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A Dual Approach for Pricing Swing Options with Bang-Bang Control

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Abstract

We discuss the pricing of swing options with bang-bang control by a dual approach. The pricing problem of American-type derivatives is generally formulated as an optimal multiple stopping problem. One method for solving the problem is a dual approach. However, for swing options with bang-bang control, the problem includes the decision of buying or selling when a right is exercised, and thus it is difficult to apply the existing dual methods. We decompose the price of the swing options into the sum of second-order differences of the price, and show that the sum of single stopping problems that correspond to the second-order differences gives an upper bound for the price of the options. We can solve a dual formulation of each single problem numerically, so that we can compute an upper bound for the price of the options. A numerical example shows that our method gives an appropriate upper bound for the price of the swing options.

1 Introduction

Swing options are American-type derivatives. These are generally traded in gas and electricity markets. When an energy buyer contracts to buy fixed amounts of energy from a seller at fixed dates, the buyer, who buys a swing option from the seller, gets rights to change the amount at some times. The amount is subject to some constraints, such as daily and annual constraints. The number of rights is also limited.

A popular method for pricing American-type derivatives is the least-squares Monte-Carlo method. The method is proposed for pricing American options by Longstaff and Schwartz [8] and Tsitsiklis and Van Roy [11]. Dörr [5] and Barrera-Esteve et al. [2] applied the method for pricing swing options.

Another method for pricing American-type derivatives is based on an optimal exercise boundary. Ibanez [7] studied in the optimal exercise boundaries of multiple American options with exercise obligations and showed that the boundaries satisfy unique and monotone properties. Using the boundary, they computed the price of the options.

The above methods are based on a sub-optimal strategy; therefore, the methods compute a lower bound for the true price of options. In contrast, a dual approach is recently advocated and it computes an upper bound for the true price of options. Rogers [10] and Haugh and Kogan [6] introduced a dual form of the pricing problem for American options. They showed that the dual form gives an upper bound for the true price of the options. Some studies [9] [1] [3] extended this method to more complicated options, such

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as multiple exercise options under volume constraints. The complicated options allow the changed volume to have only positive bias. However, on swing options the changed volume can have positive and negative values, and thus their methods cannot be used for pricing swing options.

In this paper, we discuss swing options with bang-bang control[1]. We reveal two things for the swing options. First, we discuss optimal exercise boundaries of the swing options. We show that the optimal exercise boundaries of the swing options have some good characters, such as monotonicity and uniqueness. Second, we describe a dual approach for pricing the swing options, which is our main topic. To apply a dual approach to the swing options, we introduce a second-order difference of the price of the swing options. Specifically, we decompose the price of the swing options into the sum of second-order differences of the price. We define a single optimal stopping problem for each second-order difference, and then we show that the sum of the solution of the optimal stopping problems gives an upper bound for the sum of second-order differences of the price. In the proof, we use the property of the optimal exercise boundaries. A single optimal stopping problem gives an upper bound by the classic dual approach, which enables us to calculate an upper bound for the price of the swing option.

This paper is organized as follows. Section 2 defines the swing options in this paper. In section 3, we discuss optimal exercise boundary of the swing options and show some properties of the boundary. Section 4 proposes a dual approach for pricing the swing options. Then, section 5 describes a pricing algorithm and gives a numerical result. Section 6 concludes.

2 Definition

In this section we define swing options which we consider in this paper.

There are a buyer and a seller of energy. They close a contract to trade some amount u_i of energy at a strike price of K_i at date t_i $(i = 0, 1, \ldots, T)$. A swing option is defined as a set of rights to change of delivery amount with the contract. When the buyer of energy buys a swing option from the seller and exercises a right at t_i, the buyer can change the amount from u_i to $u_i + v_i$ under some constraints. The number of rights is $L(\leq T + 1)$. The payoff upon exercise with v_i at t_i is $v_i(S_i - K_i)$ where S_i is the energy price at t_i.

In this paper, we consider swing options with bang-bang control. We assume that v_i can be equal to only $v_{\max}(\geq 0)$ or $v_{\min}(\leq 0,\ v_{\max} > v_{\min})$ at exercise dates, and the times of choosing $v_i = v_{\max}$ and $v_i = v_{\min}$ are not less than L_{b} and L_{s}, respectively. In addition, we restrict that L rights must be exercised by the maturity t_T.

Remark 1: The above setting is a specific case of swing options with daily and annual constraints. Daily constraints (often called DCQ) are generally defined by $v_{\min} \leq v_i \leq v_{\max}$ for all i and annual constraints (often called ACQ) are defined by $V_{\min} \leq \sum_{i=0}^{T} v_i \leq V_{\max}$. If $V_{\min}$ and $V_{\max}$ satisfy

$$V_{\min} = a \cdot v_{\min} + (L - a)v_{\max},$$
$$V_{\max} = b \cdot v_{\min} + (L - b)v_{\max}$$

with $a, b \in \mathbb{Z}^+$, that is, if the equality of the annual constraints can be attained by L exercises with $v_{\min}$ or $v_{\max}$, then we can show that the pricing problem under the daily and annual constraints is equivalent to that with the above bang-bang setting.

[1] "Bang-bang" control means that changed volume is only maximum or minimum in an available set.

The option with the bang-bang constraints can be interpreted as the sum of 1) $L_{\rm b}$, the number of obligations to buy, 2) $L_{\rm s}$, the number of obligations to sell and 3) $L_{\rm d}$, the number of straddles[2] ($L_{\rm b}, L_{\rm s}, L_{\rm d} \geq 0,\ L_{\rm b} + L_{\rm s} + L_{\rm d} = L$). Then, the constraints of the option are characterized as $(L_{\rm b}, L_{\rm d}, L_{\rm s})$, which means the combination of the claims. We consider $(L_{\rm b}, L_{\rm d}, L_{\rm s})$ as a state of the option. At each time, only one of three claims can be exercised. Exercise of obligations is prior to that of straddles, and this is consistent with the fact that a straddle is valuable than an obligation. When a claim is exercised, a state transits. Figure 1 shows an example of the state transition of the swing option. We call a chart like Figure 1 a transition tree.

In particular, when $L = T + 1$, the option can be decomposed into 1) L obligations to sell and 2) multiple American call options that have $L_{\rm d}$ rights and $L_{\rm b}$ obligations to buy. The obligation to sell is exercised at every time from t_0 to t_T and the call options also can be exercised.

After this section, we refer to collectively obligations and straddles as rights.

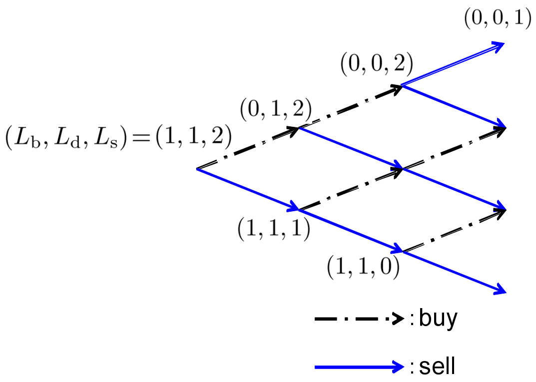

Figure 1: An example of the state transition tree of the swing option. A state $(1, 1, 2)$ transits to $(0, 1, 2)$ when a right to buy is exercised and to $(1, 1, 1)$ when a right to sell is exercised.

3 Optimal exercise strategy and boundary

In this section, we show some properties which an optimal exercise strategy and boundary of swing options satisfies.

We consider some filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t_0 \leq t \leq t_T})$ with a time horizon t_T. We define an adapted stochastic process X_t as the underlying energy price process and assume that $\mathrm{E}[X_t] < \infty$. We denote X_{t_i} as S_i. We also assume that probability P is a risk-neutral pricing probability.

Under these settings, let us formulate the pricing problem as dynamic programming. An option holder can choose an action from buying, selling and not exercising, and then $V(L_{\rm b}, L_{\rm d}, L_{\rm s}, i)$, the price of the swing option with $(L_{\rm b}, L_{\rm d}, L_{\rm s})$ at t_i, is represented as

$$\begin{aligned} V(L_{\rm b}, L_{\rm d}, L_{\rm s}, i) \;=\; & \max[Y(L_{\rm b}, L_{\rm d}, L_{\rm s}, i), Z^{\rm b}(i) + Y(L_{\rm b} - 1, L_{\rm d}, L_{\rm s}, i), \\ & Z^{\rm s}(i) + Y(L_{\rm b}, L_{\rm d}, L_{\rm s} - 1, i)] \end{aligned} \tag{1}$$

[2]Straddles are options whose holder can both buy and sell an underlying asset.

where $Y(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i)$ is a continuation value, and $Z^\mathrm{b}(i)$ and $Z^\mathrm{s}(i)$ are payoffs by exercising a right to buy and sell, respectively:

$$
\begin{gathered}
Y(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i) = \mathrm{E}[V(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i+1) \mid \mathcal{F}_{t_i}], \\
Y(0,0,0,T) = 0, \quad Z^\mathrm{b}(i) = \mathrm{e}^{-rt_i} v_{\max}(S_i - K), \quad Z^\mathrm{s}(i) = \mathrm{e}^{-rt_i} v_{\min}(S_i - K),
\end{gathered} \tag{2}
$$

where r is the spot rate, which we assume to be constant. For simplicity, we abbreviate $\mathrm{E}[\cdot \mid \mathcal{F}_{t_i}]$ to $\mathrm{E}_i[\cdot]$.

Remark 2: The notation of equation (1) is not accurate, since for example $Y(L_\mathrm{b} - 1, L_\mathrm{d}, L_\mathrm{s}, i)$ cannot be defined when $L_\mathrm{b} = 0$. But we use this notation for simplicity. Later we prove theorems in this abbreviate notation, because it is easy to extend the proofs to the general case by a similar procedure.

Let us define the difference of the value of options between different actions.

$$
\begin{aligned}
U^\mathrm{b}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i) &= Y(L_\mathrm{b} - 1, L_\mathrm{d}, L_\mathrm{s}, i) + Z^\mathrm{b}(i) - Y(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i), \\
U^\mathrm{s}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i) &= Y(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s} - 1, i) + Z^\mathrm{s}(i) - Y(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i), \\
U^\mathrm{bs}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i) &= Y(L_\mathrm{b} - 1, L_\mathrm{d}, L_\mathrm{s}, i) + Z^\mathrm{b}(i) - Z^\mathrm{s}(i) - Y(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s} - 1, i).
\end{aligned} \tag{3}
$$

Then we define an optimal exercise strategy ξ as follows:

$$
\xi(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i) = \begin{cases} 1 & (U^\mathrm{b}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i) \geq 0), \\ -1 & (U^\mathrm{s}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i) \geq 0,\ U^\mathrm{b}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i) < 0), \\ 0 & (\text{otherwise}). \end{cases} \tag{4}
$$

Optimal exercise strategies between different states intuitively have some relations. Actually, the next theorem shows monotonicity of optimal exercise strategies.

