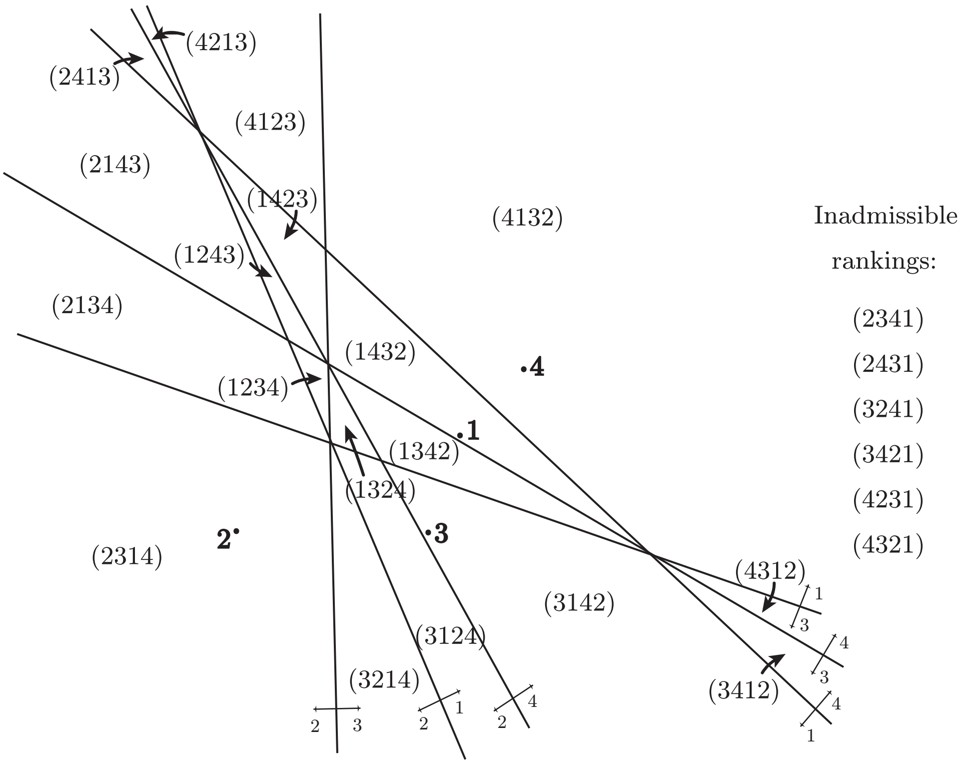

Figure 2: RP_{D_1}.

