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Takuya IIMURA, Kazuo MUROTA, and
Akihisa TAMURA

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DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

WWW page: http://www.keisu.t.u-tokyo.ac.jp/research/techrep/index.html

Sperner's Lemma and the Existence of Zero on the Discrete Simplex and Simplotope*

Takuya IIMURA† Kazuo MUROTA‡ Akihisa TAMURA§

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Abstract

In this paper we show a zero point theorem for a certain meaningful class of correspondences on a discrete simplex, which is equivalent to Sperner's lemma [*Abh. Math. Sem. Univ. Hamburg* 6 (1928) 265]. Also, we show a zero point theorem for correspondences on a discrete simplotope, which is derived from a Sperner-like theorem on the simplotope by van der Laan and Talman [*Math. Oper. Res.* 7 (1982) 1] and Freund [*Math. Oper. Res.* 11 (1986) 169]. The two discrete zero point theorems are closely related to the discrete fixed point theorem of Iimura, Murota and Tamura [*J. Math. Econ.* 41 (2005) 1030]. We also provide applications of the two theorems to economic and game models.

Keywords: Sperner's lemma, zero point, fixed point, discrete set

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JEL classification: C650, C620, C720

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†School of Business Administration, Tokyo Metropolitan University, Tokyo 192-0397, Japan, E-mail: `t.iimura@tmu.ac.jp` (T. Iimura).

‡Department of Mathematical Informatics, Graduate School of Information Science and Technology, University of Tokyo, Tokyo 113-8656, Japan. E-mail: `murota@mist.i.u-tokyo.ac.jp` (K. Murota).

§Department of Mathematics, Keio University, Yokohama 223-8522, Japan, E-mail: `aki-tamura@math.keio.ac.jp` (A. Tamura).

1 Introduction

Given a simplex and its triangulation, Sperner's lemma [8], to be stated as Theorem 2.1, says that if every vertex of the triangulation is properly labeled then there exists at least one (actually an odd number of) completely labeled subsimplex. The lemma gives a proof of Brouwer's fixed point theorem [2] (see e.g. [1]), which in turn implies the lemma (see [10]). In this sense we can say that Sperner's lemma and Brouwer's fixed point theorem are equivalent.

In this paper we show a zero point theorem for a certain (meaningful) class of correspondences on the set of vertices of the standard triangulation of unit simplex, which is equivalent to Sperner's lemma. Also, we show a zero point theorem for correspondences on the set of vertices of the standard triangulation of the direct product of unit simplices, which is derived from a Sperner-like theorem on the simplotope by van der Laan and Talman [7] and Freund [4]. Note that stated in the form of "$\forall \implies \exists$" the contraposition of an existence lemma is also an existential statement. Loosely speaking, the two zero point theorems on the simplex and simplotope are the contrapositions of Sperner's lemma on the simplex and simplotope, respectively. The two discrete zero point theorems are also closely related to the discrete fixed point theorem for direction preserving correspondences in [5], with a new, weaker, "simplexwise" version of direction preserving property.

In Section 2, we give some definitions and lemmas concerning simplices and Sperner's lemma. In Section 3, we define our correspondence, and establish the zero point theorems for correspondences on the discrete simplex and simplotope. Section 4 gives applications of our results to economic and game models, and Section 5 gives some concluding comments.

2 Definitions and lemmas

Let $\mathbf{R}^d$ be the d-dimensional Euclidean space and $\mathbf{Z}^d$ the set of integer points in $\mathbf{R}^d$. The notation $\mathbf{0}$ denotes the vectors of all zeros. The notation e^i is the ith unit vector whose jth component e^i_j is one if $j = i$ and zero if $j \neq i$. For any vectors x and y in $\mathbf{R}^d$ we denote by $x \cdot y$ the inner product of x and y. For a set X, we denote by $\operatorname{conv} X$ the convex hull of X. If X is a set of d affinely independent points then $\operatorname{conv} X$ is called a $(d-1)$-dimensional simplex. A triangulation $\mathcal{T}$ of a convex set S is a finite collection of simplices satisfying (i) $S = \bigcup_{T \in \mathcal{T}} T$, (ii) if T' is a face of $T \in \mathcal{T}$ then $T' \in \mathcal{T}$, and (iii) for $T_1, T_2 \in \mathcal{T}$ with $T_1 \cap T_2 \neq \emptyset$, $T_1 \cap T_2$ is a face of T_1 and T_2.

The set $\operatorname{conv}\{e^1, \cdots, e^d\}$ is called the $(d-1)$-dimensional unit simplex, where we assume $d > 1$ to avoid trivialities. Given any positive integer m, it admits a standard triangulation such that the components of its vertices are nonnegative integer multiples of $1/m$ that sum to one (Kuhn's regular triangulation [6]). To obtain the integrality of the triangulation, we consider its m multiple. Fix m to some positive integer, and let $S^{d-1} = \mathrm{conv}\{me^1, \cdots, me^d\}$. For brevity, let $S = S^{d-1}$. Then S admits an integral triangulation $\mathcal{T}$, any of whose $(d-1)$-dimensional element (subsimplex) $T = \mathrm{conv}\{x^1, \cdots, x^d\}$ is such that $x^1 \in \mathbf{Z}^d$ and $x^{t+1} = x^t + e^{i_{t+1}} - e^{i_t} \in \mathbf{Z}^d$, $t = 1, \cdots, d-1$, where $(i_t \mid t = 1, \cdots, d-1)$ is a permutation of $(1, \cdots, d-1)$. Then $T^0 = T \cap \mathbf{Z}^d$ is the set of vertices of T and $S^0 = S \cap \mathbf{Z}^d$ is the set of vertices of $\mathcal{T}$. We call this integral triangulation $\mathcal{T}$ the *standard triangulation of the simplex S*. Let $\lambda\colon S^0 \to \{1, \cdots, d\}$ be a function. We call λ a *labeling* of S^0.