Theorem 1: For any $(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s})$ and any i, the following monotonicity holds.

- If $\xi(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i) = 1$, then $\xi(L_\mathrm{b} + 1, L_\mathrm{d}, L_\mathrm{s} - 1, i) = 1$.
- If $\xi(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s} - 1, i) = 1$, then $\xi(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i) = 1$.
- If $\xi(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i) = -1$, then $\xi(L_\mathrm{b} - 1, L_\mathrm{d}, L_\mathrm{s} + 1, i) = -1$.
- If $\xi(L_\mathrm{b} - 1, L_\mathrm{d}, L_\mathrm{s}, i) = -1$, then $\xi(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i) = -1$.

Proofs are given in Appendix. The monotonicity is easily-interpreted on a transition tree like Figure 1. For example, in Figure 2, if a right to buy is exercised at a node a, then a right to buy is also exercised at the parent nodes and at the node b such that node a and b have a same parent and node b has more rights to buy than node a.

Next we consider optimal exercise boundaries. As Ibanez [7], we hold some assumptions about the underlying asset price process $\{X_t\}$. The following properties holds a.s. for a function $F(i, S_i) \equiv \mathrm{E}_i[\mathrm{e}^{-r\Delta t_i} f(i+1, S_{i+1})]$ and $f(i, S_i) = \max[h(S_i), F(i, S_i)]$ where $\Delta t_i = t_{i+1} - t_i$ and $h(S_i)$ is a payoff function.

(i) If f is a continuous function for S, then F is a continuous and differentiable function for S.

(ii) Let $f_S(i, S_i)$ denote the derivative of f with respect to S_i. For any c_1, c_2, if $c_1 \leq f_S(i+1, S_{i+1}) \leq c_2$ then $c_1 \leq F_S(i, S_i) \leq c_2$. In particular, if $f_S(i+1, S_{i+1}) \neq c_1, c_2$ then $c_1 < F_S(i, S_i) < c_2$.

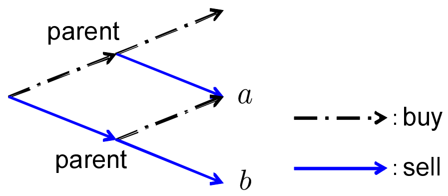

Figure 2: Monotonicity of the swing options on a transition tree

(iii) When $F_S(i, S_i) = h_S(S_i) = c$, $F(i, S_i) \neq h(S_i)$ for any S_i.

The property (iii) is a condition for excluding the case that the exercise boundary of obligations becomes indeterminate. A mean-reverting process holds these properties, and Brownian motion holds these properties in the case of $r > 0$. For Brownian motion with $r = 0$, the property (iii) is not satisfied.

Then we define optimal exercise boundaries.

Definition 1: we call $S^{\rm b}(L_{\rm b}, L_{\rm d}, L_{\rm s}, i)$, $S^{\rm s}(L_{\rm b}, L_{\rm d}, L_{\rm s}, i)$ and $S^{\rm bs}(L_{\rm b}, L_{\rm d}, L_{\rm s}, i)$ as optimal exercise boundaries of the swing option with $(L_{\rm b}, L_{\rm d}, L_{\rm s})$ at t_i when $S^{\rm b}(L_{\rm b}, L_{\rm d}, L_{\rm s}, i)$ holds $U^{\rm b}(L_{\rm b}, L_{\rm d}, L_{\rm s}, i) = 0$, $S^{\rm s}(L_{\rm b}, L_{\rm d}, L_{\rm s}, i)$ holds $U^{\rm s}(L_{\rm b}, L_{\rm d}, L_{\rm s}, i) = 0$ and $S^{\rm bs}(L_{\rm b}, L_{\rm d}, L_{\rm s}, i)$ holds $U^{\rm bs}(L_{\rm b}, L_{\rm d}, L_{\rm s}, i) = 0$.

We show that optimal exercise boundaries of the swing options are unique and monotone.

Theorem 2: Optimal exercise boundaries of the swing options satisfy following uniqueness and monotonicity.

1. $S^{\rm b}(L_{\rm b}, L_{\rm d}, L_{\rm s}, i)$ is unique, in other words, S_i which satisfies $U^{\rm b}(L_{\rm b}, L_{\rm d}, L_{\rm s}, i) = 0$ is at most one point.

2. $S^{\rm s}(L_{\rm b}, L_{\rm d}, L_{\rm s}, i)$ is unique.

3. $S^{\rm bs}(L_{\rm b}, L_{\rm d}, L_{\rm s}, i)$ is unique.

4. For any $(L_{\rm b}, L_{\rm d}, L_{\rm s})$ such that $L_{\rm b} + L_{\rm d} + L_{\rm s} < T - i + 1$, the greater the number of exercise rights is, the larger the exercise region of buying is, that is,

$$S^{\rm b}(L_{\rm b}, L_{\rm d}, L_{\rm s}, i) \leq S^{\rm b}(L_{\rm b} - 1, L_{\rm d}, L_{\rm s}, i),\ \ S^{\rm b}(L_{\rm b}, L_{\rm d}, L_{\rm s}, i) \leq S^{\rm b}(L_{\rm b}, L_{\rm d}, L_{\rm s} - 1, i).$$

   For $(L_{\rm b}, L_{\rm d}, L_{\rm s})$ such that $L_{\rm b} + L_{\rm d} + L_{\rm s} = T - i + 1$,

$$S^{\rm bs}(L_{\rm b}, L_{\rm d}, L_{\rm s}, i) \leq S^{\rm b}(L_{\rm b} - 1, L_{\rm d}, L_{\rm s}, i),\ \ S^{\rm bs}(L_{\rm b}, L_{\rm d}, L_{\rm s}, i) \leq S^{\rm b}(L_{\rm b}, L_{\rm d}, L_{\rm s} - 1, i).$$

5. For any $(L_{\rm b}, L_{\rm d}, L_{\rm s})$ such that $L_{\rm b} + L_{\rm d} + L_{\rm s} < T - i + 1$, the greater the proportion of obligations to buy, the larger the exercise region of buying is, that is,

$$S^{\rm b}(L_{\rm b}, L_{\rm d}, L_{\rm s}, i) \leq S^{\rm b}(L_{\rm b} - 1, L_{\rm d}, L_{\rm s} + 1, i).$$

   For $(L_{\rm b}, L_{\rm d}, L_{\rm s})$ such that $L_{\rm b} + L_{\rm d} + L_{\rm s} = T - i + 1$,

$$S^{\rm bs}(L_{\rm b}, L_{\rm d}, L_{\rm s}, i) \leq S^{\rm bs}(L_{\rm b} - 1, L_{\rm d}, L_{\rm s} + 1, i).$$

The properties 4-5 mean monotonicity. Similar monotonicity holds for S^{s}.

Figure 3 is an example of optimal exercise boundaries that hold Theorem 2. For example, if there is an option with $(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}})$ and $S_i \geq S^{\mathrm{b}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i)$, then exercise of a right to buy is optimal at t_i. We note that the properties (i)-(iii) about $\{X_t\}$ are necessary only for proving uniqueness and not necessary for monotonicity.

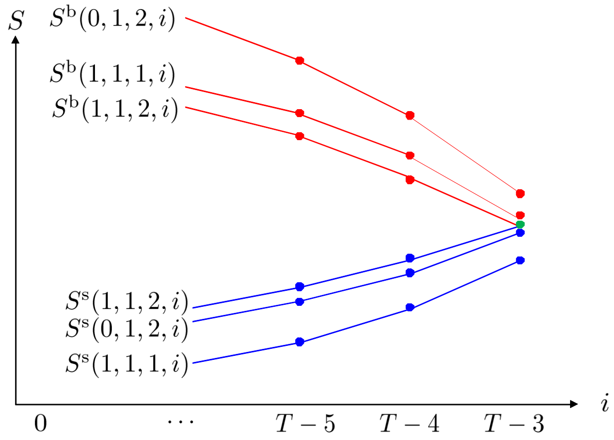

Figure 3: An example of optimal exercise boundaries

4 Main result

In this section, we show the main result that gives upper bounds for the price of the swing options.

The pricing problem of the swing option is formulated as the following multiple stopping problem:

$$\begin{aligned} V(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, 0) &= \sup_{\tau,\lambda} \sum_{j=1}^{L} \mathrm{E}[Z^{\lambda_j}(\tau_j)] \\ \text{s.t.} \quad & 0 \leq \tau_1 < \ldots < \tau_L \leq T, \quad \lambda_j \in \{\mathrm{b}, \mathrm{s}\}, \\ & \sum_{j=1}^{L} \mathbf{1}_{\{\lambda_j = \mathrm{b}\}} \geq L_{\mathrm{b}}, \quad \sum_{j=1}^{L} \mathbf{1}_{\{\lambda_j = \mathrm{s}\}} \geq L_{\mathrm{s}}. \end{aligned} \tag{5}$$

The optimal stopping time of problem (5) can be constructed from an optimal exercise strategy of the swing options. For example,

$$\tau_1^* = \inf\{i \mid U^{\mathrm{b}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i) \geq 0 \text{ or } U^{\mathrm{s}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i) \geq 0 \text{ or } i = T - L + 1\},$$

$$\lambda_1^* = \mathbf{1}_{U^{\mathrm{b}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i) \geq 0} - \mathbf{1}_{U^{\mathrm{s}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i) \geq 0} + \mathbf{1}_{i = T-L+1} \cdot (\mathbf{1}_{U^{\mathrm{bs}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i) \geq 0} - \mathbf{1}_{U^{\mathrm{bs}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i) < 0}).$$

Therefore a solution of problem (5) also has monotonicity.

Our goal is to give an upper bound of $V(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, 0)$, but we do not use the methods in the existing studies because problem (5) includes choices to buy or sell.

To solve problem (5), we aim to bound problem (5) from above by single optimal stopping problems. We define a second-order difference of the price for the number of exercise rights.

$$\begin{aligned} \Delta\Delta V(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i) &= V(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i) - V(L_{\mathrm{b}} - 1, L_{\mathrm{d}}, L_{\mathrm{s}}, i) \\ &\quad - V(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}} - 1, i) + V(L_{\mathrm{b}} - 1, L_{\mathrm{d}}, L_{\mathrm{s}} - 1, i). \end{aligned} \tag{6}$$

We call swing options with $(L_{\rm b}-1, L_{\rm d}, L_{\rm s})$, $(L_{\rm b}, L_{\rm d}, L_{\rm s}-1)$ and $(L_{\rm b}-1, L_{\rm d}, L_{\rm s}-1)$ as components of a second-order difference with $(L_{\rm b}, L_{\rm d}, L_{\rm s})$.