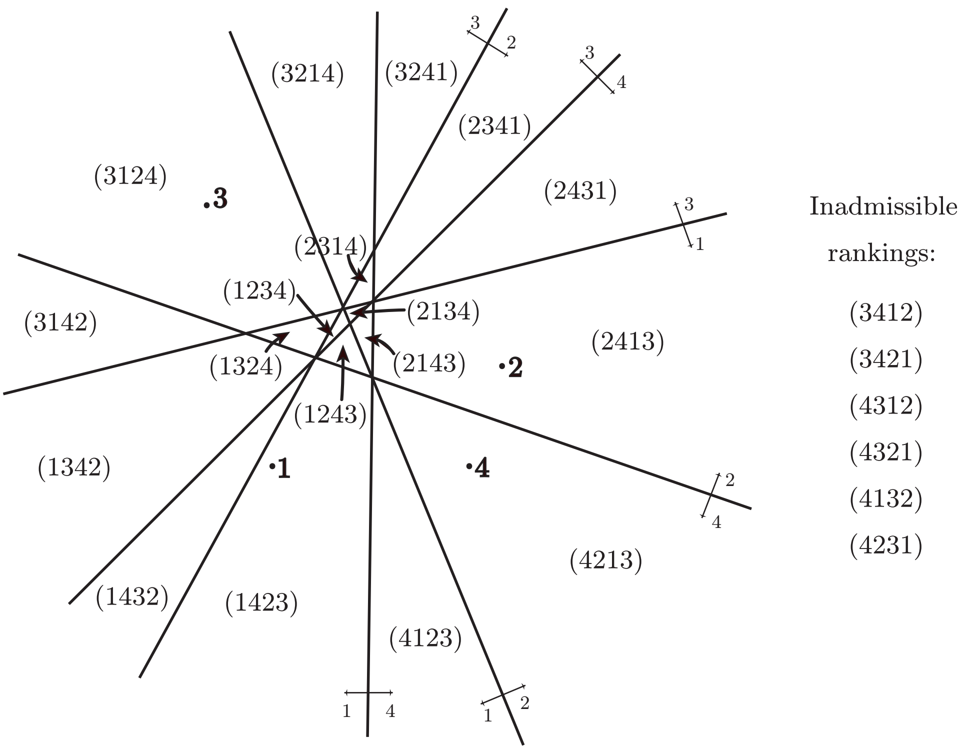

Figure 3: RP_E.

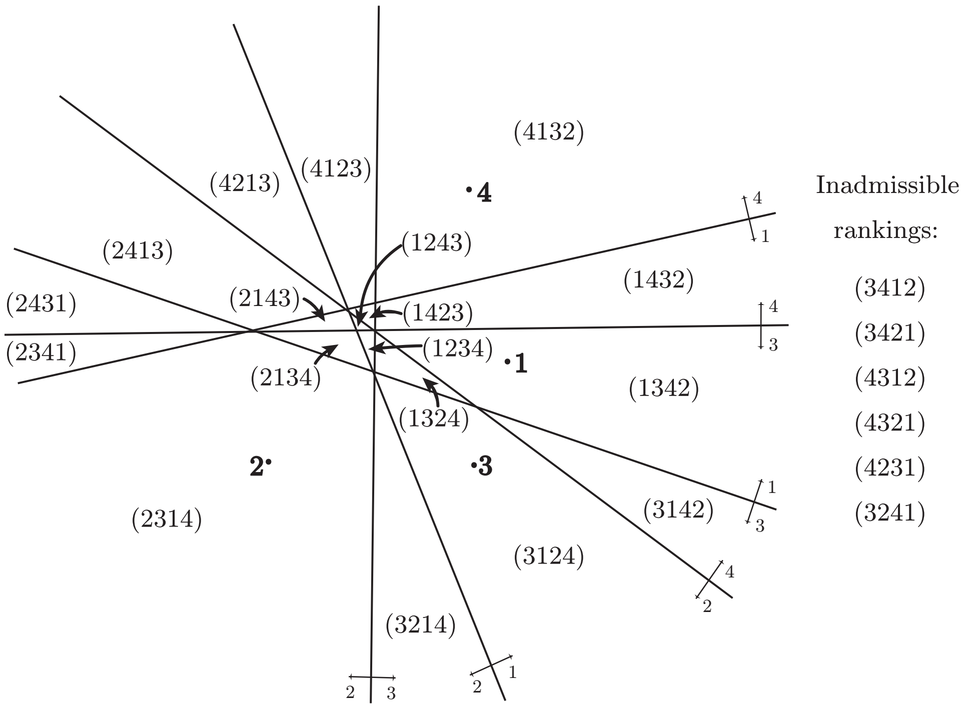

Figure 4: $\mathrm{RP}_{-\rho E}$.

6.2.3 Case $m = 5$

When $m = 5$, we have

$$\mathcal{V}_2 \cap \bar{C}_{1\cdots5} = \bigsqcup_{D \subset \mathcal{V}_2,\ D \in \mathbf{D}(\mathcal{A}_5^0)} (D \cap \bar{C}_{1\cdots5}) = R_5 \sqcup (-\rho R_5), \tag{22}$$

where

$$R_5 := \{(v_1, \ldots, v_5)^T \in \mathbb{S}^3 : v_1 \geq v_2 \geq v_3 > 0 > v_4 \geq v_5\}.$$

