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Robust Matchings and Matroid Intersections*

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Abstract

In a weighted undirected graph, a matching is said to be α-robust if for all p, the total weight of its heaviest p edges is at least α times the maximum weight of a p-matching in the graph. Here a p-matching is a matching with at most p edges. In 2002, Hassin and Rubinstein [5] showed that every graph has a $\frac{1}{\sqrt{2}}$-robust matching and it can be found by k-th power algorithm in polynomial time.

In this paper, we show that it can be extended to the matroid intersection problem, i.e., there always exists a $\frac{1}{\sqrt{2}}$-robust matroid intersection, which is polynomially computable. We also study the time complexity of the robust matching problem. We show that a 1-robust matching can be computed in polynomial time (if exists), and for any fixed number α with $\frac{1}{\sqrt{2}} < \alpha < 1$, the problem to determine whether a given weighted graph has an α-robust matching is NP-complete. These together with the positive result for $\alpha = \frac{1}{\sqrt{2}}$ in [5] give us a sharp border for the complexity for the robust matching problem. Moreover, we show that the problem is strongly NP-complete when α is a part of the input. Finally, we show the limitations of k-th power algorithm for robust matchings, i.e., for any $\epsilon > 0$, there exists a weighted graph such that no k-th power algorithm outputs a $\left(\frac{1}{\sqrt{2}} + \epsilon\right)$-approximation for computing the most robust matching.

1 Introduction

Let $G = (V, E)$ be a graph with a nonnegative weight function w on the edges. A *p-matching* is a matching with at most p edges. A matching M in G is said to be *α-robust* if for all positive integer p, it contains $\min\{p, |M|\}$ edges whose total weight is at least α times the maximum weight of a p-matching in G. The concept of the robustness was introduced in [5], and studied for several combinatorial optimization problems such as trees and paths [3, 6]. Hassin and Rubinstein [5] showed that the *k-th power algorithm* (i.e., the one for computing a maximum matching in the graph with respect to the k-th power weights w^k) provides a $\min\{2^{(1/k)-1}, 2^{-1/k}\}$-robust matching in polynomial time. In particular, when $k = 2$, this implies the existence of a $\frac{1}{\sqrt{2}}$-robust matching in any graph. They also show that the $\frac{1}{\sqrt{2}}$-robustness is the best possible for

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matchings by providing a weighted graph which does not contain α-robust matching for any $\alpha > \frac{1}{\sqrt{2}}$.

In this paper, we extend this result to the matroid intersection problem. Let $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ be two matroids with independent sets $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively, and w be a nonnegative weight on the ground set E. The matroid intersection problem is to compute a maximum common independent set $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ of two matroids $\mathcal{M}_1$ and $\mathcal{M}_2$. The matroid intersection is a natural generalization of bipartite matching, and one of the most fundamental problems in combinatorial optimization (see e.g., [8]). We show that the k-th power algorithm computes a $\min\{2^{(1/k)-1}, 2^{-1/k}\}$-robust common independent set in polynomial time, which implies that the matroid intersection problem admits a $\frac{1}{\sqrt{2}}$-robust solution, where the $\frac{1}{\sqrt{2}}$-robustness is the best possible [5]. In order to obtain the result, we make use of optimal dual values for the linear programming formulation of the matroid intersection problem.

We next consider the complexity for the robust matching problem. We show that (1) a 1-robust matching can be computed in polynomial time (if exists), and (2) for any fixed number α with $\frac{1}{\sqrt{2}} < \alpha < 1$, the problem to determine whether a given weighted graph has an α-robust matching is NP-complete. These together with the positive result for $\alpha = \frac{1}{\sqrt{2}}$ in [5] give us a sharp border for the complexity for the robust matching problem, although the NP-hardness is in the weak sense. We also show that deciding if G has an α-robust matching is strongly NP-complete when α is a part of the input. We remark that all the negative results use bipartite graphs, and hence these lead to the hardness for robust bipartite matching and matroid intersection.

We finally analyze the performance of the k-th power algorithm for the robust matching problem. Recall that the k-th power algorithm provides a $\min\{2^{(1/k)-1}, 2^{-1/k}\}$-robust matching, and we might expect that for *some* k, the k-th power algorithm provides a good approximate solution for the robust matching. However, we show that this is not the case, i.e., $\frac{1}{\sqrt{2}}$ is the best possible for the approximation of the robustness. More precisely, we show that for any $\epsilon > 0$, there exists a weighted graph such that no k-th power algorithm outputs a $\left(\frac{1}{\sqrt{2}} + \epsilon\right)$-approximation for computing the most robust matching.

The rest of the paper is organized as follows. In the next section, we recall some basic concepts and introduce notation. Section 3 shows the $\frac{1}{\sqrt{2}}$-robustness of the matroid intersection problem, and Section 4 shows the NP-hardness for the robust matching problem. In Section 5, we analyze the performance of the k-th power algorithm for the robust matching problem, which includes the polynomial solvability for the 1-robust matching. Finally, in Section 6, we give a proof for the strongly NP-hardness which is omitted in Section 4.

2 Preliminaries

For a finite set E and a nonempty collection of its subsets $\mathcal{I} \subseteq 2^E$, the pair $(E, \mathcal{I})$ is called an *independent system* if $\mathcal{I}$ satisfies the hereditary condition:

$$I' \subseteq I, I \in \mathcal{I} \Rightarrow I' \in \mathcal{I},$$

where $I \in \mathcal{I}$ of size at most p (i.e., $|I| \leq p$) is called *p-independent*. An independent system $(E, \mathcal{I})$ is a *matroid* if $\mathcal{I}$ satisfies

$$\forall I, J \in \mathcal{I},\ |I| > |J| \Rightarrow \exists i \in I \setminus J,\ J \cup \{i\} \in \mathcal{I}.$$

Given an independent system $(E, \mathcal{I})$ and a nonnegative weight function $w : E \to \mathbb{R}_+$, the *maximum* $(p\text{-})$*independent set problem* is to compute a $(p\text{-})$independent set $I \in \mathcal{I}$ that maximizes the weight $w(I) = \sum_{e \in I} w(e)$.

Let $w : E \to \mathbb{R}_+$ be a weight function on a ground set E, and let $J = \{e_1, e_2, \dots, e_q\}$ be a subset of E with $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_q)$. Define

$$J_{(p)} = \begin{cases} \{e_1, e_2, \dots, e_p\} & \text{if } p \leq q, \\ J & \text{otherwise.} \end{cases}$$

For an independence system $(E, \mathcal{I})$, we denote by $I^{(p)}$ a maximum p-independent set. An independent set J is called *α-robust* if

$$w(J_{(p)}) \geq \alpha \cdot w(I^{(p)}) \quad \text{for all } p = 1, 2, \dots, |E|.$$

Note that any matroid has a 1-robust set, since a greedy algorithm solves the maximum independent set problem for matroids. For an original weight function $w : E \to \mathbb{Z}_+$, the function $w^k : E \to \mathbb{Z}_+$ is defined by $w^k(e) = \{w(e)\}^k$.

Given two matroids $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$, the *matroid intersection* is the independent system of form $\mathcal{M}_1 \cap \mathcal{M}_2 = (E, \mathcal{I}_1 \cap \mathcal{I}_2)$, and the matroid intersection problem is to compute a maximum independent set of the matroid intersection. Let $r_i : 2^E \to \mathbb{Z}_+$ denote the *rank function* of $\mathcal{M}_i$ which is defined as $r_i(A) = \max\{|I| \mid I \subseteq A, I \in \mathcal{I}_i\}$. Edmonds [1] showed that the following linear programming solves the matroid intersection problem:

$$\begin{array}{lll} \text{maximize} & w \cdot x & \\ \text{subject to} & x(A) \leq r_1(A) & (\forall A \subseteq E) \\ & x(A) \leq r_2(A) & (\forall A \subseteq E) \\ & x_e \geq 0 & (\forall e \in E), \end{array} \tag{1}$$

where $x \in \mathbb{R}^E$ and $x(A) = \sum_{e \in A} x_e$. Consider the dual of the problem:

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{A \subseteq E} \left(r_1(A) y_A^1 + r_2(A) y_A^2\right) & \\ \text{subject to} & \displaystyle\sum_{e \in A \subseteq E} (y_A^1 + y_A^2) \geq w(e) & (\forall e \in E) \\ & y_A^1, y_A^2 \geq 0 & (\forall A \subseteq E). \end{array}$$

For an optimal solution $(\overline{y}^1, \overline{y}^2)$ of this dual, we define weight functions w_1 and w_2 as follows:

$$w_1(e) = \sum_{e \in A \subseteq E} \overline{y}_A^1, \;\; w_2(e) = \sum_{e \in A \subseteq E} \overline{y}_A^2. \tag{2}$$

Then we have the following result.

Theorem 2.1 (Edmonds [1])**.** *Let J be an optimal solution of the matroid intersection problem. Then it is a maximum independent set of $\mathcal{M}_i$ with respect to w_i, $i = 1, 2$.*

By definition of w_1 and w_2, we have $w_1(e) + w_2(e) \geq w(e)$ for any $e \in E$, and the complementary slackness implies that

$$\begin{array}{lll} x_e > 0 & \Longrightarrow & w_1(e) + w_2(e) = w(e) \quad (\forall e \in E), \\ y_A^i > 0 & \Longrightarrow & x(A) = r_i(A) \quad (\forall A \subseteq E, i = 1, 2). \end{array}$$

3 Robust Matroid Intersection

In this section, we prove the following theorem.

Theorem 3.1. *Let $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ be two matroids. For any weight function $w : E \to \mathbb{R}_+$ and $k \geq 1$, let J be a maximum common independent set of $\mathcal{M}_1$ and $\mathcal{M}_2$ with respect to w^k, i.e.,*

$$w^k(J) = \max\{w^k(I) \mid I \in \mathcal{I}_1 \cap \mathcal{I}_2\}.$$

Then J is a $\min\{2^{(1/k)-1}, 2^{-1/k}\}$-robust independent set for the matroid intersection $(E, \mathcal{I}_1 \cap \mathcal{I}_2)$.