Using equation (6), we can decompose the price of the swing option into the sum of the second-order differences:

$$V(L_{\rm b}, L_{\rm d}, L_{\rm s}, 0) = \sum_{l \in \mathcal{L}(L_{\rm b}, L_{\rm d}, L_{\rm s})} \Delta\Delta V(l_{\rm b}, l_{\rm d}, l_{\rm s}, 0), \tag{7}$$

where $l \equiv (l_{\rm b}, l_{\rm d}, l_{\rm s})$ and $\mathcal{L}(L_{\rm b}, L_{\rm d}, L_{\rm s})$ is the set of nodes in the transition tree whose root is $(L_{\rm b}, L_{\rm d}, L_{\rm s})$ like Figure 1.

Example 1: A set $\mathcal{L}(2,1,0)$ is defined as $\mathcal{L}(2,1,0) = \{(2,1,0),\ (1,1,0),\ (2,0,0),\ (1,0,0),\ (1,0,0),\ (0,1,0)\}$. Note the two occurrences of $(1,0,0)$ in $\mathcal{L}(2,1,0)$.

Next, we define a payoff function of a second-order difference. For multiple American options, Bender [3] defines a payoff function of a first-order difference. He gives an exercise strategy of a component of the first-order difference. Then the function takes a sufficiently negative value when one more right cannot be exercised at the time on the exercise strategy, and otherwise takes a normal payoff. We similarly define a payoff function, but we must add adjustment terms to the function because a second-order difference has four terms. The concrete formulations are given in Appendix.

We consider optimal stopping problems that correspond to each second-order difference in the equation (7). The next theorem shows that the sum of the optimal stopping problems gives an upper bound for the sum of the second-order differences:

Theorem 3: For any $(L_{\rm b}, L_{\rm d}, L_{\rm s})$ such that $L_{\rm b}, L_{\rm d}, L_{\rm s} \geq 0$ and $L_{\rm b} + L_{\rm d} + L_{\rm s} \leq T+1$, if an exercise strategy ξ is monotone, and satisfies some conditions related to the optimal exercise boundary ξ^*, then it holds that

$$\begin{aligned} V(L_{\rm b}, L_{\rm d}, L_{\rm s}, 0) &= \sum_{l \in \mathcal{L}(L_{\rm b}, L_{\rm d}, L_{\rm s})} \Delta\Delta V(l_{\rm b}, l_{\rm d}, l_{\rm s}, 0) \\ &\leq \sum_{l \in \mathcal{L}(L_{\rm b}, L_{\rm d}, L_{\rm s})} \sup_{0 \leq \tau \leq T} {\rm E}[Z_l^{\xi}(\tau)], \end{aligned} \tag{8}$$

where $Z_l^{\xi}(i)$ is an adjusted payoff function determined by the exercise history of components of a second-order difference with $(l_{\rm b}, l_{\rm d}, l_{\rm s})$ from t_0 to t_i, which depends on ξ. The inequality becomes an equality when $\xi = \xi^*$.

All proofs in this section and the description of the conditions are provided in Appendix. We note that Theorem 3 is about the sum of second-order differences and the inequality does not hold about a second-order difference.

The right side of equation (8) is the sum of single stopping problems, and thus we can get a dual representation of the problem that gives an upper bound for the option price:

Theorem 4: For any $(L_{\rm b}, L_{\rm d}, L_{\rm s})$ such that $L_{\rm b}, L_{\rm d}, L_{\rm s} \geq 0$ and $L_{\rm b} + L_{\rm d} + L_{\rm s} \leq T+1$, it holds for any martingale M_l with $M_l(0) = 0$ that

$$\begin{aligned} V(L_{\rm b}, L_{\rm d}, L_{\rm s}, 0) &= \sum_{l \in \mathcal{L}(L_{\rm b}, L_{\rm d}, L_{\rm s})} \sup_{0 \leq \tau \leq T} {\rm E}[Z_l^{\xi^*}(\tau)] \\ &\leq \sum_{l \in \mathcal{L}(L_{\rm b}, L_{\rm d}, L_{\rm s})} {\rm E}\left[\max_{i=0,\ldots,T}[Z_l^{\xi^*}(i) - M_l(i)]\right]. \end{aligned} \tag{9}$$

The inequality becomes an equality when $M_l(i)$ is the martingale part of the Doob decomposition of $\Delta\Delta V(\hat{l}_{\mathrm{b}}^{\xi^*}(i), \hat{l}_{\mathrm{d}}^{\xi^*}(i), \hat{l}_{\mathrm{s}}^{\xi^*}(i), i)$ where $(\hat{l}_{\mathrm{b}}^{\xi}(i), \hat{l}_{\mathrm{d}}^{\xi}(i), \hat{l}_{\mathrm{s}}^{\xi}(i))$ is residual rights determined by the exercise history of components of the second-order difference with $(l_{\mathrm{b}}, l_{\mathrm{d}}, l_{\mathrm{s}})$ from t_0 to t_i, which depends on ξ.

The problem size $|\mathcal{L}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}})|$ is $\mathrm{O}(L^2)$, so our method is robust to the number of rights.

5 Algorithm and numerical result

5.1 Algorithm

In this section we address a numerical algorithm for calculating upper and lower bounds for the price of swing options. We recall the assumption that the underlying asset price process satisfies the properties in Section 3, i.e. optimal exercise boundaries are unique[3].

For calculating an upper bound for the price of options, Andersen and Broadie [4] and Bender [3] used nested simulation. We cannot apply their methods without change, because an exercise strategy must satisfy monotonicity. Therefore we construct exercise boundaries as follows.

First we run least-squares Monte Carlo regression for all rights $(l_{\mathrm{b}}, l_{\mathrm{d}}, l_{\mathrm{s}})$ and all time t_i. Using the obtained regression coefficients, we can calculate continuation value $\hat{y}(l_{\mathrm{b}}, l_{\mathrm{d}}, l_{\mathrm{s}}, i, S_i)$ for all $(l_{\mathrm{b}}, l_{\mathrm{d}}, l_{\mathrm{s}})$ and i.

Second we construct exercise boundaries. We use the following backward algorithm for $i = T-1, \ldots, 0$ and for all $(l_{\mathrm{b}}, l_{\mathrm{d}}, l_{\mathrm{s}}) \in \mathcal{L}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}})$.

1. Set an initial value $\hat{S}_i$ as $\hat{S}^{\mathrm{b}}(l_{\mathrm{b}}, l_{\mathrm{d}}, l_{\mathrm{s}}, i)$.
2. Generate N paths from $\hat{S}_i$, i.e. $S_{i+1}^1, \ldots, S_{i+1}^N$.
3. Compute a difference $d = z^{\mathrm{b}}(i, \hat{S}_i) + \hat{Y}(l_{\mathrm{b}}-1, l_{\mathrm{d}}, l_{\mathrm{s}}, i) - \hat{Y}(l_{\mathrm{b}}, l_{\mathrm{d}}, l_{\mathrm{s}}, i)$. If $|d|$ is sufficiently small, then set $\hat{S}^{\mathrm{b}}(l_{\mathrm{b}}, l_{\mathrm{d}}, l_{\mathrm{s}}, i) = \hat{S}_i$ and go to step 4, and if $|d|$ is not small, then update $\hat{S}_i$ by the secant method and return step 2.
4. If $\hat{S}^{\mathrm{b}}(l_{\mathrm{b}}, l_{\mathrm{d}}, l_{\mathrm{s}}, i) > \hat{S}^{\mathrm{b}}(l_{\mathrm{b}}, l_{\mathrm{d}}, l_{\mathrm{s}}-1, i)$, then update $\hat{S}^{\mathrm{b}}(l_{\mathrm{b}}, l_{\mathrm{d}}, l_{\mathrm{s}}, i) = \hat{S}^{\mathrm{b}}(l'_{\mathrm{b}}, l'_{\mathrm{d}}, l'_{\mathrm{s}}, i)$. Similarly update for $\hat{S}^{\mathrm{b}}(l_{\mathrm{b}}-1, l_{\mathrm{d}}, l_{\mathrm{s}}+1, i)$,

where

$$\begin{aligned}\hat{Y}(l_{\mathrm{b}}, l_{\mathrm{d}}, l_{\mathrm{s}}, i) = \frac{1}{N}\sum_{n=1}^{N} \max \Big[& \hat{y}(l_{\mathrm{b}}, l_{\mathrm{d}}, l_{\mathrm{s}}, i+1, S_{i+1}^n), \\ & z^{\mathrm{b}}(i+1, S_{i+1}^n) + \hat{y}(l_{\mathrm{b}}-1, l_{\mathrm{d}}, l_{\mathrm{s}}, i+1, S_{i+1}^n), \\ & z^{\mathrm{s}}(i+1, S_{i+1}^n) + \hat{y}(l_{\mathrm{b}}, l_{\mathrm{d}}, l_{\mathrm{s}}-1, i+1, S_{i+1}^n) \Big]\end{aligned}$$

and $z^{\mathrm{b}}(i+1, S_{i+1})$ and $z^{\mathrm{s}}(i+1, S_{i+1})$ are precise forms of $Z^{\mathrm{b}}(i+1)$ and $Z^{\mathrm{s}}(i+1)$, respectively. Similarly, $\hat{S}^{\mathrm{s}}, \hat{S}^{\mathrm{bs}}$ are constructed. Step 4 ensures that the constructed boundary satisfies monotonicity. The condition related to optimal strategy ξ^* in Theorem 3 is not ensured, but as noted in Appendix B the condition is not severe, so is practically ensured.

[3] In the algorithm, uniqueness of exercise boundary is not necessary. However, if uniqueness is not assumed, then it is complex to estimate the region such that $\hat{U}^{\mathrm{b}}(l_{\mathrm{b}}, l_{\mathrm{d}}, l_{\mathrm{s}}, i) > 0$ and $\hat{U}^{\mathrm{s}}(l_{\mathrm{b}}, l_{\mathrm{d}}, l_{\mathrm{s}}, i) > 0$ where $\hat{U}^{\mathrm{b}}$,$\hat{U}^{\mathrm{s}}$ are estimation of U^{b},U^{s}, respectively. Thus we discuss the algorithm under the assumption of uniqueness.

Using the constructed exercise boundary, we calculate a lower bound for the price of the option by simulation with N_L paths from $i = 0$ to T.