There are five chambers $D \subset \mathcal{V}_2$ ($D \in \mathbf{D}(\mathcal{A}_5^0)$) such that $\emptyset \neq D \cap \bar{C}_{1\cdots5} \subseteq R_5$. Let $E_1, \ldots, E_5$ be those five chambers. All $D \cap \bar{C}_{1\cdots5}$ for $D = E_1, \ldots, E_5$ are listed in Table 1. The first column gives the defining inequalities of each $D \cap \bar{C}_{1\cdots5}$ (besides $v_1 \geq v_2 \geq v_3 > 0 > v_4 \geq v_5$); the second column exhibits an unnormalized representative point of each $D \cap \bar{C}_{1\cdots5}$; and the last column contains an upper bound for $|\mathfrak{S}_5 D|$ for each D. For example, the first row of Table 1 corresponds to the chamber $E_1 \ni v = (v_1, \ldots, v_5)^T$ determined by positive $v_{\{1\}}, v_{\{2\}}, v_{\{3\}}, v_{\{1,2\}}, v_{\{1,3\}}, v_{\{1,4\}}, v_{\{2,3,4\}}, v_{\{2,3\}}, v_{\{2,4\}}, v_{\{1,3,4\}}, v_{\{3,4\}}, v_{\{1,2,4\}}$ and negative $v_{\{4\}}, v_{\{5\}}, v_{\{4,5\}}$. For this E_1, we can take $v = (1/3, 1/3, 1/3, -1/6, -5/6)^T / (\sqrt{38}/6) \in E_1 \cap \bar{C}_{1\cdots5}$, which means that the cardinality of the orbit $\mathfrak{S}_5 E_1$ cannot exceed $5!/(3!) = 20 : |\mathfrak{S}_5 E_1| \leq 20$. As for $-\rho R_5$ in (22), the chambers $D \subset \mathcal{V}_2$ ($D \in \mathbf{D}(\mathcal{A}_5^0)$) such that $\emptyset \neq D \cap \bar{C}_{1\cdots5} \subseteq -\rho R_5$ are exactly those five chambers given as $D = -\rho E_i,\ i = 1, \ldots, 5$.

The discussions above imply that

$$\begin{aligned}\mathcal{V}_2 \cap \bar{C}_{1\cdots5} &= (E_1 \cap \bar{C}_{1\cdots5}) \sqcup \cdots \sqcup (E_5 \cap \bar{C}_{1\cdots5}) \\ &\quad \sqcup((-\rho E_1) \cap \bar{C}_{1\cdots5}) \sqcup \cdots \sqcup ((-\rho E_5) \cap \bar{C}_{1\cdots5}),\end{aligned}$$

Table 1: $D \cap \bar{C}_{1\cdots 5} \subseteq R_5$ $(v_1 \geq v_2 \geq v_3 > 0 > v_4 \geq v_5)$. $\epsilon > 0$ is small enough.

Defining Inequalities	Representative Point	$\lvert\mathfrak{S}_5 D\rvert$
$v_{\{3,4\}} > 0$	$(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3},\ -\frac{1}{6},\ -\frac{5}{6})$	20
$v_{\{3,4\}} < 0,\ v_{\{2,4\}} > 0$	$(\frac{1}{3}+\epsilon,\ \frac{1}{3}+\epsilon,\ \frac{1}{3}-2\epsilon,\ -\frac{1}{3},\ -\frac{2}{3})$	60
$v_{\{2,4\}} < 0,\ v_{\{1,5\}} < 0,\ v_{\{1,4\}} > 0$	$(\frac{1}{3}+2\epsilon,\ \frac{1}{3}-\epsilon,\ \frac{1}{3}-\epsilon,\ -\frac{1}{3},\ -\frac{2}{3})$	60
$v_{\{1,5\}} > 0$	$(\frac{2}{3},\ \frac{1}{6},\ \frac{1}{6},\ -\frac{1}{2},\ -\frac{1}{2})$	30
$v_{\{1,4\}} < 0$	$(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3},\ -\frac{1}{2},\ -\frac{1}{2})$	10

and this yields $\mathbf{D}_2^{1\cdots 5}(\mathcal{A}_5^0) = \{E_1, \ldots, E_5, -\rho E_1, \ldots, -\rho E_5\}$ and

$$\mathbf{D}^{1\cdots 5}(\mathcal{A}_5^0) = \{D_1, -\rho D_1, E_1, \ldots, E_5, -\rho E_1, \ldots, -\rho E_5\}. \tag{23}$$

