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Abstract

Many of practical design specifications are provided by finite frequency properties described by inequalities over restricted finite frequency intervals. A quadratic differential form (QDF) is a useful algebraic tool when we consider dissipation theory based on the behavioral approach. In this paper, we investigate time domain characterizations of the finite frequency domain inequalities (FFDIs) using QDFs. Based on QDFs, we derive a characterization of the FFDIs using quadratic differential forms as a main result. This condition leads to a physical interpretation in terms of the compensating rate, which guarantees dissipativity of some behavior with some rate constraints. Such interpretation has not been clarified by the previous studies of finite frequency properties. The aforementioned characterization yields an LMI condition whose solvability is equivalent to the FFDIs. This can be regarded as the finite frequency KYP lemma in the behavioral framework.

1 Introduction

Many of practical design specifications are provided by sets of finite frequency properties which are expressed as inequalities over restricted finite frequency intervals. Hence, the properties play an important role for dynamical system design including plant and controller design integration.

The previous works on characterizations of finite frequency properties are as follows. Iwasaki et.al [3][6] derived a linear matrix inequality (LMI) characterization for the finite frequency properties, which is called generalized Kalman-Yakubovič-Popov (KYP) lemma. Based on the lemma, a time domain characterization was derived in terms of an integral of the supply rate, called matrix valued integral quadratic constraint (IQC), for asymptotically stable state-space systems [5]. However, their physical interpretation was not fully satisfactory, when we consider the interaction between supplied power and internal energy of a system. In addition, their characterization was not essential from the view point of dissipation theory, since the characterization was derived through the generalized KYP lemma. For such reasons, it has been desired that we characterize the finite frequency properties from the dissipativity viewpoints directly.

Dissipativity is one of the most important properties when we analyze a dynamical system from the energy and power interaction with its outside environment. This interaction is expressed by an inequality called dissipation inequality. It may be important that we consider a dissipativity analysis in frequency domains. This can be verified by the following facts. It is well-known that dissipativity can be equivalently transformed to the inequality over the imaginary axis [14]. Moreover, a stability condition for a feedback system is given in terms of integrals over entire frequencies, called IQC [10]. This paper clarifies how the constraint on the frequency variable appears in the dissipation inequality.

A quadratic differential form (QDF) is a useful algebraic tool in dissipation theory based on the behavioral approach [15], because it has a one-to-one correspondence to a two-variable polynomial matrix. Since the behavioral approach is the theoretic framework which does not assume an input-output relationship in advance, we can naturally analyze and design a system described by a nonproper transfer function. Based on QDFs, Willems and Trentelman [16] has proved that dissipativity of a behavior is equivalent to a certain frequency domain inequalities on the entire frequency range. This also leads to an equivalent LMI characterization of the inequalities [13]. However, neither time domain characterization nor LMI characterization of the finite frequency properties has not been derived in the behavioral framework.

In this paper, we consider a characterization of finite frequency properties in the framework of dissipation theory. As a main result, we derive a characterization of the FFDIs in terms of the dissipation inequality described by QDFs. This characterization allows us to understand the significance of the properties directly and yields an equivalent LMI characterization as a natural result of the characterization using the inequality.

The organization of the paper is as follows. In Subsection 2.1, we review some basic definitions and results about the behavioral system theory. We introduce QDFs in Subsection 2.2 and explain dissipation theory based on QDFs in Subsection 2.3. The problem formulation is provided in Section 3. In Section 4, we derive a characterization of the finite frequency properties based on dissipation inequality as a main result. In this result, we characterize the dissipativity properties in terms of some behaviors. We restrict our attention to input-output setting in Section 5 and strengthen the characterization to the finite frequency bounded- and positive-realness with a typical mechanical example. Based on the characterization, we give a finite frequency KYP lemma for a numerical checking of the finite frequency properties in Section 6. Figure 1 explains a series of these results comparing with the previous works [3][5][6].

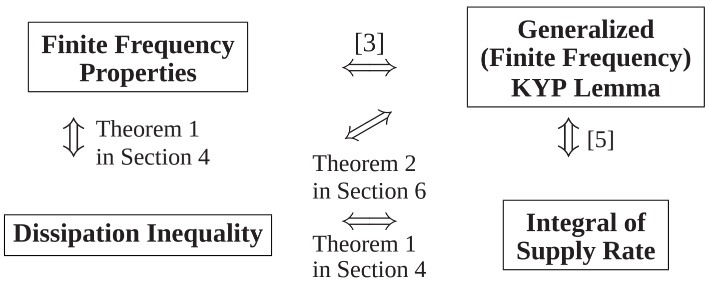

Figure 1: This figure shows the relationship between the series of our conditions and the previous works.

We use the following notations throughout this paper.

The set of $p \times q$ real and complex matrices are denoted by $\mathbb{R}^{p\times q}$ and $\mathbb{C}^{p\times q}$, respectively. We also denote $\mathbb{S}^{q\times q}$ and $\mathbb{H}^{q\times q}$ as the set of $q \times q$ real symmetric and Hermite matrices, respectively.

We denote $\mathbb{R}^{p\times q}[\xi]$ and $\mathbb{R}^{p\times q}[\zeta, \eta]$ as the set of $p\times q$ one- and two-variable polynomial matrices, respectively. The set of $p \times q$ complex coefficient one- and two-variable polynomial matrices are denoted by $\mathbb{C}^{p\times q}[\xi]$ and $\mathbb{C}^{p\times q}[\zeta, \eta]$, respectively. We denote the set of $q \times q$ Hermite two-variable polynomial matrices in the indeterminates ζ and η by $\mathbb{H}^{q\times q}[\zeta, \eta]$.

We denote $\mathbb{W}^{\mathbb{T}}$ as the set of maps from $\mathbb{T}$ to $\mathbb{W}$. Define $\mathcal{C}^{\infty}(\mathbb{R}, \mathbb{V})$ as the set of infinitely differentiable functions from $\mathbb{R}$ to the vector space $\mathbb{V}$. We also define

$$\mathcal{D}^{\infty}(\mathbb{R}, \mathbb{V}) := \{\ell \in \mathcal{C}^{\infty}(\mathbb{R}, \mathbb{V}) \mid \ell \text{ has a compact support}\}.$$

Let $\mathcal{L}_2(\mathbb{C}, \mathbb{V})$ denote the set of $\mathcal{L}_2$ functions from $\mathbb{C}$ to $\mathbb{V}$.

Finally, the row dimension of the matrix A is denoted by $\mathrm{rowdim}(A)$. We define the rank of polynomial matrix $R(\xi)$ and constant matrix $R(\lambda)$ are denoted by $\mathrm{rank}R$ and $\mathrm{rank}R(\lambda)$, respectively. We denote the matrix $\begin{bmatrix} A_1^{\top} & A_2^{\top} & \cdots & A_n^{\top} \end{bmatrix}^{\top}$ by $\mathrm{col}\,(A_1, A_2, \cdots, A_n)$. We define $\mathrm{diag}(A_1, A_2, \cdots, A_n)$ as the $q \times q$ (block) diagonal matrix with (block) diagonal elements $\{A_1, A_2, \cdots, A_n\}$. We also define $\mathrm{He}\,(A) = \frac{1}{2}\,(A + A^*)$.

2 Preliminaries

In this section, we will review the basic definitions and results from the behavioral system theory, which are taken from the references [11][15][16].

2.1 Linear Continuous-time Systems

In the behavioral system theory, a dynamical system is defined as a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, where $\mathbb{T}$ is the time axis, and $\mathbb{W}$ is the signal space in which the trajectories take their values on. The behavior $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ is the set of all possible trajectories.

In this paper, we will consider a *linear* time-invariant *continuous-time system* with $\mathbb{T} = \mathbb{R}$ and $\mathbb{W} = \mathbb{C}^q$. Such a Σ is represented by a system of linear differential-algebraic equation as

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_L \frac{d^L}{dt^L} w = 0, \tag{1}$$

where $R_i \in \mathbb{C}^{p\times q}$ $(i = 0, 1, \cdots, L)$ and $L \geq 0$. The variable $w \in \mathcal{C}^{\infty}(\mathbb{R}, \mathbb{C}^q)$ is called the *manifest variable*. We call the representation (1) a *kernel representation* of $\mathfrak{B}$. A short hand notation for (1) is

$$R\left(\frac{d}{dt}\right) w = 0, \tag{2}$$

where $R \in \mathbb{C}^{p\times q}[\xi]$ is given by

$$R(\xi) := R_0 + R_1\xi + \cdots + R_L\xi^L. \tag{3}$$

Then, the behavior is is defined as

$$\mathfrak{B} := \left\{ w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^q) \;\middle|\; R\left(\frac{d}{dt}\right) w = 0 \right\}. \tag{4}$$

The representation (2) is said to be a *minimal representation of* $\mathfrak{B}$ if $\mathrm{rowdim} R \leq \mathrm{rowdim} R'$ holds for any other $R' \in \mathbb{C}^{\bullet \times q}[\xi]$ which induces a kernel representation of $\mathfrak{B}$.

For ease of later discussion, we define the *coefficient matrix of* $R(\xi)$ in (3) as

$$\widetilde{R} := \begin{bmatrix} R_0 & R_1 & \cdots & R_L \end{bmatrix} \in \mathbb{C}^{p \times (L+1)q}.$$

The polynomial matrix $R(\xi)$ is expressed as $R(\xi) = \widetilde{R} Z_L(\xi)$ in terms of $\widetilde{R}$, where $Z_i \in \mathbb{R}^{(i+1)q \times q}[\xi]$ $(i = 0, 1, \cdots)$ is the polynomial matrix constructed by stacking the polynomial matrices $\{I_q, \xi I_q, \cdots, \xi^i I_q\}$, i.e.

$$Z_i(\xi) = \begin{bmatrix} I_q \\ \xi I_q \\ \vdots \\ \xi^i I_q \end{bmatrix}. \tag{5}$$

The behavior $\mathfrak{B}$ is called *controllable*, if for any trajectories $w_1, w_2 \in \mathfrak{B}$, there exists a time $T \geq 0$ and a trajectory $w \in \mathfrak{B}$ such that

$$w(t) = \begin{cases} w_1(t) & (t \leq 0), \\ w_2(t - T) & (t \geq T). \end{cases}$$

The behavior $\mathfrak{B}$ is controllable if and only if $\mathrm{rank} R(\lambda)$ is constant for all $\lambda \in \mathbb{C}$ [15].

Whenever $\mathfrak{B}$ is controllable, it can be described by an *image representation*

$$w = M\left(\frac{d}{dt}\right) \ell, \; M \in \mathbb{C}^{q \times m}[\xi], \tag{6}$$

where the variable $\ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^m)$ is called the *latent variable*. Then, $\mathfrak{B}$ is given as the image of the differential operator $M\left(\frac{d}{dt}\right)$ by

$$\mathfrak{B} = \{ w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^q) \mid \exists\, \ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^m) \text{ s.t. (6) } \}.$$

An image representation in (6) is a special case of the representation of $\mathfrak{B}$. The system of differential equations

$$R\left(\frac{d}{dt}\right) w = M\left(\frac{d}{dt}\right) \ell \tag{7}$$

is said to be a *latent variable representation* of $\mathfrak{B}$. In terms of the latent variable representation, $\mathfrak{B}$ can be rewritten as

$$\mathfrak{B} = \{ w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^q) \mid \exists\, \ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^m) \text{ s.t. (7) holds} \}.$$

An image representation of $\mathfrak{B}$ is called *observable* if $w = M\left(\frac{d}{dt}\right) \ell = 0$ implies $\ell = 0$. The representation (6) is observable if and only if the constant matrix $M(\lambda)$ is of full column rank for all $\lambda \in \mathbb{C}$ [15].

If (6) is an observable image representation, there exists a nonsingular permutation matrix $\Pi \in \mathbb{C}^{q\times q}$ satisfying

$$\Pi M(\xi) = \begin{bmatrix} Y(\xi) \\ U(\xi) \end{bmatrix},\ Y \in \mathbb{C}^{p\times m}[\xi],\ U \in \mathbb{C}^{m\times m}[\xi],\ p+m=q \tag{8}$$

with $U(\xi)$ nonsingular [15]. Such a partition is called an *input-output partition of* $M(\xi)$. We can regard

$$u := U\left(\frac{d}{dt}\right)\ell \text{ and } y := Y\left(\frac{d}{dt}\right)\ell$$

as input and output, respectively. In this case, corresponding to the above partition, the transfer function $G \in \mathbb{C}^{p\times m}(\xi)$ from u to y is defined by

$$G(\xi) := Y(\xi)U^{-1}(\xi). \tag{9}$$

2.2 Quadratic Differential Forms

We review the definition of a quadratic differential form (QDFs) [16] which plays a central role in this paper. We also give some basic results with respect to QDFs and dissipativity.