For computing an upper bound, we use a similar method to Bender [3]. A difference from the method of Bender is in the estimation of martingale $M(i)$ in equation (9). For the estimation, Bender generates one period nested paths. For each nested path he determines whether exercising rights or not and calculates the option value by using the least-square regression coefficients. On the other hand, we use exercise boundaries. Concretely, we generate one period nested paths and for each nested path we determine exercising or not by exercise boundaries, and the least-square regression coefficients are used only calculating the option value. This method is same burden as the method in previous studies and ensures to give an upper bound for the price.

The algorithm is as follows.

1. Generate N_U paths with T periods from S_0.
2. For $i = 0, \ldots, T-1$ and for $j = 1, \ldots, N_U$,
   (a) Generate N_X subpaths from S_i^j.
   (b) For $(l_{\rm b}, l_{\rm d}, l_{\rm s}) \in \mathcal{L}(L_{\rm b}, L_{\rm d}, L_{\rm s})$,
      i. For $k = 1, \ldots, N_X$, determine a right is exercised or not for $(l_{\rm b}, l_{\rm d}, l_{\rm s})$ on $S_{i+1}^{j,k}$ $(j = 1, \ldots, N_U)$ using exercise boundaries. Let us define
$$\hat{Z}(i+1, S_{i+1}^{j,k}) = \begin{cases} z^{\rm b}(i+1, S_{i+1}^{j,k}) & (S_{i+1}^{j,k} \geq \hat{S}^{\rm b}(l_{\rm b}, l_{\rm d}, l_{\rm s}, i)), \\ z^{\rm s}(i+1, S_{i+1}^{j,k}) & (S_{i+1}^{j,k} \leq \hat{S}^{\rm s}(l_{\rm b}, l_{\rm d}, l_{\rm s}, i)), \\ 0 & \text{(otherwise)} \end{cases}$$
      and define $(l'_{\rm b}, l'_{\rm d}, l'_{\rm s})$ as residual rights after exercise.
      ii. Compute
$$\mathrm{E}_i[\hat{V}_l^j(i+1)] = \frac{1}{N_X} \sum_{k=1}^{N_X} \left(\hat{Z}(i+1, S_{i+1}^{j,k}) + \hat{y}(l'_{\rm b}, l'_{\rm d}, l'_{\rm s}, i, S_{i+1}^{j,k}) \right).$$
   (c) Determine a right is exercised or not for $(l_{\rm b}, l_{\rm d}, l_{\rm s})$ on S_{i+1}^j and set
$$\hat{V}_l^j(i+1) = \hat{Z}(i+1, S_{i+1}^j) + \hat{y}(l'_{\rm b}, l'_{\rm d}, l'_{\rm s}, i, S_{i+1}^j).$$
   (d) Compute
$$M_l^j(i+1) = M_l^j(i) + \Delta\Delta\hat{V}_{\hat{l}}^j(i+1) - \mathrm{E}_i[\Delta\Delta\hat{V}_{\hat{l}}^j(i+1)],$$
   where $\hat{l}$ is an abbreviated form of $\hat{l}_{\rm b}^{\xi}(i+1), \hat{l}_{\rm d}^{\xi}(i+1), \hat{l}_{\rm s}^{\xi}(i+1)$.

5.2 Numerical Example

We show a numerical result that computes an upper bound for swing options. We assume that the underlying asset process $\{X_t\}$ is the following mean-reverting process:
$$\mathrm{d}X_t = -3(X_t - 40)\mathrm{d}t + 0.5\mathrm{d}W_t, \quad X_{t_0} = S_0 = 40.$$
We set parameters to $r = 0$, $K = 40$, $v_{\max} = 1$, $v_{\min} = -1$, $T = 20, 60, 100$ and $\Delta t_i \equiv t_i - t_{i-1} = 1/24$ for all i. In these settings, we price five different swing options

Table 1: A numerical example of upper and lower bounds for the price of swing options

rights	$T = 20$		$T = 60$		$T = 100$	
	lower	upper	lower	upper	lower	upper
$(2, 2, 2)$	0.8985	0.9007	1.5969	1.5987	1.9408	1.9426
	(0.0011)	(0.0006)	(0.0013)	(0.0009)	(0.0013)	(0.0015)
$(4, 4, 4)$	1.5927	1.5934	2.9912	2.9955	3.6716	3.6738
	(0.0019)	(0.0010)	(0.0024)	(0.0015)	(0.0023)	(0.0020)
$(6, 6, 6)$	2.0638	2.0692	4.2060	4.2133	5.2251	5.2421
	(0.0024)	(0.0015)	(0.0033)	(0.0034)	(0.0033)	(0.0094)
$(8, 8, 8)$	-	-	5.2584	5.2676	6.6286	6.6602
	-	-	(0.0041)	(0.0040)	(0.0041)	(0.0101)
$(10, 10, 10)$	-	-	6.1646	6.1864	7.9007	7.9364
	-	-	(0.0047)	(0.0046)	(0.0048)	(0.0079)

Notes. Standard errors are in parentheses. For $(8, 8, 8)$ and $(10, 10, 10)$, the column of $T = 20$ is blank because the number of rights exceed $T + 1$.

in the number of rights. Also, we set $N_L = 1,000,000$, $N = 5000$, $N_U = 100$ and $N_X = 10000$.

Figure 4 depicts a part of optimal exercise boundaries. We confirm that uniqueness and monotonicity holds. Table 1 indicates that for each option the difference between upper and lower bounds is less than 1% of the price. This implies that our method gives appropriate upper bounds.

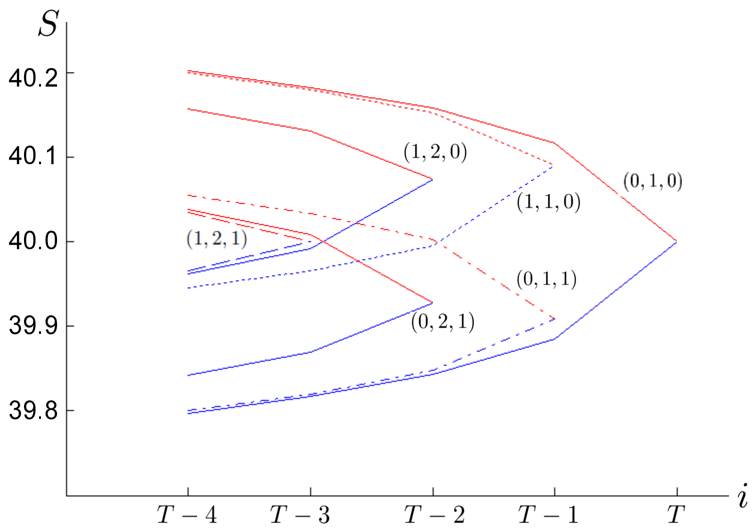

Figure 4: Optimal exercise boundaries obtained from a numerical example

6 Conclusion

In this paper, we have proposed a dual approach for pricing swing options with bang-bang control. The existing methods cannot treat options such that the holder can choose buying or selling. So, we have considered optimal stopping problems that correspond to the value of second-order differences of the swing options. We have shown that the sum of the optimal stopping problems gives a tight upper bound for the price of the options. A numerical example indicates that our methods give an appropriate upper bound with an accuracy of 1 % .

A future work is to extend the method to pricing problems with more general constraints, such as DCQ and ACQ. When DCQ is constant for all time periods, it is known that changed volumes can be limited to some discrete values in an optimal strategy. Using this character, we will be able to extend the method in a similar way.

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A Proof of Theorem in Section 3

A.1 Proof of Theorem 1

We prove Theorem 1 by backward induction on i. As described in Remark 2, the proof is an abbreviate version.

First, at $i = T$, the number of residual rights is at most 1, so we need to show the property only about $\xi(1,0,0,i), \xi(0,1,0,i), \xi(0,0,1,i)$. $\xi(1,0,0,i)$ must be equal to 1 and $\xi(0,0,1,i)$ must be equal to -1, so monotonicity trivially holds.

Next we assume that monotonicity is satisfied at $i = T-k+1, \dots, T$, and show that monotonicity is also satisfied at $i = T-k$.

Let us prove monotonicity for rights to buy, that is, $\xi(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i) = 1$. To begin with, we prove the case of $T - i + 1 > L_\mathrm{b} + L_\mathrm{d} + L_\mathrm{s}$. For simplicity, we define two difference of U.

$$\begin{aligned}
\Delta_1 U^\mathrm{b}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i) &\equiv U^\mathrm{b}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i) - U^\mathrm{b}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}-1, i), \\
\Delta_2 U^\mathrm{b}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i) &\equiv U^\mathrm{b}(L_\mathrm{b}+1, L_\mathrm{d}, L_\mathrm{s}-1, i) - U^\mathrm{b}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i).
\end{aligned}$$

Monotonicity is equivalent to $\Delta_1 U^\mathrm{b}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i), \Delta_2 U^\mathrm{b}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i) \geq 0$, so let us prove these equations. We define

$$\begin{aligned}
\mathcal{B}_1^\mathrm{b} &= \{U^\mathrm{b}(L_\mathrm{b}-1, L_\mathrm{d}, L_\mathrm{s}, i+1) \geq 0\},\ \mathcal{B}_1^\mathrm{s} = \{U^\mathrm{s}(L_\mathrm{b}-1, L_\mathrm{d}, L_\mathrm{s}, i+1) \geq 0\}, \\
\mathcal{B}_2^\mathrm{b} &= \{U^\mathrm{b}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i+1) \geq 0\},\ \mathcal{B}_2^\mathrm{s} = \{U^\mathrm{s}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i+1) \geq 0\}, \\
\mathcal{B}_3^\mathrm{b} &= \{U^\mathrm{b}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}-1, i+1) \geq 0\},\ \mathcal{B}_3^\mathrm{s} = \{U^\mathrm{s}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}-1, i+1) \geq 0\}, \\
\mathcal{B}_4^\mathrm{b} &= \{U^\mathrm{b}(L_\mathrm{b}-1, L_\mathrm{d}, L_\mathrm{s}-1, i+1) \geq 0\},\ \mathcal{B}_4^\mathrm{s} = \{U^\mathrm{s}(L_\mathrm{b}-1, L_\mathrm{d}, L_\mathrm{s}-1, i+1) \geq 0\},
\end{aligned}$$

