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Abstract

In this paper, we deal with a problem to capture a target by linear multi-agent systems where the agents behave autonomously, whereas the target escapes with a reasonable strategy. We give necessary and sufficient conditions or sufficient conditions for the success of the capturing for several cases. The conditions clarify the performance competition between the target and the agents and we propose preferable strategy for the target and the agents, respectively. We consider the problem in the case of simple or general dynamics of the target and the agents. We also demonstrate the results by using numerical simulation.

1 Introduction

In recent years, formation control composed of many agents such as a flock of robots, vehicles, aircrafts, artificial satellites or biological systems, has been actively investigated [2, 10, 11, 3, 4, 5, 6, 7]. As a topic of it, cooperative pursuit of an objective by many agents with their local information has been also discussed [9, 10, 11, 12, 13]. In particular, control method of cyclic pursuit, which models the behavior of biological systems such as a flock of birds or fish was proposed in [9, 10, 11]. This kind of research is motivated by engineering sense for applications and also by scientific interest to clarify the behaviour of biological systems. Examples of applications are found in [10, 11, 13], and search or rescue in dangerous area is a possible situation.

In the research area of pursuit or capturing, Kim and Sugie [14] proposed an effective cyclic pursuit of a target in random moving by many agents. They also demonstrated its efficiency by numerical simulations. However, theoretical condition for the success of pursuit is not given and the dynamics of the agents and the target is considerably simple. On the other hand, Hara *et al.* [17] proposed a method to analyze the stability of formation control composed of agents and a target with general dynamics. They considered a transformation of frequency variable in linear systems and gave a stability condition with the eigenvalues of a structural matrix, which represents the interchange of information between the agents and the target.

The results in this research area of pursuit or capturing are interesting, however, the assumption that the target does not move or moves regardless of the agents is artificial and it is more realistic to suppose the target takes reasonable behaviour for escape in the motion of pursuit or capture. From this point of view, in this paper, we consider a problem of capturing a target by many agents, where the target is supposed to escape along a reasonable strategy. Then, we give strict conditions for the success of the capturing or the escape with respect to their control performance indices. In particular, we consider the problem in the following steps:

(i) case of simple dynamics of a target and agents,

(ii) case of general dynamics of a target and agents.

In (i), we deal with a case that the dynamics of the target and the agents is a simple linear 1st order system. Then, we give a necessary and sufficient condition for the capturing by employing the Gershgorin theorem with detailed analysis on the coefficients of the characteristic polynomial. In (ii), we consider more general dynamics of linear systems for the target and the agents. For the purpose, we employ a notion of a transformation of frequency variable [17] and give conditions for the capturing with the eigenvalues of a structured matrix and a transformed stable/unstable region. Furthermore, we discuss the performance competition between the target and the agents with respect to the capturing or the escape and propose preferable strategy of behavior for the target or the agents.

This paper is organized as follows. In Section 2, we prepare several notions and propositions used in the following of this paper, and we formulate the cooperative capturing by multi-agent systems. We give a condition for capturing in Section 3 with a case of simple dynamics of the target and the agents. We also discuss the performance competition and propose preferable strategy for the target and the agents. In Section 4, we extend the result to the case of more general dynamics. Finally, we conclude the paper in Section 5.

Notation:

$\mathbb{R}$: real numbers, $\mathbb{R}_+$: nonnegative real numbers, $\mathbb{R}^n$: n-dimensional real vectors, $\mathbb{R}^{m\times n}$: $m \times n$-dimensional real matrices, $\mathbb{C}$: complex numbers, $\mathbb{C}_+$: right half plane including imaginary axis, $\mathbb{C}_+^c$: complement of $\mathbb{C}_+$, A^{T}: transpose of a matrix A, $\sigma_{n,r}$: rth order elementary symmetric polynomial on $k_1, \ldots, k_n$.

2 Preliminary and Formulation

In this section, at first, we introduce a stability of linear systems and the related propositions used in this paper.

In general, for nth order time invariant system;

$$\dot{x} = f(x), \quad t \geq 0, \quad x(0) = x_0, \tag{1}$$

x_e satisfying $f(x_e) = 0$ is called *equilibrium point* of (1).

Definition 2.1 *Let x_e be an equilibrium of (1). For arbitrary positive number $\epsilon > 0$, there exists a positive number $\delta(\epsilon) > 0$ and for any initial condition x_0 satisfying $\|x_0 - x_e\| \le \delta(\epsilon)$,*

$$\|x(t) - x_e\| \le \epsilon, \quad \forall t \ge 0, \tag{2}$$

then, (1) is stable on x_e.

Proposition 2.1 (e.g., see [19]) *For nth order linear time invariant system:*

$$\dot{x} = Ax, \quad t \ge 0, \quad x(0) = x_0, \tag{3}$$

the following statements are equivalent.

(i) the system (3) is stable (on $x_e = 0$).

(ii) the real part of the all eigenvalues of A are nonpositive and the eigenvalues, of which real part are zero, are simple roots of the minimal polynomial of A, if they exist.

Proposition 2.2 (Gershgorin Theorem [18]) *For a complex matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, define discs on the complex plane as*

$$\mathrm{D}_i := \left\{ s : |s - a_{ii}| \le \sum_{j=1, j\neq i}^{n} |a_{ij}| \right\}, \; i = 1,\ 2,\ \ldots,\ n. \tag{4}$$

Then, the all eigenvalues of A are in the set of $\bigcup_{i=1}^{n} \mathrm{D}_i$.

We next explain the stability of systems with a particular structure:

$$\mathcal{G}(s) = C\left(\frac{1}{v(s)}I - A\right)^{-1} B + D \tag{5}$$

where $v(s)$ is supposed to be continuous and strictly proper function of s. Let $\phi(s) := 1/v(s)$, then, we call a region, which is transformed from $\mathbb{C}_+$ by $\phi(s)$, as Ω_+, its boundary $\partial\Omega_+$. The complement of Ω_+ is called Ω_+^c.

Proposition 2.3 (e.g., see [15]) *Let $G(s)$ be a linear time invariant system. Then, a system $\mathcal{G}(s) = G(\phi(s))$ with a variable transformation $\phi(s)$ is stable iff the all poles of $G(s)$ except for single poles on $\partial\Omega_+$ are in Ω_+^c.*

Hereafter, we formulate the problem of capturing. We consider a subsystem of escape P_t called *target* and n subsystems $P_1,\ P_2,\ \ldots,\ P_n$, which are capturing the target, called *agents*. They are supposed on a two dimensional plane. We denote their coordinates by

$$p_i(t) = \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} \in \mathbb{R}^2,\ p_t(t) = \begin{bmatrix} x_t(t) \\ y_t(t) \end{bmatrix} \in \mathbb{R}^2, \tag{6}$$

and their dynamics is given by

$$\dot{p}_i = f_i(p_i, u_i), \tag{7}$$

$$\dot{p}_t = f_t(p_t, u_t), \tag{8}$$

where $f_i(\cdot)$ or $f_t(\cdot)$ is an appropriate linear function such as the equation of motion in Cartesian coordinate system.