We first consider a two-variable polynomial matrix in $\mathbb{C}^{q_1\times q_2}[\zeta,\eta]$ described by

$$\Phi(\zeta,\eta) = \sum_{i\geq 0}\sum_{j\geq 0}\Phi_{i,j}\zeta^i\eta^j,$$

where $\Phi_{i,j} \in \mathbb{C}^{q_1\times q_2}$. The above summation ranges over the non-negative integers and is assumed to be finite. The degrees of $\Phi(\zeta,\eta)$ with respect to ζ and η are defined as

$$\deg_\zeta \Phi = \max_{(i,j)\in\mathcal{I}} i \text{ and } \deg_\eta \Phi = \max_{(i,j)\in\mathcal{I}} j,$$

where $\mathcal{I} \subset \mathbb{Z}^2$ is defined by

$$\mathcal{I} := \left\{(i,j) \in \mathbb{Z}^2 \mid \Phi_{i,j} \neq 0_{q_1\times q_2}\right\}.$$

The *bilinear differential form (BDF)* $\mathsf{L}_\Phi(\ell_1,\ell_2)$ is a bilinear form of the variables $\ell_k \in \mathcal{C}^\infty(\mathbb{R},\mathbb{C}^{q_k})$ $(k=1,2)$ and their derivatives, namely

$$\mathsf{L}_\Phi : \mathcal{C}^\infty(\mathbb{R},\mathbb{C}^{q_1}) \times \mathcal{C}^\infty(\mathbb{R},\mathbb{C}^{q_2}) \to \mathcal{C}^\infty(\mathbb{R},\mathbb{R}),$$

with form

$$\mathsf{L}_\Phi(\ell_1,\ell_2) := \sum_{i=0}^{K_1}\sum_{j=0}^{K_2}\left(\frac{d^i\ell_1}{dt^i}\right)^*\Phi_{i,j}\frac{d^j\ell_2}{dt^j},$$

where $K_1 := \deg_\zeta \Phi$ and $K_2 := \deg_\eta \Phi$. There is a one-to-one correspondence between the BDF and the two-variable polynomial matrix

$$\Phi(\zeta,\eta) = \sum_{i=0}^{K_1}\sum_{j=0}^{K_2}\Phi_{i,j}\zeta^i\eta^j. \tag{10}$$

This means that ζ and η correspond to the differentiations on ℓ_1^* and ℓ_2, respectively. For $\Phi(\zeta,\eta)$ in (10), we define the mappings

$$\partial : \mathbb{C}^{q_1\times q_2}[\zeta,\eta] \to \mathbb{C}^{q_1\times q_2}[\xi],\ \partial\Phi(\xi) := \Phi(-\xi,\xi),$$
$$\star : \mathbb{C}^{q_1\times q_2}[\zeta,\eta] \to \mathbb{C}^{q_2\times q_1}[\zeta,\eta],\ \Phi^\star(\zeta,\eta) := \Phi^*(\bar\eta,\bar\zeta).$$

With every $\Phi \in \mathbb{C}^{q_1\times q_2}[\zeta,\eta]$ in (10), we define its *coefficient matrix* $\widetilde{\Phi} \in \mathbb{C}^{(K_1+1)q_1\times(K_2+1)q_2}$

$$\sim : \mathbb{C}^{q_1\times q_2}[\zeta,\eta] \to \mathbb{C}^{(K_1+1)q_1\times(K_2+1)q_2},$$
$$\widetilde{\Phi} := \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots & \Phi_{0,K_2} \\ \Phi_{1,0} & \Phi_{1,1} & \cdots & \Phi_{1,K_2} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{K_1,0} & \Phi_{K_1,1} & \cdots & \Phi_{K_1,K_2} \end{bmatrix}.$$

Then, $\Phi(\zeta,\eta)$ is expressed $\Phi(\zeta,\eta) = Z_{K_1}^\top(\xi)\widetilde{\Phi}Z_{K_2}(\eta)$ using $\widetilde{\Phi}$ and $Z_i(\xi)$ in (5).

For $\Phi \in \mathbb{C}^{q_1\times q_2}[\zeta,\eta]$ in (10), there exist $\widetilde{M}_k \in \mathbb{C}^{\mathrm{rank}\widetilde{\Phi}\times(N_k+1)q_k}$ $(k = 1,2)$ satisfying $\widetilde{\Phi} = \widetilde{M}_1^*\Sigma_\Phi\widetilde{M}_2$, where $\Sigma_\Phi \in \mathbb{S}^{\mathrm{rank}\widetilde{\Phi}\times\mathrm{rank}\widetilde{\Phi}}$, $\widetilde{M}_1, \widetilde{M}_2$ are of full row rank, and $\det\Sigma_\Phi \neq 0$. This can be proved by using the inertia theorem. In this case, we get $\mathrm{rank}\Sigma_\Phi = \mathrm{rank}\widetilde{\Phi}$. With such a factorization of $\widetilde{\Phi}$, we obtain a *canonical factorization* of $\Phi(\zeta,\eta)$ as

$$\Phi(\zeta,\eta) = F_1^\star(\zeta)\Sigma_\Phi F_2(\eta), \tag{11}$$

where $F_k \in \mathbb{C}^{\mathrm{rank}\widetilde{\Phi}\times q_k}[\xi]$ $(k = 1,2)$ is defined by $F_k(\xi) := \widetilde{F}_k Z_{N_k}(\xi)$.

We call $\Phi(\zeta,\eta)$ *Hermitian* if $\Phi(\bar\zeta,\bar\eta)^* = \Phi(\eta,\zeta)$ holds (implying $q_1 = q_2 =: q$ and $K_1 = K_2 =: K$). Then, $\Phi(\zeta,\eta)$ is expressed as

$$\Phi(\zeta,\eta) = \sum_{i=0}^{K}\sum_{j=0}^{K}\Phi_{i,j}\zeta^i\eta^j. \tag{12}$$

In this case, $\Phi(\zeta,\eta)$ induces a *quadratic differential form (QDF)* represented by

$$\mathsf{Q}_\Phi : \mathcal{C}^\infty(\mathbb{R},\mathbb{C}^q) \to \mathcal{C}^\infty(\mathbb{R},\mathbb{R}),$$
$$\mathsf{Q}_\Phi(\ell) := \mathsf{L}_\Phi(\ell,\ell).$$

The *derivative* of the VQDF $\mathsf{Q}_\Psi(\ell)$ is defined by $\frac{d}{dt}\mathsf{Q}_\Psi(\ell)$. This is also a QDF. Let $\nabla\Psi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ induce $\frac{d}{dt}\mathsf{Q}_\Psi(\ell)$, i.e. $\mathsf{Q}_{\nabla\Psi}(\ell) = \frac{d}{dt}\mathsf{Q}_\Psi(\ell)$. Then, it is given by

$$\nabla\Psi(\zeta,\eta) = (\zeta+\eta)\Psi(\zeta,\eta).$$

The nonnegativity of a QDF is characterized by its coefficient matrix as seen in the following lemma.

Lemma 1 [16][1] *Let $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ be given. Then, we have $\mathsf{Q}_\Phi(\ell) \geq 0$ for all $\ell \in \mathcal{C}^\infty(\mathbb{R},\mathbb{C}^q)$ if and only if*

$$\widetilde{\Phi} \geq 0 \tag{13}$$

holds.

2.3 Dissipation Theory

We assume that $\mathfrak{B}$ in (4) is controllable in this section. Then, $\mathfrak{B}$ has an observable image representation (6). Let $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ be given.

We give the definition of dissipativity of a behavior.

Definition 1 [16] Assume that $\mathfrak{B}$ is controllable. Let $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ be given. Then, a behavior $\mathfrak{B}$ is called *dissipative with respect to the supply rate* $\mathsf{Q}_\Phi(w)$ if

$$\int_{-\infty}^{+\infty} \mathsf{Q}_\Phi(w)dt \geq 0, \ \ \forall\, w \in \mathfrak{B} \cap \mathcal{D}^\infty(\mathbb{R},\mathbb{C}^q)$$

holds.

We may think of $\mathsf{Q}_\Phi(w)$ as the power delivered to the behavior $\mathfrak{B}$. The dissipativity implies that the net flow of energy into the system is non-negative. This shows the system dissipates energy. Hence, due to this dissipation, the rate of increase of the energy stored inside of the system does not exceed the power supplied to it. This interaction between supply, storage, and dissipation is now formalized in Definition 2 and Proposition 1 below.

We give the definition of a storage function and dissipation rate.

Definition 2 [16] Assume that $\mathfrak{B}$ is controllable. Let $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ be given.

(i) The QDF $\mathsf{Q}_\Psi(w)$ induced by $\Psi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ is called a *storage function for* $\mathfrak{B}$ *with respect to the supply rate* $\mathsf{Q}_\Phi(w)$ if

$$\frac{d}{dt}\mathsf{Q}_\Psi(w) \leq \mathsf{Q}_\Phi(w), \ \ \forall\, w \in \mathfrak{B} \tag{14}$$

holds. We call (14) the *dissipation inequality*.

(ii) The QDF $\mathsf{Q}_\Delta(w)$ induced by $\Delta \in \mathbb{H}^{q\times q}[\zeta,\eta]$ is called a *dissipation rate for* $\mathsf{Q}_\Phi(w)$ if

$$\mathsf{Q}_\Delta(w) \geq 0, \ \forall\, w \in \mathfrak{B} \tag{15}$$

and

$$\int_{-\infty}^{+\infty} \mathsf{Q}_\Phi(w)dt = \int_{-\infty}^{+\infty} \mathsf{Q}_\Delta(w)dt, \quad \forall\, w \in \mathfrak{B} \cap \mathcal{D}^\infty(\mathbb{R},\mathbb{C}^q)$$

hold.

There is a one-to-one relation between a storage function $\mathsf{Q}_\Psi(w)$ and a dissipation rate $\mathsf{Q}_\Delta(w)$ defined by

$$\frac{d}{dt}\mathsf{Q}_\Psi(w) = \mathsf{Q}_\Phi(w) - \mathsf{Q}_\Delta(w). \tag{16}$$

The equation (16) is called the *dissipation equality*.

The next proposition gives a characterization of the dissipativity in terms of a storage function and a dissipation rate.

Proposition 1 [16] *Let* $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ *be given. The following statements (i), (ii) and (iii) are equivalent.*

(i) The behavior $\mathfrak{B}$ *is dissipative with respect to the supply rate* $\mathsf{Q}_\Phi(w)$.

(ii) *There exists* $\Psi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ *satisfying the dissipation inequality* (14)*.*

(iii) *There exist* $\Psi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ *and* $\Delta \in \mathbb{H}^{q\times q}[\zeta,\eta]$ *satisfying* (15) *and the dissipation equality* (16)*.*

Consider the frequency domain inequality (FDI) expressed as

$$M(j\omega)^*\partial\Phi(j\omega)M(j\omega) \geq 0, \quad \forall\, \omega \in \mathbb{R}. \tag{17}$$

The FDI (17) is a necessary and sufficient condition for the dissipativity of $\mathfrak{B}$ from Proposition 5.2 in [16].

Proposition 2 [16] *Assume that* $\mathfrak{B}$ *is controllable. Let* (6) *be an observable image representation of* $\mathfrak{B}$ *and* $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ *induce the supply rate for* $\mathfrak{B}$*. Then, FDI* (17) *holds if and only if the behavior* $\mathfrak{B}$ *is dissipative with respect to the supply rate* $\mathsf{Q}_\Phi(w)$*.*

The above proposition shows that (17) is an inequality which interprets the dissipativity in the frequency domain.

3 Problem Formulation

In this paper, we consider a characterization of finite frequency properties in the framework of dissipation theory. We give the problem formulation in this section.

We consider a controllable linear time-invariant system $\Sigma = (\mathbb{R}, \mathbb{C}^q, \mathfrak{B})$ in this paper. Assume that $\mathfrak{B}$ is controllable. The behavior $\mathfrak{B}$ is typically represented by the kernel representation (2), where $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^q)$ is the manifest variable and $R \in \mathbb{C}^{p\times q}[\xi]$. Then, the behavior is given by (4). Assume that (2) is minimal throughout paper and suppose that an observable image representation of $\mathfrak{B}$ is described by (6) for $M \in \mathbb{C}^{q\times m}[\xi]$.

Let $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ in (12) be given. Suppose that this $\Phi(\zeta,\eta)$ induces the supply rate for $\mathfrak{B}$. Define the frequency domain Ω in the finite interval by

$$\Omega := \{\omega \in \mathbb{R} \mid \tau(\omega - \varpi_1)(\omega - \varpi_2) \leq 0\}, \tag{18}$$

where $\varpi_1, \varpi_2 \in \mathbb{R}$, $\varpi_1 \leq \varpi_2$ and $\tau \in \mathbb{Z}$ is either $+1$ or -1. Our goal is to find a characterization of the finite frequency property described by the following *finite frequency domain inequality (FFDI)* using QDFs:

$$M^*(j\omega)\partial\Phi(j\omega)M(j\omega) \geq 0, \quad \forall\, \omega \in \Omega. \tag{19}$$

The set Ω for $\tau = +1$ represents the middle frequency interval $[\varpi_1, \varpi_2]$, while Ω expresses the high frequency domain $(-\infty, \varpi_1]$ and $[\varpi_2, +\infty)$ in the case of $\tau = -1$. Moreover, Ω becomes the entire real numbers, i.e. $\Omega = \mathbb{R}$, if we choose $\varpi_1 = \varpi_2 := 0$ with $\tau = -1$.

An interpretation of the FFDI (19) from the behavioral approach is the following. Consider the QDF $\mathsf{Q}_\Phi(w)$ induced by $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ in (12). The Fourier transform of $\mathsf{Q}_\Phi(w)$ is computed as

$$\hat{w}(j\omega)^*\partial\Phi(j\omega)\hat{w}(j\omega) = \hat{\ell}(j\omega)^*M(j\omega)^*\partial\Phi(j\omega)M(j\omega)\hat{\ell}(j\omega),$$

where $\hat{w} \in \mathcal{L}_2(\mathbb{C}, \mathbb{C}^q)$ and $\hat{\ell} \in \mathcal{L}_2(\mathbb{C}, \mathbb{C}^m)$ are Fourier transform of $w \in \mathfrak{B} \cap \mathcal{D}^\infty(\mathbb{R}, \mathbb{C}^m)$ and $\ell \in \mathcal{D}^\infty(\mathbb{R}, \mathbb{C}^m)$, respectively. Since $\ell(t)$ can be taken an arbitrarily trajectory in $\mathcal{D}^\infty(\mathbb{R}, \mathbb{C}^m)$, the inequality

$$\hat{w}(j\omega)^*\partial\Phi(j\omega)\hat{w}(j\omega) \geq 0, \quad \forall\, w \in \mathfrak{B} \cap \mathcal{D}^\infty(\mathbb{R}, \mathbb{C}^m),\ \omega \in \Omega$$

is equivalent to FFDI (19). We can regard the above inequality imposes a weighted frequency constraint on $w \in \mathfrak{B}$ over the restricted frequency domain Ω. Hence, it expresses the weighted rate limitation on the trajectories contained in $\mathfrak{B}$, although the FFDI (19) is described by using $M(\xi)$.

Remark 1 In the state-space setting [3][5][6], Iwasaki et.al considered the FFDI [1]

$$\begin{bmatrix} (j\omega I_n - A)^{-1} B \\ I_m \end{bmatrix}^* \Phi_0 \begin{bmatrix} (j\omega I_n - A)^{-1} B \\ I_m \end{bmatrix} \leq 0, \quad \forall\, \omega \in \Omega, \tag{20}$$

where $\Phi_0 \in \mathbb{H}^{(n+m)\times(n+m)}$, $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$ and (A, B) is a controllable pair. We can regard the FFDI (19) as a generalization of the FFDI (20) to the behavioral approach. It is explained as follows. Let $Y \in \mathbb{C}^{p\times m}[\xi]$ and $U \in \mathbb{C}^{m\times m}[\xi]$ be defined by a right coprime factorization $(\xi I_n - A)^{-1} B = Y(\xi)U^{-1}(\xi)$. If we define $M(\xi) := \mathrm{col}\,(Y(\xi), U(\xi))$ and $\Phi(\zeta, \eta) := -\Phi_0 \in \mathbb{H}^{q\times q}$, then (19) is rewritten by the FFDI (20). In addition, define Φ_0 as

$$\Phi_0 := \frac{1}{2}\Pi^* \begin{bmatrix} 0_{m\times m} & I_m \\ I_m & 0_{m\times m} \end{bmatrix} \Pi,$$

then the FFDI (20) falls to the finite frequency positive realness [6]. Thus, the FFDI (19) can be considered as a generalization of the FFDI (20) to the behavioral approach.