and then

$$\begin{aligned}
&\Delta_1 U^\mathrm{b}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i) \\
&= -Y(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i) + Y(L_\mathrm{b}-1, L_\mathrm{d}, L_\mathrm{s}, i) + Y(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}-1, i) - Y(L_\mathrm{b}-1, L_\mathrm{d}, L_\mathrm{s}-1, i) \\
&= \mathrm{E}_i[\Delta_1 U^\mathrm{b}(L_\mathrm{b}-1, L_\mathrm{d}, L_\mathrm{s}, i+1)\mathbf{1}_{\mathcal{B}_4^\mathrm{b}} + U^\mathrm{b}(L_\mathrm{b}-1, L_\mathrm{d}, L_\mathrm{s}, i+1)\mathbf{1}_{\bar{\mathcal{B}}_4^\mathrm{b} \cup \mathcal{B}_1^\mathrm{b}} + 0\mathbf{1}_{\bar{\mathcal{B}}_1^\mathrm{b} \cup \mathcal{B}_3^\mathrm{b}} \\
&\quad - U^\mathrm{b}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}-1, i+1)\mathbf{1}_{\bar{\mathcal{B}}_3^\mathrm{b} \cup \mathcal{B}_2^\mathrm{b}} + \Delta_1 U^\mathrm{b}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i+1)\mathbf{1}_{\bar{\mathcal{B}}_2^\mathrm{b} \cup \bar{\mathcal{B}}_2^\mathrm{s}} \\
&\quad - U^\mathrm{s}(L_\mathrm{b}-1, L_\mathrm{d}, L_\mathrm{s}, i+1)\mathbf{1}_{\bar{\mathcal{B}}_1^\mathrm{s} \cup \mathcal{B}_2^\mathrm{s}} + 0\mathbf{1}_{\bar{\mathcal{B}}_3^\mathrm{s} \cup \mathcal{B}_1^\mathrm{s}} \\
&\quad + U^\mathrm{s}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}-1, i+1)\mathbf{1}_{\bar{\mathcal{B}}_4^\mathrm{s} \cup \mathcal{B}_3^\mathrm{s}} + \Delta_1 U^\mathrm{b}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}-1, i+1)\mathbf{1}_{\mathcal{B}_4^\mathrm{s}}] \\
&\geq 0,
\end{aligned}$$

where $\mathbf{1}_B$ is the indicator function[4]. The inequality is by induction. Similarly, we define $\mathcal{B}_5^\mathrm{b} = \{U^\mathrm{b}(L_\mathrm{b}+1, L_\mathrm{d}, L_\mathrm{s}-1, i+1) \geq 0\}$ and $\mathcal{B}_5^\mathrm{s} = \{U^\mathrm{s}(L_\mathrm{b}+1, L_\mathrm{d}, L_\mathrm{s}-1, i+1) \geq 0\}$, and

[4]That is to say, $\mathbf{1}_B = \begin{cases} 1 & (S_{i+1} \in B), \\ 0 & (S_{i+1} \notin B). \end{cases}$

then

$$
\begin{aligned}
&\Delta_2 U^{\mathrm{b}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i) \\
&= -Y(L_{\mathrm{b}}+1, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i) + Y(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i) + Y(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i) - Y(L_{\mathrm{b}}-1, L_{\mathrm{d}}, L_{\mathrm{s}}, i) \\
&= \mathrm{E}_i[\Delta_2 U^{\mathrm{b}}(L_{\mathrm{b}}-1, L_{\mathrm{d}}, L_{\mathrm{s}}, i+1)\mathbf{1}_{\mathcal{B}_1^{\mathrm{b}}} + U^{\mathrm{b}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i+1)\mathbf{1}_{\bar{\mathcal{B}}_1^{\mathrm{b}} \cup \mathcal{B}_3^{\mathrm{b}}} \\
&\quad + 0\mathbf{1}_{\bar{\mathcal{B}}_3^{\mathrm{b}} \cup \mathcal{B}_2^{\mathrm{b}}} - U^{\mathrm{b}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i+1)\mathbf{1}_{\bar{\mathcal{B}}_2^{\mathrm{b}} \cup \mathcal{B}_5^{\mathrm{b}}} + \Delta_2 U^{\mathrm{b}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i+1)\mathbf{1}_{\bar{\mathcal{B}}_5^{\mathrm{b}} \cup \bar{\mathcal{B}}_2^{\mathrm{s}}} \\
&\quad + (\Delta_2 U^{\mathrm{b}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i+1) + U^{\mathrm{s}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i+1))\mathbf{1}_{(\bar{\mathcal{B}}_5^{\mathrm{s}} \cap \bar{\mathcal{B}}_1^{\mathrm{s}}) \cup \mathcal{B}_2^{\mathrm{s}}} \\
&\quad + (\Delta_1 U^{\mathrm{b}}(L_{\mathrm{b}}+1, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i+1) + \Delta_2 U^{\mathrm{b}}(L_{\mathrm{b}}+1, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i+1))\mathbf{1}_{\bar{\mathcal{B}}_5^{\mathrm{s}} \cup \mathcal{B}_1^{\mathrm{s}}} \\
&\quad + (\Delta_2 U^{\mathrm{b}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i+1) + \Delta_2 U^{\mathrm{s}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i+1))\mathbf{1}_{\bar{\mathcal{B}}_1^{\mathrm{s}} \cup \mathcal{B}_5^{\mathrm{s}}} \\
&\quad + (\Delta_2 U^{\mathrm{b}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i+1) - U^{\mathrm{s}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i+1))\mathbf{1}_{\bar{\mathcal{B}}_3^{\mathrm{s}} \cup (\mathcal{B}_1^{\mathrm{s}} \cap \mathcal{B}_5^{\mathrm{s}})} \\
&\quad + \Delta_2 U^{\mathrm{b}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i+1)\mathbf{1}_{\mathcal{B}_3^{\mathrm{s}}}] \\
&\geq 0.
\end{aligned}
$$

Consequently, S^{b} is monotone in the case of $T - i + 1 > L_{\mathrm{b}} + L_{\mathrm{d}} + L_{\mathrm{s}}$.

Next we prove the case of $T - i + 1 = L_{\mathrm{b}} + L_{\mathrm{d}} + L_{\mathrm{s}}$. For monotonicity between $\xi(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i)$ and $\xi(L_{\mathrm{b}}+1, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i)$, the proof is quite similar to that in the case of $T - i + 1 > L_{\mathrm{b}} + L_{\mathrm{d}} + L_{\mathrm{s}}$, so we abbreviate the proof.

Let us prove monotonicity between $\xi(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i)$ and $\xi(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i)$. We will show that if $U^{\mathrm{b}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i) \geq 0$ then $U^{\mathrm{bs}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i) \geq 0$. Because of $U^{\mathrm{bs}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i) = U^{\mathrm{b}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i) - U^{\mathrm{s}}(L_{\mathrm{b}}-1, L_{\mathrm{d}}, L_{\mathrm{s}}, i)$, we only need to show that if $U^{\mathrm{b}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i) \geq 0$ then $U^{\mathrm{s}}(L_{\mathrm{b}}-1, L_{\mathrm{d}}, L_{\mathrm{s}}, i) \leq 0$.

Here, we use the decomposition of the swing option. As we mentioned in Section 2, $V(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i)$ with $L_{\mathrm{b}} + L_{\mathrm{d}} + L_{\mathrm{s}} = T - i + 1$ can be decomposed into the value of obligations to sell (buy) and a multiple American call (put) option. Let us denote $V^{\mathrm{c}}(L_{\mathrm{b}}, L_{\mathrm{d}}, i)$ as a multiple American call option with L_{d} rights and L_{b} obligations at t_i and denote $V^{\mathrm{p}}(L_{\mathrm{s}}, L_{\mathrm{d}}, i)$ as a multiple American put option with L_{d} rights and L_{s} obligations at t_i. Then,

$$
\begin{aligned}
&v_{\min} \cdot U^{\mathrm{b}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i) - v_{\max} \cdot U^{\mathrm{s}}(L_{\mathrm{b}}-1, L_{\mathrm{d}}, L_{\mathrm{s}}, i) \\
&= \mathrm{E}_i[-v_{\min} \cdot V(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i+1) + v_{\max} \cdot V(L_{\mathrm{b}}-1, L_{\mathrm{d}}, L_{\mathrm{s}}, i+1) \\
&\quad - (v_{\max} - v_{\min})V(L_{\mathrm{b}}-1, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i+1)] \\
&= \mathrm{E}_i[-v_{\min} \cdot (V(k, 0, 0, i+1) + (v_{\min} - v_{\max})V^{\mathrm{p}}(L_{\mathrm{s}}-1, L_{\mathrm{d}}, i+1)) \\
&\quad + v_{\max} \cdot (V(0, 0, k, i+1) + (v_{\max} - v_{\min})V^{\mathrm{c}}(L_{\mathrm{b}}-1, L_{\mathrm{d}}, i+1)) \\
&\quad - (v_{\max} - v_{\min})V(L_{\mathrm{b}}-1, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i+1)] \\
&= \mathrm{E}_i[(v_{\max} - v_{\min})(v_{\max} \cdot V^{\mathrm{c}}(L_{\mathrm{b}}-1, L_{\mathrm{d}}, i+1) + v_{\min} \cdot V^{\mathrm{p}}(L_{\mathrm{s}}-1, L_{\mathrm{d}}, i+1) \\
&\quad - V(L_{\mathrm{b}}-1, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i+1))] \\
&\geq 0.
\end{aligned}
\tag{10}
$$

The third equality is from $v_{\max} \cdot V(0, 0, k, T-k+1) - v_{\min} \cdot V(k, 0, 0, T-k+1) = 0$. From equation (10), $U^{\mathrm{s}}(L_{\mathrm{b}}-1, L_{\mathrm{d}}, L_{\mathrm{s}}, i)$ must be less than or equal to 0 when $U^{\mathrm{b}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i) \geq 0$. So it is monotone between $\xi(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i)$ and $\xi(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i)$.

Finally, the proof is completed from induction.

A.2 Proof of Theorem 2

We prove the theorem by backward induction on i. We only show uniqueness since monotonicity is trivial from uniqueness and Theorem 1.

A.2.1 Case of $i = T$

At $i = T$, the number of residual rights is at most 1, so we only need to show uniqueness of $S^{\mathrm{bs}}(1,0,0,i), S^{\mathrm{bs}}(0,1,0,i), S^{\mathrm{bs}}(0,0,1,i)$.

On our definition an obligation must be exercised until t_T, so $S^{\mathrm{bs}}(1,0,0,T) = \infty$ and $S^{\mathrm{bs}}(0,0,1,T) = -\infty$. Moreover because $S^{\mathrm{bs}}(0,1,0,T)$ satisfies $v_{\max}(S^{\mathrm{bs}}(0,1,0,T) - K) = v_{\min}(S^{\mathrm{bs}}(0,1,0,T) - K)$, $S^{\mathrm{bs}}(0,1,0,T)$ is equal to K and then uniqueness holds.

For preparation of the next section, we note that the derivative of $U^{\mathrm{bs}}(0,1,0,T) = \mathrm{e}^{-rt_T}(v_{\max} - v_{\min})(S_T - K)$ w.r.t. S_T is equal to or greater than 0.