In this paper, we define capturing as follows:

Definition 2.2 *When for any given* $\epsilon > 0$*, there exist a constant* $\delta > 0$ *and for any initial conditions* $p_i(0)$*,* $p_t(0)$ *such as* $\|p_i(0) - \{p_t(0) + l_i\}\| < \delta$*,*

$$\|p_i(t) - \{p_t(t) + l_i\}\| < \epsilon, \quad \forall t \tag{9}$$

is satisfied, then we call the capturing is attained.

The point $p_t(t) + l_i$ can be regarded as the objective for the agent P_i where each parameter $l_i := [l_{x,i} \; l_{y,i}]^{\mathrm{T}}$ is given in advance in order to surround the target. When (9) is satisfied, the agents P_i keep the relative positions around the target and succeed to track the target P_t.

3 Cooperative capturing: simple case of dynamics

3.1 State Space Representation

In this section, we deal with a simple case of the dynamics (7) and (8) as:

$$\dot{p}_i = u_i, \tag{10}$$

$$\dot{p}_t = u_t. \tag{11}$$

Then, we consider the control law of the agents in order to satisfy (9) as follows.

Control law of the agents:

$$u_i = -k_i\{p_i - (p_t + l_i)\}, \tag{12}$$

where

$$k_i := \begin{bmatrix} k_{x,i} & 0 \\ 0 & k_{y,i} \end{bmatrix}, \; k_{x,i} > 0, \; k_{y,i} > 0.$$

For the agents (10) with (12), a large k_i implies that the agent is sensitive for the move of the target and in general, the objective (9) tends to be satisfied. In this sense, k_i can be regarded as the *performance index* of the agents.

Next, we define the escape strategy; the control law, of the target as follows.

Escape strategy of the target:

$$u_t = \sum_{i=1}^{n} \alpha_i k_t (p_t - p_i), \tag{13}$$

where

$$\begin{aligned} k_t &:= \begin{bmatrix} k_{x,t} & 0 \\ 0 & k_{y,t} \end{bmatrix},\ k_{x,t} \geq 0,\ k_{y,t} \geq 0, \\ \alpha_i &:= \begin{bmatrix} \alpha_{x,i} & 0 \\ 0 & \alpha_{y,i} \end{bmatrix},\ \alpha_{x,i} > 0,\ \alpha_{y,i} > 0, \\ &\alpha_{x,1} + \alpha_{x,2} + \cdots + \alpha_{x,n} = 1, \\ &\alpha_{y,1} + \alpha_{y,2} + \cdots + \alpha_{y,n} = 1. \end{aligned} \tag{14}$$

The escape strategy (13) means there exists repulsions between the target and the agents, and k_t can be regarded as the performance index of the target. The weights α_i are considered as tuning parameters for the target. A large α_i implies that the target strongly escapes from the agent P_i. In this sense, the weights α_i assign the distribution ratios of the total performance k_t against each agent P_i.

Remark 3.1 *In general, the behavior of capturing of realistic biological systems contains many complex phases, therefore, the problem formulation given above is a considerably simplified model. However, it describes a fundamental principle of capturing and the results in the following of this paper tell us suggestive facts on the cooperative capturing and the performance competition.*

As seen from (13), the dynamics on the axis x or y is independent each other, therefore, in order to avoid the redundancy on the discussion and the notations, in the following of this paper, we refer to the x axis only and omit the subscript x in the notations (e.g., $k_{x,i} \to k_i$, $l_{x,i} \to l_i$, $\alpha_{x,i} \to \alpha_i$).

We arrange the dynamics of the whole system (10)–(13) in the following state space representation (pick up x-axis only).

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{l}, \\ \boldsymbol{x} &= \begin{bmatrix} x_t & x_1 & x_2 & \cdots & x_n \end{bmatrix}^{\mathrm{T}}, \\ \boldsymbol{l} &= \begin{bmatrix} 0 & l_1 & l_2 & \cdots & l_n \end{bmatrix}^{\mathrm{T}}, \end{aligned} \tag{15}$$

where

$$A := \left[\begin{array}{c|cccc} k_t & -\alpha_1 k_t & -\alpha_2 k_t & \cdots & -\alpha_n k_t \\ \hline k_1 & -k_1 & & & O \\ k_2 & & -k_2 & & \\ \vdots & & & \ddots & \\ k_n & O & & & -k_n \end{array}\right], \tag{16}$$

$$B := \left[\begin{array}{c|cccc} k_t & 0 & 0 & \cdots & 0 \\ \hline 0 & -k_1 & & & O \\ 0 & & -k_2 & & \\ \vdots & & & \ddots & \\ 0 & O & & & -k_n \end{array}\right].$$

The block diagram of the whole system is given in Fig. 1.

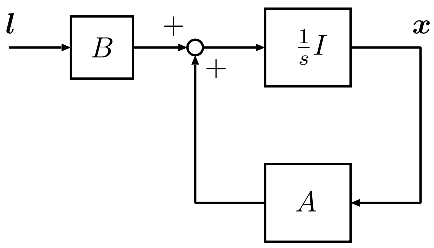

Fig. 1: The block diagram of system (15): simple case of dynamics

3.2 Condition for Capturing

We get the following result:

Theorem 3.1 *In the system (15) composed of a target P_t and n agents P_i, a necessary and sufficient condition for the capturing is*

$$k_t \sum_{i=1}^{n} \frac{\alpha_i}{k_i} < 1. \tag{17}$$

The proof is given in Section 3.4.

The condition (17) gives us insight on performance competition between the agents and the target and preferable strategy for them:

Performance as a group

The condition (17) can be deformed into

$$k_t < \left(\sum_{i=1}^{n} \frac{\alpha_i}{k_i} \right)^{-1}. \tag{18}$$

The right hand side is the weighted harmonic mean of k_i and it represents the performance of the agents as a whole.

Strategy for the agents

In general, according to the relationship between the weighted harmonic mean and arithmetical mean, $\left(\sum_{i=1}^{n} \frac{\alpha_i}{k_i}\right)^{-1}$ is bounded by

$$\left(\sum_{i=1}^{n} \frac{\alpha_i}{k_i} \right)^{-1} \leq \sum_{i=1}^{n} \alpha_i k_i,$$

and the equality is held at $k_1 = k_2 = \cdots = k_n =: k$. Therefore, when α_i are fixed and $\sum_{i=1}^{n} \alpha_i k_i$ is constant, $k_1 = k_2 = \cdots = k_n = k$ is the best strategy for the agents. This implies the agents should form "homogeneous" group. When $k_1 = k_2 = \cdots = k_n = k$, (17) or (18) becomes

$$k_t < k \tag{19}$$

and the condition for the capture or the escape is a simple comparison between their individual performances.