Since $\mathbf{D}^{1\cdots 5}(\mathcal{A}_5^0)$ includes a cross section under the action of $\mathfrak{S}_5$ on $\mathbf{D}(\mathcal{A}_5^0)$, we know

$$\begin{aligned} \mathbf{D}(\mathcal{A}_5^0) &= \mathfrak{S}_5 D_1 \cup \mathfrak{S}_5(-\rho D_1) \cup \mathfrak{S}_5 E_1 \cup \cdots \cup \mathfrak{S}_5 E_5 \\ &\qquad \cup \mathfrak{S}_5(-\rho E_1) \cup \cdots \cup \mathfrak{S}_5(-\rho E_5), \end{aligned} \tag{24}$$

which implies

$$\begin{aligned} |\mathbf{D}(\mathcal{A}_5^0)| &\leq |\mathfrak{S}_5 D_1| + |\mathfrak{S}_5(-\rho D_1)| + \sum_{i=1}^{5} |\mathfrak{S}_5 E_i| + \sum_{i=1}^{5} |\mathfrak{S}_5(-\rho E_i)| \\ &= 2(|\mathfrak{S}_5 D_1| + \sum_{i=1}^{5} |\mathfrak{S}_5 E_i|) = 2(5 + \sum_{i=1}^{5} |\mathfrak{S}_5 E_i|). \end{aligned} \tag{25}$$

From the last column of Table 1, wee can see $\sum_{i=1}^{5} |\mathfrak{S}_5 E_i| \leq 20+60+60+30+10 = 180$, so $|\mathbf{D}(\mathcal{A}_5^0)| \leq 2(5+180) = 370$ by (25). But since 370 is equal to $|\mathbf{D}(\mathcal{A}_5^0)| = |\mathbf{Ch}(\mathcal{A}_5^0)|$ (see Lemma 5.3), the inequality in (25) is actually an equality. This means the $|\mathbf{D}^{1\cdots 5}(\mathcal{A}_5^0)| = 2(1+5) = 12$ orbits on the right-hand side of (24) are all distinct:

$$\begin{aligned} \mathbf{D}(\mathcal{A}_5^0) &= \mathfrak{S}_5 D_1 \sqcup \mathfrak{S}_5(-\rho D_1) \sqcup \mathfrak{S}_5 E_1 \sqcup \cdots \sqcup \mathfrak{S}_5 E_5 \\ &\qquad \sqcup \mathfrak{S}_5(-\rho E_1) \sqcup \cdots \sqcup \mathfrak{S}_5(-\rho E_5). \end{aligned}$$

Hence, $\mathbf{D}^{1\cdots 5}(\mathcal{A}_5^0)$ in (23) is a cross section. Therefore, $|\mathbf{D}(\mathcal{A}_5^0)/\mathfrak{S}_5| = |\mathbf{D}^{1\cdots 5}(\mathcal{A}_5^0)| = 12$, and the number of inequivalent ranking patterns of unfolding models of codimension one is $12 - 1 = 11 = 1 + 2 \cdot 5$.

Notice, in passing, $\sum_{i=1}^{5} |\mathfrak{S}_5 E_i| = 180$ so that the upper bounds in the last column of Table 1 are actually exact numbers.

6.2.4 Case $m = 6$

When $m = 6$, we have

$$\mathcal{V}_2 \cap \bar{C}_{1\cdots 6} = \bigsqcup_{D \subset \mathcal{V}_2,\ D \in \mathbf{D}(\mathcal{A}_6^0)} (D \cap \bar{C}_{1\cdots 6}) = R_{6,1} \sqcup (-\rho R_{6,1}) \sqcup R_{6,2} \sqcup (-\rho R_{6,2}),$$

Table 2: $D \cap \bar{C}_{1\cdots 6} \subseteq R_{6,1}$ $(v_1 \geq v_2 \geq v_3 \geq v_4 > 0 > v_5 \geq v_6)$. $\epsilon > 0$ is small enough.