4 Characterization of Finite Frequency Properties

This section derives a characterization of finite frequency properties in terms of a dissipation inequality and an integral of the supply rate using QDFs as a main result.

4.1 Main Theorem

We define $\varpi_- \in \mathbb{R}$ and $\varpi_+ \in \mathbb{R}$ by

$$\varpi_- := \frac{\varpi_2 - \varpi_1}{2} \ \text{ and } \ \varpi_+ := \frac{\varpi_1 + \varpi_2}{2} \tag{21}$$

and the set $\mathcal{G}$ by

$$\mathcal{G} := \left\{ \Gamma \in \mathbb{H}^{q\times q}[\zeta,\eta] \;\middle|\; \begin{array}{l} \Gamma(\zeta,\eta) := \begin{bmatrix} 1 \\ \zeta \end{bmatrix}^* \begin{bmatrix} -\varpi_1\varpi_2 & -j\varpi_+ \\ j\varpi_+ & -1 \end{bmatrix} \begin{bmatrix} 1 \\ \eta \end{bmatrix} \Upsilon(\zeta,\eta) \\ \text{for some } \Upsilon \in \mathbb{H}^{q\times q}[\zeta,\eta] \text{ such that} \\ \tau \mathsf{Q}_\Upsilon(w) \geq 0,\ \forall\, w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^q) \end{array} \right\}. \tag{22}$$

We see that

$$\begin{aligned} \partial\Gamma(j\omega) &= -\tau\,(\omega - \varpi_1)\,(\omega - \varpi_2) \cdot \tau\partial\Upsilon(j\omega) \\ &\geq 0 \end{aligned} \tag{23}$$

holds for all $\omega \in \Omega$.

[1] Iwasaki and Hara [3] considered a unified FFDI which can describe the FFDIs in both continuous- and discrete-time systems. The FFDI (20) is the continuous-time version of the FFDI in [3]

We have seen from Proposition 1 that the FDI (17) is equivalent to the dissipation inequality (14). As we consider the FDI (17) restricted to Ω, we can imagine that there holds an analogous relationship to Proposition 1. This is explained as follows.

Assume that there exist two-variable polynomial matrices $\Psi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ and $\Gamma \in \mathcal{G}$ satisfying

$$\frac{d}{dt}\mathsf{Q}_\Psi(w) \le \mathsf{Q}_\Phi(w) - \mathsf{Q}_\Gamma(w), \quad \forall\, w \in \mathfrak{B}. \tag{24}$$

The above inequality corresponds to the dissipation inequality (14). Inequality (24) is equivalent to the existence of $\Delta \in \mathbb{H}^{q\times q}[\zeta,\eta]$ satisfying a two-variable polynomial matrix equation

$$\begin{aligned}&(\zeta+\eta)M^\star(\zeta)\Psi(\zeta,\eta)M(\eta)\\ &\quad = M^\star(\zeta)\Phi(\zeta,\eta)M(\eta) - M^\star(\zeta)\Gamma(\zeta,\eta)M(\eta) - M^\star(\zeta)\Delta(\zeta,\eta)M(\eta)\end{aligned} \tag{25}$$

and $\mathsf{Q}_\Delta(w) \ge 0$, $\forall\, w \in \mathfrak{B}$. Substituting $\zeta = -j\omega$ and $\eta = j\omega$ into (25), we obtain the FFDI

$$\begin{aligned}M(j\omega)^*\partial\Phi(j\omega)M(j\omega) &= M(j\omega)^*\partial\Gamma(j\omega)M(j\omega) + M(j\omega)^*\partial\Delta(j\omega)M(j\omega)\\ &\ge 0, \quad \forall\, \omega \in \Omega\end{aligned}$$

from (23). The above inequality guarantees the FFDI (19).

Inequality (24) also gives a necessary condition for the finite frequency property. Thus, we obtain the following main result which provides a necessary and sufficient condition for the property.

Theorem 1 *Assume that $\mathfrak{B}$ in* (4) *is controllable and that $\mathfrak{B}$ is represented by an observable image representation* (6). *Let $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ be given. Define Ω by* (18) *and $\mathcal{G}$ by* (22). *Then, the following statements (i), (ii) and (iii) are equivalent.*

(i) *FFDI* (19) *holds for all $\omega \in \Omega$.*

(ii) *There exist $\Psi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ and $\Gamma \in \mathcal{G}$ satisfying inequality* (24).

(iii) *Inequality*

$$\int_{-\infty}^{+\infty} \mathsf{Q}_\Phi(w)dt \ge 0 \tag{26}$$

holds for all $w \in \mathfrak{B}$ satisfying

$$\tau \mathrm{He}\left((\dot z - j\varpi_1 z)(\dot z - j\varpi_2 z)^*\right) \le 0, \tag{27}$$

where $z \in \mathcal{D}^\infty(\mathbb{R}, \mathbb{C}^{(N+1)q})$ is defined by

$$z := Z_N\left(\frac{d}{dt}\right)w \tag{28}$$

with some nonnegative integer $N \in \mathbb{Z}$.

Proof See Appendix B.1 for the proof. □

We call the QDF $\mathsf{Q}_\Gamma(w)$ satisfying (24) a *compensation rate for $\mathfrak{B}$ with respect to the frequency domain Ω*. Namely, $\mathsf{Q}_\Gamma(w)$ guarantees the dissipativity of some behavior related to $\mathfrak{B}$ and Ω. The detail of this claim is explained in Subsection 4.2.

Remark 2 It should be noted that the characterization in Theorem 1 is representation-free. Namely, it does not suppose any particular representation of $\mathfrak{B}$. In this sense, this theorem gives a more general result than the previous works done by Iwasaki et.al [5].

Remark 3 The equivalence of (i) and (iii) corresponds to the result if we restrict Theorem 3 in [5] to continuous-time systems. The statement (iii) shows that the integral of the power supplied to the system is nonnegative for the manifest variable which varies in the frequency contained in Ω.

Remark 4 The two-variable polynomial

$$\begin{bmatrix}1\\ \zeta\end{bmatrix}^* \begin{bmatrix}-\varpi_1\varpi_2 & -j\varpi_+\\ j\varpi_+ & -1\end{bmatrix} \begin{bmatrix}1\\ \eta\end{bmatrix}$$

is a real coefficient polynomial if Ω is symmetric about the origin, i.e. low frequency domain

$$\Omega_{\rm low} := \{\, \omega \in \mathbb{R} \mid\ |\omega| \leq \varpi \} \tag{29}$$

and high frequency domain

$$\Omega_{\rm high} := \{\, \omega \in \mathbb{R} \mid\ |\omega| \geq \varpi \} \tag{30}$$

for example, where $\varpi \in \mathbb{R}$ is a given scalar satisfying $\varpi \geq 0$. If $M(\xi)$ and $\Phi(\zeta,\eta)$ are all real polynomial matrices, we can restrict $\Psi(\zeta,\eta)$ and $\Gamma(\zeta,\eta)$ in Theorem 1 to real symmetric two-variable polynomial matrices without loss of generality.

4.2 Physical Interpretation

In this subsection, we clarify the physical interpretation of Theorem 1 from the viewpoint of dissipation theory.

Define the subbehavior $\mathfrak{B}_\Omega \subset \mathfrak{B}$ by

$$\mathfrak{B}_\Omega := \left\{ w \in \mathfrak{B} \;\middle|\; w \text{ satisfies (27) for } z \in \mathcal{D}^\infty(\mathbb{R}, \mathbb{C}^{(N+1)q}) \text{ in (28)} \right\}. \tag{31}$$

Since $\mathsf{Q}_\Gamma(w)$ can be rewritten by

$$\mathsf{Q}_\Gamma(w) = -\dot{z}^*\widetilde{\Upsilon}\dot{z} + j\varpi_+\left(\dot{z}^*\widetilde{\Upsilon}z - z^*\widetilde{\Upsilon}\dot{z}\right) - \varpi_1\varpi_2 z^*\widetilde{\Upsilon}z,$$

we have

$$\begin{aligned}\mathsf{Q}_\Gamma(w) &= \mathrm{tr}\left[\widetilde{\Upsilon}\left\{-\dot{z}\dot{z}^* + j\varpi_+\left(\dot{z}z^* - z\dot{z}^*\right) - \varpi_1\varpi_2 zz^*\right\}\right]\\ &= \mathrm{tr}\left[\tau\widetilde{\Upsilon}\cdot\left\{-\tau\mathrm{He}\left(\left(\dot{z} - j\varpi_1 z\right)\left(\dot{z} - j\varpi_2 z\right)^*\right)\right\}\right].\end{aligned} \tag{32}$$

It follows from $\tau\widetilde{\Upsilon} \geq 0$ that $\mathsf{Q}_\Gamma(w) \geq 0, \forall\, w \in \mathfrak{B}_\Omega$ holds if and only if z satisfies (27). Hence, we can regard $\mathfrak{B}_\Omega$ as the set of all trajectories in $\mathfrak{B}$ which vary in the frequency contained in Ω. Namely, $\mathfrak{B}_\Omega$ has a rate constraint determined by $\Phi(\zeta,\eta)$ and Ω.

We can obtain the following corollary, which shows the physical interpretation of Theorem 1 in terms of the dissipation inequality.

Corollary 1 *Assume that $\mathfrak{B}$ in* (4) *and that $\mathfrak{B}$ is represented by an observable image representation* (6)*. Let $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ be given. Define Ω by* (18)*, $\mathcal{G}$ by* (22) *and $\mathfrak{B}_\Omega$ by* (31)*. Then, the following statements (i), (ii) and (iii) are equivalent.*

(i) *FFDI* (19) *holds for all* $\omega \in \Omega$.

(ii) *There exists* $\Psi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ *satisfying the dissipation inequality*

$$\frac{d}{dt}\mathsf{Q}_\Psi(w) \le \mathsf{Q}_\Phi(w),\ \forall\, w \in \mathfrak{B}_\Omega.$$

(iii) *There exist* $\Psi \in \mathbb{H}^{q\times q}[\zeta,\eta]$, $\Delta \in \mathbb{H}^{q\times q}[\zeta,\eta]$ *and* $\Gamma \in \mathcal{G}$ *satisfying the dissipation equality*

$$\frac{d}{dt}\mathsf{Q}_\Psi(w) = \mathsf{Q}_\Phi(w) - \mathsf{Q}_{\Delta+\Gamma}(w),\ \forall\, w \in \mathfrak{B}_\Omega \tag{33}$$

and

$$\mathsf{Q}_{\Delta+\Gamma}(w) \ge 0,\ \forall\, w \in \mathfrak{B}_\Omega. \tag{34}$$

This implies that the QDF $\mathsf{Q}_{\Delta+\Gamma}(w)$ *is a dissipation rate for* $\mathfrak{B}_\Omega$.

(iv) *The behavior* $\mathfrak{B}_\Omega$ *is dissipative with respect to the supply rate* $\mathsf{Q}_\Phi(w)$.

Proof See Appendix B.2 for the proof. □

Corollary 1 provides us a physical interpretation of the compensating rate as explained below.

It is not difficult to see that $\mathfrak{B}$ is not necessarily dissipative with respect to the supply rate $\mathsf{Q}_\Phi(w)$ from Proposition 1. However, Corollary 1 (iv) states that, if we concentrate ourselves to the trajectories to those varying in the frequency contained in Ω, then $\mathfrak{B}_\Omega$ becomes dissipative. Namely, the compensating rate guarantees the dissipativity of the subbehavior which has a constraint on the rate of change. We describe this interpretation after an intuitive example. Such an observation has not been considered in the previous works by Iwasaki et.al [5].

Consider the latent variable $\ell_\omega \in \mathcal{C}^\infty(\mathbb{R},\mathbb{C}^m)$ by $\ell_\omega(t) := e^{j\omega t}v$, $v \in \mathbb{C}^m$ for a given $\omega \in \Omega$. We easily get

$$w(t) = M\left(\frac{d}{dt}\right)e^{j\omega t}v = e^{j\omega t}M(j\omega)v, \tag{35}$$

which implies $w \in \mathfrak{B}_\Omega$. Since $\tau\mathsf{Q}_\Upsilon(w) \ge 0$ holds for all $w \in \mathfrak{B}$, we have

$$\mathsf{Q}_\Gamma(w) = -\tau(\omega-\varpi_1)(\omega-\varpi_2)\cdot\tau\mathsf{Q}_\Upsilon(w) \ge 0,\ \forall\, w \in \mathfrak{B}_\Omega \text{ s.t. (35)}.$$

From (24) and the above inequality, the inequality

$$\mathsf{Q}_\Phi(w) \ge \mathsf{Q}_\Gamma(w) + \frac{d}{dt}\mathsf{Q}_\Psi(w) \ge \frac{d}{dt}\mathsf{Q}_\Psi(w) \tag{36}$$

holds for all $w \in \mathfrak{B}_\Omega$ such that (35). On the other hand, we have $\mathsf{Q}_\Gamma(w) \le 0$ for all $w \in \mathfrak{B}_\Omega$ such that (35) if $\omega \notin \Omega$. Hence, (36) does not always hold, which concludes the intuitive explanation.

We generalize the above intuitive explanation to a physical interpretation in the dissipativity theory. We can see from Corollary 1 that QDF $\mathsf{Q}_\Gamma(w)$ satisfies the equality

$$\int_{-\infty}^{+\infty}\mathsf{Q}_{\Delta+\Gamma}(w)dt = \int_{-\infty}^{+\infty}\mathsf{Q}_\Phi(w)dt, \quad \forall\, w \in \mathfrak{B}_\Omega \cap \mathcal{D}^\infty(\mathbb{R},\mathbb{C}^q).$$

Also, we observe that

$$
\begin{aligned}
\mathsf{Q}_{\Delta+\Gamma}(w) &= \mathsf{Q}_{\Delta}(w) + \mathsf{Q}_{\Gamma}(w) \\
&= \operatorname{tr}\left[\widetilde{\Delta}\cdot zz^*\right] + \operatorname{tr}\left[\tau\widetilde{\Upsilon}\cdot\left\{-\tau\mathrm{He}\left((\dot{z}-j\varpi_1 z)\,(\dot{z}-j\varpi_2 z)^*\right]\right\}\right) \\
&= \operatorname{tr}\left[\widetilde{\Delta}\cdot zz^* + \tau\widetilde{\Upsilon}\cdot\left\{-\tau\mathrm{He}\left((\dot{z}-j\varpi_1 z)\,(\dot{z}-j\varpi_2 z)^*\right)\right\}\right] \\
&= \operatorname{tr}\left[\begin{bmatrix}\widetilde{\Delta} & 0\\ 0 & \tau\widetilde{\Upsilon}\end{bmatrix}\begin{bmatrix}zz^* & 0\\ 0 & -\tau\mathrm{He}\left((\dot{z}-j\varpi_1 z)\,(\dot{z}-j\varpi_2 z)^*\right)\end{bmatrix}\right]
\end{aligned}
$$

holds. This implies that $\mathsf{Q}_{\Delta+\Gamma}(w)$ satisfies

$$\mathsf{Q}_{\Delta+\Gamma}(w) \geq 0, \quad \forall\, w \in \mathfrak{B}_\Omega$$

and

$$\mathsf{Q}_{\Delta+\Gamma}(w) \ngeq 0, \quad \forall\, w \in \mathfrak{B}\setminus\mathfrak{B}_\Omega.$$

If we regard QDFs $\mathsf{Q}_\Phi(w)$ and $\mathsf{Q}_\Psi(w)$ as the supply rate and the storage function in (33) along the line of Definition 2, the above observation shows that QDF $\mathsf{Q}_{\Delta+\Gamma}(w)$ becomes the dissipation rate of $\mathfrak{B}_\Omega$ for supply rate $\mathsf{Q}_\Phi(w)$ from Definition 2 (ii). Therefore, $\mathsf{Q}_\Gamma(w)$ can be regarded as a compensating power which guarantees the dissipativity of $\mathfrak{B}_\Omega$.