Strategy for the target

Suppose α_i is a tuning parameter for the target. When $k_1 > k_2 > \cdots > k_n$, $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ is a preferable setting of the weights for the target in order to decrease the right hand side of (18). This implies that setting large weight α_i corresponding to the small k_i, i.e. a weak agent, and strong escape from it, is a preferable strategy for the target.

3.3 Numerical Simulation I

We show numerical simulations to demonstrate the result. Let $n = 4$ and

$$\begin{aligned} k_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ k_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\ k_3 &= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix},\ k_4 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \\ l_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ l_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ l_3 &= \begin{bmatrix} -1 \\ -1 \end{bmatrix},\ l_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ p_1(0) &= \begin{bmatrix} -8 \\ -2 \end{bmatrix},\ p_2(0) = \begin{bmatrix} 3 \\ -5 \end{bmatrix} \\ p_3(0) &= \begin{bmatrix} 6 \\ 4 \end{bmatrix},\ p_4(0) = \begin{bmatrix} -4 \\ 1 \end{bmatrix}. \end{aligned} \tag{20}$$

We also set k_t and the weights α_i as

$$k_t = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}, \tag{21}$$

$$\begin{aligned} \alpha_1 &= \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},\ \alpha_2 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}, \\ \alpha_3 &= \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},\ \alpha_4 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}. \end{aligned} \tag{22}$$

Note that the weights are normalized as $\sum \alpha_i = I$. Then,

$$k_{x,t} \sum_{i=1}^{4} \frac{\alpha_{x,i}}{k_{x,i}} = k_{y,t} \sum_{i=1}^{4} \frac{\alpha_{y,i}}{k_{y,i}} = 0.7375 < 1, \tag{23}$$

therefore, the condition (17) for capturing is attained on x-axis and y-axis, simultaneously. Fig. 2 shows the loci of the agents P_i ($i = 1, 2, 3, 4$) and the target P_t, and it is known that actually the capturing is attained.

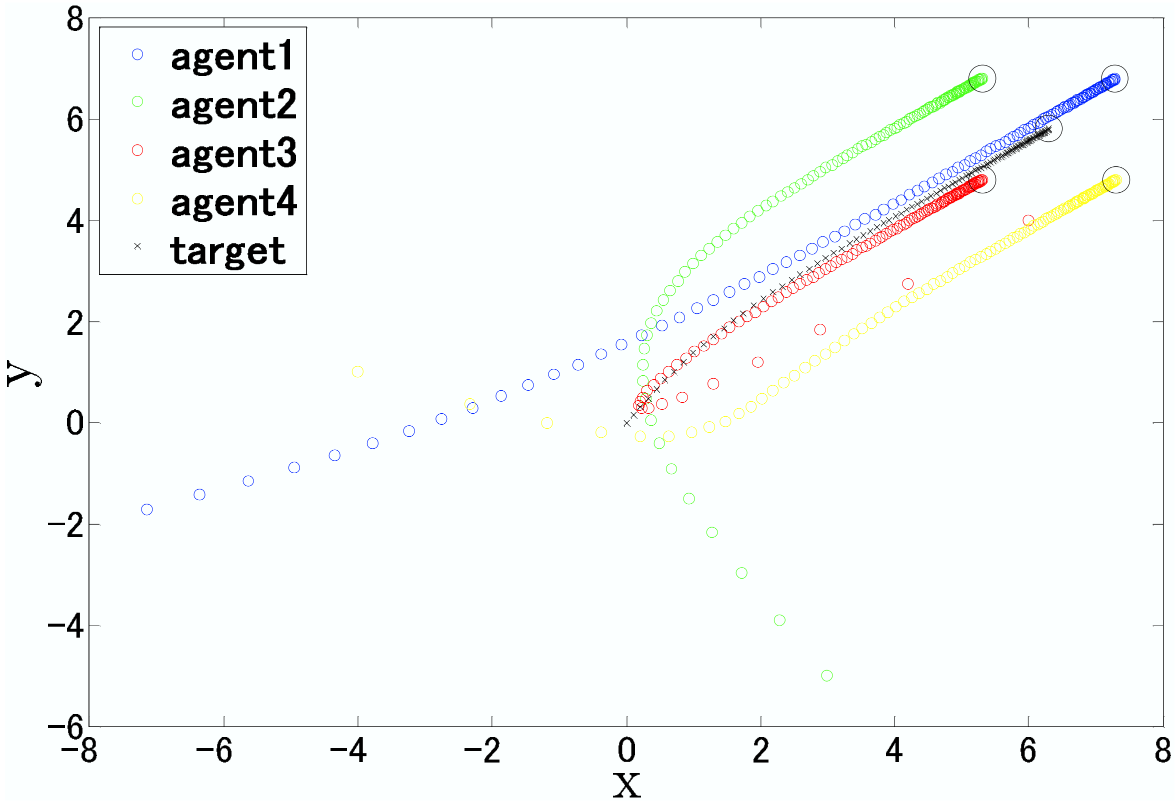

Fig. 2: The loci of the agents P_i ($i = 1, 2, 3, 4$, marked by '∘') and the target P_t (marked by '×') (case of capture)

Next, since the performance k_1 of the agent P_1 is the weakest among the agents, tune the weights α_i by

$$\begin{aligned} \alpha_1 &= \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \ \alpha_2 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}, \\ \alpha_3 &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \ \alpha_4 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \end{aligned} \tag{24}$$

whereas to keep $\sum \alpha_i = I$. This means the target escapes from the weakest agent P_1 strongly. In this case,

$$k_{x,t} \sum_{i=1}^{4} \frac{\alpha_{x,i}}{k_{x,i}} = k_{y,t} \sum_{i=1}^{4} \frac{\alpha_{y,i}}{k_{y,i}} = 1.0625 > 1 \tag{25}$$

and the condition (17) is not satisfied. Actually, Fig. 3 shows that the capturing fails and the target succeeds to escape.