Defining Inequalities	Representative Point	$\lvert\mathfrak{S}_6 D\rvert$
$v_{\{4,5\}} > 0$	$(\frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ -\frac{1}{4}+\epsilon,\ -\frac{3}{4}-\epsilon)$	30
$v_{\{4,5\}} < 0,\ v_{\{3,5\}} > 0$	$(\frac{1}{4}+\epsilon,\ \frac{1}{4}+\epsilon,\ \frac{1}{4}+\epsilon,\ \frac{1}{4}-3\epsilon,\ -\frac{1}{4},\ -\frac{3}{4})$	120
$v_{\{3,5\}} < 0,\ v_{\{2,5\}} > 0,\ v_{\{3,4,5\}} > 0$	$(\frac{1}{4}+\epsilon,\ \frac{1}{4}+\epsilon,\ \frac{1}{4}-\epsilon,\ \frac{1}{4}-\epsilon,\ -\frac{1}{4},\ -\frac{3}{4})$	180
$v_{\{2,5\}} < 0,\ v_{\{3,4,5\}} > 0,\ v_{\{1,5\}} > 0$	$(\frac{3}{4},\ \frac{1}{12},\ \frac{1}{12},\ \frac{1}{12},\ -\frac{1}{8},\ -\frac{7}{8})$	120
$v_{\{2,5\}} < 0,\ v_{\{3,4,5\}} < 0,\ v_{\{2,4,5\}} > 0,\ v_{\{1,5\}} > 0$	$(\frac{3}{4}-\epsilon,\ \frac{1}{12}+\epsilon,\ \frac{1}{12},\ \frac{1}{12},\ -\frac{1}{6}-\frac{\epsilon}{2},\ -\frac{5}{6}+\frac{\epsilon}{2})$	360
$v_{\{2,4,5\}} < 0,\ v_{\{2,3,5\}} > 0,\ v_{\{1,5\}} > 0$	$(\frac{3}{4}+\epsilon,\ \frac{1}{12},\ \frac{1}{12},\ \frac{1}{12}-\epsilon,\ -\frac{1}{6}+\frac{\epsilon}{2},\ -\frac{5}{6}-\frac{\epsilon}{2})$	360
$v_{\{2,3,5\}} < 0,\ v_{\{1,6\}} < 0,\ v_{\{1,5\}} > 0$	$(\frac{3}{4},\ \frac{1}{12},\ \frac{1}{12},\ \frac{1}{12},\ -\frac{1}{5},\ -\frac{4}{5})$	120
$v_{\{1,6\}} > 0$	$(\frac{3}{4},\ \frac{1}{12},\ \frac{1}{12},\ \frac{1}{12},\ -\frac{1}{2},\ -\frac{1}{2})$	60
$v_{\{3,4,5\}} < 0,\ v_{\{2,5\}} > 0$	$(\frac{5}{12},\ \frac{5}{12},\ \frac{1}{12},\ \frac{1}{12},\ -\frac{1}{4},\ -\frac{3}{4})$	180
$v_{\{1,5\}} < 0,\ v_{\{3,4,5\}} > 0$	$(\frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ -\frac{1}{2}+\epsilon,\ -\frac{1}{2}-\epsilon)$	30
$v_{\{1,5\}} < 0,\ v_{\{3,4,5\}} < 0,\ v_{\{2,4,5\}} > 0$	$(\frac{1}{4}+\epsilon,\ \frac{1}{4}+\epsilon,\ \frac{1}{4}-\epsilon,\ \frac{1}{4}-\epsilon,\ -\frac{1}{2}+\epsilon,\ -\frac{1}{2}-\epsilon)$	180
$v_{\{1,5\}} < 0,\ v_{\{2,4,5\}} < 0,\ v_{\{2,3,5\}} > 0,\ v_{\{1,4,5\}} > 0$	$(\frac{1}{4}+\epsilon,\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}-\epsilon,\ -\frac{1}{2}+\frac{\epsilon}{2},\ -\frac{1}{2}-\frac{\epsilon}{2})$	360
$v_{\{1,4,5\}} < 0$	$(\frac{1}{4}+\epsilon,\ \frac{1}{4}+\epsilon,\ \frac{1}{4}+\epsilon,\ \frac{1}{4}-3\epsilon,\ -\frac{1}{2},\ -\frac{1}{2})$	60
$v_{\{2,3,5\}} < 0,\ v_{\{1,5\}} < 0$	$(\frac{1}{4}+3\epsilon,\ \frac{1}{4}-\epsilon,\ \frac{1}{4}-\epsilon,\ \frac{1}{4}-\epsilon,\ -\frac{1}{2},\ -\frac{1}{2})$	60