Definition 2.1. A vertex $x \in S^0$ is *properly labeled* by λ if $x_i = 0$ implies $\lambda(x) \neq i$. A subsimplex T in $\mathcal{T}$ is *completely labeled* by λ if the set of labels of the vertices is $\{1, \cdots, d\}$, i.e., if $\lambda(T^0) = \{1, \cdots, d\}$, where $\lambda(T^0) = \{\lambda(x) \mid x \in T^0\}$.

Theorem 2.1 (Sperner's lemma [8])**.** *Let S^0 be the set of vertices of the standard triangulation $\mathcal{T}$ of the simplex S. If every vertex in S^0 is properly labeled by λ, then there exists a subsimplex in $\mathcal{T}$ completely labeled by λ.*

The direct product of simplices is called a *simplotope*. Let $S = S^{d_1-1} \times \cdots \times S^{d_n-1}$, where $S^{d_i-1} = \mathrm{conv}\{m_i e^{i,1}, \cdots, m_i e^{i,d_i}\} \subset \mathbf{R}^{d_i}$, in which d_i and m_i are some given positive integers ($d_i > 1$) and $e^{i,j}$ is the jth unit vector in $\mathbf{R}^{d_i}$, $j = 1, \cdots, d_i$, $i = 1, \cdots, n$. The simplotope $S \subset \mathbf{R}^{d_1+\cdots+d_n}$ has $\prod_{i=1}^n d_i$ vertices and dimension $\sum_{i=1}^n (d_i - 1)$. Let d be the number of the vertices of $\sum_{i=1}^n (d_i-1)$-dimensional simplex, i.e., $d = (\sum_{i=1}^n (d_i-1)) + 1$. Let $\tilde{e}^{i,j}$ be a unit vector in $\mathbf{R}^{d_1+\cdots+d_n}$, given by a concatenation of $i-1$ zero vectors in $\mathbf{R}^{d_k}$ $(k = 1, \cdots, i-1)$, the jth unit vector $e^{i,j}$ in $\mathbf{R}^{d_i}$, and $n-i$ zero vectors in $\mathbf{R}^{d_k}$ $(k = i+1, \cdots, n)$, in this order. We regard any vector $x \in \mathbf{R}^{d_1+\cdots+d_n}$ as such a concatenation of n vectors in $\mathbf{R}^{d_i}$, $i = 1, \cdots, n$, and denote the jth component of the ith subvector of x by $x_{i,j}$. In this notation $\tilde{e}^{i,j}$ is a vector such that $\tilde{e}^{i,j}_{i,j} = 1$ with all other components being zeros.

Let $S^0 = S \cap \mathbf{Z}^{d_1+\cdots+d_n}$. The simplotope S admits an integral triangulation $\mathcal{T}$ similar to those of component simplices, any of whose $(d-1)$-dimensional subsimplex $T = \mathrm{conv}\{x^1, \cdots, x^d\}$ is such that $x^1 \in S^0$ and $x^{t+1} = x^t + \tilde{e}^{i_t, j_t+1} - \tilde{e}^{i_t, j_t} \in S^0$, $t = 1, \cdots, d-1$, where $((i_t, j_t) \mid t = 1, \cdots, d-1)$ is a permutation of $((1,1), (1,2), \cdots, (1, d_1 - 1); (2,1), (2,2), \cdots, (2, d_2-1); \cdots; (n,1), (n,2), \cdots, (n, d_n - 1))$ (see [3]). We call this integral triangulation $\mathcal{T}$ the *standard triangulation of the simplotope* S. Let $\lambda^i\colon S^0 \to \{(i,1),\cdots,(i,d_i)\}$ be a function, $i=1,\cdots,n$. We also call λ^i a labeling of S^0.

Definition 2.2. Let $i \in \{1,\cdots,n\}$. A vertex $x \in S^0$ is *properly labeled* by λ^i if $x_{i,j}=0$ implies $\lambda^i(x) \neq (i,j)$. A subsimplex T in $\mathcal{T}$ is *completely labeled* by λ^i if $\lambda^i(T^0) = \{(i,1),\cdots,(i,d_i)\}$, where $\lambda^i(T^0)=\{\lambda^i(x) \mid x \in T^0\}$.

In [7] and [4], a Sperner-like theorem (Theorem 2.2 below) is proved using a slightly different labeling function $L\colon S^0 \to \bigcup_{i=1}^n \{(i,1),\cdots,(i,d_i)\}$. Recall that $x_{i,k}$ denotes the kth component of the ith subvector of $x \in S^0$.