Since the matroid intersection problem is polynomially solvable, as a corollary ($k = 2$), we have the following result.

Corollary 3.2. *The matroid intersection problem admits a $\frac{1}{\sqrt{2}}$-robust solution, and furthermore, it can be computed in polynomial time.*

Let J be a maximum common independent set of $\mathcal{M}_1$ and $\mathcal{M}_2$ with respect to w^k. For $p \geq 1$, we show that $w(J_{(p)}) \geq 2^{(1/k)-1} \cdot w(I^{(p)})$ if $|J| \leq |I^{(p)}|$, and $w(J_{(p)}) \geq \min\{2^{(1/k)-1}, 2^{-1/k}\} \cdot w(I^{(p)})$, otherwise.

3.1 The case when $|J| \leq |I^{(p)}|$

Let $q = |I^{(p)}| - |J| \geq 0$. To make the discussion clear, let us modify the problem instance by adding q new elements $F = \{f_1, \dots, f_q\}$ to E:

$$\begin{aligned} E &:= E \cup F, \\ \mathcal{I}_i &:= \{I \cup F' \mid I \in \mathcal{I}_i, F' \subseteq F\}, \\ w(f_j) &:= 0 \quad (j = 1, \dots, q), \\ J &:= J \cup F. \end{aligned}$$

We furthermore truncate the two matroids by $|I^{(p)}|$:

$$\mathcal{I}_i := \{I \in \mathcal{I}_i \mid |I| \leq |I^{(p)}|\}.$$

It is not difficult to see that after this transformation, $\mathcal{M}_1$ and $\mathcal{M}_2$ are still matroids, J and $I^{(p)}$ are common bases (i.e., maximal independent sets for both $\mathcal{M}_1$ and $\mathcal{M}_2$). Hence it suffices to show $w(J_{(p)}) = w(J) \geq 2^{(1/k)-1} \cdot w(I^{(p)})$. We show this by proving

$$w(J \setminus I^{(p)}) \geq 2^{(1/k)-1} \cdot w(I^{(p)} \setminus J).$$

For a common base B and a common independent set L of $\mathcal{M}_1$ and $\mathcal{M}_2$, let us construct a bipartite graph $G(B, L) = (V, A = A_1 \cup A_2)$ as follows.

$$\begin{aligned} V &= (B \setminus L) \cup (L \setminus B), \\ A_i &= \{(x, y) \mid x \in B \setminus L,\ y \in L \setminus B,\ (B \setminus \{x\}) \cup \{y\} \in \mathcal{I}_i\} \quad (i = 1, 2). \end{aligned} \tag{3}$$

For a vertex $x \in V$ and for $X \subseteq V$, we define

$$\begin{aligned} \delta_i(x) &= \{v \in V \mid (x, v) \in A_i\}, \\ \delta_i(X) &= \bigcup_{x \in X} \delta_i(x) \qquad (i = 1, 2). \end{aligned}$$

Lemma 3.3. *For any $X \subseteq L \setminus B$, $(B \setminus \delta_1(X)) \cup X \in \mathcal{I}_1$ and $(B \setminus \delta_2(X)) \cup X \in \mathcal{I}_2$.*

Proof. Assume a contrary that $(B \setminus \delta_1(X)) \cup X \notin \mathcal{I}_1$, that is, there exists an $\mathcal{M}_1$-circuit $C_1 \subseteq (B \setminus \delta_1(X)) \cup X$. Then we have $C_1 \cap X \neq \emptyset$, since B is independent. Furthermore, $|C_1 \cap X| \geq 2$ holds by the definition of $\delta_1(X)$. For $e \in C_1 \cap X$, let C_2 be the unique circuit included in $B \cup \{e\}$. Since $e \in C_1 \cap C_2$ and $C_1 \neq C_2$, by the circuit axiom of matroids, there exists an $\mathcal{M}_1$-circuit $C_1' \subseteq (C_1 \cup C_2) \setminus \{e\}$. This C_1' satisfies $|C_1' \cap X| < |C_1 \cap X|$. By repeatedly applying the above argument to C_1', we can obtain an $\mathcal{M}_1$-circuit $C \subseteq (B \setminus \delta_1(X)) \cup X$ with $|C \cap X| \leq 1$, which contradicts the definition of $\delta_1(X)$. Hence $(B \setminus \delta_1(X)) \cup X$ is $\mathcal{M}_1$-independent.

Similarly, it can be shown that $(B \setminus \delta_2(X)) \cup X$ is $\mathcal{M}_2$-independent. □

The following lemma follows from Lemma 3.3 and Hall's theorem.

Lemma 3.4. *Bipartite graph $G(B, L)$ has two matchings $M_i \subseteq A_i$ $(i = 1, 2)$ which cover $L \setminus B$.*

Proof. Recall Hall's theorem for bipartite matchings, i.e., an undirected bipartite graph $G = (V_1 \cup V_2, E)$ has a a matching which covers V_1 if and only if $|\delta(X)| \geq |X|$ holds for all $X \subseteq V_1$, where $\delta(X)$ denotes the set of neighbors of X.

Note that B is a common base, and Lemma 3.3 shows that

$$\forall X \subseteq L \setminus B,\ |\delta_1(X)| \geq |X|,\ |\delta_2(X)| \geq |X|,$$

which together with Hall's theorem proves the lemma. □

Let us now consider the bipartite graph $G(J, I^{(p)})$. Since J and $I^{(p)}$ are common bases, Lemma 3.4 implies that $G(J, I^{(p)})$ contains two perfect matchings $M_1 \subseteq A_1$ and $M_2 \subseteq A_2$. Therefore, each connected component of $M_1 \cup M_2$ forms an alternating cycle with A_1 and A_2. Consider one of these cycles with vertices $e_1, f_1, \ldots, e_d, f_d, e_1$ such that $e_j \in J \setminus I^{(p)}$, $f_j \in I^{(p)} \setminus J$, $(e_j, f_j) \in A_1$ $(j = 1, \ldots, d)$, $(f_j, e_{j+1}) \in A_2$ $(j = 1, \ldots, d-1)$, and $(f_d, e_1) \in A_2$. For every such cycle, we shall show

$$\frac{\sum_{j=1}^{d} w(e_j)}{\sum_{j=1}^{d} w(f_j)} \geq \frac{\sum_{j=1}^{d} (w_1(e_j) + w_2(e_j))^{1/k}}{\sum_{j=1}^{d} (w_1(f_j) + w_2(f_j))^{1/k}} \geq 2^{(1/k)-1}, \tag{4}$$

where w_1 and w_2 are weight functions constructed by an optimal dual solution for LP formulation (1) of the matroid intersection problem with respect to the weight w^k (see (2)). This completes the proof for the case in which $|J| \leq |I^{(p)}|$.

Note that the first inequality in (4) holds by $w_1(e) + w_2(e) \geq w^k(e)$ for every $e \in E$ and the complementary slackness (i.e., $w_1(e) + w_2(e) = w^k(e)$ for every $e \in J$). To show the second inequality in (4), we introduce $4d$ variables $x_0, \ldots, x_{2d-1}, y_1, \ldots, y_{2d}$, where x_{2j-1}, x_{2j-2}, y_{2j-1} and y_{2j} $(j = 1, \ldots, d)$ correspond to $w_1(e_j)$, $w_2(e_j)$, $w_1(f_j)$ and $w_2(f_j)$, respectively. Since $(e_j, f_j) \in A_1$, by Theorem 2.1, we have $w_1(e_j) \geq w_1(f_j)$ $(j = 1, \ldots, d)$. Similarly, we have $w_2(e_{j+1}) \geq w_2(f_j)$ $(j = 1, \ldots, d-1)$, and $w_2(e_1) \geq w_2(f_d)$. Therefore, we consider the following optimization problem to prove (4).

$$\begin{array}{lll} \text{minimize} & Z = \dfrac{(x_0 + x_1)^{1/k} + (x_2 + x_3)^{1/k} + \cdots + (x_{2d-2} + x_{2d-1})^{1/k}}{(y_1 + y_2)^{1/k} + (y_3 + y_4)^{1/k} + \cdots + (y_{2d-1} + y_{2d})^{1/k}} \\ \text{subject to} & x_j \geq y_j \geq 0 \quad (j = 1, \ldots, 2d-1) \\ & x_0 \geq y_{2d} \geq 0 \\ & \text{the denominator of } Z \text{ is positive.} \end{array}$$

Since Z is clearly minimized when $x_j = y_j$ $(1 \le j \le 2d-1)$ and $x_0 = y_{2d}$, it is consequently enough to prove

$$\begin{array}{lrcl} \text{minimize} & Z(x) & = & \dfrac{(x_0+x_1)^{1/k}+(x_2+x_3)^{1/k}+\cdots+(x_{2d-2}+x_{2d-1})^{1/k}}{(x_1+x_2)^{1/k}+(x_3+x_4)^{1/k}+\cdots+(x_{2d-1}+x_0)^{1/k}} \\ \text{subject to} & x_j & \ge & 0 \quad (j=0,\ldots,2d-1) \\ & & & \text{the denominator of } Z(x) \text{ is positive} \end{array} \tag{5}$$

is at least $2^{(1/k)-1}$. Let us consider the following operation **REDUCE**(j) which transforms a nonnegative x without increasing $Z(x)$.

REDUCE(j)
If $x_{2j-2} + x_{2j-1} \ge x_{2j} + x_{2j+1}$, then $x_{2j-1} := x_{2j-1} + x_{2j}$ and $x_{2j} := 0$.
Otherwise, $x_{2j} := x_{2j-1} + x_{2j}$ and $x_{2j-1} := 0$.