A.2.2 Case of $i = T - k$ $(k > 1)$

We assume that uniqueness and monotonicity are satisfied at $i = T-k+1, \ldots, T$, and show that the properties are also satisfied at $i = T - k$.

First we show uniqueness of S^{b} and S^{bs}. For S^{b}, we define $\mathcal{B}_1^{\mathrm{b}} = \{S^{\mathrm{b}}(L_{\mathrm{b}}-1, L_{\mathrm{d}}, L_{\mathrm{s}}, i+1) \le S_{i+1}\}$, $\mathcal{B}_1^{\mathrm{s}} = \{S^{\mathrm{s}}(L_{\mathrm{b}}-1, L_{\mathrm{d}}, L_{\mathrm{s}}, i+1) \ge S_{i+1}\}$, $\mathcal{B}_2^{\mathrm{b}} = \{S^{\mathrm{b}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i+1) \le S_{i+1}\}$ and $\mathcal{B}_2^{\mathrm{s}} = \{S^{\mathrm{s}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i+1) \ge S_{i+1}\}$, and then

$$
\begin{aligned}
U^{\mathrm{b}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i) &= Y(L_{\mathrm{b}}-1, L_{\mathrm{d}}, L_{\mathrm{s}}, i) + Z^{\mathrm{b}}(i) - Y(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i) \\
&= Z^{\mathrm{b}}(i) + \mathrm{E}_i[(Y(L_{\mathrm{b}}-2, L_{\mathrm{d}}, L_{\mathrm{s}}, i+1) - Y(L_{\mathrm{b}}-1, L_{\mathrm{d}}, L_{\mathrm{s}}, i+1))\mathbf{1}_{\mathcal{B}_1^{\mathrm{b}}} \\
&\quad - Z^{\mathrm{b}}(i+1)\mathbf{1}_{\bar{\mathcal{B}}_1^{\mathrm{b}} \cup \mathcal{B}_2^{\mathrm{b}}} \\
&\quad + (Y(L_{\mathrm{b}}-1, L_{\mathrm{d}}, L_{\mathrm{s}}, i+1) - Y(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i+1))\mathbf{1}_{\bar{\mathcal{B}}_2^{\mathrm{b}} \cup \bar{\mathcal{B}}_2^{\mathrm{s}}} \\
&\quad + (Y(L_{\mathrm{b}}-1, L_{\mathrm{d}}, L_{\mathrm{s}}, i+1) - Y(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i+1) - Z^{\mathrm{s}}(i+1))\mathbf{1}_{\mathcal{B}_2^{\mathrm{s}} \cup \bar{\mathcal{B}}_1^{\mathrm{s}}} \\
&\quad + (Y(L_{\mathrm{b}}-1, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i+1) - Y(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i+1))\mathbf{1}_{\mathcal{B}_1^{\mathrm{s}}}] \\
&= Z^{\mathrm{b}}(i) - \mathrm{E}_i[Z^{\mathrm{b}}(i+1)] \\
&\quad + \mathrm{E}_i[U^{\mathrm{b}}(L_{\mathrm{b}}-1, L_{\mathrm{d}}, L_{\mathrm{s}}, i+1)\mathbf{1}_{\mathcal{B}_1^{\mathrm{b}}} + 0\mathbf{1}_{\bar{\mathcal{B}}_1^{\mathrm{b}} \cup \mathcal{B}_2^{\mathrm{b}}} + U^{\mathrm{b}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i+1)\mathbf{1}_{\bar{\mathcal{B}}_2^{\mathrm{b}} \cup \bar{\mathcal{B}}_2^{\mathrm{s}}} \\
&\quad + U^{\mathrm{bs}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i+1)\mathbf{1}_{\mathcal{B}_2^{\mathrm{s}} \cup \bar{\mathcal{B}}_1^{\mathrm{s}}} + U^{\mathrm{b}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i+1)\mathbf{1}_{\mathcal{B}_1^{\mathrm{s}}}].
\end{aligned}
$$

A derivative of the third term of the right side of the equation w.r.t. S_i is greater than or equal to 0 from induction and property (ii), and the derivative of $Z^{\mathrm{b}}(i) - \mathrm{E}_i[Z^{\mathrm{b}}(i+1)]$ w.r.t. S_i is also greater than or equal to 0 from property (ii), so the derivative of $U^{\mathrm{b}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i)$ is greater than or equal to 0 and S^{b} is unique from property (iii) .

Similarly for U^{bs}, we define $\mathcal{B}_3^{\mathrm{b}} = \{S^{\mathrm{bs}}(L_{\mathrm{b}}-1, L_{\mathrm{d}}, L_{\mathrm{s}}, i+1) \le S_{i+1}\}$ and $\mathcal{B}_3^{\mathrm{s}} = \{S^{\mathrm{bs}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i+1) \ge S_{i+1}\}$, and then

$$
\begin{aligned}
U^{\mathrm{bs}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i) &= Y(L_{\mathrm{b}}-1, L_{\mathrm{d}}, L_{\mathrm{s}}, i+1) + Z^{\mathrm{b}}(i) - Y(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i+1) - Z^{\mathrm{s}}(i) \\
&= Z^{\mathrm{b}}(i) - Z^{\mathrm{s}}(i) - \mathrm{E}_i[Z^{\mathrm{b}}(i+1) - Z^{\mathrm{s}}(i+1)] \\
&\quad + \mathrm{E}_i[U^{\mathrm{bs}}(L_{\mathrm{b}}-1, L_{\mathrm{d}}, L_{\mathrm{s}}, i+1)\mathbf{1}_{\mathcal{B}_3^{\mathrm{b}}} + 0\mathbf{1}_{\bar{\mathcal{B}}_3^{\mathrm{b}} \cup \bar{\mathcal{B}}_3^{\mathrm{s}}} \\
&\quad + U^{\mathrm{bs}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}-1, i+1)\mathbf{1}_{\mathcal{B}_3^{\mathrm{s}}}]
\end{aligned}
$$

so S^{bs} is unique in a similar way to the case of S^{b}.

B Proof of Theorems in Section 4

B.1 Detail of Theorem 3

First we define the condition that an exercise strategy ξ must satisfy. The condition claims that ξ must be an appropriate approximation of ξ^*, that is, ξ must satisfy the following condition.

- If $U^{\mathrm{bs}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i) = 0$, then $\hat{U}^{\mathrm{b}}(L_{\mathrm{b}} - 1, L_{\mathrm{d}}, L_{\mathrm{s}}, i) \geq 0$ and $\hat{U}^{\mathrm{s}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}} - 1, i) \geq 0$,

where $\hat{U}^{\mathrm{b}}$ and $\hat{U}^{\mathrm{s}}$ are the estimate value of U^{b} and U^{s} on ξ, respectively. In the context of optimal exercise boundaries, the condition is rewritten as

$$\begin{aligned}\hat{S}^{\mathrm{b}}(L_{\mathrm{b}} - 1, L_{\mathrm{d}}, L_{\mathrm{s}}, i) &\geq S^{\mathrm{bs}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i),\\ \hat{S}^{\mathrm{s}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}} - 1, i) &\leq S^{\mathrm{bs}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i),\end{aligned} \tag{11}$$

where $\hat{S}^{\mathrm{b}}$ and $\hat{S}^{\mathrm{s}}$ are exercise boundaries constructed by ξ.

Note that ξ does not have to satisfy the condition that

- If $U^{\mathrm{bs}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i) = 0$, then $\hat{U}^{\mathrm{b}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}} - 1, i) \geq 0$ and $\hat{U}^{\mathrm{s}}(L_{\mathrm{b}} - 1, L_{\mathrm{d}}, L_{\mathrm{s}}, i) \geq 0$,

which is equivalent to

$$\begin{aligned}\hat{S}^{\mathrm{b}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}} - 1, i) &\geq S^{\mathrm{bs}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i),\\ \hat{S}^{\mathrm{s}}(L_{\mathrm{b}} - 1, L_{\mathrm{d}}, L_{\mathrm{s}}, i) &\leq S^{\mathrm{bs}}(L_{\mathrm{b}}, L_{\mathrm{d}}, L_{\mathrm{s}}, i).\end{aligned} \tag{12}$$

This reduces the accuracy of estimation of ξ that is necessary to give a true upper bound of the options. For example, in Figure 4, $\hat{S}^{\mathrm{b}}(1, 2, 0, T - 3)$ and $\hat{S}^{\mathrm{bs}}(1, 2, 1, T - 3)$ are very close, so $\hat{S}^{\mathrm{b}}(1, 2, 0, T - 3)$ and $S^{\mathrm{bs}}(1, 2, 1, T - 3)$ will be close. Therefore if equation (12) must hold, then the accuracy approximation is necessary.

Next we define an adjusted payoff function $Z^{\xi}_{(l_{\mathrm{b}}, l_{\mathrm{d}}, l_{\mathrm{s}})}(i)$. The function $Z^{\xi}_{(l_{\mathrm{b}}, l_{\mathrm{d}}, l_{\mathrm{s}})}(i)$ means the value of a second-order difference of a swing option with $(l_{\mathrm{b}}, l_{\mathrm{d}}, l_{\mathrm{s}})$. If components of the second-order difference are exercised at t_i, then we define $Z^{\xi}_{(l_{\mathrm{b}}, l_{\mathrm{d}}, l_{\mathrm{s}})}(i) = -C$ such that C is a sufficient large constant value.

Example 2: We consider $Z^{\xi}_{(1,1,2)}(i)$. At $i = 0$, suppose $\xi(0, 1, 1, 0) = 1$. Then $\xi(1, 1, 2, 0) = \xi(1, 1, 1, 0) = \xi(0, 1, 2, 0) = 1$ from monotonicity. These executions are shown by solid arrows in Figure 5. In this case, we interpret that a right to buy is automatically exercised and $Z^{\xi}_{(1,1,2)}(0) = -C$, and this auto execution is shown by a dotted arrow in Figure 5. Moreover, at $i = 1$, $Z^{\xi}_{(1,1,2)}(i)$ indicates a second-order difference of a swing option with $(0, 1, 2)$.