Theorem 2.2 (van der Laan and Talman [7], Freund [4, Theorem 3])**.** *Let* $L\colon S^0 \to \bigcup_{i=1}^n \{(i,1),\cdots,(i,d_i)\}$. *If every* x *in* S^0 *is properly labeled by* L *in the sense that* $x_{i,k}=0$ *implies* $L(x) \neq (i,k)$, $k=1,\cdots,d_i$, $i=1,\cdots,n$, *then there exists a subsimplex* T *in* $\mathcal{T}$ *such that* $L(T^0)=\{(j,1),\cdots,(j,d_j)\}$ *for some* $j \in \{1,\cdots,n\}$.

3 The main results

3.1 A zero point theorem on the discrete simplex

Let $S=\mathrm{conv}\{me^1,\cdots,me^d\} \subset \mathbf{R}^d$, $S^0 = S \cap \mathbf{Z}^d$, and $\mathcal{T}$ the standard triangulation of the simplex S. For any $T \in \mathcal{T}$, $T^0 = T \cap \mathbf{Z}^d$ is the set of vertices of T. We are concerned with a correspondence (set-valued function) $\Delta\colon S^0 \to\to \{e^j - e^i \in \mathbf{Z}^d \mid i,j=1,\cdots,d\}$ (possibly $i=j$), which is "pointwise one-directional" and "simplexwise direction preserving" in the following sense.

Definition 3.1. A correspondence $\Delta\colon S^0 \to\to \{e^j - e^i \in \mathbf{Z}^d \mid i,j=1,\cdots,d\}$ is (negatively) *pointwise one-directional* if, for each $x \in S^0$ such that $\Delta(x)\setminus\{\mathbf{0}\} \neq \emptyset$, there exists one and only one $i_x \in \{1,\cdots,d\}$ such that $\delta_{i_x} < 0$ for all $\delta \in \Delta(x)\setminus\{\mathbf{0}\}$ (i.e. $\delta = e^j - e^{i_x}$ for some j if $\delta \in \Delta(x)$ and $\delta \neq \mathbf{0}$); it is *simplexwise direction preserving on* $\mathcal{T}$ if, for each $(d-1)$-dimensional subsimplex T in $\mathcal{T}$, there exists a family $\mathcal{D}_T$ of $\delta(x) \in \Delta(x)$ indexed by $x \in T^0$, i.e., $\mathcal{D}_T = (\delta(x) \in \Delta(x) \mid x \in T^0)$, such that $\delta_i(x)\delta_i(x') \geq 0$ for all $i \in \{1,\cdots,d\}$ for any x and x' in T^0.

Remark 3.1. In the definition of simplexwise direction preservingness we may choose different $\delta(x) \in \Delta(x)$ for $\mathcal{D}_T$ for different $T \ni x$. As a special case of singleton-valued Δ (i.e. if Δ is a function), the condition reduces to the "simplicial local direction preserving" condition of [9]. Also we remark that $\delta(x)\cdot\delta(x') \geq 0$ is equivalent to $\delta_i(x)\delta_i(x') \geq 0$ for all $i=1,\cdots,d$ by the form of $\delta(x)$ and $\delta(x')$ (somewhat similarly to [5]).

Now, let us denote $\{x\}+\Delta(x)$ by $x+\Delta(x)$, for short. We claim that the next theorem holds, and is equivalent to Sperner's lemma (Theorem 2.1).

Theorem 3.1. *Let S^0 be the set of vertices of the standard triangulation $\mathcal{T}$ of the simplex S. If $\Delta\colon S^0 \to\!\!\to \{e^j - e^i \in \mathbf{Z}^d \mid i,j = 1,\cdots,d\}$ is pointwise one-directional and simplexwise direction preserving on $\mathcal{T}$, then there exists an $x \in S^0$ such that $\mathbf{0} \in \Delta(x)$ or $x + \Delta(x) \not\subseteq S^0$. (Hence $\mathbf{0} \in \Delta(x)$ if Δ points inward at the boundary of S.)*

Proof of [Sperner's lemma $\Longrightarrow$ Theorem 3.1]. Given the correspondence Δ of the theorem, define a function $\lambda_\Delta\colon S^0 \to \{1,\cdots,d,0\}$ by $\lambda_\Delta(x) = 0$ if $\mathbf{0} \in \Delta(x)$, and $\lambda_\Delta(x) = i_x$ if $\mathbf{0} \notin \Delta(x)$ and $\delta_{i_x} < 0$ for all $\delta \in \Delta(x)$, for each $x \in S^0$. If $\lambda_\Delta(x) = 0$ for some $x \in S^0$ then $\mathbf{0} \in \Delta(x)$ and we are done. So assume in the following that $\lambda_\Delta(x) \neq 0$ for every $x \in S^0$. Then $\lambda_\Delta(T^0) \neq \{1,\cdots,d\}$ for every $T \in \mathcal{T}$. To see this, suppose $\lambda_\Delta(T^0) = \{1,\cdots,d\}$ for some $T \in \mathcal{T}$, and let $x^i \in T^0$ be such that $\lambda_\Delta(x^i) = i$, $i = 1,\cdots,d$. Then every δ in $\Delta(x^i)$ is written as $\delta = e^j - e^i$ for some $j \in \{1,\cdots,d\} \setminus \{i\}$. In particular, for each $\delta = e^j - e^1 \in \Delta(x^1)$, $\delta \cdot \delta' < 0$ for all δ' in $\Delta(x^j)$ since δ' is written as $\delta' = e^h - e^j$ ($h \in \{1,\cdots,d\} \setminus \{j\}$). This contradicts the simplexwise direction preservingness of Δ. Hence $\lambda_\Delta(T^0) \neq \{1,\cdots,d\}$ for every $T \in \mathcal{T}$. Since λ_Δ is a labeling of S^0 and every T in $\mathcal{T}$ is incompletely labeled by λ_Δ, the contraposition of Sperner's lemma then says that there is an x in S^0 improperly labeled by λ_Δ, i.e., $x \in S^0$ such that $x_i = 0$ and $\lambda_\Delta(x) = i$ for some i. Then we have $x + \Delta(x) \not\subseteq S^0$. □