Corollary 1 (iii) also gives a time domain characterization of sum-of-squares (SoS) decomposition by similar discussion made by Hara and Iwasaki [2]. This is explained as follows.

Since (6) is an observable image representation of $\mathfrak{B}$, (33) can be equivalently rewritten by a two-variable polynomial matrix equation

$$
\begin{aligned}
\Phi'(\zeta,\eta) =& M^\star(\zeta)\Delta(\zeta,\eta)M(\eta) + M^\star(\zeta)\Gamma(\zeta,\eta)M(\eta) \\
&+ (\zeta+\eta)M^\star(\zeta)\Psi(\zeta,\eta)M(\eta),
\end{aligned}
$$

where $\Phi' \in \mathbb{H}^{m\times m}[\zeta,\eta]$ is defined by

$$\Phi'(\zeta,\eta) := M^\star(\zeta)\Phi(\zeta,\eta)M(\eta). \tag{37}$$

From (22), we obtain

$$
\begin{aligned}
\Phi'(\zeta,\eta) =& M^\star(\zeta)\Delta(\zeta,\eta)M(\eta) \\
&+ \tau\left\{-\zeta\eta + j\varpi_+(\zeta-\eta) - \varpi_1\varpi_2\right\}\cdot\tau M^\star(\zeta)\Upsilon(\zeta,\eta)M(\eta) \\
&+ (\zeta+\eta)M^\star(\zeta)\Psi(\zeta,\eta)M(\eta).
\end{aligned}
$$

By substituting $\zeta = -j\omega$, $\eta = j\omega$, $\varpi_1 = \varpi_2 =: \varpi$ and $\tau = -1$ into the above equation, it follows from (34) that

$$
\begin{aligned}
\partial\Phi'(j\omega) &= M(j\omega)^*\partial\Delta(j\omega)M(j\omega) + (\omega-\varpi)^2\left\{-M(j\omega)^*\partial\Upsilon(j\omega)M(j\omega)\right\} \\
&\geq 0, \quad \forall\,\omega\in\Omega
\end{aligned} \tag{38}
$$

holds. Since $M(j\omega)^*\partial\Delta(j\omega)M(j\omega) \geq 0$ and $-M(j\omega)^*\partial\Upsilon(j\omega)M(j\omega) \geq 0$ hold for all $\omega\in\mathbb{R}$, we see that (38) gives an SoS decomposition of the polynomial matrix $\partial\Phi'(j\omega)$ in the indeterminate ω. Hence, statement (iii) gives a time domain interpretation of the SoS factorization derived in [2] for continuous-time systems.

4.3 Numerical Example

In this subsection, we demonstrate a simple numerical example to show how FFDI (19) is characterized in terms of QDFs based on Theorem 1 and Corollary 1.

Consider the behavior $\mathfrak{B}$ given by a kernel representation

$$\begin{bmatrix} \frac{d^2}{dt^2}+2 \\ 0 \end{bmatrix} w_1 + \begin{bmatrix} \frac{d}{dt} \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} -1 \\ \frac{d^2}{dt^2} \end{bmatrix} w_3 = 0,$$

where $w := \mathrm{col}\,(w_1, w_2, w_3)$ is the manifest variable. This representation is is induced by a polynomial matrix

$$R(\xi) = \begin{bmatrix} \xi^2+2 & \xi & -1 \\ 0 & 1 & \xi^2 \end{bmatrix}. \tag{39}$$

We see that $\mathfrak{B}$ has an observable image representation

$$w = M\left(\frac{d}{dt}\right)\ell,\ M(\xi) = \begin{bmatrix} \xi^3+1 \\ -\xi^2(\xi^2+2) \\ \xi^2+2 \end{bmatrix},\ \ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}).$$

We introduce a two-variable polynomial matrix $\Phi \in \mathbb{H}^{3\times 3}[\zeta, \eta]$ defined by

$$\Phi(\zeta, \eta) := \begin{bmatrix} 1-\zeta\eta & \zeta\eta+\eta^2 & 0 \\ \zeta\eta+\zeta^2 & 2\zeta+2\eta+1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which induces the the supply rate for $\mathfrak{B}$ given by

$$\mathsf{Q}_\Phi(w) = w_1^2 - \dot{w}_1^2 + 2\dot{w}_1\dot{w}_2 + 2w_1\dot{w}_2 + 2w_2\dot{w}_2 + 2\dot{w}_2^2 + w_2^2 + w_3^2.$$

We analyze a finite frequency property based on the above $M(\xi)$ and $\Phi(\zeta, \eta)$, where we set the (low) frequency domain $\Omega := [-1, 1]$. We have the following FFDI

$$\begin{aligned} M(j\omega)^*\partial\Phi(j\omega)M(j\omega) &= -3\omega^6 + 5\omega^4 - 5\omega^2 + 5 \\ &\geq 0, \quad \forall\, \omega \in \Omega, \end{aligned}$$

since $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$ and

$$\partial\Phi(j\omega) = \begin{bmatrix} 1-\omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \geq 0, \quad \forall\, \omega \in \Omega.$$

Define two-variable polynomial matrices $\Psi, \Delta, \Gamma \in \mathbb{H}^{3\times 3}[\zeta, \eta]$ by

$$\Psi(\zeta, \eta) := \begin{bmatrix} 0 & \eta & 0 \\ \zeta & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \Delta(\zeta, \eta) := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta\eta \end{bmatrix},$$

$$\Gamma(\zeta, \eta) := (1-\zeta\eta)\Upsilon(\zeta, \eta),\ \Upsilon(\zeta, \eta) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, QDFs $\mathsf{Q}_\Psi(w)$ and $\mathsf{Q}_\Gamma(w)$ are computed as

$$\begin{aligned} \mathsf{Q}_\Psi(w) &= 2w_1\dot{w}_2 + 2w_2^2, \\ \mathsf{Q}_\Gamma(w) &= \dot{w}_1^2 - w_1^2 + \dot{w}_3^2 - w_3^2, \end{aligned}$$

respectively, and hence we have

$$\mathsf{Q}_\Phi(w) - \frac{d}{dt}\mathsf{Q}_\Psi(w) = \mathsf{Q}_\Delta(w) + \mathsf{Q}_\Gamma(w) = w_1^2 - \dot{w}_1^2 + w_2^2 + w_3^2.$$

We easily see that

$$\frac{d}{dt}\mathsf{Q}_\Psi(w) \not\leq \mathsf{Q}_\Phi(w), \quad \forall\, w \in \mathfrak{B}$$

holds. In addition, if we add $\mathsf{Q}_\Gamma(w)$ to the left-hand side of the above inequality, we get

$$\mathsf{Q}_\Gamma(w) + \frac{d}{dt}\mathsf{Q}_\Psi(w) \leq \mathsf{Q}_\Phi(w) \quad \forall\, w \in \mathfrak{B},$$

because $\mathsf{Q}_\Delta(w) = w_2^2 + \dot{w}_3^2 \geq 0$ holds for all $w \in \mathfrak{B}$. Hence, we can see from Theorem 1 (ii) that FFDI (19) holds.

Moreover, focusing on inequality

$$\mathsf{Q}_\Delta(w) + \mathsf{Q}_\Gamma(w) = w_1^2 - \dot{w}_1^2 + w_2^2 + w_3^2 \geq 0, \quad \forall\, w \in \mathfrak{B}_\Omega.$$

yields the dissipation inequality

$$\frac{d}{dt}\mathsf{Q}_\Psi(w) \leq \mathsf{Q}_\Phi(w), \quad \forall\, w \in \mathfrak{B}_\Omega.$$

This shows that $\mathfrak{B}_\Omega$ is dissipative with respect to the supply rate $\mathsf{Q}_\Phi(w)$ from Corollary 1 (ii). This is guaranteed by the existence of the compensation rate $\mathsf{Q}_\Gamma(w)$.

5 Finite Frequency Bounded- and Positive-Realness

In Section 5, we consider the case where the results of the previous section is applied to the finite frequency bounded- and positive- realness [6] under the input-output setting.

We also consider a controllable linear time-invariant system $\Sigma = (\mathbb{R}, \mathbb{C}^q, \mathfrak{B})$ as in Section 4. Assume that $\mathfrak{B}$ is controllable. Let (6) be an observable image representation of $\mathfrak{B}$. Let (8) be an input-output partition of $M(\xi)$, where $Y \in \mathbb{R}^{p\times m}[\xi]$ and $U \in \mathbb{R}^{m\times m}[\xi]$ is nonsingular. Then, w is partitioned as $w = \mathrm{col}\,(y, u)$, where $u := U\left(\frac{d}{dt}\right)\ell$ and $y := Y\left(\frac{d}{dt}\right)\ell$ are an input and output, respectively. Such a partition always exists by the observability assumption of (6). Then, the transfer function $G \in \mathbb{C}^{p\times m}(\xi)$ from u to y is given by (9). Define the low frequency domain by

$$\Omega'_{\mathrm{low}} := \{\omega \in \mathbb{R} \mid |\omega| \leq \varpi \text{ and } \det\left(U(j\omega)\right) \neq 0\} \tag{40}$$

for a given $\varpi \in \mathbb{R}$, $\varpi \geq 0$.

5.1 Finite Frequency Bounded Realness

We characterize the finite frequency bounded realness of $G(\xi)$ in this subsection.

We give the definition of the finite frequency bounded realness.

Definition 3 Assume that $\mathfrak{B}$ in (4) and that $\mathfrak{B}$ is represented by an observable image representation (6). Let (8) be an input-output partition of $M(\xi)$, where $Y \in \mathbb{R}^{p\times m}[\xi]$ and $U \in \mathbb{R}^{m\times m}[\xi]$ is nonsingular. Let $G \in \mathbb{C}^{p\times m}(\xi)$ in (9) be given. Define Ω'_{low} by (40). Then, $G(\xi)$ is called *finite frequency bounded real (FFBR) with bandwidth* ϖ if

$$G(j\omega)^* G(j\omega) \le \gamma^2 I \tag{41}$$

holds for all $\omega \in \Omega'_{\mathrm{low}}$, where $\gamma \in \mathbb{R}$ is a given positive number.

Define $\Phi \in \mathbb{H}^{q\times q}$ by

$$\Phi := \Pi^* \begin{bmatrix} -I_p & 0_{p\times m} \\ 0_{m\times p} & \gamma^2 I_m \end{bmatrix} \Pi, \tag{42}$$

where $\Pi \in \mathbb{C}^{q\times q}$ is a nonsingular permutation matrix in (8). Then, we immediately obtain the following corollary from Theorem 1.

Corollary 2 *Assume that* $\mathfrak{B}$ *in* (4) *and that* $\mathfrak{B}$ *is represented by an observable image representation* (6)*. Let* (8) *be an input-output partition of* $M(\xi)$*, where* $Y \in \mathbb{R}^{p\times m}[\xi]$ *and* $U \in \mathbb{R}^{m\times m}[\xi]$ *is nonsingular. Define* $G \in \mathbb{C}^{p\times m}(\xi)$ *by* (9)*. Let* $\Phi \in \mathbb{H}^{q\times q}$ *be given by* (42)*. Define* Ω'_{low} *by* (40)*. Then, the following statements (i), (ii), (iii) and (iv) are equivalent.*

(i) The transfer function $G(\xi)$ *is FFBR with bandwidth* ϖ*.*

(ii) FFDI (20) *holds for all* $\omega \in \Omega'_{\mathrm{low}}$*.*

(iii) There exist $\Psi \in \mathbb{H}^{q\times q}[\zeta, \eta]$ *and* $\Gamma \in \mathcal{G}$ *satisfying*

$$\mathsf{Q}_\Gamma(w) + \frac{d}{dt}\mathsf{Q}_\Psi(w) \le \gamma^2 \|u\|^2 - \|y\|^2, \quad \forall\, w \in \mathfrak{B}.$$

(iv) Inequality

$$\gamma^2 \int_{-\infty}^{+\infty} \|u\|^2\, dt \ge \int_{-\infty}^{+\infty} \|y\|^2\, dt$$

holds for all $u \in \mathcal{D}^\infty(\mathbb{R}, \mathbb{C}^m)$ *and* $y \in \mathcal{D}^\infty(\mathbb{R}, \mathbb{C}^p)$ *satisfying*

$$\dot{z}\dot{z}^* \le \varpi^2 z z^*, \tag{43}$$

where $z \in \mathcal{D}^\infty(\mathbb{R}, \mathbb{C}^{(N+1)q})$ *is defined by* (28) *with* $w = \mathrm{col}\,(y, u)$ *and some nonnegative integer* $N \in \mathbb{Z}$*.*

Proof See Appendix B.3 for the proof. □

5.2 Finite Frequency Positive Realness

In the following, we characterize the finite frequency positive realness of $G(\xi)$ in the case where $G(\xi)$ is square, i.e. $p = m$. This property is one of the key properties for the integrated design [6].

Definition 4 Assume that $\mathfrak{B}$ in (4) and that $\mathfrak{B}$ is represented by an observable image representation (6). Let (8) be an input-output partition of $M(\xi)$, where $Y \in \mathbb{R}^{m\times m}[\xi]$ and $U \in \mathbb{R}^{m\times m}[\xi]$ is nonsingular. Let $G \in \mathbb{C}^{m\times m}(\xi)$ in (9) be given. Define $\Omega'_{\rm low}$ by (40). Then, $G(\xi)$ is called *finite frequency positive real (FFPR) with bandwidth* ϖ if

$$G(j\omega) + G(j\omega)^* \geq 0, \quad \forall\, \omega \in \Omega'_{\rm low} \tag{44}$$

holds.