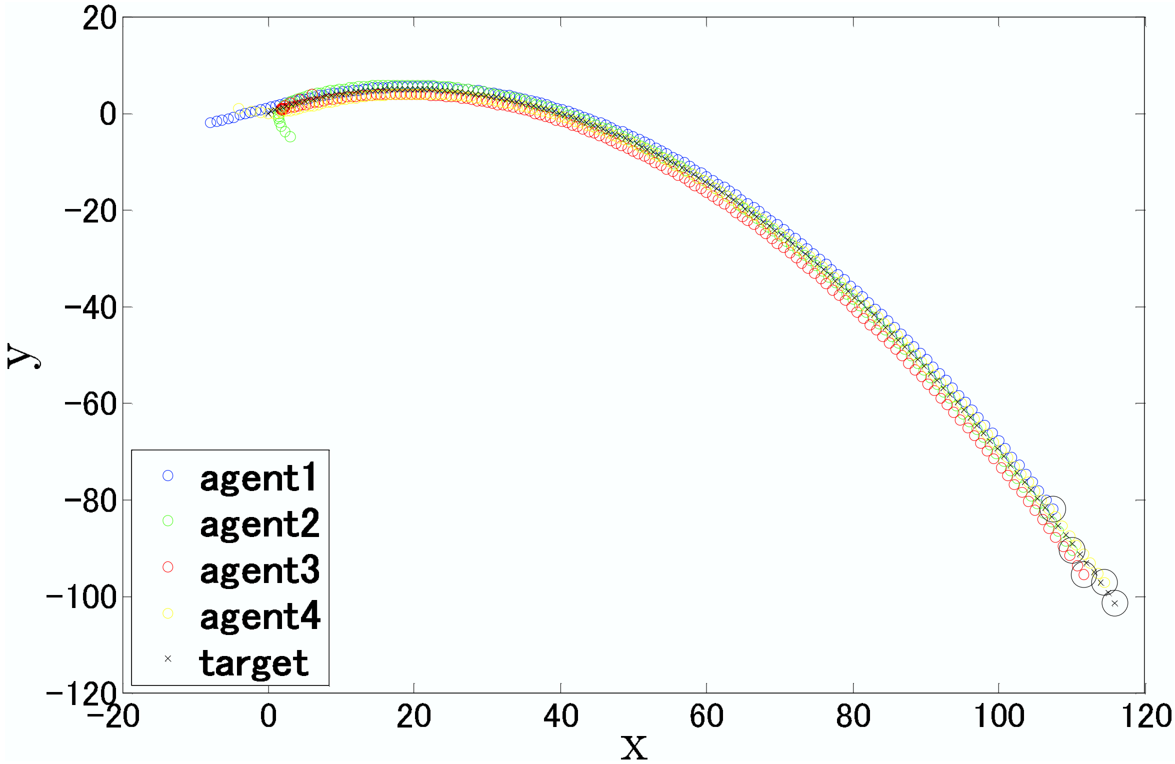

Fig. 3: The loci of the agents P_i ($i = 1, 2, 3, 4$, marked by '$\circ$') and the target P_t (marked by '$\times$') (case of escape)

The above two cases demonstrate that the tuning of the weights α_i is important for the target to escape.

3.4 Proof of Theorem 3.1

We check the location of the eigenvalues of A with respect to $k_t(\geq 0)$ for fixed $k_i(> 0)$.

From the Gershgorin Theorem (Proposition 2.2), the all eigenvalues of A are located in the region of the sum of a disc centered at $(k_t, 0)$ with radius k_t (call D_t hereafter) and discs centered at $(-k_i, 0)$ with radius k_i ($i = 1, 2, \cdots, n$) (call D_i hereafter) in the complex plane. This implies that the all discs touch the imaginary axis only at the origin for any k_t and k_i, and the discs D_i exist in the left half plane for any $k_i > 0$. When $k_t = 0$, the disc D_t becomes a point located at the origin, therefore, the all discs exist in the closed left half plane. Next, when $k_t > 0$, only the disc D_t extends in the right half plane which still touch the imaginary axis only at the origin (see Fig. 4).

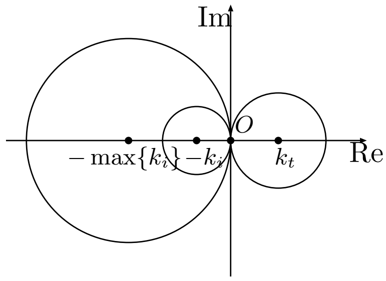

Fig. 4: The location of the Gershgorin discs in the case $k_t > 0$ and $k_i > 0$

From the above facts, we show the condition (17) for the stability with the following steps:

(i) The eigenvalues of A are continuous functions of k_t (their loci on the complex plane with respect to k_t are continuous).

(ii) When the eigenvalues move from the left half plane to the right half plane (and vice versa) by the continuous change of k_t, they necessarily pass through the origin.

(iii) The matrix A always has an eigenvalue at the origin.

(iv) the other eigenvalues should be asymptotically stable for (9).

(v) When $k_t = 0$, the other eigenvalues exists in the open left half plane, that is, capturing is attained.

(vi) The value of k_t, which satisfies the condition that one of the eigenvalues in (v) is located on the origin, is unique (call $\bar{k}_t$). Moreover, when $0 \le k_t < \bar{k}_t$, the eigenvalues in (v) is asymptotically stable, otherwise they are marginally stable or unstable.

The statement (i) is obvious from the definition of A. With (i), the statement (ii) is also obvious since the Gershgorin discs D_t and D_i touch the imaginary axis only at the origin from Proposition 2.2.

Next, by using a nonsingular matrix

$$T = \left[\begin{array}{c|cccc} 1 & 0 & 0 & \cdots & 0 \\ \hline 1 & 1 & & & O \\ 1 & & 1 & & \\ \vdots & & & \ddots & \\ 1 & O & & & 1 \end{array}\right],$$

transform the state variable x in (15) as

$$\begin{aligned} \boldsymbol{x} =& T\boldsymbol{z}, \\ \boldsymbol{z} :=& \left[\begin{array}{ccccc} z_t & z_1 & z_2 & \cdots & z_n \end{array}\right]^{\mathrm{T}}, \end{aligned} \tag{26}$$

then,

$$\dot{\boldsymbol{z}} = T^{-1}AT\boldsymbol{z} + T^{-1}B =: \tilde{A}\boldsymbol{z} + T^{-1}B,$$

where $\tilde{A}$ is given by

$$
\begin{aligned}
\tilde{A} &= T^{-1}AT \\
&= \left[\begin{array}{c|cccc}
0 & -w_1 & -w_2 & \cdots & -w_n \\
\hline
0 & -k_1 + w_1 & w_2 & \cdots & w_n \\
0 & w_1 & -k_2 + w_2 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & w_n \\
0 & w_1 & \cdots & w_{n-1} & -k_n + w_n
\end{array}\right],
\end{aligned} \tag{27}
$$

and

$$
w_i := \alpha_i k_t, \quad i = 1, 2, \cdots, n. \tag{28}
$$

From this, (iii) is given. The statement (iv) is from (iii) and Proposition 2.1.