Here we define $x_{2d} = x_0$ and $x_{2d+1} = x_1$. Note that for any nonnegative a, b and c with $a \ge b \ge c$, we have

$$a^{1/k} + b^{1/k} \ge (a+c)^{1/k} + (b-c)^{1/k}$$

by the concavity of function $f(x) = x^{1/k}$ for $k \ge 1$. Thus **REDUCE**(j) creates a new feasible solution x to (5) without increasing $Z(x)$. By repeatedly applying **REDUCE**(j), $j = 1, \ldots, d$, we obtain x with at least one 0 for each pair of $(x_1, x_2), (x_3, x_4), \ldots, (x_{2d-1}, x_0)$. Then, by removing variables of value 0, $Z(x)$ can be represented as

$$\begin{aligned} Z(x) &= \frac{\sum_{j\in J_1}(x_{2j}+x_{2j+1})^{1/k} + \sum_{j\in J_2} x_{2j}^{1/k} + \sum_{j\in J_3} x_{2j+1}^{1/k}}{\sum_{j\in J_1}\left(x_{2j}^{1/k} + x_{2j+1}^{1/k}\right) + \sum_{j\in J_2} x_{2j}^{1/k} + \sum_{j\in J_3} x_{2j+1}^{1/k}} \\ &\ge \min\left\{ \min_{j\in J_1} \frac{(x_{2j}+x_{2j+1})^{1/k}}{x_{2j}^{1/k} + x_{2j+1}^{1/k}}, 1 \right\}. \end{aligned}$$

for some sets $J_1, J_2, J_3 \subseteq \{1, \ldots, d\}$. Since we have $\frac{(a+b)^{1/k}}{a^{1/k}+b^{1/k}} \ge \frac{(a+b)^{1/k}}{((a+b)/2)^{1/k}+((a+b)/2)^{1/k}} = 2^{(1/k)-1}$ for any positive a and b, we can conclude that $Z(x) \ge 2^{(1/k)-1}$.

3.2 The case when $|J| > |I^{(p)}|$

Let J be a maximum common independent set of $\mathcal{M}_1$ and $\mathcal{M}_2$ with respect to w^k. we shall show that $w(J_{(p)}) \ge \min\{2^{(1/k)-1}, 2^{-1/k}\} \cdot w(I^{(p)})$ if $|J| > |I^{(p)}|$.

Let us construct a set $K \subseteq J$ such that $|K| = |I^{(p)}|$ and

$$w(K \setminus I^{(p)}) \ge \min\{2^{(1/k)-1}, 2^{-1/k}\} \cdot w(I^{(p)} \setminus K), \tag{6}$$

which completes the proof.

Similarly to the case in which $|J| \le |I^{(p)}|$, we truncate matroids $\mathcal{M}_i$ $(i = 1, 2)$ by $|J|$:

$$\mathcal{I}_i := \{I \in \mathcal{I}_i \mid |I| \le |J|\}.$$

Note that J becomes a common base by this transformation. Consider a bipartite graph $G(J, I^{(p)}) = (V, A_1 \cup A_2)$ defined as (3). By Lemma 3.4, $G(J, I^{(p)})$ has two matchings $M_1 \subseteq A_1$ and $M_2 \subseteq A_2$ that both cover $I^{(p)} \setminus J$. $M_1 \cup M_2$ consists of alternating cycles and paths, where each path starts and ends with vertices in $J \setminus I^{(p)}$. Let $V_1, \ldots, V_q$ denote the vertex set of connected components of $M_1 \cup M_2$. Note that $\bigcup_\ell V_\ell$ contains $I^{(p)} \setminus J$. For each component, define K_ℓ by $K_\ell = V_\ell \cap J$ if it forms a cycle; otherwise, $K_\ell = V_\ell \cap (J \setminus \{e\})$, where e denotes a vertex in $V_\ell \cap J$ with the minimum $w(e)$. We then define $K = (I^{(p)} \cap J) \cup \bigcup_\ell K_\ell$. We note that $K \subseteq J$ and $|K| = |I^{(p)}|$. In order to prove (6), we show

$$w(K_\ell) \geq \min\{2^{(1/k)-1}, 2^{-1/k}\} \cdot w(V_\ell \setminus K_\ell) \quad \text{for all } \ell. \tag{7}$$

If the connected component forms a cycle, then (7) holds by a similar argument to the case where $|J| \geq |I^{(p)}|$. On the other hand, if it forms a path $e_1, f_1, \ldots, e_d, f_d, e_{d+1}$, where $e_i \in J \setminus I^{(p)}$ and $f_i \in I^{(p)} \setminus J$, then (7) can be restated as

$$\frac{\sum_{j=1}^{d+1} w(e_j) - \min_{1 \leq j \leq d+1} w(e_j)}{\sum_{j=1}^{d} w(f_j)} \geq \min\{2^{(1/k)-1}, 2^{-1/k}\}. \tag{8}$$

Let w_1 and w_2 denote weight functions constructed by an optimal dual solution for LP formulation (1) of the matroid intersection problem with respect to the weight w^k (see (2)). Then the left-hand-side of (8) is at least

$$\frac{\sum_{j=1}^{d+1} (w_1(e_j) + w_2(e_j))^{1/k} - \min_{1 \leq j \leq d+1} (w_1(e_j) + w_2(e_j))^{1/k}}{\sum_{j=1}^{d} (w_1(f_j) + w_2(f_j))^{1/k}},$$

which follows from the dual feasibility and complementary slackness. By Theorem 2.1, we have $w_1(e_j) \geq w_1(f_j) \geq 0$ and $w_2(e_{j+1}) \geq w_2(f_j) \geq 0$ for all j $(1 \leq j \leq d)$ and $w_1(e_{d+1}), w_2(e_1) \geq 0$. Thus it is sufficient to consider the case in which $w_1(e_{d+1}) = w_2(e_1) = 0$, and $w_1(e_j) = w_1(f_j)$ and $w_2(e_{j+1}) = w_2(f_j)$ for $j = 1, \ldots, d$. Namely we consider the following optimization problem:

$$\begin{array}{lll} \text{minimize} & Z(x) = \dfrac{x_1^{1/k} + (x_2 + x_3)^{1/k} + \cdots + (x_{2d-2} + x_{2d-1})^{1/k} + x_{2d}^{1/k} - s_{\min}^{1/k}}{(x_1 + x_2)^{1/k} + (x_3 + x_4)^{1/k} + \cdots + (x_{2d-1} + x_{2d})^{1/k}} \\ \text{subject to} & x_j \geq 0 \quad (j = 1, \ldots, 2d) \\ & \text{the denominator of } Z(x) \text{ is positive,} \end{array} \tag{9}$$

where $s_{\min} = \min\{x_1, x_2 + x_3, \ldots, x_{2d-2} + x_{2d-1}, x_{2d}\}$ and x_{2j-1} and x_{2j}, respectively, correspond to $w_1(e_j) = w_1(f_j)$ and $w_2(e_{j+1}) = w_2(f_j)$. We prove by induction on d that the optimal value of problem (9) is at least $\min\{2^{(1/k)-1}, 2^{-1/k}\}$.

If $d = 1$, then the claim is true, since

$$Z(x) = \frac{(\max\{x_1, x_2\})^{1/k}}{(x_1 + x_2)^{1/k}} \geq \left(\frac{1}{2}\right)^{1/k}$$

for any $x_1, x_2 \in \mathbb{R}_+$ with $x_1 + x_2 > 0$.

Supposing that the claim is true for any $d' \leq d - 1$, we consider the claim for $d > 1$. Define s_j $(1 \leq j \leq k + 1)$ by $s_j = x_1$ if $j = 1$, $x_{2j-2} + x_{2j-1}$ if $2 \leq j \leq d$, and x_{2d} if $j = d + 1$. Then, we have

$$Z(x) = \frac{\sum_{j=1}^{d+1} s_j^{1/k} - s_{\min}^{1/k}}{\sum_{j=1}^{d} (x_{2j-1} + x_{2j})^{1/k}}.$$

We first show the following case.

Lemma 3.5. *If $s_{\min} = s_j$ for some j with $2 \leq j \leq d$, then we have $Z(x) \geq \min\{2^{(1/k)-1}, 2^{-1/k}\}$.*

Proof. Let $s_h = s_{\min}$ $(2 \le h \le d)$. Then we have

$$Z(x) = \frac{\sum_{j=1}^{h} s_j^{1/k} - s_h^{1/k} + \sum_{j=h}^{d+1} s_j^{1/k} - s_h^{1/k}}{\sum_{j=1}^{h-1}(x_{2j-1}+x_{2j})^{1/k} + \sum_{j=h}^{d}(x_{2j-1}+x_{2j})^{1/k}}.$$

If $\sum_{j=1}^{h-1}(x_{2j-1}+x_{2j})^{1/k} = 0$, then $Z(x) = \frac{\sum_{j=h}^{d+1} s_j^{1/k} - s_h^{1/k}}{\sum_{j=h}^{d}(x_{2j-1}+x_{2j})^{1/k}}$. By the inductive hypothesis, this implies the claim. Similarly, we can prove the claim if $\sum_{j=h}^{d}(x_{2j-1}+x_{2j})^{1/k} = 0$. On the other hand, if $\sum_{j=1}^{h-1}(x_{2j-1}+x_{2j})^{1/k}$, $\sum_{j=h}^{d}(x_{2j-1}+x_{2j})^{1/k} \neq 0$, we obtain

$$Z(x) \ge \min\left\{ \frac{\sum_{j=1}^{h} s_j^{1/k} - s_h^{1/k}}{\sum_{j=1}^{h-1}(x_{2j-1}+x_{2j})^{1/k}}, \frac{\sum_{j=h}^{d+1} s_j^{1/k} - s_h^{1/k}}{\sum_{j=h}^{d}(x_{2j-1}+x_{2j})^{1/k}} \right\}.$$

By the inductive hypothesis, this again implies the claim. □

By the lemma, we assume without loss of generality that $s_1 = s_{\min}$, and separately consider two cases: (1) $s_{j-1} \ge s_j \le s_{j+1}$ for some j, and (2) $s_1 \le \cdots \le s_j \ge \cdots \ge s_{d+1}$ for some j.