where

$$\begin{aligned} R_{6,1} &:= \{(v_1,\ldots,v_6)^T \in \mathbb{S}^4 : v_1 \geq v_2 \geq v_3 \geq v_4 > 0 > v_5 \geq v_6\}, \\ R_{6,2} &:= \{(v_1,\ldots,v_6)^T \in \mathbb{S}^4 : v_1 \geq v_2 \geq v_3 > 0 > v_4 \geq v_5 \geq v_6,\ v_3 > -v_4\}. \end{aligned}$$

All nonempty $D \cap \bar{C}_{1\cdots 6}$ ($D \subset \mathcal{V}_2$, $D \in \mathbf{D}(\mathcal{A}_6^0)$) that are included in $R_{6,1}$ and in $R_{6,2}$ are listed in Tables 2 and 3, respectively. The first columns provide the defining inequalities of each $D \cap \bar{C}_{1\cdots 6}$ (besides $v_1 \geq v_2 \geq v_3 \geq v_4 > 0 > v_5 \geq v_6$ and $v_1 \geq v_2 \geq v_3 > 0 > v_4 \geq v_5 \geq v_6$, $v_3 > -v_4$, respectively); the second columns show an unnormalized representative point of each $D \cap \bar{C}_{1\cdots 6}$; and the last columns include an upper bound for $|\mathfrak{S}_6 D|$ for each D.

There are 14 (resp. 13) rows in Table 2 (resp. Table 3), and the sum of the upper bounds for $|\mathfrak{S}_6 D|$ in the last column of the table is 2220 (resp. 3420). Since the value $2(6+2220+3420) = 11292$ equals $|\mathbf{Ch}(\mathcal{A}_6^0)|$ (Lemma 5.3), we can conclude, by the same reasoning as in the case of $m = 5$, that the number of inequivalent ranking patterns of unfolding models of codimension one is $1 + 2(14+13) = 55$.

Open problem: We have seen that for any $m \leq 6$, subset $\mathbf{D}^{1\cdots m}(\mathcal{A}^0) \subset \mathbf{D}(\mathcal{A}^0)$ is a cross section so that the upper bound in (20) is actually the exact number. Does this hold true for all m?

References

[1] C. A. Athanasiadis, *Algebraic Combinatorics of Graph Spectra, Subspace Arrangements and Tutte Polynomials,* Ph.D. thesis, MIT, 1996.

Table 3: $D \cap \bar{C}_{1\cdots 6} \subseteq R_{6,2}$ $(v_1 \geq v_2 \geq v_3 > 0 > v_4 \geq v_5 \geq v_6,\ v_3 > -v_4)$. $\epsilon > 0$ is small enough.