Next, denote the 2-2 block matrix of $\tilde{A}$ as

$$
\tilde{A}_{22} = \begin{bmatrix}
-k_1 + w_1 & w_2 & \cdots & w_n \\
w_1 & -k_2 + w_2 & \ddots & \vdots \\
\vdots & \ddots & \ddots & w_n \\
w_1 & \cdots & w_{n-1} & -k_n + w_n
\end{bmatrix}. \tag{29}
$$

Let $\psi(s)$ be the characteristic polynomial of $\tilde{A}_{22}$, then a direct calculation gives

$$
\begin{aligned}
\psi(s) &= \det(sI - \tilde{A}_{22}) \\
&= \prod_{i=1}^{n}(s + k_i) - \sum_{i=1}^{n}\left(w_i \prod_{j=1, j\neq i}^{n}(s + k_j)\right) \\
&=: s^n + c_{n-1}s^{n-1} + c_{n-2}s^{n-2} + \cdots + c_1 s + c_0,
\end{aligned} \tag{30}
$$

where the coefficient c_0 is described by

$$
\begin{aligned}
c_0 =& \sigma_{n,n} - \sum_{i=1}^{n}(\sigma_{n,n-1} - k_i\sigma_{n,n-2} + k_i^2\sigma_{n,n-3} - \\
& \cdots + (-1)^{n-1}k_i^{n-1})w_i
\end{aligned}
$$

with elementary symmetric polynomials $\sigma_{n,j}$ on k_1–k_n defined in Definition A.1. From the definition of $\sigma_{n,j}$, it is obvious that $\sigma_{n,j} > 0$ for all j.

When $k_t = 0$,

$$
\psi(0) = \sigma_{n,n} > 0. \tag{31}
$$

This implies $\tilde{A}_{22}$ does not have an eigenvalue at origin when $k_t = 0$. However, recall that when $k_t = 0$, the all Gershgorin discs exist in the close left half plane and they only touch the imaginary axis at the origin. Therefore, (v) is concluded.

Finally, we show (vi). At first, note that the signatures of w_i and k_t are same for all i from the definition of w_i. On the other hand, we can also show that

$$
\sigma_{n,p-1} - k_i\sigma_{n,p-2} + k_i^2\sigma_{n,p-3} - \cdots + (-1)^{p-1}k_i^{p-1} > 0
$$

for any p and i by Lemma A.1. This implies c_0 is a uniformly decreasing number with respect to k_t for fixed k_i. Therefore,

$$\bar{k}_t \text{ s.t. } \psi(0) = c_0 = 0 \tag{32}$$

is unique. From the fact (i), (ii) and (v), when $0 \leq k_t < \bar{k}_t$, $\psi(0) = c_0 > 0$ and $\tilde{A}_{22}$ is asymptotically stable. Contrary, when $k_t \geq \bar{k}_t$, $\psi(0) = c_0 \leq 0$ and $\tilde{A}_{22}$ is unstable or marginaly stable. This concludes (vi).

From (iii), (iv) and (vi), $0 \leq k_t < \bar{k}_t$ or equivalently $\psi(0) = c_0 > 0$ is the necessary and sufficient condition for the capturing.

Finally, we express the condition

$$\psi(0) = c_0 > 0 \tag{33}$$

by using k_t, k_i and α_i. A direct calculation gives:

$$\begin{aligned}
(33) \Longleftrightarrow & \sigma_{n,n} - \sum_{i=1}^{n} (\sigma_{n,n-1} - k_i \sigma_{n,n-2} + k_i^2 \sigma_{n,n-3} - \\
& \cdots + (-1)^{n-1} k_i^{n-1}) w_i > 0 && (34) \\
\Longleftrightarrow & \prod_{i=1}^{n} k_i - w_1 \prod_{i=2}^{n} k_i - w_2 \prod_{\substack{i=1 \\ i \neq 2}}^{n} k_i - w_3 \prod_{\substack{i=1 \\ i \neq 3}}^{n} k_i - \\
& \cdots - w_n \prod_{i=1}^{n-1} k_i > 0 && (35) \\
\Longleftrightarrow & w_1 \prod_{i=2}^{n} k_i + w_2 \prod_{\substack{i=1 \\ i \neq 2}}^{n} k_i + w_3 \prod_{\substack{i=1 \\ i \neq 3}}^{n} k_i + \\
& \cdots + w_n \prod_{i=1}^{n-1} k_i < \prod_{i=1}^{n} k_i && (36) \\
\Longleftrightarrow & \sum_{i=1}^{n} \frac{w_i}{k_i} = k_t \sum_{i=1}^{n} \frac{\alpha_i}{k_i} < 1. && (37)
\end{aligned}$$

This concludes the statement of the theorem. ■

4 Cooperative capturing: general case of dynamics

In Section 3, we deal with a simple case of the dynamics of the target and the agents. In order to correspond to more realistic situations, we extend the class in this section. Note that we consider a case:

$$\alpha_i = \frac{1}{n}, \ \forall i, \tag{38}$$

in this section for simplifying the problem.

4.1 State Space Representation with a Transformed Frequency Variable

In this section, we consider the following dynamics for the agents and the target.

$$p_i = \begin{bmatrix} v(s) & 0 \\ 0 & v(s) \end{bmatrix} u_i \tag{39}$$

$$p_t = \begin{bmatrix} v(s) & 0 \\ 0 & v(s) \end{bmatrix} u_t \tag{40}$$

The control law of the agents and the escape strategy of the target are the same of (12) and (13), respectively.

We also describe the dynamics on x-axis only such as in Section 3, then we get the following equation corresponding to (15):

$$\begin{aligned} \frac{1}{v(s)}\boldsymbol{x} =& A\boldsymbol{x} + B\boldsymbol{l}, \\ \boldsymbol{x} =& \begin{bmatrix} x_t & x_1 & x_2 & \cdots & x_n \end{bmatrix}^{\mathrm{T}}, \\ \boldsymbol{l} =& \begin{bmatrix} 0 & l_1 & l_2 & \cdots & l_n \end{bmatrix}^{\mathrm{T}}, \end{aligned} \tag{41}$$

where A is defined in (16) with $\alpha_i = \frac{1}{n}$, $\forall i$. The corresponding block diagram is given in Fig. 5.