Lemma 3.6. *If there exists some* j $(2 \le j \le d)$ *with* $s_{j-1} \ge s_j \le s_{j+1}$*, then we have* $Z(x) \ge \min\{2^{(1/k)-1}, 2^{-1/k}\}$.

Proof. Let us modify x as follows.

If $x_{2j-2} \ge x_1$, then $x_{2j-3} := x_{2j-3} + x_{2j-2} - x_1$, $x_{2j-2} := x_1$, $x_{2j} := x_{2j-1} + x_{2j}$, and $x_{2j-1} := 0$.
Otherwise, $x_{2j} := x_{2j-2} + x_{2j-1} + x_{2j} - x_1$ and $x_{2j-1} := x_1 - x_{2j-2}$.

After this operation, we have $s_1 = s_j = s_{\min}$ without increasing $Z(x)$. By Lemma 3.5, we have $Z(x) \ge \min\{2^{(1/k)-1}, 2^{-1/k}\}$. □

For case (2), we further separate it to the following three cases.

(2a) $j \ge 3$ (i.e., $s_1 \le s_2 \le s_3$),
(2b) $j = 2$ and $d \ge 3$ (i.e., $s_1 \le s_2 > s_3 \ge \cdots \ge s_{d+1}$),
(2c) $d = 2$ (i.e., $s_1 \le s_3 \le s_2$).

Case (2a): we consider the following modification of x without increasing $Z(x)$.

If $x_2 \ge x_1$, then $x_4 := x_3 + x_4$ and $x_3 := 0$.
Otherwise, $x_4 := x_2 + x_3 + x_4 - x_1$ and $x_3 := x_1 - x_2$.

If $x_2 < x_1$, we have $s_2 = s_1 = s_{\min}$. It follows from Lemma 3.5 that $Z(x) \ge \min\{2^{(1/k)-1}, 2^{-1/k}\}$. Otherwise (i.e. $x_2 \ge x_1$), we have $x_3 = 0$ and $s_1 = s_{\min}$. Since $x_3 = 0$, it holds that

$$Z(x) = \frac{x_2^{1/k} + \sum_{j=3}^{d+1} s_j^{1/k}}{(x_1+x_2)^{1/k} + \sum_{j=2}^{d}(x_{2j-1}+x_{2j})^{1/k}}.$$

If $(x_1+x_2)^{1/k} \neq 0$ and $\sum_{j=2}^{d}(x_{2j-1}+x_{2j})^{1/k} \neq 0$, then

$$Z(x) \ge \min\left\{ \frac{x_2^{1/k}}{(x_1+x_2)^{1/k}}, \frac{\sum_{j=3}^{d+1} s_j^{1/k}}{\sum_{j=2}^{d}(x_{2j-1}+x_{2j})^{1/k}} \right\},$$

which is at least $\min\{2^{(1/k)-1}, 2^{-1/k}\}$ by $x_2 \ge x_1$ and the inductive hypothesis. Similarly, we can deal with the case when $(x_1+x_2)^{1/k} = 0$ or $\sum_{j=2}^{d}(x_{2j-1}+x_{2j})^{1/k} = 0$.

Case (2b): In this case, transform x by

$$x_{2d-1} := x_{2d-1} + x_{2d} - x_1 \text{ and } x_{2d} := x_1.$$

By this transformation, we have $s_{d+1} = s_1 = s_{\min}$ and $s_d \geq s_{d+1}$. Thus, if $s_{d-1} \geq s_d$ holds, then by symmetry to **Case (2a)**, we can obtain the claim. Otherwise (i.e., $s_{d-1} < s_d$), if $d = 3$, we have $s_1 \leq s_2 \leq s_3$, which implies the claim by **Case (2a)**. On the other hand, if $d \geq 4$, we have $s_{d-2} \geq s_{d-1} \leq s_d$, which again implies the claim by Lemma 3.6.

Case (2c): We prove the claim by a direct calculation. In this case,

$$Z(x) = \frac{(x_2 + x_3)^{1/k} + x_4^{1/k}}{(x_1 + x_2)^{1/k} + (x_3 + x_4)^{1/k}}.$$

If we fix the values of $s_2 = x_2 + x_3$ and x_4, then $Z(x)$ is minimized when $x_1 = x_4$ and $x_2 = x_3$, because of $0 \leq x_1 \leq x_4$ and the concavity of $f(x) = x^{1/k}$. Thus, it suffices to consider the minimum value of

$$\frac{(2x_2)^{1/k} + x_4^{1/k}}{2(x_2 + x_4)^{1/k}}$$

under the condition $2x_2 \geq x_4 \geq 0$. By setting $x := x_4/x_2$, this is equal to the minimal value of

$$g(x) = \frac{2^{1/k} + x^{1/k}}{2(1 + x)^{1/k}}$$

under the condition $2 \geq x \geq 0$. Since

$$g'(x) = \frac{1}{2k}(1 + x)^{-1-(1/k)}(x^{(1/k)-1} - 2^{1/k}),$$

g is minimized either when $x = 0$ or $x = 2$. In the former case, $g = 2^{(1/k)-1}$, and in the latter case, $g = (2/3)^{1/k}(\geq 2^{-1/k})$, which completes the proof of **Case (2c)**.

4 Complexity of α-Robust Matching

In this section, we study the time complexity of the following problem.

> **α-ROBUST-MATCHING**
>
> **Instance:** A graph $G = (V, E)$ and a weight $w(e) \in \mathbb{Z}_+$ for each $e \in E$.
>
> **Question:** Is there an α-robust matching in G ?

Theorem 4.1. α-ROBUST-MATCHING *is* NP-*complete when* $\frac{1}{\sqrt{2}} < \alpha < 1$, *and it is polynomially solvable when* $\alpha \leq \frac{1}{\sqrt{2}}$ *or* $\alpha = 1$.

Note that the polynomial result for $\alpha \leq \frac{1}{\sqrt{2}}$ is given in [5]. This theorem gives us a sharp border for the complexity of α-ROBUST-MATCHING. The proof of Theorem 4.1 consists of the following three parts. In Sections 4.1 and 4.2, we deal with the cases when $\frac{2+\sqrt{2}}{4} < \alpha < 1$ and $\frac{1}{\sqrt{2}} < \alpha \leq \frac{2+\sqrt{2}}{4}$, respectively. A polynomial-time algorithm for $\alpha = 1$ is presented in Section 5.1 (see Theorem 5.1).

We also show that when α is a part of the input, detecting an α-robust matching is NP-complete in the strong sense. For a precise description, we introduce the following problem.

ROBUST-MATCHING

Instance: A graph $G = (V, E)$, a weight $w(e) \in \mathbb{Z}_+$ for each $e \in E$, and positive integers α_1 and α_2.

Question: Is there an $\frac{\alpha_1}{\alpha_2}$-robust matching in G ?

Theorem 4.2. ROBUST-MATCHING *is* NP-*complete in the strong sense, that is, it is* NP-*complete even if the size of the input is* $\Theta(|V| + |E| + w(E) + \alpha_1 + \alpha_2)$.

The proof of Theorem 4.2 is given in Section 6.

4.1 NP-hardness for α-Robust Matching when $\frac{2+\sqrt{2}}{4} < \alpha < 1$

In this subsection, we deal with the case when $\frac{2+\sqrt{2}}{4} < \alpha < 1$. For a concise description, we first discuss the case when $\alpha = \frac{7}{8}$, and then show how to modify the reduction for a general α with $\frac{2+\sqrt{2}}{4} < \alpha < 1$.

Our proof of Theorem 4.1 is based on the NP-completeness of PARTITION. For a finite set S and a function $f : S \to \mathbb{R}_+$, PARTITION is the problem to find a partition (A, B) of S with $f(A) = f(B)$, and it is one of Karp's NP-complete problems [7]. By NP-completeness of PARTITION, we can see that the following problem is also NP-complete.

PARTITION′

Instance: A finite set S and a non-negative number $f(e)$ for each $e \in S$.

Question: Is there a partition (A, B) of S such that $|A| = |B|$ and $f(A) = f(B)$?

Proposition 4.3. PARTITION′ *is* NP-*complete.*

Proof. PARTITION′ is obviously in NP. Given an instance S of PARTITION, construct an instance of PARTITION′ by adding $|S|$ new elements e with $f(e) = 0$ to S. Then it is not difficult to see that S has a solution if and only if so does the corresponding instance of PARTITION′, which implies that PARTITION′ is NP-hard. □

In what follows, we show that there exists a polynomial-reduction from PARTITION′ to $\frac{7}{8}$-ROBUST-MATCHING.

Suppose a set S with $|S| = n$ and a function $f : S \to \mathbb{R}_+$ are an instance of PARTITION′. We construct an instance of $\frac{7}{8}$-ROBUST-MATCHING as follows. Let $G = (V, E)$ be a graph defined by

$$V = \{v_{i,j} \mid i = 1, 2, 3, 4,\ j \in S\},$$
$$E = \{e_{i,j} \mid e_{i,j} = (v_{i,j}, v_{i+1,j}),\ i = 1, 2, 3,\ j \in S\}.$$

Let L be a large enough integer relative to $f(S)$ (for example, $L = 7f(S) + 1$) and define the weights of edges as

$$w(e_{1,j}) = 7(L + f(j)),$$
$$w(e_{2,j}) = 12(L + f(j)),$$
$$w(e_{3,j}) = 9(L + f(j))$$

for $j \in S$.