Proof of [Theorem 3.1 $\Longrightarrow$ Sperner's lemma]. Assume that every $(d-1)$-dimensional subsimplex T of $\mathcal{T}$ is incompletely labeled by $\lambda\colon S^0 \to \{1,\cdots,d\}$, and define $\Delta_\lambda\colon S^0 \to\!\!\to \{e^j - e^i \in \mathbf{Z}^d \mid i,j = 1,\cdots,d\}$ by $\Delta_\lambda(x) = \{e^j - e^{\lambda(x)} \mid j \neq \lambda(x)\ (j = 1,\cdots,d)\}$, for each $x \in S^0$ (the cardinality of $\Delta_\lambda(x)$ is $d-1$). Then Δ_λ is clearly pointwise one-directional. It is also simplexwise direction preserving, since, for $\mu(T) \in \{1,\cdots,d\} \setminus \lambda(T^0)$ (a missing label of T), we can take $\mathcal{D}_T = (\delta(x) = e^{\mu(T)} - e^{\lambda(x)} \mid x \in T^0)$ to let $\delta_i(x)\delta_i(x') \geq 0$ hold for all $i \in \{1,\cdots,d\}$ for any x and x' in T^0. Hence there is an x such that $x+\Delta_\lambda(x) \not\subseteq S^0$ by Theorem 3.1. Since $x + \Delta_\lambda(x) \not\subseteq S^0$ if and only if $x_i = 0$ and $\lambda(x) = i$ for some i, we can conclude that there exists an x in S^0 improperly labeled by λ, which proves the lemma. □

Given a pointwise one-directional $\Delta\colon S^0 \to\!\!\to \{e^j - e^i \in \mathbf{Z}^d \mid i,j = 1,\cdots,d\}$ (possibly

$i = j$), define a function $\lambda_\Delta \colon S^0 \to \{1, \cdots, d, 0\}$ by

$$\lambda_\Delta(x) = \begin{cases} 0 & \text{if } \mathbf{0} \in \Delta(x), \text{ and} \\ i_x & \text{if } \mathbf{0} \notin \Delta(x) \text{ and } \delta_{i_x} < 0 \text{ for all } \delta \in \Delta(x). \end{cases} \tag{1}$$

Also, given a $\lambda \colon S^0 \to \{1, \cdots, d, 0\}$, define a pointwise one-directional $\Delta_\lambda \colon S^0 \to\to \{e^j - e^i \in \mathbf{Z}^d \mid i, j = 1, \cdots, d\}$ by

$$\Delta_\lambda(x) = \begin{cases} \{\mathbf{0}\} & \text{if } \lambda(x) = 0, \text{ and} \\ \{e^j - e^{\lambda(x)} \mid j \neq \lambda(x) \ (j = 1, \cdots, d)\} & \text{otherwise.} \end{cases} \tag{2}$$

Then the set of λ_Δ and the set of Δ_λ are one-to-one each other. If we classify $x \in S^0$ such that $\lambda_\Delta(x) = 0$ also as "improperly labeled" by λ_Δ, and continue to call $(d-1)$-dimensional $T \in \mathcal{T}$ such that $\lambda_\Delta(T^0) \neq \{1, \cdots, d\}$ "incompletely labeled" by λ_Δ, then the equivalence established above may be summarized as in Figure 1. Theorem 3.1 is then of the form of the contraposition of Sperner's lemma.

	Sperner's lemma	
$\forall$ vertices (properly labeled)	$\Longrightarrow$	$\exists$ subsimplex (completely labeled)
	(contraposition of Sperner's lemma)	
$\exists x$[improperly labeled by λ_Δ]	$\Longleftarrow$	$\forall T$[incompletely labeled by λ_Δ]
$\Updownarrow$	*Theorem 3.1*	$\Updownarrow$
$\exists x[\mathbf{0} \in \Delta_\lambda(x) \mid x + \Delta_\lambda(x) \not\subseteq S^0]$	$\Longleftarrow$	Δ_λ pointwise 1-d & simplexwise dp

Figure 1: The relationship between Sperner's lemma and Theorem 3.1

3.2 A zero point theorem on the discrete simplotope

Let $S^{d_i-1} = \text{conv}\{m_i e^{i,1}, \cdots, m_i e^{i,d_i}\} \subset \mathbf{R}^{d_i}$, $i = 1, \cdots, n$, $S = S^{d_1-1} \times \cdots \times S^{d_n-1}$, $S^0 = S \cap \mathbf{Z}^{d_1+\cdots+d_n}$, and let $\mathcal{T}$ be the standard triangulation of the simplotope S. For any $T \in \mathcal{T}$, $T^0 = T \cap \mathbf{Z}^{d_1+\cdots+d_n}$ is the set of vertices of T. Let $\Delta^i \colon S^0 \to\to \{e^{i,k} - e^{i,j} \in \mathbf{Z}^{d_i} \mid j, k = 1, \cdots, d_i\}$ (possibly $j = k$), $i = 1, \cdots, n$. The pointwise one-directional property and the simplexwise direction preserving property of Δ^i are similarly defined as follows. Let $d = (\sum_{i=1}^n (d_i - 1)) + 1$.