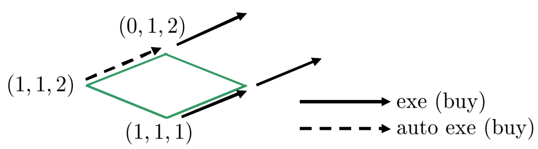

Figure 5: An example of auto exercise

Let us define $\hat{l}^{\xi}_{\mathrm{b}}(i), \hat{l}^{\xi}_{\mathrm{d}}(i), \hat{l}^{\xi}_{\mathrm{s}}(i)$ as the state of a second-order difference at t_i.

$$\begin{aligned}&(\hat{l}^{\xi}_{\mathrm{b}}(0), \hat{l}^{\xi}_{\mathrm{d}}(0), \hat{l}^{\xi}_{\mathrm{s}}(0)) = (l_{\mathrm{b}}, l_{\mathrm{d}}, l_{\mathrm{s}}),\\ &\hat{l}^{\xi}_{\mathrm{b}}(i+1) = \max[0,\ \hat{l}^{\xi}_{\mathrm{b}}(i) - \mathbf{1}_{\xi(\hat{l}_{\mathrm{b}}, \hat{l}_{\mathrm{d}}, \hat{l}_{\mathrm{s}} - 1, i) = 1}],\\ &\hat{l}^{\xi}_{\mathrm{d}}(i+1) = \max[0,\ \hat{l}^{\xi}_{\mathrm{d}}(i) - \mathbf{1}_{\xi(\hat{l}_{\mathrm{b}}, \hat{l}_{\mathrm{d}}, \hat{l}_{\mathrm{s}} - 1, i) = 1} \cdot \mathbf{1}_{\hat{l}^{\xi}_{\mathrm{b}}(i) = 0} - \mathbf{1}_{\xi(\hat{l}_{\mathrm{b}} - 1, \hat{l}_{\mathrm{d}}, \hat{l}_{\mathrm{s}}, i) = -1} \cdot \mathbf{1}_{\hat{l}^{\xi}_{\mathrm{s}}(i) = 0}],\\ &\hat{l}^{\xi}_{\mathrm{s}}(i+1) = \max[0,\ \hat{l}^{\xi}_{\mathrm{s}}(i) - \mathbf{1}_{\xi(\hat{l}_{\mathrm{b}} - 1, \hat{l}_{\mathrm{d}}, \hat{l}_{\mathrm{s}}, i) = -1}],\end{aligned} \tag{13}$$

where $\hat{l}_{\rm b},\hat{l}_{\rm d},\hat{l}_{\rm s}$ is an abbreviated form of $\hat{l}^{\xi}_{\rm b}(i),\hat{l}^{\xi}_{\rm d}(i),\hat{l}^{\xi}_{\rm s}(i)$. Then the value of $Z^{\xi}_{(l_{\rm b},l_{\rm d},l_{\rm s})}(i)$ depends on whether options with $(\hat{l}_{\rm b}-1,\hat{l}_{\rm d},\hat{l}_{\rm s})$ and $(\hat{l}_{\rm b},\hat{l}_{\rm d},\hat{l}_{\rm s}-1)$ are exercised or not exercised at t_i on an exercise strategy ξ:

$$\begin{aligned}
Z^{\xi}_{(l_{\rm b},l_{\rm d},l_{\rm s})}(i) =\ & D^{\xi}_{(l_{\rm b},l_{\rm d},l_{\rm s})}(i) \\
& + \begin{cases}
\hat{U}_{\max}(\hat{l}_{\rm b},\hat{l}_{\rm d},\hat{l}_{\rm s},i) & \left(\xi(\hat{l}_{\rm b},\hat{l}_{\rm d},\hat{l}_{\rm s}-1,i)=\xi(\hat{l}_{\rm b}-1,\hat{l}_{\rm d},\hat{l}_{\rm s},i)=0\right), \\
0 & \left(\left(\xi(\hat{l}_{\rm b},\hat{l}_{\rm d},\hat{l}_{\rm s}-1,i)=1,\ \xi(\hat{l}_{\rm b}-1,\hat{l}_{\rm d},\hat{l}_{\rm s},i)=0\right) \text{ or} \right. \\
& \left. \left(\xi(\hat{l}_{\rm b}-1,\hat{l}_{\rm d},\hat{l}_{\rm s},i)=-1,\ \xi(\hat{l}_{\rm b},\hat{l}_{\rm d},\hat{l}_{\rm s}-1,i)=0\right)\right), \\
-C+\Delta\hat{U}(\hat{l}_{\rm b}-1,\hat{l}_{\rm d},\hat{l}_{\rm s},i) & (\xi(\hat{l}_{\rm b}-1,\hat{l}_{\rm d},\hat{l}_{\rm s},i)=1), \\
-C+\Delta\hat{U}(\hat{l}_{\rm b},\hat{l}_{\rm d},\hat{l}_{\rm s}-1,i) & (\xi(\hat{l}_{\rm b},\hat{l}_{\rm d},\hat{l}_{\rm s}-1,i)=-1),
\end{cases} \\
\hat{U}_{\max}(\hat{l}_{\rm b},\hat{l}_{\rm d},\hat{l}_{\rm s},i) =\ & \max\left[\hat{U}^{\rm b}(\hat{l}_{\rm b},\hat{l}_{\rm d},\hat{l}_{\rm s}-1,i),\hat{U}^{\rm s}(\hat{l}_{\rm b}-1,\hat{l}_{\rm d},\hat{l}_{\rm s},i)\right], \\
\Delta\hat{U}(\hat{l}_{\rm b},\hat{l}_{\rm d},\hat{l}_{\rm s},i) =\ & \hat{U}^{\rm b}(\hat{l}_{\rm b},\hat{l}_{\rm d},\hat{l}_{\rm s},i)-\hat{U}^{\rm b}(\hat{l}_{\rm b},\hat{l}_{\rm d},\hat{l}_{\rm s}-1,i),
\end{aligned} \tag{14}$$

where $D^{\xi}_{(l_{\rm b},l_{\rm d},l_{\rm s})}(i)$ is the following adjustment term which depends on an exercise history by ξ:

$$\begin{aligned}
D^{\xi}_{(l_{\rm b},l_{\rm d},l_{\rm s})}(i+1) =\ & D^{\xi}_{(l_{\rm b},l_{\rm d},l_{\rm s})}(i) \\
& + \begin{cases}
-\hat{U}^{\rm s}(\hat{l}_{\rm b}-1,\hat{l}_{\rm d},\hat{l}_{\rm s}-1,i) & \left(\xi(\hat{l}_{\rm b},\hat{l}_{\rm d},\hat{l}_{\rm s}-1,i)=-1,\ \xi(\hat{l}_{\rm b}-1,\hat{l}_{\rm d},\hat{l}_{\rm s}-1,i)=0\right), \\
-\hat{U}^{\rm b}(\hat{l}_{\rm b}-1,\hat{l}_{\rm d},\hat{l}_{\rm s}-1,i) & \left(\xi(\hat{l}_{\rm b}-1,\hat{l}_{\rm d},\hat{l}_{\rm s},i)=1,\ \xi(\hat{l}_{\rm b}-1,\hat{l}_{\rm d},\hat{l}_{\rm s}-1,i)=0\right), \\
-C-\Delta\hat{U}(\hat{l}_{\rm b}-1,\hat{l}_{\rm d},\hat{l}_{\rm s},i) & \left(\xi(\hat{l}_{\rm b},\hat{l}_{\rm d},\hat{l}_{\rm s}-1,i)=1,\ \xi(\hat{l}_{\rm b}-1,\hat{l}_{\rm d},\hat{l}_{\rm s},i)=0\right) \\
-C-\Delta\hat{U}(\hat{l}_{\rm b},\hat{l}_{\rm d},\hat{l}_{\rm s}-1,i) & \left(\xi(\hat{l}_{\rm b}-1,\hat{l}_{\rm d},\hat{l}_{\rm s},i)=-1,\ \xi(\hat{l}_{\rm b},\hat{l}_{\rm d},\hat{l}_{\rm s}-1,i)=0\right), \\
0 & (\text{otherwise}).
\end{cases}
\end{aligned} \tag{15}$$

In equation (14) and (15), the term added to $-C$ is necessary for the inequality in equation (8) to become an equality.

B.2 Proof of Theorem 3

As a preparation, we define a part transition tree and the set of the trees.

Definition 2: We call a tree $\mathcal{K}$ as a part transition tree of $\mathcal{L}(L_{\rm b},L_{\rm d},L_{\rm s})$, if the tree $\mathcal{K}$ includes all leafs[5] of $\mathcal{L}(L_{\rm b},L_{\rm d},L_{\rm s})$. We also denote $\mathcal{T}(L_{\rm b},L_{\rm d},L_{\rm s})$ as the set of all part transition trees of $\mathcal{L}(L_{\rm b},L_{\rm d},L_{\rm s})$.

For simplicity, we denote time-shifted problems as follows:

$$\begin{aligned}
SV^{\xi}(l_{\rm b},l_{\rm d},l_{\rm s},i) &= \operatorname{ess}\sup_{0\le\tau\le T-i} {\rm E}_i[Z^{\xi^i}_l(\tau)]\cdot\exp^{-r(t_i-t_0)}, \\
SY^{\xi}(l_{\rm b},l_{\rm d},l_{\rm s},i) &= \operatorname{ess}\sup_{0\le\tau\le T-i+1} {\rm E}_i[Z^{\xi^{i+1}}_l(\tau)]\cdot\exp^{-r(t_i-t_0)}, \\
\xi^i(l_{\rm b},l_{\rm d},l_{\rm s},k) &= \xi(l_{\rm b},l_{\rm d},l_{\rm s},i+k) \qquad ((l_{\rm b},l_{\rm d},l_{\rm s})\in\mathcal{L}(L_{\rm b},L_{\rm d},L_{\rm s}),\ k=0,\dots,T-i).
\end{aligned}$$

We prove the theorem in two parts. First we show the following Lemma.

[5] A leaf is a node at the bottom of the tree, which corresponds to a swing option with one residual right.

Lemma 1: For $i = 0, \dots, T-1$, any $\mathcal{K} \in \mathcal{T}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s})$ and any ξ that satisfies the condition of Theorem 3, there is a $\mathcal{K}' \in \mathcal{T}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s})$ and the following equation holds:

$$\sum_{l \in \mathcal{K}} SV^\xi(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, i) \geq \sum_{l \in \mathcal{K}'} SY^\xi(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, i). \tag{16}$$

Proof. To begin with, if $\xi(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}) = 0$, then it is trivial that equation (16) has inequality for $\mathcal{K}' = \mathcal{K}$.