For a maximal matching $M \subseteq E$, we define a partition (A, B) of S by

$$A = \{j \mid j \in S,\ e_{1,j} \in M,\ e_{3,j} \in M\}, \tag{10}$$
$$B = \{j \mid j \in S,\ e_{2,j} \in M\}. \tag{11}$$

Conversely, a partition (A, B) of S defines a maximal matching M in G in a similar way. We say that such a maximal matching M corresponds to a partition (A, B) of S. Recall that $M_{(p)}$ is a set of heaviest $\min\{p, |M|\}$ edges of M and $M^{(p)}$ is a maximum p-matching.

Lemma 4.4. *If a maximal matching M in G is $\frac{7}{8}$-robust, then its corresponding partition (A, B) of S satisfies that $|A| = |B| = \frac{n}{2}$ and $f(A) = f(B)$.*

Proof of Lemma 4.4. Suppose that M is a $\frac{7}{8}$-robust matching of G. Since

$$w(M_{(n)}) = 9|A|L + 9f(A) + 12|B|L + 12f(B),$$
$$w(M^{(n)}) = 12nL + 12f(S),$$

it holds that

$$w(M_{(n)}) - \frac{7}{8}w(M^{(n)}) = 3\left(|B|L + f(B) - \frac{1}{2}nL - \frac{1}{2}f(S)\right) \geq 0. \tag{12}$$

On the other hand, since

$$w(M_{(2n)}) = 16|A|L + 16f(A) + 12|B|L + 12f(B),$$
$$w(M^{(2n)}) = 16nL + 16f(S),$$

it holds that

$$w(M_{(2n)}) - \frac{7}{8}w(M^{(2n)}) = -4\left(|B|L + f(B) - \frac{1}{2}nL - \frac{1}{2}f(S)\right) \geq 0. \tag{13}$$

By (12) and (13), we have $|B|L + f(B) - \frac{1}{2}nL - \frac{1}{2}f(S) = 0$. Since L is large enough, this means that $|B| = \frac{1}{2}n$ and $f(B) = \frac{1}{2}f(S)$, which shows the present lemma. □

For a set $T = \{j_1, j_2, \ldots, j_{|T|}\} \subseteq S$ with $f(j_1) \geq f(j_2) \geq \cdots \geq f(j_{|T|})$ and for $0 \leq p \leq |T|$, define $T_{(p)}$ by

$$T_{(p)} = \{j_1, j_2, \ldots, j_p\}. \tag{14}$$

Lemma 4.5. *If a partition (A, B) of S satisfies that $|A| = |B|$ and $f(A) = f(B)$, then its corresponding maximal matching M in G is $\frac{7}{8}$-robust.*

Proof of Lemma 4.5. Suppose that $|A| = |B| = \frac{n}{2}$ and $f(A) = f(B)$. It suffices to show that $w(M_{(p)}) \geq \frac{7}{8}w(M^{(p)})$ for any $1 \leq p \leq 2n$. We consider the following cases:

Case 1: When $p = n$ or $p \geq 2n$, by the same calculation as the proof of Lemma 4.4, we have $w(M_{(p)}) = \frac{7}{8}w(M^{(p)})$.

Case 2: When $1 \leq p \leq \frac{n}{2}$, we have $w(M_{(p)}) - \frac{7}{8}w(M^{(p)}) = 12pL + 12f(B_{(p)}) - \frac{21}{2}pL - \frac{21}{2}f(S_{(p)})$, which is at least $\frac{3}{2}pL - \frac{21}{2}f(S_{(p)}) > 0$.

Case 3: When $\frac{n}{2}+1 \le p \le n-1$, it holds that $w(M_{(p)}) - \frac{7}{8}w(M^{(p)}) = w(M_{(n)}) - 9(n-p)L - 9f(A \setminus A_{(p-\frac{n}{2})}) - \frac{7}{8}\left(w(M^{(n)}) - 12(n-p)L - 12f(S \setminus S_{(p-\frac{n}{2})})\right)$, which is at least $\frac{3}{2}(n-p)L - 9f(A \setminus A_{(p-\frac{n}{2})}) > 0$.

Case 4: When $n+1 \le p \le \frac{3}{2}n$, it holds that $w(M_{(p)}) - \frac{7}{8}w(M^{(p)}) = w(M_{(n)}) + 7(p-n)L + 7f(A_{(p-n)}) - \frac{7}{8}\left(w(M^{(n)}) + 4(p-n)L + 4f(S_{(p-n)})\right)$, which is at least $\frac{7}{2}(p-n)L - \frac{7}{2}f(S_{(p-n)}) > 0$.

Case 5: When $\frac{3}{2}n+1 \le p \le 2n-1$, we have $w(M_{(p)}) - \frac{7}{8}w(M^{(p)}) > w(M_{(2n)}) - \frac{7}{8}w(M^{(2n)}) = 0$.

Therefore, $w(M_{(p)}) \ge \frac{7}{8}w(M^{(p)})$ for any p, which shows the lemma. □

Since it suffices to deal with maximal matchings in $\frac{7}{8}$-ROBUST-MATCHING, Lemmata 4.4 and 4.5 imply that there exists a partition (A, B) of S such that $|A| = |B|$ and $f(A) = f(B)$ if and only if G has a $\frac{7}{8}$-robust matching with respect to the weight function w. This shows that PARTITION′ can be reduced to $\frac{7}{8}$-ROBUST-MATCHING, and hence $\frac{7}{8}$-ROBUST-MATCHING is NP-hard.

In order to show the NP-hardness of α-ROBUST-MATCHING for a general number α with $\frac{2+\sqrt{2}}{4} < \alpha < 1$, define the weight of the edges of G as

$$\begin{aligned} w(e_{1,j}) &= a(L + f(j)), \\ w(e_{2,j}) &= b(L + f(j)), \\ w(e_{3,j}) &= c(L + f(j)), \end{aligned}$$

where $0 < a < c < b < a + c$,

$$\frac{a+b+c}{2(a+c)} = \frac{b+c}{2b} = \alpha,$$

and L is a large enough integer relative to $f(S)$. For example, $a = 4\alpha(1-\alpha)$, $b = 2\alpha - 1$, and $c = (2\alpha-1)^2$ satisfy the above conditions if $\frac{2+\sqrt{2}}{4} < \alpha < 1$. By the same argument as the case $\alpha = \frac{7}{8}$, we obtain the NP-hardness for α-ROBUST-MATCHING when $\frac{2+\sqrt{2}}{4} < \alpha < 1$.

4.2 NP-hardness for α-Robust Matching when $\frac{1}{\sqrt{2}} < \alpha \le \frac{2+\sqrt{2}}{4}$

In this subsection, we modify the proof in the previous subsection to apply the case when $\frac{1}{\sqrt{2}} < \alpha \le \frac{2+\sqrt{2}}{4}$.

As with the previous subsection, we reduce an instance of PARTITION′ to the problem. Let a set S with $|S| = n$ and $f : S \to \mathbb{R}_+$ be an instance of PARTITION′. We construct an instance of α-ROBUST-MATCHING as follows.

Let $G = (V, E)$ be a graph defined by

$$\begin{aligned} V &= \{v_{i,j} \mid i = 1,2,3,4,\ j \in S\} \cup \{u_1, u_2, u_3, u_4, u_5\}, \\ E &= \{e_{i,j} \mid e_{i,j} = (v_{i,j}, v_{i+1,j}),\ i = 1,2,3,\ j \in S\} \\ &\quad \cup \{(u_1,u_2),(u_2,u_3),(u_3,u_4),(u_4,u_5)\}. \end{aligned}$$

Define the weight of the edges of G as

$$\begin{aligned} w(e_{1,j}) &= a(L + f(j)), \\ w(e_{2,j}) &= b(L + f(j)), \\ w(e_{3,j}) &= c(L + f(j)), \end{aligned}$$

for $j \in S$, and

$$\begin{aligned}
w(u_1, u_2) &= w(u_3, u_4) = N(nL + f(S)), \\
w(u_2, u_3) &= (\frac{1}{\alpha} + \epsilon)N(nL + f(S)), \\
w(u_4, u_5) &= (\sqrt{2} - \frac{1}{\alpha} - \epsilon)N(nL + f(S)),
\end{aligned}$$

where $0 < \epsilon < \sqrt{2} - \frac{1}{\alpha}$, $0 < a < c < b < a + c$, $N > 0$, $w(u_4, u_5) > w(e_{i,j})$ for every i, j, L is large enough relative to $f(S)$, and

$$\frac{\sqrt{2}N + (a + b + c)/2}{2N + (a + c)} = \frac{\sqrt{2}N + (b + c)/2}{2N + b} = \alpha.$$

For example, $a = 2\alpha$, $b = 10$, $c = 11 - 2\alpha$, $\epsilon = \frac{\sqrt{2}-1/\alpha}{2}$, and $N = \frac{21-22\alpha}{4\alpha-2\sqrt{2}}$ satisfy the conditions if n is large enough. Then we can show that S has a desired partition if and only if G has an α-robust matching.

If a matching M does not contain (u_2, u_3), then

$$w(M_{(1)}) \leq N(nL + f(S)) < \alpha w(u_2, u_3) = \alpha w(M^{(1)}),$$

and hence M is not α-robust. Thus, when we detect an α-robust matching, we only consider matchings containing (u_2, u_3). Furthermore, we may assume that the matchings contain (u_4, u_5).

In the same way as the previous subsection, we define by (10) and (11) the correspondence between a maximal matching M containing both (u_2, u_3) and (u_4, u_5) in G and a partition (A, B) of S. In order to show the equivalence of the instance of PARTITION′ and that of α-ROBUST MATCHING, we use the following proposition.

Proposition 4.6. *Let M be a maximal matching in G containing (u_2, u_3) and (u_4, u_5), and (A, B) be a partition of S corresponding to M. Then, M is α-robust if and only if $|A| = |B| = \frac{n}{2}$ and $f(A) = f(B)$.*

Proof. We only show the "only if" part of the proposition. The "if" part is obtained from a simple case analysis in the same way as the proof of Lemma 4.5.