Definition 3.2. A correspondence $\Delta^i \colon S^0 \to\to \{e^{i,k} - e^{i,j} \in \mathbf{Z}^{d_i} \mid j, k = 1, \cdots, d_i\}$ is (negatively) *pointwise one-directional* if, for each $x \in S^0$ such that $\Delta^i(x) \setminus \{\mathbf{0}\} \neq \emptyset$, there exists one and only one $j_x \in \{1, \cdots, d_i\}$ such that $\delta^i_{j_x} < 0$ for all $\delta^i \in \Delta^i(x) \setminus \{\mathbf{0}\}$

(i.e. $\delta^i = e^{i,k} - e^{i,j_x}$ for some k if $\delta^i \in \Delta^i(x)$ and $\delta^i \neq \mathbf{0}$); it is *simplexwise direction preserving on* $\mathcal{T}$ if, for each $(d-1)$-dimensional subsimplex T in $\mathcal{T}$, there exists a family $\mathcal{D}^i_T$ of $\delta^i(x) \in \Delta^i(x)$ indexed by $x \in T^0$, i.e., $\mathcal{D}^i_T = (\delta^i(x) \in \Delta^i(x) \mid x \in T^0)$, such that $\delta^i_j(x)\delta^i_j(x') \geq 0$ for all $j \in \{1, \cdots, d_i\}$ for any x and x' in T^0.

We claim that the next theorem is derived from the Sperner-like theorem (Theorem 2.2).

Theorem 3.2. *Let S^0 be the set of vertices of the standard triangulation $\mathcal{T}$ of the simplotope S. If $\Delta^i \colon S^0 \to\to \{e^{i,k} - e^{i,j} \mid j, k = 1, \cdots, d_i\}$ is pointwise one-directional and simplexwise direction preserving on $\mathcal{T}$ for all $i = 1, \cdots, n$, then, for $\Delta \colon S^0 \to\to \mathbf{Z}^{d_1+\cdots+d_n}$ defined by $\Delta(x) = \{(\delta^1, \cdots, \delta^n) \mid \delta^i \in \Delta^i(x),\ i = 1, \cdots, n\}$, there exists an $x \in S^0$ such that $\mathbf{0} \in \Delta(x)$ or $x + \Delta(x) \not\subseteq S^0$. (Hence $\mathbf{0} \in \Delta(x)$ if Δ points inward at the boundary of S.)*

Proof. Given the correspondences Δ^i of the theorem, define $\lambda^i_\Delta \colon S^0 \to \{(i,1), \cdots, (i,d_i), (i,0)\}$ by $\lambda^i_\Delta(x) = (i,0)$ if $\mathbf{0} \in \Delta^i(x)$, and $\lambda^i_\Delta(x) = (i, j_x)$ if $\mathbf{0} \notin \Delta^i(x)$ and $\delta^i_{j_x} < 0$ for all $\delta^i \in \Delta^i(x)$, for each $x \in S^0$, for every $i = 1, \cdots, n$. If $\lambda^i_\Delta(x) = (i,0)$ for all $i = 1, \cdots, n$ for some $x \in S^0$ then $\mathbf{0} \in \Delta(x)$ and we are done. So assume in the following that $\lambda^i_\Delta(x) \neq (i,0)$ for some $i \in \{1, \cdots, n\}$ for all $x \in S^0$. Then $\lambda^i_\Delta(T^0) \neq \{(i,1), \cdots, (i,d_i)\}$ for every $T \in \mathcal{T}$ due to the simplexwise direction preservingness of Δ^i, for every $i = 1, \cdots, n$ (the proof is similar to a part of *Proof of [Sperner's lemma $\Longrightarrow$ Theorem 3.1]*). Define $L_\Delta \colon S^0 \to \bigcup_{i=1}^n \{(i,1), \cdots, (i,d_i)\}$ by $L_\Delta(x) = \lambda^i_\Delta(x)$ for each $x \in S^0$ with some $i \in \{1, \cdots, n\}$ such that $\lambda^i_\Delta(x) \neq (i,0)$. Then L_Δ is a labeling function of S^0 that appears in the Sperner-like theorem (Theorem 2.2), and $L_\Delta(T^0) \neq \{(i,1), \cdots, (i,d_i)\}$ for all $i = 1, \cdots, n$ for every $T \in \mathcal{T}$, since otherwise $\lambda^i(T^0) = \{(i,1), \cdots, (i,d_i)\}$ for some $i \in \{1, \cdots, n\}$ and $T \in \mathcal{T}$ (here T^0 is the set vertices of T and $L_\Delta(T^0) = \{L_\Delta(x) \mid x \in T^0\}$). The contraposition of the Sperner-like theorem then says that there is an x in S^0 improperly labeled by L_Δ, i.e., $x \in S^0$ such that $x_{i,j} = 0$ and $\lambda^i_\Delta(x) = (i,j)$ for some $i \in \{1, \cdots, n\}$ and $j \in \{1, \cdots, d_i\}$. Then we have $x + \Delta(x) \not\subseteq S^0$. □