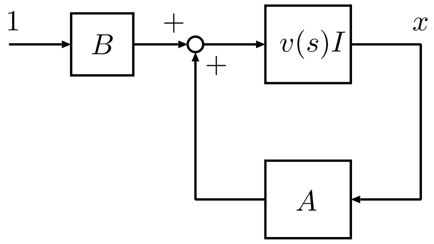

Fig. 5: The block diagram of system (41): general case of dynamics

The transfer function from the input 1 of B to x is given by

$$\mathcal{G}(s) = \left(\frac{1}{v(s)}I - A\right)^{-1} B. \tag{42}$$

By using the stability analysis explained in Section 2, (42) can be regarded as a linear system with a transformed frequency variable $\phi(s) := 1/v(s)$. In the following, we give conditions for capturing under the cases that $v(s)$ is in given classes. We consider a class for $v(s)$ as follows:

Definition 4.1 *Denote the set of $\phi(s)$, which satisfies Ω_+ $(:= \phi(\mathbb{C}_+))$ does not contain $\phi \in (-\mu,\, 0]$, $\mu > 0$ on the real axis, by Φ_μ. Also define*

$$\mathcal{V}_\mu := \left\{ v(s) = \frac{1}{\phi(s)},\ \phi(s) \in \Phi_\mu, v(s) : \textit{strictly proper} \right\}. \tag{43}$$

For example,

$$v(s) = \frac{1}{s(s+a)},\ a > 0 \tag{44}$$

is in $\mathcal{V}_\infty$. This is the typical case that the dynamics of the agents and the target obeys a second order system composed of mass and viscosity. In this case, the complement region Ω_+^c of Ω_+ at $a = 1$ is described as the shaded area in Fig. 6.

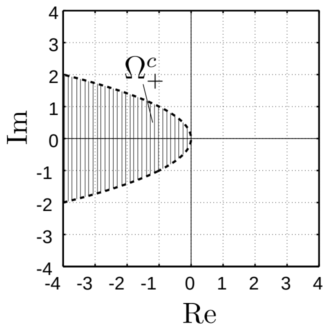

Fig. 6: The region Ω_+^c in the case $v(s) = \frac{1}{s(s+1)}$

The boundary of the region Ω_+^c in Fig. 6 is given by

$$\phi(j\omega) = j\omega(j\omega + a) = -\omega^2 + j\omega a \tag{45}$$

and the set Ω_+^c can be regarded as a transformed stable region for the eigenvalues of A in (42). It is also known that Ω_+ does not contain $(-\infty, 0]$ on the real axis.

On the other hand,

$$v(s) = \frac{1}{s} \cdot \frac{1}{ms^2 + ds + \lambda},\ m,\ d,\ \lambda > 0 \tag{46}$$

is in $\mathcal{V}_\mu$, where $0 < \mu < \infty$. This is another typical case of the dynamics composed of an integrater, mass, spring and damper. In this case, the complement region Ω_+^c of Ω_+ at $m = 1$, $d = 1.5$, $\lambda = 1.8$ is described as the shaded area in Fig. 7. It is also known that Ω_+ does not contain $(-\mu, 0] \simeq (-2.7, 0]$ on the real axis.

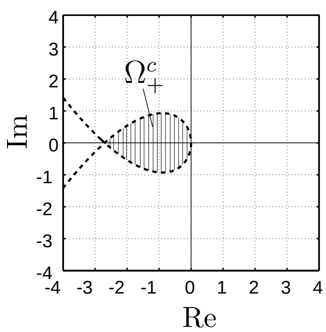

Fig. 7: The region Ω_+^c in the case $v(s) = \frac{1}{s(s^2 + 1.5s + 1.8)}$

4.2 Condition for Capturing

Now, we give a condition for capturing:

Theorem 4.1 *Assume*

$$v(s) \in \mathcal{V}_\mu, \tag{47}$$

then, the following are held.

(i) When $0 < \mu < \infty$*, a sufficient condition that the system (41) attains capturing is*

$$\frac{k_t}{n} \sum_{i=1}^{n} \frac{1}{k_i} < 1 \tag{48}$$

and

$$-\mu < -2k_{\max}, \tag{49}$$

where $k_{\max} := \max\{k_1,\ k_2,\ \cdots,\ k_n\}$.

(ii) When $\mu = \infty$*, a necessary and sufficient condition that the system (41) attains capturing is (48).*

Proof Proof of (i): From Proposition 2.3, a condition that the all eigenvalues except for a zero eigenvalue of A are in Ω_+^c and the system attains capturing (9) are equivalent. On the other hand, Theorem 3.1 gives a necessary and sufficient condition (17) for that the all eigenvalues except for a zero eigenvalue of A are in open left half plane. Note that when $\alpha_i = \frac{1}{n}$, $\forall i$, the matrix $\tilde{A}_{22}$ given in (29) is symmetry, therefore, the all eigenvalues of A are real. Therefore, their minimum is larger than or equal to $-2k_{\max}$ from the Gershgorin theorem. On the other hand, the unstable region Ω_+ for $v(s) \in \mathcal{V}_\mu$ does not contain $\phi \in (-\mu,\ 0]$ on the real axis. Therefore, (48) and (49) imply the all eigenvalues except a zero eigenvalue of A are in the stable region Ω_+^c. This concludes the statement.

Proof of (ii): In the case of $v(s) \in \mathcal{V}_\infty$, the unstable region Ω_+ on the complex plane does not contain the negative part of the real axis, therefore, the stability of the system (41) is equivalent to (48). □

From Theorem 4.1 and 3.1, it is known that the conditions for capturing are identical for the first order system (10), (11) and the system (39) and (40) where $v(s) \in \mathcal{V}_\infty$.

4.3 Numerical Simulation II

We show numerical simulations in a case of (44) with $a = 1$, that is, $f(s) \in \mathcal{V}_\infty$, (20) and (21). Note that the weights are

$$\alpha_i = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix},\ \forall i. \tag{50}$$

In this case, the necessary and sufficient condition (48) for capturing in Theorem 4.1-(ii) is satisfied as

$$\frac{k_{x,t}}{4}\sum_{i=1}^{4}\frac{1}{k_{x,i}}=\frac{k_{y,t}}{4}\sum_{i=1}^{4}\frac{1}{k_{y,i}}=0.7812<1. \tag{51}$$

Fig. 8 shows the loci of the agents P_i $(i = 1, 2, 3, 4)$ and the target P_t. From the simulation, it is known that the capturing succeed.