We consider the case of $\xi(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}) = 1$. We decompose $\mathcal{K}$ into the following four groups:

$$\begin{aligned}
\mathcal{K}_1 &= \mathcal{K} \cap \{(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}) \mid \xi(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}-1, i) = 0\}, \\
\mathcal{K}_2 &= \mathcal{K} \cap \{(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}) \mid \xi(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}-1, i) = 1 \text{ and } \xi(l_\mathrm{b}-1, l_\mathrm{d}, l_\mathrm{s}, i) = 0\}, \\
\mathcal{K}_3 &= \mathcal{K} \cap \{(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}) \mid \xi(l_\mathrm{b}-1, l_\mathrm{d}, l_\mathrm{s}, i) = 1 \text{ and } \xi(l_\mathrm{b}-1, l_\mathrm{d}, l_\mathrm{s}-1, i) = 0\}, \\
\mathcal{K}_4 &= \mathcal{K} \cap \{(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}) \mid \xi(l_\mathrm{b}-1, l_\mathrm{d}, l_\mathrm{s}-1, i) = 1\}.
\end{aligned} \tag{17}$$

Moreover, we decompose $\mathcal{K}_1$ into two groups:

$$\begin{aligned}
\mathcal{K}_{11} &= \mathcal{K}_1 \cap \{(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}) \mid (l_\mathrm{b}+1, l_\mathrm{d}, l_\mathrm{s}) \in \mathcal{K}_3 \cap \mathcal{K}\}, \\
\mathcal{K}_{12} &= \mathcal{K}_1 \cap \{(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}) \mid (l_\mathrm{b}+1, l_\mathrm{d}, l_\mathrm{s}) \notin \mathcal{K}_3 \cap \mathcal{K}\}.
\end{aligned} \tag{18}$$

For a second-order difference, possible combinations of $\mathcal{K}_{11}, \mathcal{K}_{12}, \mathcal{K}_2, \mathcal{K}_3$ and $\mathcal{K}_4$ are de-

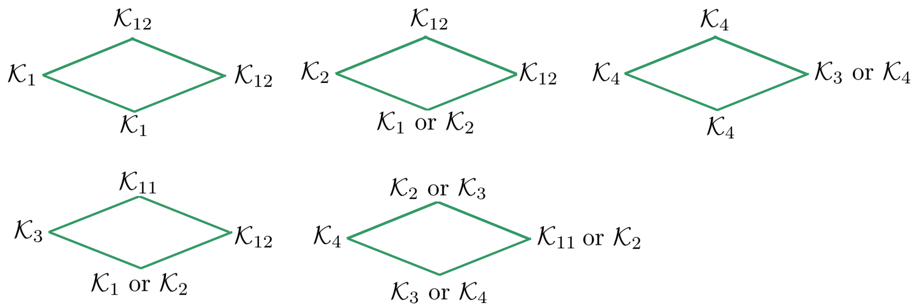

Figure 6: Possible combinations of $\mathcal{K}$

picted in Figure 6. We induce equation (16) as follows:

$$\begin{aligned}
&\sum_{l \in \mathcal{K}} SV^\xi(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, i) = \sum_{j=1}^{4} \sum_{l \in \mathcal{K}_j} SV^\xi(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, i) \\
&\geq \sum_{l \in \mathcal{K}_4} SY^\xi(l_\mathrm{b}-1, l_\mathrm{d}, l_\mathrm{s}, i) + \sum_{l \in \mathcal{K}_2} 0 \\
&\quad + \sum_{l \in \mathcal{K}_3} \Big(SY^\xi(l_\mathrm{b}-1, l_\mathrm{d}, l_\mathrm{s}, i) - \hat{U}^\mathrm{b}(l_\mathrm{b}-1, l_\mathrm{d}, l_\mathrm{s}-1, i) \Big) + \sum_{l \in \mathcal{K}_1} SV^\xi(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, i) \\
&\geq \sum_{l \in \mathcal{K}_4} SY^\xi(l_\mathrm{b}-1, l_\mathrm{d}, l_\mathrm{s}, i) + \sum_{l \in \mathcal{K}_3} \Big(SY^\xi(l_\mathrm{b}-1, l_\mathrm{d}, l_\mathrm{s}, i) - \hat{U}^\mathrm{b}(l_\mathrm{b}-1, l_\mathrm{d}, l_\mathrm{s}-1, i) \Big) \\
&\quad + \sum_{l \in \mathcal{K}_{11}} \hat{U}^\mathrm{b}(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}-1, i) + \sum_{l \in \mathcal{K}_{12}} SY^\xi(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, i) \\
&= \sum_{l \in \mathcal{K}_4} SY^\xi(l_\mathrm{b}-1, l_\mathrm{d}, l_\mathrm{s}, i) + \sum_{l \in \mathcal{K}_3} SY^\xi(l_\mathrm{b}-1, l_\mathrm{d}, l_\mathrm{s}, i) + \sum_{l \in \mathcal{K}_{12}} SY^\xi(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, i) \\
&= \sum_{l \in \mathcal{K}'} SY^\xi(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, i).
\end{aligned} \tag{19}$$

The last equality is from relations of a second-order difference in Figure 6.

In the case of $\xi(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}) = -1$, equation (16) holds for symmetry, and thus the proof is finished. □

Lemma 2: For $i = 0, \ldots, T$, any $\mathcal{K} \in \mathcal{T}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s})$ and any ξ,

$$V(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, i) = \sum_{l \in \mathcal{L}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s})} \Delta\Delta V(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, i) \leq \sum_{l \in \mathcal{K}} SV^{\xi}(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, i) \tag{20}$$

Proof. We proof by backward induction on i. For $i = T$, from the definition of $\mathcal{K}$, $\mathcal{K}$ must be equal to $\mathcal{L}(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s})$ and $|\mathcal{K}|$ is equal to 1, so that equation (20) has equality.

Let us assume that equation (20) holds for $i = k+1$ and proof equation (20) for $i = k$. First, if $\xi^*(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, k) = 0$, then equation (20) holds by induction and Lemma 1. Thus we consider the case of $\xi^*(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, k) = 1$. We decompose $\mathcal{K}$ into the following groups:

$$\begin{aligned}
&\mathcal{M}_1 = \mathcal{K} \cap \{(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}) \mid (l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}) \in \mathcal{L}(L_\mathrm{b} - 1, L_\mathrm{d}, L_\mathrm{s})\}, \\
&\mathcal{M}_2 = \mathcal{K} \setminus \mathcal{M}_1, \\
&\mathcal{M}_{21} = \mathcal{M}_2 \cap \{(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}) \mid \xi(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s} - 1, i) = 0\}, \\
&\mathcal{M}_{22} = \mathcal{M}_2 \cap \{(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}) \mid \xi(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s} - 1, i) = 1 \text{ and } \xi(l_\mathrm{b} - 1, l_\mathrm{d}, l_\mathrm{s}, i) = 0\}, \\
&\mathcal{M}_{23} = \mathcal{M}_2 \cap \{(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}) \mid \xi(l_\mathrm{b} - 1, l_\mathrm{d}, l_\mathrm{s}, i) = 1\}.
\end{aligned} \tag{21}$$

From equation (11) and equation (12), $\xi(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, k) \neq -1$ holds for $l \in \mathcal{M}_2 \setminus (L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s})$, and then

$$\begin{aligned}
&\sum_{l \in \mathcal{K}} SV^{\xi}(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, k) \\
&= \mathbf{1}_{\xi(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, k) = -1} \sum_{l \in \mathcal{K}} SV^{\xi}(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, k) + \mathbf{1}_{\xi(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, k) \neq -1} \sum_{l \in \mathcal{K}} SV^{\xi}(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, k) \\
&\geq \mathbf{1}_{\xi(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, k) = -1} \left(\sum_{l \in \mathcal{M}_1} SV^{\xi}(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, k) + \sum_{l \in \mathcal{M}_{21}} SV^{\xi}(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, k) \right) \\
&\quad + \mathbf{1}_{\xi(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, k) \neq -1} \left(\left(\sum_{l \in \mathcal{M}_1} SV^{\xi}(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, k) + \sum_{l \in \mathcal{M}_{23}} SV^{\xi}(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, k) \right) + \sum_{l \in \mathcal{M}_{21}} SV^{\xi}(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, k) \right) \\
&\geq \mathbf{1}_{\xi(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, k) = -1} \left(\sum_{l \in \mathcal{M}_1} SV^{\xi}(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, k) + Z^\mathrm{b}(k) \right) \\
&\quad + \mathbf{1}_{\xi(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, k) \neq -1} \left(\left(\sum_{l \in \mathcal{M}_1} SV^{\xi}(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, k) + \sum_{l \in \mathcal{M}_{23}} SV^{\xi}(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, k) \right) + Z^\mathrm{b}(k) \right) \\
&\geq \mathbf{1}_{\xi(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, k) = -1} \left(\sum_{l \in \mathcal{K}'} SY^{\xi}(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, k) + Z^\mathrm{b}(k) \right) \\
&\quad + \mathbf{1}_{\xi(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, k) \neq -1} \left(\sum_{l \in \mathcal{K}''} SY^{\xi}(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, k) + Z^\mathrm{b}(k) \right) \\
&\geq Y(L_\mathrm{b} - 1, L_\mathrm{d}, L_\mathrm{s}, k) + Z^\mathrm{b}(k) \\
&= V(L_\mathrm{b}, L_\mathrm{d}, L_\mathrm{s}, k),
\end{aligned} \tag{22}$$

where $\mathcal{K}', \mathcal{K}'' \in \mathcal{T}(L_\mathrm{b} - 1, L_\mathrm{d}, L_\mathrm{s})$. The third inequality is obtained by the proof of Lemma 1 and the fourth inequality is by induction.

In the case of $\xi^*(l_\mathrm{b}, l_\mathrm{d}, l_\mathrm{s}, k) = -1$, we can similarly prove so the proof is completed. □

From Lemma 2 and $\xi^0 = \xi$, the inequality of equation (8) is trivial. Additionally, from the definition of $Z_l^{\xi^*}(i)$ and the optimality of ξ^*, $\operatorname{ess\,sup}_{0\leq\tau\leq T} \mathrm{E}_i[Z_l^{\xi^*}(\tau)] = \Delta\Delta V(l_{\mathrm{b}}, l_{\mathrm{d}}, l_{\mathrm{s}}, 0)$, and then the equality of equation (8) holds. Therefore, Theorem 3 is proved.

C The proof of Theorem 4

The left side of equation (9) is the set of single stopping problems, and thus we can apply the theorem of Rogers [10] to the left side of equation (9). Then we prove the inequality of equation (9).

From the theorem of Rogers [10], the inequality in equation (9) becomes an equality when $M_l(i)$ is the martingale part of the Doob decomposition of Snell envelope, which is equal to $\operatorname{ess\,sup}_{i\leq\tau\leq T} \mathrm{E}_i[Z_l^{\xi^*}(\tau)]$. From the definition of $Z_l^{\xi}(i)$, the martingale part is equal to the martingale part of the Doob decomposition of $\Delta\Delta V(\hat{l}_{\mathrm{b}}^{\xi^*}(i), \hat{l}_{\mathrm{d}}^{\xi^*}(i), \hat{l}_{\mathrm{s}}^{\xi^*}(i), i)$, and thus Theorem 4 holds.