Observe that the Sperner-like theorem is summarized as

$$\forall x[\exists i[x \text{ is properly labeled by } \lambda^i]] \Longrightarrow \exists T[\exists i[T \text{ is completely labeled by } \lambda^i]],$$

using a set of labelings $\lambda^i \colon S^0 \to \{(i,1), \cdots, (i,d_i)\}$ $(i = 1, \cdots, n)$ satisfying $L(x) = \lambda^i(x)$ for every $x \in S^0$ with some $i \in \{1, \cdots, n\}$ such that $\lambda^i(x) \neq (i,0)$, instead of $L \colon S^0 \to$

$\bigcup_{i=1}^{n}\{(i,1),\cdots,(i,d_i)\}$. Figure 2 shows the relationship between the Sperner-like theorem and Theorem 3.2. Note that x with $\lambda^i_\Delta(x) = (i,0)$ is classified as "improperly labeled" therein.

	Sperner-like theorem	
$\forall x[\exists i[x \text{ is properly labeled by } \lambda^i]]$	$\Longrightarrow$	$\exists T[\exists i[T \text{ is completely labeled by } \lambda^i]]$
	(contraposition of Sperner-like theorem)	
$\exists x[\forall i[x \text{ is improperly labeled by } \lambda^i_\Delta]]$	$\Longleftarrow$	$\forall T[\forall i[T \text{ is incompletely labeled by } \lambda^i_\Delta]]$
$\Downarrow$	*Theorem 3.2*	$\Uparrow$
$\exists x[\mathbf{0} \in \Delta(x) \mid x + \Delta(x) \not\subseteq S^0]$	$\Longleftarrow$	$\forall i[\Delta^i \text{ pointwise 1-d \& simplexwise dp}]$

Figure 2: The relationship between Theorem 2.2 (Sperner-like theorem) and Theorem 3.2

4 Applications

In this section we give the applications of our results, which are the simple adaptations of Theorems 3.1 and 3.2 in some specific context, respectively.

(1) Walrasian equilibrium in integer prices

Let $\mathbf{Z}^d_+$ be the nonnegative orthant of $\mathbf{Z}^d$ and $\zeta\colon \mathbf{Z}^d_+ \setminus \{\mathbf{0}\} \to\!\to \mathbf{Z}^d$ a nonempty-valued correspondence satisfying (i) $\zeta(tp) = \zeta(p)$ if $p, tp \in \mathbf{Z}^d_+ \setminus \{\mathbf{0}\}$ (homogeneous of degree zero), (ii) $p \cdot z \leq 0$ for all $z \in \zeta(p)$ and all $p \in \mathbf{Z}^d_+ \setminus \{\mathbf{0}\}$ (weak Walras's law), and (iii) $z_i > 0$ if $z \in \zeta(p)$ and $p_i = 0$ (boundary condition). We call ζ an (aggregate) *excess demand correspondence* that assigns for each nonnegative nonzero integer price vector $p \in \mathbf{Z}^d_+ \setminus \{\mathbf{0}\}$ a nonempty set $\zeta(p)$ of integer excess demand vectors $z \in \mathbf{Z}^d$. By the zero-homogeneity, the set $S^0 = S \cap \mathbf{Z}^d$ with $S = \mathrm{conv}\{me^1, \cdots, me^d\}$ is a natural subset of integer price vectors, for which we consider the existence of a Walrasian equilibrium price vector $p^* \in S^0$ such that $\mathbf{0} \in \zeta(p^*)$.

For $p \in S^0$ such that $p_k > 0$ for all $k = 1, \cdots, d$, the weak Walras's law implies that $z_i < 0$ for at least one i if $z \in \zeta(p)$ and $z \neq \mathbf{0}$, so assign $i_p \in \{1, \cdots, d\}$ such that $z_{i_p} < 0$ for some $z \in \zeta(p)$ if $\mathbf{0} \notin \zeta(p)$; assign 0 if $\mathbf{0} \in \zeta(p)$. For $p \in S^0$ such that $p_k = 0$ for some k, the boundary condition implies that $p_i > 0$ if $z_i \leq 0$, so assign $i_p \in \{1, \cdots, d\}$ such that $z_{i_p} \leq 0$ for some $z \in \zeta(p)$ if $\mathbf{0} \notin \zeta(p)$; assign 0 if $\mathbf{0} \in \zeta(p)$. The assigned integer i_p represents a good whose price is to be lowered at p. Let $\mathcal{T}$ be the standard triangulation of the simplex S.