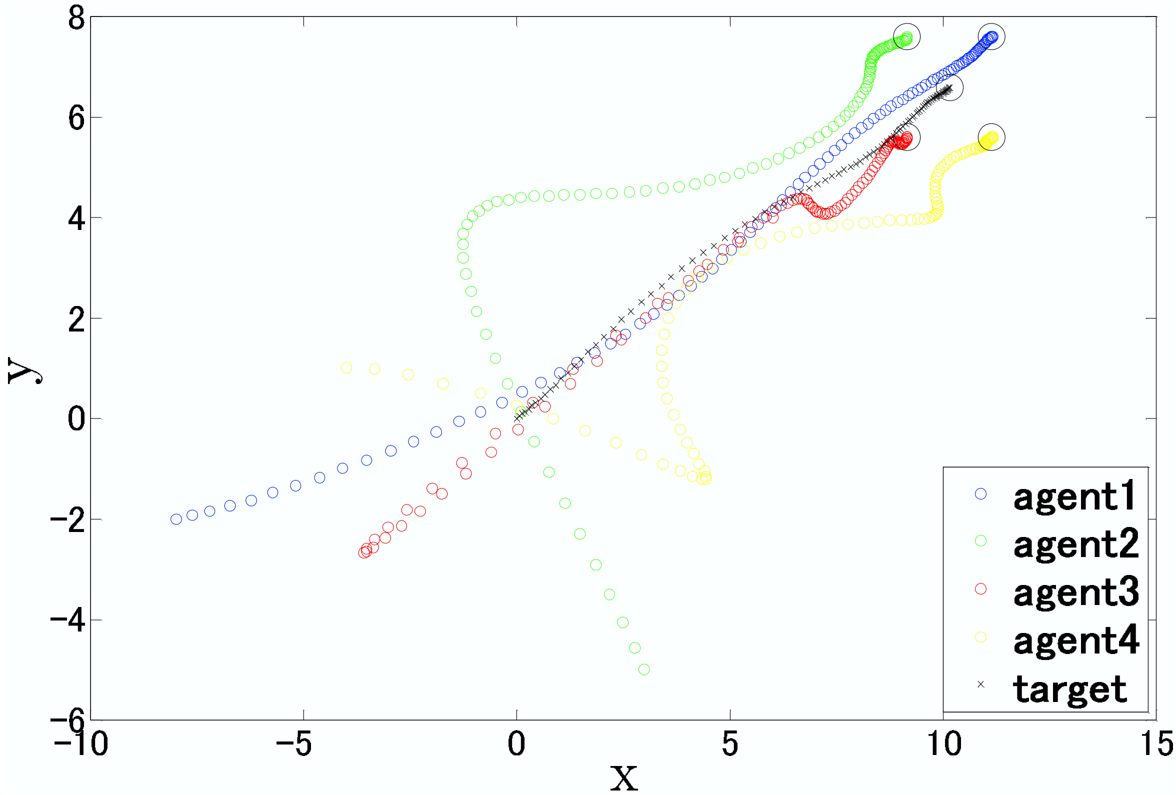

Fig. 8: The loci of the agents P_i $(i = 1, 2, 3, 4$, marked by '$\circ$') and the target P_t (marked by '$\times$') (case of $v(s) \in \mathcal{V}_\infty$ and success of capture)

Next, we set k_t as

$$k_t=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}, \tag{52}$$

then,

$$\frac{k_{x,t}}{4}\sum_{i=1}^{4}\frac{1}{k_{x,i}}=\frac{k_{y,t}}{4}\sum_{i=1}^{4}\frac{1}{k_{y,i}}=1.0417>1. \tag{53}$$

This implies (48) is not satisfied. Fig. 9 shows the numerical simulation of the loci of the agents and the target in this case. From the figure, it is known that the capturing fails.

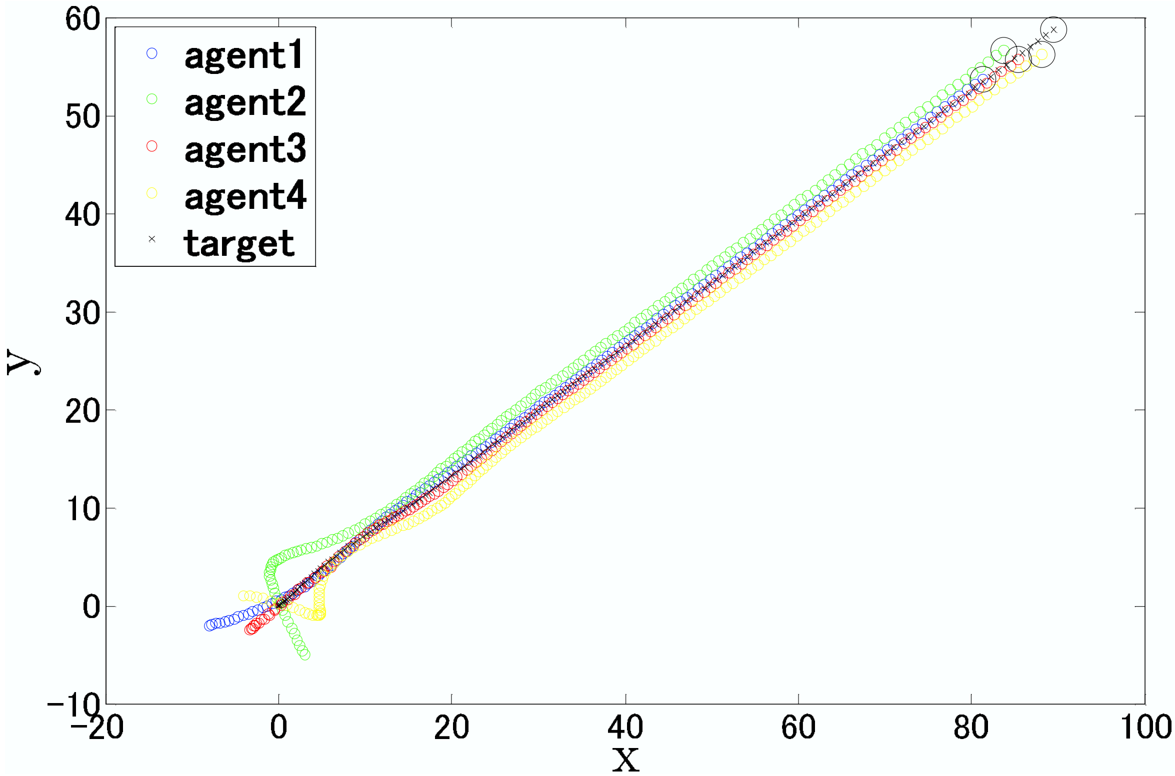

Fig. 9: The loci of the agents P_i ($i = 1, 2, 3, 4$, marked by '$\circ$') and the target P_t (marked by '$\times$') (case of $v(s) \in \mathcal{V}_\infty$ and escape)

5 Conclusion

In this paper, we considered cooperative capturing problem by multi-agent systems where the target escapes with a reasonable strategy. We gave necessary and sufficient conditions or a sufficient condition for the case of simple dynamics or general dynamics of the target and the agents. We furthermore discussed the meaning of the condition with respect to the performance competition between the target and the agents. In particular, we explained that a reasonable strategy for the target is to escape strongly from weak agents, and contrary, for the agents it is reasonable to form a homogeneous group.

References

[1] Y. Zou, P.R. Pagilla, and E. Misawa, "Formation of a group of vehicles with full information using constraint forces," *ASME Jornal of Dynamic Systems, Measurement, and Control*, Vol. 129, pp. 654–661, 2007.