Suppose that M is an α-robust matching of G. Since

$$\begin{aligned}
w(M_{(n+2)}) &= \sqrt{2}N(nL + f(S)) + c|A|L + cf(A) + b|B|L + bf(B), \\
w(M^{(n+2)}) &= 2N(nL + f(S)) + bnL + bf(S),
\end{aligned}$$

it holds that

$$w(M_{(n+2)}) - \alpha w(M^{(n+2)}) = (b - c)\left(|B|L + f(B) - \frac{1}{2}nL - \frac{1}{2}f(S)\right) \geq 0. \tag{15}$$

On the other hand, since

$$\begin{aligned}
w(M_{(2n+2)}) &= \sqrt{2}N(nL + f(S)) + (a + c)|A|L + (a + c)f(A) + b|B|L + bf(B), \\
w(M^{(2n+2)}) &= 2N(nL + f(S)) + (a + c)nL + (a + c)f(S),
\end{aligned}$$

it holds that

$$w(M_{(2n+2)}) - \alpha w(M^{(2n+2)}) = -(a + c - b)\left(|B|L + f(B) - \frac{1}{2}nL - \frac{1}{2}f(S)\right) \geq 0. \tag{16}$$

By (15) and (16), we have $|B|L + f(B) - \frac{1}{2}nL - \frac{1}{2}f(S) = 0$. Since L is large enough, this means that $|B| = \frac{1}{2}n$ and $f(B) = \frac{1}{2}f(S)$, which shows the "only if" part of the proposition. $\square$

By this proposition, PARTITION′ can be reduced to α-ROBUST-MATCHING, which shows Theorem 4.1 when $\frac{1}{\sqrt{2}} < \alpha \leq \frac{2+\sqrt{2}}{4}$.

5 k-th power algorithm

As shown in the previous section, α-ROBUST-MATCHING is NP-complete when $\frac{1}{\sqrt{2}} < \alpha < 1$. However, it is known [5] that every graph has a $\frac{1}{\sqrt{2}}$-robust matching which can be computed by the *k-th power algorithm*.

k-th power algorithm

Step 1. For a weight function $w : E \to \mathbb{Z}_+$, define $w^k : E \to \mathbb{Z}_+$ by $w^k(e) = \{w(e)\}^k$.

Step 2. Find a matching (or an independent set) M maximizing $w^k(M)$, and output M.

In this section, we analyze the performance of the k-th power algorithm.

5.1 1-robust matching

When $k = 1$ and $k = 2$, the k-th power algorithm finds an ordinary maximum matching and a $\frac{1}{\sqrt{2}}$-robust matching, respectively. In this section, we show that when k is large enough, the k-th power algorithm finds a 1-robust matching, if one exists.

Theorem 5.1. *Let $(E, \mathcal{I})$ be an independent system. Suppose that k is large enough to satisfy $w^k(e) > |E|w^k(e')$ if $w(e) > w(e')$. If $(E, \mathcal{I})$ has a 1-robust independent set, then it can be found by the k-th power algorithm.*

Proof. Let $F = \{f_1, f_2, \ldots, f_s\}$ be a 1-robust independent set for $(E, \mathcal{I})$, and $G = \{g_1, g_2, \ldots, g_t\}$ be output by the k-th power algorithm, where $w(f_1) \geq w(f_2) \geq \cdots \geq w(f_s)$ and $w(g_1) \geq w(g_2) \geq \cdots \geq w(g_t)$. Note that $w(I^{(p)}) = w(F_{(p)})$ holds for any p by the definition of the 1-robustness, where $I^{(p)}$ denotes a maximum p-independent set. Assume that $w(I^{(p-1)}) = w(G_{(p-1)})$ and $w(I^{(p)}) > w(G_{(p)})$ for some $p\,(\geq 1)$. Then we have

$$\begin{aligned} w^k(I^{(p)}) - w^k(G_{(p)}) &= w^k(f_p) - w^k(g_p) \\ &> (|E| - 1)w^k(g_p) \\ &> \sum_{j \geq p+1} w^k(g_j), \end{aligned}$$

which implies $w^k(I^{(p)}) > w^k(G)$. This contradicts the maximality of $w^k(G)$. Thus, $w^k(I^{(p)}) = w^k(G_{(p)})$ holds for every p. □

As a corollary of Theorem 5.1, for sufficiently large k, the k-power algorithm compute in polynomial time a 1-robust matching and a 1-robust common independent set of two matroids, for example.

We remark that, instead of using the k-th power of the original weight, we can use *any* weight function w' that satisfies $w'(e_1) > |E| \cdot w'(e_2)$ for all pairs of edges e_1 and e_2 with $w(e_1) > w(e_2)$.

For example, when we have t different weights $w_1 < w_2 < \cdots < w_t$, let $f(w_i) = (|E| + 1)^i$ and $w'(e) = f(w(e))$. Then $w'(e)$ satisfies this condition. We can find a 1-robust independent set (if exists) by finding a maximum independent set with respect to the weight function w'.

5.2 Negative results

We have already seen that the k-th power algorithm outputs a meaningful solution when $k = 1, 2, +\infty$, and so it might be expected that by choosing an appropriate parameter k depending on an instance, the k-th power algorithm outputs a good approximate solution for the robustness. However, in this subsection, we give a result against this expectation.

We consider the following optimization problem corresponding to α-ROBUST-MATCHING.

MAX-ROBUST-MATCHING

Instance: A graph $G = (V, E)$ and a weight $w(e) \in \mathbb{Z}_+$ for each $e \in E$.

Find: The maximum α such that G has an α-robust matching.

Since this problem is NP-hard by Theorem 4.1, we consider approximation algorithms for the problem. For an instance of MAX-ROBUST-MATCHING whose maximum value is α^* and for $0 < \beta < 1$, a matching M in G is *β-approximately robust* if M is $(\alpha^*\beta)$-robust. Obviously, for any instance of the problem, the k-th power algorithm finds a $\frac{1}{\sqrt{2}}$-approximately robust matching when $k = 2$. The following theorem shows that $\frac{1}{\sqrt{2}}$ is the best approximation ratio of the k-th power algorithm.

Theorem 5.2. *For any $\epsilon > 0$, there exists an instance of* MAX-ROBUST-MATCHING *such that the k-th power algorithm does not output $\left(\frac{1}{\sqrt{2}} + \epsilon\right)$-approximately robust matching for any k.*

Proof. It suffices to show that for any small $\epsilon' > 0$, there exists an instance of MAX-ROBUST-MATCHING satisfying the following conditions:

(A) There exists a $(1 - \epsilon')$-robust matching.

(B) For any k, the output of the k-th power algorithm is not $\left(\frac{1}{\sqrt{2}} + \epsilon'\right)$-robust.

We consider the following instance of the problem. Define $\gamma = \frac{1}{\sqrt{2}}$ for a concise description, and let L be an integer such that $L > \frac{5}{\epsilon'}$. Let $S_0, S_1, \ldots, S_L$ be finite sets with $|S_0| = L$ and $|S_t| = \lfloor(\sqrt{2})^t\rfloor$ for $t = 1, 2, \ldots, L^2$. Let $G = (V, E)$ be a graph defined by

$$\begin{aligned}
V_0 &= \{v_{i,j} \mid i = 1, 2,\ j \in S_0\}, \\
V_t &= \{v_{i,j} \mid i = 1, 2, 3, 4,\ j \in S_t\} \quad \text{for } t = 1, 2, \ldots, L^2, \\
V &= \left(\bigcup_{t=0}^{L^2} V_t\right) \cup \{u_1, u_2, u_3, u_4\}, \\
E_0 &= \{(v_{1,j}, v_{2,j}) \mid j \in S_0\}, \\
E_t &= \{e_{i,j} \mid e_{i,j} = (v_{i,j}, v_{i+1,j}),\ i = 1, 2, 3,\ j \in S_t\} \quad \text{for } t = 1, 2, \ldots, L^2, \\
E &= \left(\bigcup_{t=0}^{L^2} E_t\right) \cup \{(u_1, u_2), (u_2, u_3), (u_3, u_4)\}.
\end{aligned}$$

Define a weight function $w : E \to \mathbb{R}_+$ by

$$w(e) = \begin{cases} \sqrt{2} - \epsilon' & \text{if } e = (u_2, u_3), \\ 1 & \text{if } e \in E_0 \cup \{(u_1, u_2), (u_3, u_4)\}, \\ \gamma^t & \text{if } e = e_{2,j} \text{ for } j \in S_t, \\ \gamma^{t+1} & \text{if } e = e_{1,j} \text{ or } e = e_{3,j} \text{ for } j \in S_t, \end{cases}$$

for $t = 1, 2, \ldots, L^2$.

Then, (A) and (B) hold in this instance by Lemmata 5.3 and 5.5 below, which yields Theorem 5.2. ☐

Lemma 5.3. *There exists a $(1-\epsilon')$-robust matching in G.*

Proof. We show that a matching $M = \{(u_2, u_3)\} \cup E_0 \cup \{e_{1,j}, e_{3,j} \mid j \in S_1 \cup S_2 \cup \cdots \cup S_{L^2}\}$ is $(1-\epsilon')$-robust. It is obvious that $w(M_{(p)}) = w(M^{(p)})$ for $p = 1, 2, \ldots, L+1$.

For $p \geq L+2$, we use the following claim.