Proposition 4.1. *Suppose that, for every $(d-1)$-dimensional $T \in \mathcal{T}$, the goods whose prices are to be lowered at the vertices of T are not completely different. Then there exists a Walrasian equilibrium price vector.*

Proof. Let $\lambda \colon S^0 \to \{1, \cdots, d, 0\}$ be the assignment rule. Then $\lambda(T^0) \neq \{1, \cdots, d\}$ by the assumption, where $T^0 = T \cap \mathbf{Z}^d$, the set of vertices of T. Let $\mu(T) \in \{1, \cdots, d\} \setminus \lambda(T^0)$ for each $T \in \mathcal{T}$. Then $\Delta \colon S^0 \to\!\!\to \{e^j - e^i \mid i, j = 1, \cdots, d\}$ defined by $\Delta(p) = \{e^{\mu(T)} - e^{\lambda(p)} \mid T^0 \text{ contains } p\}$ if $\lambda(p) \neq 0$ and $\Delta(p) = \{\mathbf{0}\}$ otherwise is a price adjustment correspondence that is pointwise one-directional and simplexwise direction preserving. Also Δ points inward at $p \in S^0$ on the boundary of S due to the boundary condition of ζ. Hence there exists a $p^* \in S^0$ such that $\mathbf{0} \in \Delta(p^*)$ by Theorem 3.1, which is a Walrasian equilibrium price vector since $\mathbf{0} \in \Delta(p^*) \iff \lambda(p^*) = 0 \iff \mathbf{0} \in \zeta(p^*)$. □

(2) Nash equilibrium in rational mixed strategies

It is known that every finite n-person game has a Nash equilibrium point in mixed strategies. It is not known, however, when it is obtained as an n-tuple of rational mixed strategies. We address this issue here.

Let $d_i > 1$ be the number of pure strategies of player i, $i = 1, \cdots, n$. We identify the kth unit vector $e^{i,k}$ in $\mathbf{R}^{d_i}$ with the kth pure strategy of player i. Then the set of (mixed) strategies of player i is $\mathrm{conv}\{e^{i,1}, \cdots, e^{i,d_i}\}$, the $(d_i - 1)$-dimensional unit simplex. Given the payoff $\pi_i(e^{1,k_1}, \cdots, e^{n,k_n}) \in \mathbf{R}$ of player i for each pure strategy profile $(e^{1,k_1}, \cdots, e^{n,k_n})$, the payoff $P_i(s)$ of player i for each strategy profile $s = (s^1, \cdots, s^n)$, $s^j \in \mathrm{conv}\{e^{j,1}, \cdots, e^{j,d_j}\}$ for each $j = 1, \cdots, n$, is $P_i(s) = \sum_{k_1=1}^{d_1} \cdots \sum_{k_n=1}^{d_n} s^1_{k_1} \cdots s^n_{k_n} \pi_i(e^{1,k_1}, \cdots, e^{n,k_n})$. We denote by $P_i(s \setminus t^i)$ the payoff of player i when the strategy t^i is used by player i in the profile s.

Let $S^{d_i-1} = \mathrm{conv}\{m_i e^{i,1}, \cdots, m_i e^{i,d_i}\}$ with some positive integer m_i, $i = 1, \cdots, n$, and $S = S^{d_1-1} \times \cdots \times S^{d_n-1}$. We call $S^0 = S \cap \mathbf{Z}^{d_1+\cdots+d_n}$ the *discretized strategy profile space*. We identify $x = (x^1, \cdots, x^n) \in S^0$ with the n-tuple of rational vectors of strategy $s = (x^1/m_1, \cdots, x^n/m_n)$, for which we put $s = x/m$, for brevity. Then $P_i(x/m) = \sum_{k_1=1}^{d_1} \cdots \sum_{k_n=1}^{d_n} (x^1_{k_1}/m_1) \cdots (x^n_{k_n}/m_n) \pi_i(e^{1,k_1}, \cdots, e^{n,k_n})$. Let $\beta^i \colon S^0 \to\!\!\to S^{d_i-1} \cap \mathbf{Z}^{d_i}$ be the best reply correspondence of player i such that $y^i \in \beta^i(x)$ if and only if $P_i((x/m) \setminus (y^i/m_i)) \geq P_i((x/m) \setminus (z^i/m_i))$ for all $z^i \in S^{d_i-1} \cap \mathbf{Z}^{d_i}$. Then $\mathrm{conv}\, \beta^i(x)$ is a nonempty face of S^{d_i-1} due to the multilinear form of payoff function. We say that a pure strategy $e^{i,k}$ is *acceptable* for player i at x if $m_i e^{i,k}$ is a vertex of $\mathrm{conv}\, \beta^i(x)$. Define

$\beta\colon S^0 \to\!\!\to S^0$ by $\beta(x) = \beta^1(x)\times\cdots\times\beta^n(x)$ for each $x \in S^0$. A discretized strategy profile $x^* \in S^0$ is a Nash equilibrium point if $x^* \in \beta(x^*)$. Let $\mathcal{T}$ be the standard triangulation of the simplotope S, whose dimension is $d-1$ with $d = (\sum_{i=1}^n (d_i - 1)) + 1$.

Proposition 4.2. *Suppose that, for every $(d-1)$-dimensional $T \in \mathcal{T}$, there is at least one pure strategy for each player acceptable for him at all the vertices of T. Then there exists a Nash equilibrium point in the discretized strategy profile space.*