Suppose that $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ is described by

$$\Phi := \frac{1}{2}\Pi \begin{bmatrix} 0_{m\times m} & I_m \\ I_m & 0_{m\times m} \end{bmatrix} \Pi, \tag{45}$$

where $\Pi \in \mathbb{C}^{q\times q}$ is a nonsingular permutation matrix in (8). Then, we obtain the following corollary which characterizes the FFPR property.

Corollary 3 *Assume that* $\mathfrak{B}$ *in* (4) *and that* $\mathfrak{B}$ *is represented by an observable image representation* (6)*. Let* $\Phi \in \mathbb{H}^{2m\times 2m}$ *be given by* (45)*. Let* (8) *be an input-output partition of* $M(\xi)$*, where* $Y \in \mathbb{R}^{p\times m}[\xi]$ *and* $U \in \mathbb{R}^{m\times m}[\xi]$ *is nonsingular. Define* $G \in \mathbb{C}^{m\times m}(\xi)$ *by* (9)*. Let* $\Phi \in \mathbb{H}^{q\times q}$ *be given by* (45)*. Define* $\Omega'_{\rm low}$ *by* (40)*. Then, the following statements (i), (ii), (iii) and (iv) are equivalent.*

(i) The transfer function $G(\xi)$ *is FFPR with bandwidth* ϖ*.*

(ii) FFDI (20) *holds for all* $\omega \in \Omega'_{\rm low}$*.*

(iii) There exist $\Psi \in \mathbb{H}^{2m\times 2m}[\zeta,\eta]$ *and* $\Gamma \in \mathcal{G}$ *satisfying*

$$\mathsf{Q}_\Gamma(w) + \frac{d}{dt}\mathsf{Q}_\Psi(w) \leq u^* y \quad \forall w \in \mathfrak{B}.$$

(iv) Inequality

$$\int_{-\infty}^{+\infty} u^* y dt \geq 0$$

holds for all $u, y \in \mathcal{D}^\infty(\mathbb{R}, \mathbb{C}^m)$ *satisfying* (43)*, where* $z \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^{(N+1)q})$ *is defined by* (28) *with* $w = \mathrm{col}\,(y, u)$ *some nonnegative integer* $N \in \mathbb{Z}$*.*

Proof See Appendix B.4 for the proof. □

5.3 Numerical Example: Mechanical System

In this subsection, we apply Corollary 3 to a typical mechanical system and confirm the efficiency of the result.

Consider a mass-spring-damper mechanism depicted in Fig. 2. The mechanism consists of two carts with mass $m \in \mathbb{R}$ linked with a spring $k \in \mathbb{R}$ and a damper $c \in \mathbb{R}$. We apply a force $F \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ to cart 1 and measure the velocity of cart 2. In this setting, we examine that the system has the FFPR property.

Let $w := \mathrm{col}(\dot{z}_2, F)$ and $\ell := z_1$ be the manifest and latent variable, respectively. We can regard F and $\dot{z}_2$ as an input and output variable, respectively. Then, the behavior is given by

$$\mathfrak{B} = \left\{ w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^2) \mid \exists\, \ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \text{ s.t. (46)} \right\},$$

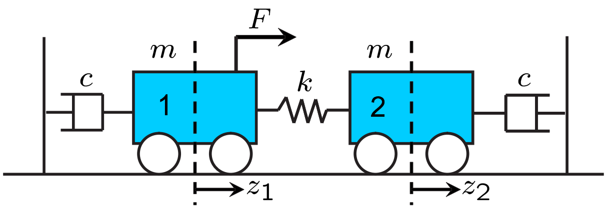

Figure 2: mass-spring-damper system

which has a latent variable representation

$$\begin{bmatrix} k & \frac{d}{dt} \\ m\frac{d^2}{dt^2}+c\frac{d}{dt}+k & 0 \end{bmatrix} w = \begin{bmatrix} m\frac{d^2}{dt^2}+c\frac{d}{dt}+k \\ k \end{bmatrix} \dot{\ell}. \tag{46}$$

By eliminating ℓ, the kernel representation of $\mathfrak{B}$ is induced by

$$R(\xi) = \begin{bmatrix} \left(m\xi^2+c\xi+k\right)^2 - k^2 & -k\xi \end{bmatrix}.$$

Hence, $\mathfrak{B}$ is represented by the image representation

$$w = M\left(\frac{d}{dt}\right)\ell,\ M(\xi) = \begin{bmatrix} k \\ m^2\xi^3+2mc\xi^2+\left(2mk+c^2\right)\xi+2kc \end{bmatrix} \in \mathbb{R}^{2\times 1}[\xi].$$

Let $G \in \mathbb{R}(\xi)$ be the transfer function from F to $\dot{z}_2$. Then, $G(\xi)$ can be computed as

$$G(\xi) = \frac{k}{m^2\xi^3+2mc\xi^2+(2mk+c^2)\,\xi+2kc}.$$

In order to check the FFPR property, define $\Phi \in \mathbb{S}^{2\times 2}$ in (45) by

$$\Phi := \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then, the supply rate for $\mathfrak{B}$ is given by $\mathsf{Q}_\Phi(w) = F\dot{z}_2$. We see that

$$M^*(j\omega)\partial\Phi(j\omega)M(j\omega) = \frac{2mc}{k}\left(-\omega^2+\frac{k}{m}\right)$$

holds. Then, $G(\xi)$ is FFPR in the frequency domain (40) for

$$\varpi := \alpha\sqrt{\frac{k}{m}},\quad \alpha \in [0,1] \ \text{ and } \ U(\xi) := m^2\xi^3+2mc\xi^2+\left(2mk+c^2\right)\xi+2kc.$$

Define $\Psi, \Gamma \in \mathbb{H}^{2\times 2}[\zeta,\eta]$ in Corollary 3 by

$$\Psi(\zeta,\eta) := \frac{m^2\left(\zeta^2+\eta^2-\zeta\eta\right)+2mc\,(\zeta+\eta)+2km+c^2}{2k}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Gamma(\zeta,\eta) := (\varpi^2-\zeta\eta)\Upsilon(\zeta,\eta),\ \Upsilon(\zeta,\eta) := \frac{2mc}{k}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{H}^{2\times 2}[\zeta,\eta].$$

Then, $\mathsf{Q}_\Psi(w)$ and $\mathsf{Q}_\Gamma(w)$ are computed as

$$\mathsf{Q}_\Psi(w) = m\dot{z}_1\dot{z}_2 - c\,(z_1 - z_2)\,\dot{z}_2 + \frac{c_2^2}{2k}\dot{z}_2{}^2,$$
$$\mathsf{Q}_\Gamma(w) = \frac{2mc}{k}\left(-\ddot{z}_2{}^2 + \alpha^2\frac{k}{m}\dot{z}_2{}^2\right),$$

respectively. Thus, we obtain the equality

$$\mathsf{Q}_\Phi(w) - \frac{d}{dt}\mathsf{Q}_\Psi(w) = 2c(1-\alpha)z_1^2 + \mathsf{Q}_\Gamma(w).$$

Hence, we see that

$$\frac{d}{dt}\mathsf{Q}_\Psi(w) \not\leq \mathsf{Q}_\Phi(w),\ \forall\, w \in \mathfrak{B}$$

holds. However, if we add the QDF $\mathsf{Q}_\Gamma(w)$ to the left-hand side of the above inequality, we get

$$\mathsf{Q}_\Gamma(w) + \frac{d}{dt}\mathsf{Q}_\Psi(w) \leq \mathsf{Q}_\Phi(w), \quad \forall\, w \in \mathfrak{B},$$

because $2c(1-\alpha)z_1^2 \geq 0$ holds for all z_1 from $\alpha \in [0, 1]$. This shows that inequality (24) is satisfied for all $w \in \mathfrak{B}$.

6 Finite Frequency KYP Lemma

In this section, we give an LMI characterization of FFDI (19) or the finite frequency KYP lemma for a numerical checking of the finite frequency properties. We first derive a finite frequency property characterization in terms of $\mathfrak{B}$-canonical polynomial matrices as a preliminary result. This yields the finite frequency KYP lemma in the behavioral framework. See Appendix A.1 for the definition and basic properties of $\mathfrak{B}$-canonical polynomial matrices.

6.1 Finite Frequency KYP Lemma

We here assume that $R \in \mathbb{C}^{p\times q}[\xi]$ in (2) is row reduced [7]. This assumption does not lose the generality, because there always exists a unimodular polynomial matrix $U \in \mathbb{C}^{p\times p}[\xi]$ satisfying

$$R_{\rm red}(\xi) = U(\xi)R(\xi),$$

where $R_{\rm red} \in \mathbb{C}^{q\times q}[\xi]$ is row reduced. It should be noted that $R_{\rm red}(\xi)$ may be obtained by the command `rowred` of Polynomial Toolbox [12] for MATLAB. In addition, we set the following degree constraint

$$\deg R \geq \deg_\zeta \Phi = \deg_\eta \Phi. \tag{47}$$

This constraint does not lose the generality. If (47) does not hold, i.e. $\deg R < \deg_\zeta \Phi = \deg_\eta \Phi$, we can reduce it to (47) by taking $R_{L+1} = R_{L+2} = \cdots = R_K = 0_{p\times q}$. Hence, it is sufficient to prove under the assumption (47).

From Theorem 1 and Lemma A.2, we obtain a characterization for the finite frequency property using $\mathfrak{B}$-canonical polynomial matrices.

Proposition 3 *Assume that* $\mathfrak{B}$ *in* (4) *is controllable and that* $R \in \mathbb{C}^{p\times q}[\xi]$ *is row-reduced. Suppose that* $\mathfrak{B}$ *is represented by an observable image representation* (6)*. Let* $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ *be given by* (12) *and satisfy* (47)*. Define* Ω *by* (18) *and* $\mathcal{G}$ *by* (22)*. Then, the following statements (i), (ii) and (iii) are equivalent.*

(i) FFDI (19) *holds for all* $\omega \in \Omega$*.*

(ii) There exist unique $\mathfrak{B}$*-canonical* $\Psi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ *and* $\Gamma \in \mathcal{G}$ *with* $\mathfrak{B}$*-canonical* $\Upsilon \in \mathbb{H}^{q\times q}[\zeta,\eta]$ *satisfying*

$$\frac{d}{dt}\mathsf{Q}_\Psi(w) \le \mathsf{Q}_\Phi(w) - \mathsf{Q}_\Gamma(w),\ \forall\, w \in \mathfrak{B}. \tag{48}$$

(iii) Inequality (26) *holds for all* $w \in \mathfrak{B}$ *satisfying* (27)*, where* $z \in \mathcal{D}^\infty(\mathbb{R},\mathbb{C}^{(N+1)q})$ *is defined by* (28) *for nonnegative integer such that* $N \le \deg R - 1$*.*

Proof See Appendix B.5 for the proof. □

In Theorem 1 and Corollary 1, we do not know the degree of $\Psi(\zeta,\eta)$ and $\Gamma(\zeta,\eta)$ in advance. Although Proposition 3 is a preliminary result for the finite frequency KYP lemma, it shows that the upper bounds of the degree are determined by that of $R(\xi)$.

As we have established the preliminary result, we give the finite frequency KYP lemma. In the following, we transform the kernel representation (2) into a latent variable representation with a first-order differential-algebraic equation (A.2) along the same line in [13]. Let $r_i \in \mathbb{C}^{1\times q}[\xi]$ $(i = 1,\cdots,p)$ denote the ith row of $R(\xi)$, i.e.

$$R(\xi) = \begin{bmatrix} r_1(\xi) \\ r_2(\xi) \\ \vdots \\ r_p(\xi) \end{bmatrix}.$$

For these r_i's, define $R_\mathrm{e} \in \mathbb{C}^{\sum_{i=1}^p (L+1-\rho_i)\times q}[\xi]$ by

$$R_\mathrm{e}(\xi) := \begin{bmatrix} R_\mathrm{e}^1(\xi) \\ R_\mathrm{e}^2(\xi) \\ \vdots \\ R_\mathrm{e}^p(\xi) \end{bmatrix},\ R_\mathrm{e}^i(\xi) := \begin{bmatrix} r_i(\xi) \\ \xi r_i(\xi) \\ \vdots \\ \xi^{L-\rho_i} r_i(\xi) \end{bmatrix}, \tag{49}$$

where $\rho_i \in \mathbb{Z}$ $(i = 1,2,\cdots,p)$ denotes the maximal degree of the elements of $r_i(\xi)$. We easily see that $R_\mathrm{e}(\xi)$ satisfies $\deg R_\mathrm{e} = L$. Define the variable $v \in \mathcal{C}^\infty(\mathbb{R},\mathbb{C}^{(L+1)q})$ by stacking w and its derivatives as

$$v := Z_L\left(\frac{d}{dt}\right) w,$$

where $Z_L \in \mathbb{R}^{(L+1)q\times q}[\xi]$ is defined by (5). Then, the kernel representation (2) can be rewritten by a first-order differential-algebraic equation expressed as

$$\begin{aligned} \begin{bmatrix} I_{Lq} & 0_{Lq\times q} \end{bmatrix} \frac{d}{dt} v &= \begin{bmatrix} 0_{Lq\times q} & I_{Lq} \end{bmatrix} v, \\ \widetilde{R}_\mathrm{e} v &= 0, \\ w &= \begin{bmatrix} I_q & 0_{q\times Lq} \end{bmatrix} v, \end{aligned}$$

where $\widetilde{R}_{\rm e} \in \mathbb{C}^{\sum_{i=1}^{p}(L+1-\rho_i)\times(L+1)q}$ denotes the coefficient matrix of $R_{\rm e}(\xi)$. We can see from this expression that $\widetilde{R}_{\rm e}v = 0$ holds if and only if

$$v = \widetilde{R}_{\rm e}^{\perp}k,\ k \in \mathcal{C}^{\infty}(\mathbb{R}, \mathbb{C}^d),\ d := (L+1)q - \sum_{i=1}^{p}(L+1-\rho_i) \tag{50}$$

holds, where $\widetilde{R}_{\rm e}^{\perp} \in \mathbb{C}^{(L+1)q\times d}$ is the constant matrix satisfying

$$\operatorname{im}\left(\widetilde{R}_{\rm e}^{\perp}\right) = \ker\left(\widetilde{R}_{\rm e}\right). \tag{51}$$

This implies that $\widetilde{R}_{\rm e}$ can define the first-order latent variable representation with manifest variable w and the latent variable k as

$$w = \begin{bmatrix} I_q & 0_{q\times Lq} \end{bmatrix} \widetilde{R}_{\rm e}^{\perp} k, \tag{52}$$

$$E\frac{d}{dt}k = Fk, \tag{53}$$

where

$$E := \begin{bmatrix} I_{Lq} & 0_{Lq\times q} \end{bmatrix} \widetilde{R}_{\rm e}^{\perp} \in \mathbb{C}^{Lq\times d}, \tag{54}$$

$$F := \begin{bmatrix} 0_{Lq\times q} & I_{Lq} \end{bmatrix} \widetilde{R}_{\rm e}^{\perp} \in \mathbb{C}^{Lq\times d}. \tag{55}$$

Using (50) and (51), $\mathfrak{B}$ in (4) coincides with the set of trajectories given by (52) and (53) (see pp. 287 in [13]), i.e.

$$\mathfrak{B} = \left\{ w \in \mathcal{C}^{\infty}(\mathbb{R}, \mathbb{C}^q) \,\middle|\, \exists\, k \in \mathcal{C}^{\infty}(\mathbb{R}, \mathbb{C}^d) \text{ s.t. (52) and (53) } \right\}.$$

Using $\widetilde{R}_{\rm e}^{\perp}$ and $\widetilde{\Phi}$, we define $\Phi_0 \in \mathbb{H}^{d\times d}$ by

$$\Phi_0 := \left(\widetilde{R}_{\rm e}^{\perp}\right)^* \begin{bmatrix} \widetilde{\Phi} & 0_{(L+1)q\times(L-K)q} \\ 0_{(L-K)q\times(L+1)q} & 0_{(L-K)q\times(L-K)q} \end{bmatrix} \widetilde{R}_{\rm e}^{\perp}, \tag{56}$$

where $\tilde{\Phi} \in \mathbb{H}^{(K+1)q\times(K+1)q}$ is the coefficient matrix of $\Phi(\zeta, \eta)$. Consequently, we obtain the finite frequency KYP lemma in the behavioral framework. This is a natural result which follows from Lemma 1 and Proposition 3.