Defining Inequalities	Representative Point	$\lvert\mathfrak{S}_6 D\rvert$
$v_{\{3,4,5\}} > 0$	$(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3},\ -\frac{1}{6}+\epsilon,\ -\frac{1}{6}+\epsilon,\ -\frac{2}{3}-2\epsilon)$	60
$v_{\{3,5\}} > 0,\ v_{\{3,4,5\}} < 0,\ v_{\{2,4,5\}} > 0$	$(\frac{1}{3}+\epsilon,\ \frac{1}{3}+\epsilon,\ \frac{1}{3}-2\epsilon,\ -\frac{1}{6},\ -\frac{1}{6},\ -\frac{2}{3})$	180
$v_{\{3,5\}} > 0,\ v_{\{2,4,5\}} < 0,\ v_{\{1,6\}} < 0,\ v_{\{1,4,5\}} > 0$	$(\frac{2}{3},\ \frac{1}{6},\ \frac{1}{6},\ -\frac{1}{8},\ -\frac{1}{8},\ -\frac{3}{4})$	180
$v_{\{3,5\}} < 0,\ v_{\{2,4,5\}} > 0$	$(\frac{1}{3}+\epsilon,\ \frac{1}{3}+\epsilon,\ \frac{1}{3}-2\epsilon,\ -\epsilon,\ -\frac{1}{3}+\epsilon,\ -\frac{2}{3})$	360
$v_{\{3,5\}} < 0, v_{\{2,5\}} > 0, v_{\{2,4,5\}} < 0, v_{\{1,6\}} < 0, v_{\{1,4,5\}} > 0$	$(\frac{2}{3},\ \frac{1}{6}+\epsilon,\ \frac{1}{6}-\epsilon,\ -2\epsilon,\ -\frac{1}{6},\ -\frac{5}{6}+2\epsilon)$	720
$v_{\{2,5\}} < 0,\ v_{\{1,6\}} < 0,\ v_{\{1,4,5\}} > 0$	$(\frac{2}{3},\ \frac{1}{6},\ \frac{1}{6},\ -\epsilon,\ -\frac{1}{4},\ -\frac{3}{4}+\epsilon)$	360
$v_{\{2,3,5\}} > 0,\ v_{\{1,6\}} > 0,\ v_{\{2,3,6\}} < 0$	$(\frac{2}{3},\ \frac{1}{6},\ \frac{1}{6},\ -2\epsilon,\ -\frac{1}{3}+\epsilon,\ -\frac{2}{3}+\epsilon)$	360
$v_{\{2,3,6\}} > 0,\ v_{\{1,6\}} > 0$	$(\frac{1}{2},\ \frac{1}{4},\ \frac{1}{4},\ -\epsilon,\ -\frac{1}{2}+\frac{\epsilon}{2},\ -\frac{1}{2}+\frac{\epsilon}{2})$	180
$v_{\{2,3,5\}} < 0$	$(\frac{2}{3},\ \frac{1}{6},\ \frac{1}{6},\ -\epsilon,\ -\frac{1}{2}+\frac{\epsilon}{2},\ -\frac{1}{2}+\frac{\epsilon}{2})$	180
$v_{\{3,5\}} > 0,\ v_{\{1,4,5\}} < 0,\ v_{\{1,6\}} < 0$	$(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3},\ -\frac{1}{6}-\epsilon,\ -\frac{1}{6}-\epsilon,\ -\frac{2}{3}+2\epsilon)$	60
$v_{\{3,5\}} < 0,\ v_{\{2,5\}} > 0,\ v_{\{1,4,5\}} < 0,\ v_{\{1,6\}} < 0$	$(\frac{1}{3}+\epsilon,\ \frac{1}{3}+\epsilon,\ \frac{1}{3}-2\epsilon,\ -3\epsilon,\ -\frac{1}{3}+\epsilon,\ -\frac{2}{3}+2\epsilon)$	360
$v_{\{2,5\}} < 0,\ v_{\{1,5\}} > 0,\ v_{\{1,4,5\}} < 0,\ v_{\{1,6\}} < 0$	$(\frac{1}{3}+2\epsilon,\ \frac{1}{3}-\epsilon,\ \frac{1}{3}-\epsilon,\ -3\epsilon,\ -\frac{1}{3},\ -\frac{2}{3}+3\epsilon)$	360
$v_{\{1,5\}} < 0$	$(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3},\ -2\epsilon,\ -\frac{1}{2}+\epsilon,\ -\frac{1}{2}+\epsilon)$	60

[2] C. A. Athanasiadis, Characteristic polynomials of subspace arrangements and finite fields, *Adv. Math.* **122** (1996), 193–233.

[3] C. H. Coombs, Psychological scaling without a unit of measurement, *Psychol. Rev.* **57** (1950), 145–158.

[4] H. Crapo and G.-C. Rota, *On the Foundations of Combinatorial Theory: Combinatorial Geometries,* preliminary edition, MIT Press, Cambridge, MA, 1970.