Claim 5.4. *Let p and t be integers with $p \geq L+2$ and $1 \leq t \leq L^2 - 3$. If there exist $t' \geq t+3$ and $j \in S_{t'}$ such that $e_{2,j}$ is contained in a maximum p-matching $M^{(p)}$, then $e_{2,j} \notin M^{(p)}$ for any $j \in S_t$.*

Proof of Claim 5.4. Assume that $e_{2,j_1}, e_{2,j_2} \in M^{(p)}$ for $j_1 \in S_t$ and $j_2 \in S_{t'}$. Then,

$$w(e_{2,j_1}) + w(e_{2,j_2}) \leq w(e_{1,j_1}) + w(e_{3,j_1}),$$

because $1 + \gamma^3 \leq 2\gamma$. This contradicts the maximality of $w(M^{(p)})$. ☐

By this claim, it holds that

$$M^{(p)} \setminus M \subseteq \{(u_1, u_2), (u_3, u_4)\} \cup \{w(e_{2,j}) \mid j \in S_t \cup S_{t+1} \cup S_{t+2}\}$$

for some t, and hence

$$w(M^{(p)} \setminus M) \leq 2 + \gamma^t(\sqrt{2})^t + \gamma^{t+1}(\sqrt{2})^{t+1} + \gamma^{t+2}(\sqrt{2})^{t+2} \leq 5.$$

Since $w(M^{(p)}) > L$ for $p \geq L+2$, we have

$$w(M^{(p)}) - w(M_{(p)}) \leq 5 < \epsilon' w(M^{(p)}),$$

and hence, M is $(1-\epsilon')$-robust. ☐

Lemma 5.5. *For any k, the output of the k-th power algorithm in G is not $\left(\frac{1}{\sqrt{2}} + \epsilon'\right)$-robust.*

Proof. Suppose that a matching M does not contain (u_2, u_3). Then, $(\frac{1}{\sqrt{2}} + \epsilon')M^{(1)} > 1 \geq M_{(1)}$, which means that M is not $\left(\frac{1}{\sqrt{2}} + \epsilon'\right)$-robust. Thus, it suffices to consider the case when the algorithm outputs a matching containing (u_2, u_3).

By the definition of the k-th power algorithm, if the output of the algorithm contains (u_2, u_3), then it coincides with $M^* = \{(u_2, u_3)\} \cup E_0 \cup \{e_{2,j} \mid j \in S_1 \cup S_2 \cup \ldots S_{L^2}\}$. Then, we have

$$\begin{aligned} w(M^*_{(|E|)}) &= w(u_2, u_3) + w(E_0) + \sum_{t=1}^{L^2} w(M^* \cap E_t) \\ &< L + \sqrt{2} + \sum_{t=1}^{L^2} \gamma^t(\sqrt{2})^t < L^2 + L + 2. \end{aligned}$$

On the other hand,

$$\begin{aligned} w(M^{(|E|)}) &= w(u_1, u_2) + w(u_3, u_4) + w(E_0) \\ &\quad + 2\sum \{w(e_{1,j}) + w(e_{3,j}) \mid j \in S_1 \cup S_2 \cup \cdots \cup S_{L^2}\} \\ &\geq L + 2 + \sum_{t=1}^{L^2} 2\gamma^{t+1}((\sqrt{2})^t - 1) > \sqrt{2}L^2 + L - 2. \end{aligned}$$

Hence,

$$\left(\frac{1}{\sqrt{2}} + \epsilon'\right) w(M^{(|E|)}) > L^2 + \epsilon' L^2 > w(M^*_{(|E|)}),$$

which means that M^* is not $\left(\frac{1}{\sqrt{2}} + \epsilon'\right)$-robust. $\square$

6 Proof of Theorem 4.2

In this section, we give a proof of Theorem 4.2.

Theorem 4.2. ROBUST-MATCHING *is* NP*-complete in the strong sense, that is, it is* NP*-complete even if the size of the input is* $\Theta(|V| + |E| + w(E) + \alpha_1 + \alpha_2)$.

ROBUST-MATCHING is obviously in NP. In order to show the NP-hardness in the strong sense, we consider the following problem.

3-PARTITION

Instance: A set S with $|S| = 3m$, a bound $B \in \mathbb{Z}_+$, a size $f(j) \in \mathbb{Z}_+$ for each $j \in S$ such that $f(S) = mB$.

Question: Can S partitioned into m disjoint sets $S_1, S_2, \ldots, S_m$ such that $|S_p| = 3$ and $f(S_p) = B$ for each $1 \leq p \leq m$?

3-PARTITION is known to be NP-complete in the strong sense [4], that is, 3-PARTITION is NP-complete even if the size of the input is $\Theta(|S| + f(S))$.

In what follows in this section, we show that 3-PARTITION can be reduced to ROBUST-MATCHING.

6.1 Reduction to ROBUST-MATCHING

Given an instance of 3-PARTITION with $m \geq 6$, we construct an instance of ROBUST-MATCHING as follows.

As shown in Fig. 1, let $G = (V, E)$ be a graph given by $V = \bigcup_{j \in S} V_j$ and $E = \bigcup_{j \in S} E_j$, where

$$V_j = \bigcup_{i=1}^{m} \{s_{i,j}, t_{i,j}, u_{i,j}, v_{i,j}\} \cup \{r_j\} \quad (j \in S),$$
$$E_j = \bigcup_{i=1}^{m} \{(r_j, s_{i,j}), (s_{i,j}, t_{i,j}), (t_{i,j}, u_{i,j}), (u_{i,j}, v_{i,j})\} \quad (j \in S),$$

For $i = 1, \ldots, m$, let k_i be an integer defined by

$$k_m = m^2 \quad \text{and} \quad k_i = \left\lfloor \left(1 + \frac{4}{m}\right) k_{i+1} \right\rfloor \quad (i = 1, \ldots, m-1).$$

Define the weight of the edges by

$$w(r_j, s_{i,j}) = (m^2 - 5)k_i(L + f(j)),$$
$$w(s_{i,j}, t_{i,j}) = (m^2 + 2m - 7)k_i(L + f(j)),$$
$$w(t_{i,j}, u_{i,j}) = (m^2 + 3m - 4)k_i(L + f(j)),$$
$$w(u_{i,j}, v_{i,j}) = (m^2 + 3m - 5)k_i(L + f(j)),$$

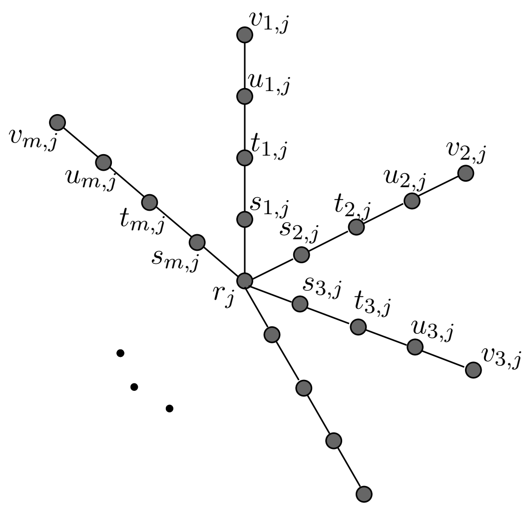

Figure 1: Construction of V_j and E_j.

where $L = 2m^3k_1f(S)$ which is a large enough integer. Note that $w(e)$ is an integer for every $e \in E$. Define α_1 and α_2 as $\alpha_1 = m^2 + 4m - 1$ and $\alpha_2 = m^2 + 4m$, and let $\alpha(m) = \frac{\alpha_1}{\alpha_2}$.

Then we can see the following proposition.

Proposition 6.1. *The input size of the instance of* ROBUST-MATCHING *defined above is a polynomial of m and $f(S)$.*

Proof. Since

$$k_i \leq \left(1 + \frac{4}{m}\right)^{m-i} k_m < \mathrm{e}^4 m^2$$

for $i = 1, \ldots, m$, there exists a polynomial of m, L, and $f(S)$ which is greater than $w(E)$. Since L is a polynomial of m and $f(S)$, the result follows. $\square$

Now we show that the instance of ROBUST-MATCHING defined as above is equivalent to the original instance of 3-PARTITION.

6.2 Correctness

In this section, we show the equivalence of the instance of ROBUST-MATCHING defined in Section 6.1 and the original instance of 3-PARTITION when $m \geq 6$.

First, we observe the following.

Lemma 6.2. *Let $m \geq 6$ be an integer. For $j \in S$ and for $i = 1, \ldots, m-1$, $w(t_{i+1,j}, u_{i+1,j}) < w(r_j, s_{i,j})$.*

Proof. It suffices to show that $(m^2 + 3m - 4)k_{i+1} < (m^2 - 5)k_i$ for $i = 1, \ldots, m-1$. Since $m^2 + 3m - 3 < \left(1 + \frac{4}{m}\right)(m^2 - 5)$ for $m \geq 6$, we have that

$$(m^2 - 5)k_i = (m^2 - 5)\left\lfloor \left(1 + \frac{4}{m}\right) k_{i+1} \right\rfloor > (m^2 + 3m - 4)k_{i+1},$$

which completes the proof. $\square$

When we detect an $\alpha(m)$-robust matching it suffices to deal with maximal matchings. Furthermore, we may assume that

(C) a maximal matching M contains one of $\{(t_{i,j}, u_{i,j})\}$ and $\{(s_{i,j}, t_{i,j}),\ (u_{i,j}, v_{i,j})\}$ for $i = 1, \dots, m$ and for $j \in S$,

because $w(r_j, s_{i,j}) < w(s_{i,j}, t_{i,j})$. For a maximal matching $M \subseteq E$, define a set $S_i \subseteq S$ by

$$S_i = \{j \mid j \in S,\ (t_{i,j}, u_{i,j}) \in M\}$$

for $i = 1, \dots, m$. In this case, we say that M corresponds to the subsets $S_1, \dots, S_m$ of S. Conversely, if subsets $S_1, \dots, S_m$ of S are mutually disjoint, then $S_1, \dots, S_m$ define a maximal matching M in G satisfying (C) uniquely, that is, M contains $\{(t_{i,j}, u_{i,j}), (r_j, s_{i,j})\}$ for every $j \in S_i$ and M contains $\{(s_{i,j}, t_{i,j}),\ (u_{i,j}, v_{i,j})\}$ for every $j \in \bar{S}_i$, where $\bar{S}_i = S \setminus S_i$ for $i = 1, \dots, m$.