Theorem 2 *Assume that $\mathfrak{B}$ in* (4) *is controllable and that $R \in \mathbb{C}^{p\times q}[\xi]$ is row reduced. Suppose that $\mathfrak{B}$ is represented by an image representation* (6). *Let $\Phi \in \mathbb{H}^{q\times q}[\zeta, \eta]$ be given by* (12) *and satisfy* (47). *Define Ω by* (18). *Then, the following statements (i) and (ii) are equivalent.*

(i) FFDI (19) *holds for all $\omega \in \Omega$.*

(ii) There exist $\widetilde{\Psi} \in \mathbb{H}^{Lq\times Lq}$ and $\widetilde{\Upsilon} \in \mathbb{H}^{Lq\times Lq}$ satisfying

$$\tau\widetilde{\Upsilon} \geq 0, \tag{57}$$

$$E^*\widetilde{\Psi}F + F^*\widetilde{\Psi}E + (\varpi_-^2 - {\varpi_+}^2)E^*\widetilde{\Upsilon}E - F^*\widetilde{\Upsilon}F + \left\{(j\varpi_+F^*\widetilde{\Upsilon}E) + (j\varpi_+F^*\widetilde{\Upsilon}E)^*\right\} \leq \Phi_0, \tag{58}$$

where $E, F \in \mathbb{C}^{Lq\times d}$ and $\Phi_0 \in \mathbb{H}^{d\times d}$ are defined by (54), (55) *and* (56), *respectively, and $\varpi_-, \varpi_+ \in \mathbb{R}$ are defined by* (21).

Proof See Appendix B.6 for the proof. □

We now explain relationship between Theorem 2 and the previous works [3][13] associated with KYP lemma.

Theorem 2 is not a new result because this is a special case of the generalized KYP lemma [3] if we restrict ourselves to continuous-time systems and the curve in the complex plane to Ω. The lemma was derived based on the input-output setting, however, Theorem 2 does not assume such a relation in advance.

On the other hand, Theorem 2 includes the KYP lemma derived in [13] in a sense that we can deal with the LMIs over the restricted frequency domain. This is explained as follows.

If we choose the parameters $\tau = -1$ and $\varpi_1 = \varpi_2 = 0$, Ω coincides with the set of real numbers $\mathbb{R}$. Hence, Theorem 2 falls to the KYP lemma [13] in the behavioral framework. Since $\varpi = \varpi_+ = 0$ holds, the LMI (58) is equivalent to

$$E^*\widetilde{\Psi}F + F^*\widetilde{\Psi}E + F^*\left(-\widetilde{\Upsilon}\right)F \leq \Phi_0. \tag{59}$$

It follows from $-\widetilde{\Upsilon} \geq 0$ that the above LMI is equivalently rewritten by rewritten by

$$E^*\widetilde{\Psi}F + F^*\widetilde{\Psi}E \leq \Phi_0.$$

If there exist $\widetilde{\Psi}$ and $\widetilde{\Upsilon}$ satisfying (59), $\widetilde{\Upsilon} = 0$ also satisfies (59) from $-\widetilde{\Upsilon} \geq 0$. Then, the LMI (59) can be rewritten by the following LMIs

$$\left(\widetilde{R_{\rm e}}^{\perp}\right)^* \left(\begin{bmatrix} \widetilde{\Phi} & 0_{(L+1)q\times(L-K)q} \\ 0_{(L-K)q\times(L+1)q} & 0_{(L-K)q\times(L-K)q}\end{bmatrix} - \begin{bmatrix} 0_{Lq\times q} & \widetilde{\Psi} \\ 0_{q\times q} & 0_{q\times Lq}\end{bmatrix} - \begin{bmatrix} 0_{q\times Lq} & 0_{q\times q} \\ \widetilde{\Psi} & 0_{Lq\times q}\end{bmatrix}\right)\widetilde{R_{\rm e}}^{\perp} \geq 0,$$

which was proposed in Theorem 4.2 in [13]. Hence, Theorem 2 includes the KYP lemma in previous works in behavioral approach.

6.2 Numerical Example

In the subsection, we will see how to check the finite frequency property based on the LMI conditions in Theorem 2.

Consider the behavior $\mathfrak{B}$ whose kernel representation is induced by $R(\xi)$ in (39). We see that $R_{\rm e}(\xi)$ coincides with $R(\xi)$. This polynomial matrix has the coefficient matrix $\widetilde{R}_{\rm e} \in \mathbb{R}^{2\times 9}$ which is given by

$$\widetilde{R}_{\rm e} = \left[\begin{array}{ccc:ccc:ccc} 2 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ \hdashline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

From (51), we compute

$$\widetilde{R}_{\rm e}^{\perp} = \left[\begin{array}{ccccccc} 0.3780 & 0 & -0.3780 & 0 & -0.3780 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.7071 \\ 0.9186 & 0 & 0.0814 & 0 & 0.0814 & 0 & 0 \\ \hdashline 0 & 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0.0814 & 0 & 0.9186 & 0 & -0.0814 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 & 0 \\ \hdashline 0.0814 & 0 & -0.0814 & 0 & 0.9186 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.7071 \end{array}\right].$$

We can solve the LMIs (57) and (58) by using the solver in Robust Control Toolbox [9] of MATLAB 2009a. One of the feasible solutions obtained is given by

$$\widetilde{\Psi} = \left[\begin{array}{rrr:rrr} -0.3105 & 0.2767 & 0.1751 & -0.8558 & 0.8161 & -0.2541 \\ 0.2767 & 2.0634 & -0.1511 & 0.0649 & -0.1580 & 0.1172 \\ 0.1751 & -0.1511 & -0.1025 & 0.3220 & 0.0249 & -0.2742 \\ \hdashline -0.8558 & 0.0649 & 0.3220 & -0.1485 & -0.1451 & 0.1666 \\ 0.8161 & -0.1580 & 0.0249 & -0.1451 & -0.0705 & 0.0098 \\ -0.2541 & 0.1172 & -0.2742 & 0.1666 & 0.0098 & -0.1527 \end{array}\right],$$

$$\widetilde{\Upsilon} = \left[\begin{array}{rrr:rrr} 0.8302 & -0.0438 & -0.1069 & -0.0040 & -0.0348 & -0.0104 \\ -0.0438 & 0.7088 & -0.0847 & 0.0499 & 0.0018 & -0.0022 \\ -0.1069 & -0.0847 & 0.5867 & 0.0077 & 0.0044 & 0.0009 \\ \hdashline -0.0040 & 0.0499 & 0.0077 & 1.1633 & 0.0154 & -0.0792 \\ -0.0348 & 0.0018 & 0.0044 & 0.0154 & 0.6566 & -0.0132 \\ -0.0104 & -0.0022 & 0.0009 & -0.0792 & -0.0132 & 0.6100 \end{array}\right].$$

Note that the symmetric matrix $\widetilde{\Upsilon}$ is nonnegative definite, since the eigenvalues are located at

$$\{0.5014,\ 0.5956,\ 0.6540,\ 0.7475,\ 0.8767,\ 1.1804\}.$$

We can therefore conclude from Theorem 2 that the finite frequency property holds.

7 Conclusions

In this paper, we have characterized the finite frequency properties in terms of the dissipation inequality and the integral of the supply rate based on QDFs. This leads the finite frequency KYP lemma which characterizes the FFDI (19) as a natural result.

As a future work, our results should be applied to a synthesis of a controller with frequency domain specifications in the framework of dissipation theory. Partial solutions for such problems have been derived by Iwasaki and Hara [4] based on state-space and descriptor system. However, our result may be efficient to solve these problems, since we can deal with systems described by artless high-order differential algebraic equations.

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Appendix A Background Materials

In this section, we collect the background materials which are used in the proofs. They relate LMIs with QDFs and play important roles in Section 6.

Appendix A.1 $\mathfrak{B}$-canonical Polynomial Matrices

We introduce $\mathfrak{B}$-canonicity of polynomial matrices in this appendix, which are taken from the reference [7][8].

We assume that $R \in \mathbb{C}^{q\times q}[\xi]$ in (2) is row reduced [7] in this section. The assumption does not lose the generality as we have explained in Section 6. We define the $\mathfrak{B}$-canonicity of polynomial matrices.

Definition A.1 [8] Let $\mathfrak{B}$ be represented by a kernel representation (2) for $R \in \mathbb{C}^{p\times q}[\xi]$. Assume that $R(\xi)$ is row reduced. Let $D \in \mathbb{C}^{p\times q}[\xi]$ be given. Let $r_i \in \mathbb{C}^{1\times q}[\xi]$ and $d_i \in \mathbb{C}^{1\times q}[\xi]$ $(i = 1, \cdots, p)$ denote the ith rows of $R(\xi)$ and $D(\xi)$, respectively. A polynomial matrix $D(\xi)$ is called $\mathfrak{B}$-*canonical* if

$$\deg d_i \leq \deg r_i - 1, \quad \forall\, i = 1, \cdots, p$$

holds.

The next lemma ensures the uniqueness of an R-canonical polynomial matrix up to $\mathfrak{B}$-equivalence.

Lemma A.1 [8] *Let $\mathfrak{B}$ be represented by a kernel representation* (2) *for $R \in \mathbb{C}^{p\times q}[\xi]$. Assume that $R(\xi)$ is row reduced. For any $D \in \mathbb{C}^{p\times q}[\xi]$, there exists a unique $\mathfrak{B}$-canonical $D' \in \mathbb{C}^{p\times q}[\xi]$ satisfying*

$$D\left(\frac{d}{dt}\right) w = D'\left(\frac{d}{dt}\right) w, \quad \forall\, w \in \mathfrak{B}.$$

We now define the $\mathfrak{B}$-canonicity of two-variable polynomial matrices. Let $\Phi \in \mathbb{H}^{q\times q}[\zeta, \eta]$ be given. Suppose that $\Phi(\zeta, \eta)$ has a (symmetric) canonical factorization

$$\Phi(\zeta, \eta) = F^\star(\zeta)\Sigma_\Phi F(\eta) \tag{A.1}$$

with $\Sigma_\Phi \in \mathbb{S}^{\mathrm{rank}\widetilde{\Phi}\times\mathrm{rank}\widetilde{\Phi}}$, $\det \Sigma_\Phi \neq 0$ and $F \in \mathbb{C}^{\mathrm{rank}\widetilde{\Phi}\times m}[\xi]$.

Definition A.2 [8] Let $\mathfrak{B}$ be represented by a kernel representation (2) for $R \in \mathbb{C}^{p\times q}[\xi]$. Assume that $R(\xi)$ is row reduced. Let $\Phi \in \mathbb{H}^{q\times q}[\zeta, \eta]$ be given. Let $F \in \mathbb{C}^{\mathrm{rank}\widetilde{\Phi}\times m}[\xi]$ be defined by the canonical factorization (A.1). Then, $\Phi(\zeta, \eta)$ is called $\mathfrak{B}$-*canonical* if $F(\xi)$ is $\mathfrak{B}$-canonical.

The following result is an immediate consequence of the uniqueness of the canonical factorization of $\Phi(\zeta, \eta)$ and of Lemma A.1.

Lemma A.2 [8] *Let $\mathfrak{B}$ be represented by a kernel representation* (2) *for $R \in \mathbb{C}^{p\times q}[\xi]$. Assume that $R(\xi)$ is row reduced. Let $\Phi \in \mathbb{H}^{q\times q}[\zeta, \eta]$ be given. Then, for any $\Phi(\zeta, \eta)$, there exists a unique $\mathfrak{B}$-canonical $\Phi' \in \mathbb{H}^{q\times q}[\zeta, \eta]$ satisfying*

$$\mathsf{Q}_{\Phi'}(w) = \mathsf{Q}_\Phi(w), \quad \forall\, w \in \mathfrak{B}.$$

Appendix A.2 Trimness

We summarize the definition and some basic result of the trimness of first order kernel representation in this appendix, which are taken from the reference [13].

Definition A.3 [13] Let $\mathfrak{B}_\mathrm{f}$ be the behavior whose kernel representation is described by a first-order differential-algebraic equation

$$E\dot{w} = Fw,\ E, F \in \mathbb{C}^{p\times q}. \tag{A.2}$$

Define the *set of consisting points* of (A.2) by

$$\mathcal{W}_0 := \{w_0 \in \mathbb{C}^q \mid \exists\, w \in \mathfrak{B}_\mathrm{f} \text{ s.t. } w(0) = w_0\}.$$

The representation given by (A.2) is called *trim* if $\mathcal{W}_0 = \mathbb{C}^q$ holds.

The next lemma ensures that the representation given by (52) is trim, which is used to prove the the finite-frequency KYP Lemma. This lemma was taken from Lemma 4.1 in [13].