[2] C. W. Reynolds, "Flocks, herds and schools: A distributed behavioral model," *Computer Graphics*, Vol. 21, No.4, pp. 25–34, 1987.

[3] J. P. Desai, J. P. Ostrowski, and V. Kumar, "Modeling and control of formations of nonholonomic mobile robots," *IEEE Transactions on Robotics and Automation*, Vol. 17, pp. 905–908, 2001.

[4] M. Egerstedt and X. Hu, "Formation constrained multiagent control," *IEEE Transactions on Robotics and Automation*, Vol. 17, pp. 947–951, 2001.

[5] N. E. Leonard and E. Fiorello, "Virtual leader, artificial potentials and coordinated control of groups," *Proceedings of the 40th IEEE Conference on Decision and Control*, pp. 2968–2973, 2001.

[6] R. Olfati-Saber and R. M. Murray, "Distributed cooperative control of multiple vehicle formations using structual potential functions," *The 15th IFAC World Congress*, 2002.

[7] H. G. Tanner, A. Jadbabaie, and G. J. Pappas, "Stable flocking of mobile agents, Part I: Fixed topology," *42nd IEEE Conference on Decision and Control*, pp. 2010–2015, 2003.

[8] E. Rimon and D. E. Koditschec, "Exact robot navigation using artificial potential function," *IEEE Transactions on Robotics and Automation*, Vol. 8, pp. 501–518, 1992.

[9] A. Sinha and D. Ghose, "Generalization of linear cyclic pursuit with application to rendezvous of multiple autonomous agents," *IEEE Transactions on Automatic Control*, Vol. 51, No. 11, pp. 1819–1824, 2006.

[10] J. A. Marshall, M. E. Broucke, and B. A. Francis, "Formations of vehicles in cyclic pursuit," *IEEE Transactions on Automatic Control*, Vol. 49, No. 11, pp. 1963–1974, 2004.

[11] J. A. Marshall, M. E. Broucke, and B. A. Francis, "Pursuit formations of unicycles," *Automatica*, Vol. 42 pp. 3–12, 2006.

[12] K. Kobayashi, K. Otsubo, and S. Hosoe, "Design of decentralized capturing behavior by multiple robots," *IEEE Workshop on Distributed Intelligent Systems: Collective Intelligence and its Applications*, pp. 463–468, 2006.

[13] R. Sepulchre, D. A. Paley, and N. E. Leonard, "Group coodination and cooperative control of steered particles in the plane," *Group Coordination and Cooperative Control*, pp. 217–232, Springer-Verlag, Lecture Notes in Control and Information Sciences, 2006.

[14] T.-H. Kim and T. Sugie, "Cooperative control for target-capturing task based on a cyclic pursuit strategy," *Automatica*, Vol. 43, No. 8, pp. 1426–1431, 2007.

[15] S. Hara, T. Hayakawa, and H. Sugata, "Stability analysis of linear systems with generalized frequency variables and its application to formation control, *46th IEEE Conference on Decision and Control*, pp. 1459–1466, 2007.

[16] J. Cortés, S. Martínez, T. Karatas, and F. Bullo, "Converage control for mobile sensing networks," *IEEE Transactions on Robotics and Automation*, vol. 20, No. 2, pp. 243–255, 2004.

[17] S. Hara, T.-H. Kim and Y. Hori, "Distributed formation control for target-enclosing operation by multiple dynamic agents based on a cyclic pursuit strategy," Technical Reports, The University of Tokyo, 2007. (available at `http://www.keisu.t.u-tokyo.ac.jp/research/techrep/index.html`)

[18] G. H. Golub and C. F. van Loan, "Matrix computations," Third edition, Johns Hopkins, 1996.

[19] C.-T. Chen, "Linear system theory and design," Third edition, Oxford, 1999.

A Appendix

We give the definition of elementary symmetric polynomial and the related lemmas.

Definition A.1 (elementary symmetric polynomial) *For given k_1, k_2, ..., k_n, consider the expansion of $\prod_{i=1}^{n}(x+k_i)$ as*

$$\begin{aligned}&\prod_{i=1}^{n}(x+k_i)\\&=x^n+\sigma_{n,1}x^{n-1}+\sigma_{n,2}x^{n-2}+\cdots+\sigma_{n,n-1}x+\sigma_{n,n}.\end{aligned} \tag{54}$$

Then, $\sigma_{n,r}$ is called an elementary symmetric polynomial on k_1, k_2, ..., k_n.

For example, when $n=3$,

$$\begin{aligned}\sigma_{3,1}&=k_1+k_2+k_3,\\\sigma_{3,2}&=k_1k_2+k_2k_3+k_3k_1,\\\sigma_{3,3}&=k_1k_2k_3.\end{aligned}$$

Lemma A.1 *For arbitrary $p=1,\ \dots,\ n$ and $i=1,\ \dots,\ n$,*

$$\sigma_{n,p-1}-k_i\sigma_{n,p-2}+k_i^2\sigma_{n,p-3}-\cdots+(-1)^{p-1}k_i^{p-1}>0. \tag{55}$$

Proof At first, the left hand side of the inequality can be deformed to

$$\begin{aligned}&\sigma_{n,p-1}-k_i\sigma_{n,p-2}+k_i^2\sigma_{n,p-3}-\cdots+(-1)^{p-1}k_i^{p-1}\\=&\sigma_{n,p-1}-k_i(\sigma_{n,p-2}-k_i(\sigma_{n,p-3}-\\&\cdots-k_i(\sigma_{n,2}-k_i(\sigma_{n,1}-k_i))\cdots)).\end{aligned}$$

The most inside term of the brackets is positive as

$$\begin{aligned}&\sigma_{n,1}-k_i\\&=k_1+k_2+\cdots+k_{i-1}+k_{i+1}+\cdots+k_n\\&>0\quad(\because\ k_i>0,\ \forall i).\end{aligned}$$

Next,

$$\begin{aligned}&\sigma_{n,2}-k_i(\sigma_{n,1}-k_i)\\=&\quad k_1k_2+k_1k_3+\cdots k_1k_{i-1}+k_1k_{i+1}+\cdots k_1k_n\\&+k_2k_3+k_2k_4+\cdots k_2k_{i-1}+k_2k_{i+1}+\cdots k_2k_n\\&\vdots\\>&0.\end{aligned}$$

Repeat this and get

$$
\begin{aligned}
&\sigma_{n,p-1} - k_i(\sigma_{n,p-2} - k_i(\sigma_{n,p-3} - \\
&\cdots - k_i(\sigma_{n,2} - k_i(\sigma_{n,1} - k_i))\cdots)) \\
=&\sigma_{n,p-1} \text{ except for the terms which contains } k_i \\
>&0.
\end{aligned}
$$

This concludes the statement. □