Lemma 6.3. *If a maximal matching M in G satisfying (C) is $\alpha(m)$-robust, then its corresponding subsets $S_1, \dots, S_m$ of S satisfy that $|S_i| = 3$ and $f(S_i) = B$ for $i = 1, \dots, m$ and $S_1, \dots, S_m$ are mutually disjoint.*

Proof. Suppose that M is an $\alpha(m)$-robust matching. Since

$$\begin{aligned} w(M_{(3m)}) &= k_1 \left\{(m^2 + 3m - 5)(|S|L + f(S)) + |S_1|L + f(S_1)\right\}, \\ w(M^{(3m)}) &= k_1(m^2 + 3m - 4)(|S|L + f(S)), \end{aligned}$$

it holds that

$$\frac{k_1}{m}\big\{ - |S|L - f(S) + m(|S_1|L + f(S_1))\big\} = w(M_{(3m)}) - \alpha(m)w(M^{(3m)}) \geq 0. \tag{17}$$

On the other hand, since

$$\begin{aligned} w(M_{(6m)}) &\leq k_1 \left\{(2m^2 + 5m - 12)(|S|L + f(S)) - (2m - 3)(|S_1|L + f(S_1))\right\}, \\ w(M^{(6m)}) &= k_1(2m^2 + 5m - 12)(|S|L + f(S)), \end{aligned}$$

by Lemma 6.2, it holds that

$$\frac{k_1(2m-3)}{m}\left\{|S|L + f(S) - m(|S_1|L + f(S_1))\right\} \geq w(M_{(6m)}) - \alpha(m)w(M^{(6m)}) \geq 0. \tag{18}$$

By (17) and (18), we have $|S|L + f(S) - m(|S_1|L + f(S_1)) = 0$, and $(r_j, s_{1,j}) \in M$ for every $j \in S_1$. Since L is large enough, this means that $|S_1| = \frac{1}{m}|S| = 3$ and $f(S_1) = \frac{1}{m}f(S) = B$. Note that it also holds that

$$w(M_{(6m)}) - \alpha(m)w(M^{(6m)}) = 0.$$

Now we show that

$$\begin{aligned} &|S_i| = 3, && (19) \\ &f(S_i) = B, && (20) \\ &(r_j, s_{i,j}) \in M \text{ for every } j \in S_i, \text{ and} && (21) \\ &w(M_{(6im)}) - \alpha(m)w(M^{(6im)}) = 0 && (22) \end{aligned}$$

for $i = 1, 2, \dots, m$ by induction on i. The above arguments show that these conditions hold for $i = 1$.

Assume that (22) holds for $i - 1$, that is,

$$w(M_{(6(i-1)m)}) - \alpha(m)w(M^{(6(i-1)m)}) = 0.$$

Then, by a similar calculation as the case $i = 1$, we have

$$\frac{k_i}{m}\{-|S|L - f(S) + m(|S_i|L + f(S_i))\}$$
$$= w(M_{(6(i-1)m+3m)}) - \alpha(m)w(M^{(6(i-1)m+3m)}) \geq 0, \tag{23}$$
$$\frac{k_i(2m-3)}{m}\{|S|L + f(S) - m(|S_i|L + f(S_i))\}$$
$$\geq w(M_{(6im)}) - \alpha(m)w(M^{(6im)}) \geq 0. \tag{24}$$

By (23) and (24), we have $|S|L + f(S) - m(|S_i|L + f(S_i)) = 0$ and $(r_j, s_{i,j}) \in M$ for every $j \in S_i$. Hence, we also have $|S_i| = \frac{1}{m}|S| = 3$, $f(S_i) = \frac{1}{m}f(S) = B$, and $w(M_{(6im)}) - \alpha(m)w(M^{(6im)}) = 0$.

Therefore, (19), (20), (21), and (22) hold for $i = 1, 2, \ldots, m$ by induction on i. Since (21) implies that $S_1, \ldots, S_m$ are mutually disjoint, we complete the proof. □

Lemma 6.4. *If mutually disjoint subsets $S_1, \ldots, S_m$ of S satisfy that $|S_i| = 3$ and $f(S_i) = B$ for $i = 1, \ldots, m$, then its corresponding maximal matching M in G satisfying (C) is $\alpha(m)$-robust.*

Proof. Suppose that $|S_i| = 3$ and $f(S_i) = B$ for $i = 1, \ldots, m$. Recall that for a set $T \subseteq S$, $T_{(p)}$ is a subset of T defined as (14). It suffices to show that $w(M_{(p)}) \geq \alpha(m)w(M^{(p)})$ for any $1 \leq p \leq 6m^2$. We consider the following cases:

Case 1: When $p = 6(q-1)m + 3m$ or $p = 6qm$ for $q = 1, \ldots, m$, by the proof of Lemma 6.3, we have $w(M_{(p)}) = \alpha(m)w(M^{(p)})$.

Case 2: Suppose that $p = 6(q-1)m + p'$ with $1 \leq p' \leq 3$ and $1 \leq q \leq m$. Since

$$w(M_{(p)}) = w(M_{(6(q-1)m)}) + (m^2 + 3m - 4)k_q(p'L + f((S_q)_{(p')})),$$
$$w(M^{(p)}) = w(M^{(6(q-1)m)}) + (m^2 + 3m - 4)k_q(p'L + f(S_{(p')})),$$

it holds that

$$w(M_{(p)}) - \alpha(m)w(M^{(p)}) \geq (m^2 + 3m - 4)k_q\left\{\frac{p'}{m^2 + 4m}L - \alpha(m)f(S_{(p')})\right\} > 0.$$

Case 3: Suppose that $p = 6(q-1)m + p'$ with $4 \leq p' \leq 3m - 1$ and $1 \leq q \leq m$. Since

$$w(M_{(p)}) = w(M_{(6(q-1)m+3m)}) - (m^2 + 3m - 5)k_q((3m - p')L + f(\bar{S}_q \setminus (\bar{S}_q)_{(p'-3)})),$$
$$w(M^{(p)}) = w(M^{(6(q-1)m+3m)}) - (m^2 + 3m - 4)k_q((3m - p')L + f(S \setminus S_{(p')})),$$

it holds that

$$w(M_{(p)}) - \alpha(m)w(M^{(p)})$$
$$\geq k_q\left\{\frac{1}{m}(3m - p')L - (m^2 + 3m - 5)f(\bar{S}_q \setminus (\bar{S}_q)_{(p'-3)})\right\}$$
$$> 0.$$

Case 4: Suppose that $p = 6(q-1)m + p'$ with $3m + 1 \leq p' \leq 6m - 3$ and $1 \leq q \leq m$. Since

$$w(M_{(p)}) = w(M_{(6(q-1)m+3m)}) + (m^2 + 2m - 7)k_q((p' - 3m)L + f((\bar{S}_q)_{(p'-3m)})),$$
$$w(M^{(p)}) = w(M^{(6(q-1)m+3m)}) + (m^2 + 2m - 8)k_q((p' - 3m)L + f(S_{(p'-3m)})),$$

it holds that

$$\begin{aligned} w(M_{(p)}) - \alpha(m)w(M^{(p)}) \\ &\geq k_q \left\{ \frac{2m-2}{m}(p'-3m)L - (m^2+2m-8)\alpha(m)f(S_{(p'-3m)}) \right\} \\ &> 0. \end{aligned}$$

Case 5: Suppose that $p = 6(q-1)m + p'$ with $6m-2 \leq p' \leq 6m-1$ and $1 \leq q \leq m$. Since

$$\begin{aligned} w(M_{(p)}) &= w(M_{(6qm)}) - (m^2-5)k_q((6m-p')L + f(S_q \setminus (S_q)_{(p'-6m+3)})), \\ w(M^{(p)}) &= w(M^{(6qm)}) - (m^2+2m-8)k_q((6m-p')L + f(S \setminus S_{(p'-6m+3)})), \end{aligned}$$

it holds that

$$\begin{aligned} w(M_{(p)}) - \alpha(m)w(M^{(p)}) \\ &\geq k_q \left\{ \frac{2m^2-4m+2}{m}(6m-p')L - (m^2-5)f(S_q \setminus (S_q)_{(p'-6m+3)}) \right\} \\ &> 0. \end{aligned}$$

Therefore $w(M_{(p)}) \geq \alpha(m)w(M^{(p)})$ for any $1 \leq p \leq 6m^2$, which completes the proof. □

Lemmata 6.3 and 6.4 imply that there exists a partition $(S_0, S_1, \ldots, S_m)$ of S such that $|S_i| = 3$ and $f(S_i) = B$ for $i = 1, \ldots, m$ if and only if G has an $\alpha(m)$-robust matching with respect to the weight function w. This shows that the instance of ROBUST-MATCHING defined in Section 6.1 is equivalent to the original instance of 3-PARTITION.

Combining this result and Proposition 6.1, we obtain Theorem 4.2. □

7 Concluding Remarks

In the present paper, we show that for any α with $\frac{1}{\sqrt{2}} < \alpha < 1$, α-ROBUST-MATCHING is NP-complete (Theorem 4.1) and that detecting an α-robust matching is NP-complete in the strong sense when α is a part of the input (Theorem 4.2). It is still open whether detecting an α-robust matching is NP-complete in the strong sense when α is fixed.

We also investigate the performance of the k-th power algorithm. In particular, we give an algorithm to find a 1-robust matching, and show the hardness to find a β-approximately robust matching when $\frac{1}{\sqrt{2}} < \beta \leq 1$. It is open whether other algorithms can find a β-approximately robust matching for some $\frac{1}{\sqrt{2}} < \beta \leq 1$ in polynomial time.

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