Lemma A.3 [13] *Let $\mathfrak{B}$ be represented by a kernel representation* (2) *for $R \in \mathbb{C}^{p\times q}[\xi]$. Assume that $R(\xi)$ is row reduced. Then, the kernel representation* (52) *is trim.*

Appendix B Proofs

Appendix B.1 Proof of Theorem 1

The proof consists of three steps. We first show the characterization for the low frequency property in Appendix B.1.1. This leads to the high frequency case in Appendix B.1.2. Finally, we conclude the proof in Appendix B.1.3 for the general frequency property. Note that the most part of the proof are devoted to Appendix B.1.1.

Appendix B.1.1 Low Frequency Case

In this appendix, we restrict our attention to the low frequency property and derive a characterization of the property as preliminary result.

Define the low frequency domain $\Omega_{\rm low} \subset \mathbb{R}$ in the restricted interval by (29). We remark that τ in (18) is equal to $+1$ for the low frequency property. We consider a characterization of the following FFDI based on QDFs.

$$M(j\omega)^*\partial\Phi(j\omega)M(j\omega) \geq 0, \quad \forall\, \omega \in \Omega_{\rm low} \tag{B.1}$$

For this purpose, define the set of two-variable matrices for the frequency domain given by

$$\mathcal{G}_{\rm low} := \left\{ \Gamma \in \mathbb{H}^{q\times q}[\zeta,\eta] \;\middle|\; \begin{array}{c} \Gamma(\zeta,\eta) := \left(\varpi^2 - \zeta\eta\right)\Upsilon(\zeta,\eta) \\ \text{for some } \Upsilon \in \mathbb{H}^{q\times q}[\zeta,\eta] \text{ such that} \\ \mathsf{Q}_\Upsilon(w) \geq 0,\ \forall\, w \in \mathcal{C}^\infty(\mathbb{R},\mathbb{C}^q) \end{array} \right\}. \tag{B.2}$$

We see that

$$\partial\Gamma(j\omega) = \left(\varpi^2 - \omega^2\right)\partial\Upsilon(j\omega) \geq 0, \quad \forall\, \omega \in \Omega_{\rm low}.$$

holds for any $\Gamma \in \mathcal{G}$. We obtain a necessary and sufficient condition for the low frequency property.

Lemma B.1 *Assume that $\mathfrak{B}$ in* (4) *is controllable and that $\mathfrak{B}$ is represented by an observable image representation* (6)*. Let $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ be given. Define $\Omega_{\rm low}$ by* (29) *and define $\mathcal{G}_{\rm low}$ by* (B.2)*. Then, the following statements (i), (ii) and (iii) are equivalent.*

(i) *FFDI* (B.1) *holds for all $\omega \in \Omega_{\rm low}$.*

(ii) *There exist $\Psi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ and $\Gamma \in \mathcal{G}_{\rm low}$ satisfying* (24).

(iii) *Inequality* (26) *holds for all $w \in \mathfrak{B}$ satisfying*

$$\dot{z}\dot{z}^* \leq \varpi^2 zz^*, \tag{B.3}$$

where $z \in \mathcal{D}^\infty(\mathbb{R},\mathbb{C}^{(N+1)q})$ is defined by (28) *with some nonnegative integer $N \in \mathbb{Z}$.*

Proof (i)⇒(ii) Assume that the statement (ii) does not hold. This is the case if and only if there do not exist $\Psi(\zeta,\eta)$ and $\Gamma \in \mathcal{G}_{\rm low}$ satisfying

$$\frac{d}{dt}\mathsf{Q}_\Psi(w) - \mathsf{Q}_\Phi(w) + \mathsf{Q}_\Gamma(w) \le \varepsilon \left\|w\right\|^2, \quad \forall\, w \in \mathfrak{B}$$

for some $\varepsilon > 0$. The above inequality is equivalent to the two-variable polynomial matrix equation

$$\begin{aligned} &M^\star(\zeta)\Gamma(\zeta,\eta)M(\eta) + (\zeta+\eta)M^\star(\zeta)\Psi(\zeta,\eta)M(\eta) - M^\star(\zeta)\Phi(\zeta,\eta)M(\eta) \\ &\quad - \varepsilon M^\star(\zeta)M(\eta) + M^\star(\zeta)\Delta(\zeta,\eta)M(\eta) = 0. \end{aligned} \tag{B.4}$$

for some $\Delta \in \mathbb{H}^{q\times q}[\zeta,\eta]$ such that $\mathsf{Q}_\Delta(w) \ge 0$, $\forall\, w \in \mathfrak{B}$. Substituting $\zeta = -j\omega$ and $\eta = j\omega$ into (B.4), we get

$$\begin{aligned} &M(j\omega)^*\partial\Gamma(j\omega)M(j\omega) - M(j\omega)^*\partial\Phi(j\omega)M(j\omega) - \varepsilon M(j\omega)^*M(j\omega) \\ &\quad + \partial\Delta(j\omega)M(j\omega) = 0, \quad \forall\, \omega \in \Omega_{\rm low}. \end{aligned}$$

Since $\partial\Delta(j\omega) \ge 0$ holds for all $\omega \in \Omega_{\rm low}$, we obtain the matrix inequality

$$M(j\omega)^*\partial\Gamma(j\omega)M(j\omega) - M(j\omega)^*\partial\Phi(j\omega)M(j\omega) \le \varepsilon M(j\omega)^*M(j\omega), \quad \forall\, \omega \in \Omega_{\rm low}.$$

This shows that there does not exist $\Gamma \in \mathcal{G}_{\rm low}$ satisfying

$$M(j\omega)^*\partial\Gamma(j\omega)M(j\omega) - M(j\omega)^*\partial\Phi(j\omega)M(j\omega) \le 0, \quad \forall\, \omega \in \Omega_{\rm low}.$$

From the definition of $\Gamma(\zeta,\eta)$, there does not exist $\Upsilon(\zeta,\eta)$ satisfying $\mathsf{Q}_\Upsilon(w) \ge 0$, $\forall\, w \in \mathfrak{B}$ and

$$\begin{aligned} M(j\omega)^*\partial\Phi(j\omega)M(j\omega) &\ge \left(\varpi^2 - \omega^2\right) M(j\omega)^*\partial\Upsilon(j\omega)M(j\omega) \\ &\ge 0, \quad \forall\, \omega \in \Omega_{\rm low}. \end{aligned}$$

Hence, the statement (i) does not hold, which completes the proof of the claim.

(ii)⇒(iii) By integrating (24) from $t = -\infty$ to $t = +\infty$ along $w \in \mathfrak{B} \cap \mathcal{D}^\infty(\mathbb{R},\mathbb{C}^q)$, we get the inequality

$$\int_{-\infty}^{+\infty} \mathsf{Q}_\Gamma(w)dt \le \int_{-\infty}^{+\infty} \mathsf{Q}_\Phi(w)dt, \quad \forall\, w \in \mathfrak{B} \cap \mathcal{D}^\infty(\mathbb{R},\mathbb{C}^q).$$

The QDF $\mathsf{Q}_\Upsilon(w)$ is expressed as $\mathsf{Q}_\Upsilon(w) = z^*\widetilde{\Upsilon}z$ from (28), where $\widetilde{\Upsilon} \in \mathbb{H}^{(N+1)q\times(N+1)q}$ is the coefficient matrix of $\Upsilon(\zeta,\eta)$. Hence, $\mathsf{Q}_\Gamma(w)$ can be rewritten as

$$\mathsf{Q}_\Gamma(w) = -\dot{z}^*\widetilde{\Upsilon}\dot{z} + \varpi^2 z^*\widetilde{\Upsilon}z.$$

By integrating the above equation from $t = -\infty$ to $t = +\infty$, we obtain

$$\begin{aligned} \int_{-\infty}^{+\infty} \mathsf{Q}_\Gamma(w)dt &= \int_{-\infty}^{+\infty} \left(-\dot{z}^*\widetilde{\Upsilon}\dot{z} + \varpi^2 z^*\widetilde{\Upsilon}z\right) dt \\ &= \int_{-\infty}^{+\infty} \left\{-\mathrm{tr}\left[\widetilde{\Upsilon}\left(\dot{z}\dot{z}^* - \varpi^2 zz^*\right)\right]\right\} dt \\ &= \int_{-\infty}^{+\infty} \mathrm{tr}\left[\widetilde{\Upsilon}\left\{-\left(\dot{z}\dot{z}^* - \varpi^2 zz^*\right)\right\}\right] dt. \end{aligned}$$

Since $\tau\widetilde{\Upsilon} \geq 0$ holds from (B.2) and Lemma 1, we have (26) for all $w \in \mathfrak{B}$ satisfying (B.3). This concludes the claim.

(iii)⇒(i) We prove the statement (i) by showing a contraposition. Define $\Phi' \in \mathbb{H}^{m\times m}[\zeta,\eta]$ by (37).

Assume that there exists an $\omega_0 \in \Omega_{\text{low}}$ such that $\omega_0 \geq 0$ and $\partial\Phi'(j\omega_0) \not\geq 0$, i.e. the minimum eigenvalue of $\partial\Phi'(j\omega_0)$ is negative. We can assume $\omega_k' > 0$ because it can be proved the case where $\omega_k' = 0$ by replacing ω_k' with $\omega_k' + \varepsilon$ $(\varepsilon > 0)$ and taking the limitation $\varepsilon \to 0$. Let $v \in \mathbb{C}^m$ be the eigenvector corresponding to the eigenvalue. Then, we have

$$\mathsf{Q}_{\Phi'}(e^{j\omega_0 t}v) = v^*\partial\Phi'(j\omega_0)v < 0. \tag{B.5}$$

Let $\ell_n \in \mathcal{D}^\infty(\mathbb{R},\mathbb{C}^m)$ be a latent variable satisfying

$$\ell_n(t) = \begin{cases} e^{j\omega_0 t}v & \left(|t| \leq \frac{2\pi n}{\omega_0}\right) \\ \widetilde{\ell}_n\left(t + \frac{2\pi n}{\omega_0}\right) & \left(t < -\frac{2\pi n}{\omega_0}\right) \\ \widetilde{\ell}_n\left(t - \frac{2\pi n}{\omega_0}\right) & \left(t > \frac{2\pi n}{\omega_0}\right) \end{cases}$$

with some nonnegative integer $n \in \mathbb{Z}$, where $\widetilde{\ell}_n \in \mathcal{D}^\infty(\mathbb{R},\mathbb{C}^m)$ is chosen as a function which does not depend on n and be such that ℓ_n is a smooth function for n. For the above ℓ_n, define $z_n \in \mathcal{D}^\infty(\mathbb{R},\mathbb{C}^{(N+1)q})$ by

$$z_n := Z_N\left(\frac{d}{dt}\right)w_n,\ w_n := M\left(\frac{d}{dt}\right)\ell_n \in \mathfrak{B}.$$

We can compute

$$\dot{z}_n\dot{z}_n^* - \varpi^2 z_n z_n^* = \left(\omega_0^2 - \varpi^2\right) Z_N(j\omega_0)M(j\omega_0)vv^*M(j\omega_0)^*Z_N(j\omega_0)^*$$

Since we assumed $|\omega_0| \leq \varpi$, we get

$$\dot{z}_n\dot{z}_n^* - \varpi^2 z_n z_n^* \leq 0.$$

On the other hand, we observe that

$$\begin{aligned} \int_{-\infty}^{+\infty} \mathsf{Q}_\Phi(w_n)dt &= \int_{-\infty}^{+\infty} \mathsf{Q}_{\Phi'}(\ell_n)dt \\ &= \frac{4\pi n}{\varpi_0}v^*\partial\Phi'(j\omega_0)v + A_1 \end{aligned}$$

holds, where $A_1 \in \mathbb{R}$ is a constant which does not depend on n. Hence, if we choose n as a sufficiently large number, from (B.5), we have

$$\int_{-\infty}^{+\infty} \mathsf{Q}_\Phi(w_n)dt < 0.$$

This implies that the statement (iii) does not hold, which completes the proof. □

Appendix B.1.2 High Frequency Case

As we have completed a characterization of the low frequency property, we consider to characterize the high frequency property in this appendix.

Define the high frequency domain $\Omega_{\mathrm{high}} \subset \mathbb{R}$ in the domain given by (30). Note that τ in (18) is equal to -1 for this case. We derive a characterization of the following FFDI in terms of QDFs.

$$M(j\omega)^* \partial\Phi(j\omega) M(j\omega) \geq 0, \quad \forall\, \omega \in \Omega_{\mathrm{high}} \tag{B.6}$$

Similarly to the low frequency case, we define the set of two-variable polynomial matrices for Ω_{high} by

$$\mathcal{G}_{\mathrm{high}} := \left\{ \Gamma \in \mathbb{H}^{q\times q}[\zeta,\eta] \;\middle|\; \begin{array}{c} \Gamma(\zeta,\eta) := \left(\varpi^2 - \zeta\eta\right) \Upsilon(\zeta,\eta) \\ \text{for some } \Upsilon \in \mathbb{H}^{q\times q}[\zeta,\eta] \text{ such that} \\ \mathsf{Q}_\Upsilon(w) \geq 0,\ \forall\, w \in \mathcal{C}^\infty(\mathbb{R},\mathbb{C}^q) \end{array} \right\}. \tag{B.7}$$

We obtain a necessary and sufficient condition for the high frequency domain property by using a similar discussion to the low frequency case.

Lemma B.2 *Assume that* $\mathfrak{B}$ *in* (4) *is controllable and that* $\mathfrak{B}$ *is represented by an observable image representation* (6). *Let* $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ *be given. Define* Ω_{high} *by* (30) *and define* $\mathcal{G}_{\mathrm{high}}$ *by* (B.7). *Then, the following statements (i), (ii) and (iii) are equivalent.*

(i) FFDI (B.6) *holds for all* $\omega \in \Omega_{\mathrm{high}}$.

(ii) There exist $\Psi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ *and* $\Gamma \in \mathcal{G}_{\mathrm{high}}$ *satisfying* (24).

(iii) Inequality (26) *holds for all* $w \in \mathfrak{B}$ *satisfying*

$$\dot{z}\dot{z}^* \geq \varpi^2 z z^*,$$

where $z \in \mathcal{D}^\infty(\mathbb{R},\mathbb{C}^{(N+1)q})$ *is defined by* (28) *with some nonnegative integer* $N \in \mathbb{Z}$.

Proof The proof follows immediately by the same discussion to the low frequency case. □

Appendix B.1.3 Proof of Theorem 1

We conclude the proof of Theorem 1 in this appendix.