Proof. For each $(d-1)$-dimensional $T \in \mathcal{T}$ and player $i = 1, \cdots, n$, let $\mu^i(T) \in \{(i,1), \cdots, (i, d_i)\}$ be the index of an acceptable pure strategy for player i at T, which always exists due to the assumption. For each $x = (x^1, \cdots, x^n) \in S^0$ and player $i = 1, \cdots, n$, let $\lambda^i(x) = (i, k)$ if there is $e^{i,k}$ not acceptable for i at x and $x^i_k > 0$, and $\lambda^i(x) = (i, 0)$ otherwise (i.e. if all the pure strategies are acceptable or every unacceptable one is already "dropped" in that $x^i_k = 0$; $\lambda^i(x)$ signifies a pure strategy to be dropped, if any). Note that $\lambda^i(x) = (i, 0)$ if and only if $x^i \in \beta^i(x)$. Now, define $\Delta^i\colon S^0 \to\!\!\to \{e^{i,k} - e^{i,j} \mid j, k = 1, \cdots, d_i\}$ (possibly $j = k$) for each player $i = 1, \cdots, n$ by $\Delta^i(x) = \{\mathbf{0}\}$ if $\lambda^i(x) = (i, 0)$, and $\Delta^i(x) = \{e^{\mu^i(T)} - e^{\lambda^i(x)} \mid T^0 \text{ contains } x\}$ otherwise ($T^0 = T \cap \mathbf{Z}^{d_1+\cdots+d_n}$ is the set of vertices of T). Then every Δ^i is pointwise one-directional, simplexwise direction preserving, and $\Delta\colon S^0 \to\!\!\to S^0$ defined by $\Delta(x) = \Delta^1(x) \times \cdots \times \Delta^n(x)$ for each $x \in S^0$ satisfies $x + \Delta(x) \subseteq S^0$. Hence there exists an $x^* \in S^0$ such that $\mathbf{0} \in \Delta(x^*)$ by Theorem 3.2, which is a Nash equilibrium point since $\mathbf{0} \in \Delta(x^*) \iff \lambda^i(x^*) = (i, 0)$ $(\forall i = 1, \cdots, n) \iff x^* \in \beta(x^*)$. □

5 Concluding comments

Let S^0 be the set of vertices of the standard triangulation $\mathcal{T}$ of the simplex S. The zero point theorem on the simplex (Theorem 3.1) gives a discrete *fixed point* theorem for functions $f\colon S^0 \to S^0$ such that $f(x) = x + e^j - e^i$, with the condition that $(f_i(x) - x_i)(f_i(x') - x'_i) \geq 0$ for all $i = 1, \cdots, d$ (or equivalently $(f(x) - x)\cdot(f(x') - x') \geq 0$ in this case) for any x and x' in the same subsimplex of $\mathcal{T}$. The proof of the existence of an $x \in S^0$ such that $x = f(x)$ is immediate from Theorem 3.1 if we define $\Delta\colon S^0 \to\!\!\to \{e^j - e^i \mid i, j = 1, \cdots, d\}$ by $\Delta(x) = \{f(x) - x\}$. We note that this is a special case of the theorem in [5].

Theorem 3.1 also gives a proof of continuous fixed point theorem by Brouwer. If $S \subset \mathbf{R}^d$ is the unit simplex and $f\colon S \to S$ is continuous, the proof is direct and goes as follows (this is a slightly modified, typical proof of "Sperner $\Longrightarrow$ Brouwer"; see e.g. [1] for this type of proof of Brouwer's theorem). For each positive integer m, let $\mathcal{T}_m$ be the standard triangulation of S such that $S^0_m = S \cap (\frac{1}{m}\mathbf{Z}^d)$ is the set of vertices of $\mathcal{T}_m$. Define $\Delta^m \colon S^0_m \to\to \{\frac{1}{m}e^j - \frac{1}{m}e^i \mid i, j = 1, \cdots, d\}$ by $\Delta^m(x) = \{\frac{1}{m}e^j - \frac{1}{m}e^i \mid j \neq i \ (j = 1, \cdots, d)\}$ with an $i \in \{k \mid x_k > f_k(x)\}$ if $x \neq f(x)$, and $\Delta^m(x) = \{\mathbf{0}\}$ if $x = f(x)$, for each $x \in S^0_m$. Then, for each m, Δ^m is pointwise one-directional and $x + \Delta^m(x) \subseteq S^0_m$ for all $x \in S^0_m$. If Δ^m is also simplexwise direction preserving then $\mathbf{0} \in \Delta^m(x)$ for some $x \in S^0_m$. If not, there exists a completely labeled subsimplex, since Δ^m is simplexwise direction preserving if and only if all the subsimplices of $\mathcal{T}_m$ are incompletely labeled. Hence we have either a finite m such that $\mathbf{0} \in \Delta^m(x)$ for some $x = f(x) \in S^0_m$, or a sequence of completely labeled subsimplices converging to some $x = f(x) \in S$.

In this paper we started from a unit simplex $\{x \in \mathbf{R}^d \mid \mathbf{1} \cdot x = 1, \ x \geq \mathbf{0}\}$ (where $\mathbf{1}$ is the vector of all ones), and used the "negative" one-directional condition "$\delta_{i_x} < 0$ for all $\delta \in \Delta(x) \setminus \{\mathbf{0}\}$ for some i_x". A similar argument is possible, starting from a non-unit simplex $\{x \in \mathbf{R}^d \mid \mathbf{1} \cdot x = 0, \ x \leq \mathbf{1}\}$ ([6]) using a "positive" one-directional condition "$\delta_{i_x} > 0$ for all $\delta \in \Delta(x) \setminus \{\mathbf{0}\}$ for some i_x". The convention of sign is determined so as to obtain $x + \Delta(x) \subseteq S^0$ at the extreme points of the simplex S at hand. The same comment also applies to the theorem on the simplotope.

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