[5] W. S. DeSarbo and D. L. Hoffman, Constructing MDS joint spaces from binary choice data: A multidimensional unfolding threshold model for marketing research, *J. Marketing Res.* **24** (1987), 40–54.

[6] G. De Soete, H. Feger and K. C. Klauer, *New Developments in Psychological Choice Modeling,* North-Holland, Amsterdam, 1989.

[7] T. S. Evans, N-point finite temperature expectation values at real times, in *Thermal Field Theories* (H. Ezawa, T. Arimitsu, Y. Hashimoto, eds.), Nuclear Physics B **374**, North-Holland, Amsterdam, 1992, pp. 340–370.

[8] T. S. Evans, What is being calculated with thermal field theory?, in *Particle Physics and Cosmology,* Proceedings of the 9th Lake Louise Winter Institute, Lake Louise, Alberta, Canada, 20–26 February 1994 (A. Astbury, B. A. Campbell, W. Israel, F. C. Khanna, D. Page, J. L. Pinfold, eds.), World Scientific Publishing, Singapore, 1995, pp. 343–352.

[9] I. J. Good and T. N. Tideman, Stirling numbers and a geometric structure from voting theory, *J. Combin. Theory* A **23** (1977), 34–45.

[10] M. J. Hinich and M. C. Munger, *Analytical Politics,* Cambridge University Press, Cambridge, 1997.

[11] H. Kamiya, P. Orlik, A. Takemura and H. Terao, Arrangements and ranking patterns, *Ann. Combin.* **10** (2006), 219–235.

[12] H. Kamiya and A. Takemura, On rankings generated by pairwise linear discriminant analysis of m populations, *J. Multivariate Anal.* **61** (1997), 1–28.

[13] H. Kamiya and A. Takemura, Characterization of rankings generated by linear discriminant analysis, *J. Multivariate Anal.* **92** (2005), 343–358.

[14] H. Kamiya, A. Takemura and H. Terao, Periodicity of hyperplane arrangements with integral coefficients modulo positive integers, *J. Algebraic Combin.* **27** (2008), 317–330.

[15] H. Kamiya, A. Takemura and H. Terao, The characteristic quasi-polynomials of the arrangements of root systems and mid-hyperplane arrangements, in *Arrangements, Local Systems and Singularities,* CIMPA Summer School, Galatasaray University, Istanbul, 2007 (F. El Zein, A. I. Suciu, M. Tosun, A. M. Uludağ, S. Yuzvinsky, eds.), Progress in Mathematics **283**, Birkhäuser Verlag, Basel, 2009, pp. 177–190.

[16] H. Kamiya, A. Takemura and H. Terao, Periodicity of non-central integral arrangements modulo positive integers, arXiv:0803.2755v1 [math.CO], to appear in *Ann. Combin.*

[17] D. B. MacKay, R. F. Easley and J. L. Zinnes, A single ideal point model for market structure analysis, *J. Marketing Res.* **32** (1995), 433–443.

[18] P. Orlik and H. Terao, *Arrangements of Hyperplanes,* Springer-Verlag, Berlin, 1992.

[19] R. Stanley, Ordering events in Minkowski space, *Adv. Appl. Math.* **37** (2006), 514–525.

[20] R. Stanley, An introduction to hyperplane arrangements, in *Geometric Combinatorics* (E. Miller, V. Reiner, B. Sturmfels, eds.), IAS/Park City Mathematics Series **13**, American Mathematical Society, Providence, RI, 2007, pp. 389–496.

[21] R. M. Thrall, A combinatorial problem, *Michigan Math. J.* **1** (1952), 81–88.

[22] M. A. van Eijck, *Thermal Field Theory and the Finite-Temperature Renormalization Group,* Ph.D. thesis, University of Amsterdam, 1995.

[23] T. Zaslavsky, Facing up to arrangements: Face-count formulas for partitions of space by hyperplanes, *Mem. Amer. Math. Soc.* vol. 1, no. 154 (1975).

[24] T. Zaslavsky, Perpendicular dissection of space, *Discrete Comput. Geom.* **27** (2002), 303–351.