Define $\omega' \in \mathbb{R}$ and $\varpi \in \mathbb{R}$ by

$$\omega' := \omega + \varpi_+ \ \text{ and } \ \varpi := \varpi_-.$$

Then, we have $\omega' \in \Omega$ if and only if there hold $\omega \in \Omega_{\mathrm{low}}$ $(\tau = +1)$ and $\omega \in \Omega_{\mathrm{high}}$ $(\tau = -1)$. Hence, the claim follows immediately for $\tau = +1$ and $\tau = -1$ from Lemmas B.1 and B.2, respectively. This completes the proof of Theorem 1. □

Appendix B.2 Proof of Corollary 1

(i)⇒(iii) We assume that the statement (i) holds. From Theorem 1, there exist two-variable polynomial matrices $\Psi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ and $\Gamma \in \mathcal{G}$ satisfying the inequality (24) for all $w \in \mathfrak{B}$. This is the case if and only if (33) holds for some $\Delta \in \mathbb{H}^{q\times q}[\zeta,\eta]$ such that $\mathsf{Q}_\Delta(w) \geq 0$, $\forall\, w \in \mathfrak{B}$. Since $\mathsf{Q}_\Gamma(w)$ satisfies $\mathsf{Q}_\Gamma(w) \geq 0$ for all $w \in \mathfrak{B}_\Omega$ from (32), we get

$$\mathsf{Q}_{\Delta+\Gamma}(w) = \mathsf{Q}_\Delta(w) + \mathsf{Q}_\Gamma(w) \geq 0, \quad \forall\, w \in \mathfrak{B}_\Omega.$$

Hence, the first part of (iii) follows.

In addition, by integrating (33) from $t = -\infty$ to $t = +\infty$ along $\mathfrak{B}_\Omega \cap \mathcal{D}^\infty(\mathbb{R},\mathbb{C}^q)$, we obtain

$$\int_{-\infty}^{+\infty} \mathsf{Q}_{\Gamma+\Delta}(w)dt = \int_{-\infty}^{+\infty} \mathsf{Q}_\Phi(w)dt, \quad \forall\, w \in \mathfrak{B}_\Omega \cap \mathcal{D}^\infty(\mathbb{R},\mathbb{C}^q). \tag{B.8}$$

This shows that $\mathsf{Q}_{\Delta+\Gamma}(w)$ becomes the dissipation rate for $\mathfrak{B}_\Omega$ with respect to the supply rate $\mathsf{Q}_\Phi(w)$ from Definition 2 (ii). This completes the proof.

(iii)⇒(iv) Integrating (33) from $t = -\infty$ to $t = +\infty$ along $\mathfrak{B}_\Omega \cap \mathcal{D}^\infty(\mathbb{R},\mathbb{C}^q)$ yields (B.8). Since $\mathsf{Q}_{\Delta+\Gamma}(w)$ satisfies $\mathsf{Q}_{\Delta+\Gamma}(w) \geq 0$, $\forall\, w \in \mathfrak{B}_\Omega$, we get

$$\int_{-\infty}^{+\infty} \mathsf{Q}_\Phi(w)dt = \int_{-\infty}^{+\infty} \mathsf{Q}_{\Gamma+\Delta}(w)dt \geq 0, \quad \forall\, w \in \mathfrak{B}_\Omega \cap \mathcal{D}^\infty(\mathbb{R},\mathbb{C}^q).$$

Hence, the statement (iv) holds.

(iv)⇒(i) Since the statement (iv) is equivalent to the statement (iii) of Theorem 1, the proof follows immediately from Proposition 1.

(ii)⇔(iii) The proof is straightforward from Proposition 1. □

Appendix B.3 Proof of Corollary 2

The equivalence of (ii), (iii) and (iv) follows immediately from Theorem 1 since $\mathsf{Q}_\Phi(w) = \gamma^2 \|u\|^2 - \|y\|^2$ holds for all $w = \mathrm{col}(y,u) \in \mathfrak{B}$. Hence, we have only to show the statement (i) is equivalent to (ii).

Assume that (i) holds. Pre- and post-multiplying (41) by $U(j\omega)^*$ and $U(j\omega)$, respectively, we get

$$Y(j\omega)^* Y(j\omega) \leq \gamma^2 U(j\omega)^* U(j\omega), \quad \forall\, \omega \in \Omega'_{\mathrm{low}}.$$

From the definition of $G(\xi)$, the above inequality is equivalent to the FFDI (20) for Φ in (42). □

Appendix B.4 Proof of Corollary 3

Assume that the statement (i) holds. Pre- and post-multiplying (44) by $U(j\omega)^*$ and $U(j\omega)$, respectively, we get

$$U(j\omega)^* Y(j\omega) + Y(j\omega)^* U(j\omega) \geq 0, \quad \forall\, \omega \in \Omega'_{\mathrm{low}}.$$

It follows from (45) and the definition of $G(\xi)$ that the above inequality is equivalent to (ii). The equivalence of (ii), (iii) and (iv) follows immediately from Theorem 1 since $\mathsf{Q}_\Phi(w) = u^* y$ holds for all $w = \mathrm{col}(y,u) \in \mathfrak{B}$. □

Appendix B.5 Proof of Proposition 3

(i)⇒(ii) Assume that the statement (i) holds. Then, there exist $\Psi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ and $\Gamma \in \mathcal{G}$ satisfying (24). We can choose these matrices as $\mathfrak{B}$-canonical two-variable polynomial matrices from Lemma A.2. This concludes the claim.

(ii)⇒(iii) Assume that there exist $\Psi(\zeta,\eta)$ and $\Gamma(\zeta,\eta)$ satisfying the statement (ii). Since $\Upsilon(\zeta,\eta)$ is $\mathfrak{B}$-canonical, $\mathsf{Q}_\Upsilon(w)$ is expressed as

$$\mathsf{Q}_\Upsilon(w) = z^*\widetilde{\Upsilon}z,\ \widetilde{\Upsilon} \in \mathbb{H}^{(N+1)q\times(N+1)q},\ z := Z_N\left(\frac{d}{dt}\right)w$$

for $N \le \deg R - 1$. Then, we obtain

$$\int_{-\infty}^{+\infty}\mathsf{Q}_\Gamma(w)dt = \mathrm{tr}\left[\tau\widetilde{\Upsilon}\left\{-\tau\int_{-\infty}^{+\infty}\mathrm{He}\left((\dot{z}-j\varpi_1 z)\left(\dot{z}-j\varpi_2 z\right)^*\right]dt\right\}\right).$$

Since $\tau\widetilde{\Upsilon} \ge 0$ holds from (22) and Lemma 1, we have (26) for all $w \in \mathfrak{B}$ satisfying (B.3). This concludes the claim.

(iii)⇒(i) The proof is straightforward from Theorem 1. □

Appendix B.6 Proof of Theorem 2

Assume that the FFDI (19) holds. From Proposition 3, there exist unique $\mathfrak{B}$-canonical $\Psi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ and $\Upsilon \in \mathbb{H}^{q\times q}[\zeta,\eta]$ satisfying

$$\begin{aligned}\frac{d}{dt}\mathsf{Q}_\Psi(w) &+ \left(\varpi_-^2 - \varpi_+{}^2\right)\mathsf{Q}_\Upsilon(w) - \mathsf{Q}_\Upsilon(\dot{w}) + j\varpi_+\left\{\mathsf{L}_\Upsilon(\dot{w},w) + \mathsf{L}_\Upsilon(w,\dot{w})\right\}\\ &\le \mathsf{Q}_\Phi(w), \quad \forall\, w \in \mathfrak{B}.\end{aligned} \tag{B.9}$$

It follows from the $\mathfrak{B}$-canonicity of $\Psi(\zeta,\eta)$ that $\mathsf{Q}_\Psi(w)$ is expressed as

$$\begin{aligned}\mathsf{Q}_\Psi(w) &= \left\{Z_{L-1}\left(\frac{d}{dt}\right)w\right\}^*\widetilde{\Psi}Z_{L-1}\left(\frac{d}{dt}\right)w\\ &= \left\{Z_L\left(\frac{d}{dt}\right)w\right\}^*\begin{bmatrix}\widetilde{\Psi} & 0_{Lq\times q}\\ 0_{q\times Lq} & 0_{q\times q}\end{bmatrix}Z_L\left(\frac{d}{dt}\right)w\\ &= \left\{Z_L\left(\frac{d}{dt}\right)w\right\}^*\begin{bmatrix}I_{Lq} & 0_{Lq\times q}\end{bmatrix}^\top\widetilde{\Psi}\begin{bmatrix}I_{Lq} & 0_{Lq\times q}\end{bmatrix}Z_L\left(\frac{d}{dt}\right)w.\end{aligned}$$

Since $Z_L\left(\frac{d}{dt}\right)w = \widetilde{R}_\mathrm{e}^\perp k$ holds from (50), we get

$$\mathsf{Q}_\Psi(w) = k^*\left(\widetilde{R}_\mathrm{e}^\perp\right)^*\begin{bmatrix}I_{Lq} & 0_{Lq\times q}\end{bmatrix}^\top\widetilde{\Psi}\begin{bmatrix}I_{Lq} & 0_{Lq\times q}\end{bmatrix}\widetilde{R}_\mathrm{e}^\perp k. \tag{B.10}$$

Substituting (55) into (B.10), $\mathsf{Q}_\Psi(w)$ is rewritten by

$$\mathsf{Q}_\Psi(w) = k^*E^*\widetilde{\Psi}Ek.$$

Similarly, from the $\mathfrak{B}$-canonicity of $\Upsilon(\zeta,\eta)$ and $\Phi(\zeta,\eta)$, we obtain

$$\begin{aligned}
\mathsf{Q}_\Upsilon(w) &= k^* E^* \widetilde{\Upsilon} E k,\\
\mathsf{Q}_\Upsilon(\dot{w}) &= \dot{k}^* E^* \widetilde{\Upsilon} E \dot{k},\\
\mathsf{L}_\Upsilon(\dot{w}, w) &= \dot{k}^* E^* \widetilde{\Upsilon} E k,\\
\mathsf{Q}_\Phi(w) &= k^* \left(\widetilde{R}_{\mathrm{e}}^{\perp}\right)^* \begin{bmatrix} \widetilde{\Phi} & 0_{(L+1)q\times(L-K)q} \\ 0_{(L-K)q\times(L+1)q} & 0_{(L-K)q\times(L-K)q} \end{bmatrix} \widetilde{R}_{\mathrm{e}}^{\perp} k \\
&= k^* \Phi_0 k.
\end{aligned}$$

Hence, (B.9) is equivalently rewritten by

$$\begin{aligned}
\frac{d}{dt} k^* E^* \widetilde{\Psi} E k + \left(\varpi_-^2 - \varpi_+{}^2\right) k^* E^* \widetilde{\Upsilon} E k - \dot{k}^* E^* \widetilde{\Upsilon} E \dot{k} \\
+ j\varpi_+ \left(\dot{k}^* E^* \widetilde{\Upsilon} E k + w^* E^* \widetilde{\Upsilon} E \dot{k}\right) \le k^* \Phi_0 k
\end{aligned}$$

for all $k \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^d)$ such that (53). This is the case if and only if the following inequality holds from (53) and the product rule.

$$\begin{aligned}
k^* F^* \widetilde{\Psi} E k + k^* E^* \widetilde{\Psi} F k + \left(\varpi_-^2 - \varpi_+{}^2\right) k^* E^* \widetilde{\Upsilon} E k - \dot{k}^* E^* \widetilde{\Upsilon} E \dot{k} \\
+ j\varpi_+ \left(k^* F^* \widetilde{\Upsilon} E k + w^* E^* \widetilde{\Upsilon} F k\right) \le k^* \Phi_0 k.
\end{aligned}$$

Since the kernel representation (53) is trim from Lemma A.3, the above inequality is equivalent to the LMI (58). This completes the proof. □

References

[1] T. Cotroneo and J.C. Willems. An LMI Condition for Nonnegativity of a Quadratic Differential Form along a Behavior. In *Proceedings of the 14-th International Symposium on Mathematical Theory of Networks and Systems*, Perpignan, France, 2000.

[2] S. Hara and T. Iwasaki. Sum-of-squares decomposition via generalized kyp lemma. *IEEE Transactions on Automatic Control*, Vol. 54, pp. 1025–1029, 2009.

[3] T. Iwasaki and S. Hara. Generalized KYP Lemma: Unified Frequency Domain Inequalities With Design Applications. *IEEE Transactions on Automatic Control*, Vol. AC-50, pp. 41–59, January 2005.

[4] T. Iwasaki and S. Hara. Feedback control synthesis of multiple frequency domain specifications via generalized KYP lemma. *International Journal of Robust and Nonlinear Control*, Vol. 17, pp. 415–434, 2007.

[5] T. Iwasaki, S. Hara, and A.L. Fradkov. Time Domain Interpretations of Frequency Domain Inequalities on (Semi) Finite Ranges. *Systems and Control Letters*, Vol. 54, No. 7, pp. 681–691, 2005.

[6] T. Iwasaki, S. Hara, and H. Yamauchi. Dynamical System Design From a Control Perspective: Finite Frequency Positive-Realness Approach. *IEEE Transactions on Automatic Control*, Vol. AC-48, No. 8, pp. 1337–1354, 2003.

[7] T. Kailath. *Linear Systems*. Prentice-Hall, 1980.

[8] C. Kojima, P. Rapisarda, and K. Takaba. Canonical Forms for Polynomial and Quadratic Differential Operators. *Systems and Control Letters*, 2007.

[9] Mathworks. *Robust Control Toolbox: Users Guide*. (http://www.mathworks.com/access/helpdesk/toolbox/robust/robust.shtml).

[10] A. Megretski and A. Rantzer. System Analysis via Integral Quadratic Constraints. *IEEE Transactions on Automatic Control*, Vol. 42, No. 6, pp. 819–830, 1997.

[11] J.W. Polderman and J.C. Willems. *Introduction to Mathematical Systems Theory: A Behavioral Approach*. Springer-Verlag, 1998.

[12] Polyx. *The Polynomial Toolbox for MATLAB: Manual*, 1999. (`http://www.polyx.com/`).

[13] R. van der Geest and H.L. Trentelman. The Kalman-Yakubovich-Popov Lemma in a Behavioural Framework. *Systems and Control Letters*, Vol. 32, pp. 283–290, 1997.

[14] J.C. Willems. Least Squares Stationary Optimal Control and the Algebraic Riccati Equation. *IEEE Transactions on Automatic Control*, Vol. AC-16, No. 6, pp. 621–634, 1971.

[15] J.C. Willems. Paradigms and Puzzles in the Theory of Dynamical Systems. *IEEE Transactions on Automatic Control*, Vol. AC-36, No. 11, pp. 259–294, 1991.

[16] J.C. Willems and H.L. Trentelman. On Quadratic Differential Forms. *SIAM Journal on Control and Optimization*, Vol. 36, No. 5, pp. 1703–1749, 1998.