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Self-Organizing Hexagons in Economic Agglomeration: Core–Periphery Models and Central Place Theory

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Abstract

Self-organization of hexagonal population distributions from uniformly inhabited state is predicted by group-theoretic bifurcation theory, and its existence is demonstrated by computational bifurcation analysis. A system of places periodically distributed on an infinite two-dimensional domain is modeled using core–periphery models and by an infinite-periodic domain assumption. Computationally obtained distributions represent those envisaged by central place theory in economic geography based on a normative and geometrical approach, and were inferred to emerge by Krugman (1996) in new economic geography for his core–periphery model in two dimensions. The missing link between central place theory and new economic geography has been provided in light of bifurcation theory, and the horizon of core–periphery models has been extended.

Keywords: bifurcation, central place theory, core–periphery models, group-theoretic bifurcation theory, hexagons, new economic geography, self organization

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1 Introduction

Self-organization of hexagonal population distributions[1] from a uniformly inhabited state was envisioned by central place theory in economic geography based on a normative and geometrical approach. It was inferred to emerge by Krugman (1996) in the new economic geography for core–periphery models in two dimensions. Herein, we verify the existence of such distributions for these models theoretically and demonstrate their existence computationally based on an interdisciplinary study synthesizing three independent mainstreams: central place theory, new economic geography, and group-theoretic bifurcation theory.

In central place theory of economic geography,[2] self-organization of hexagonal market areas of three kinds shown in Fig. 1 was proposed by Christaller (1966) based on market, traffic, and administrative principles. The assemblage of hexagonal market area with different sizes is expected to produce hierarchical hexagonal distributions of the population of places (cities, towns, villages, etc.). Yet these are based on a normative and geometrical approach; they are not derived from market equilibrium conditions, as stated by Fujita et al. (1999, p.27):

> Unfortunately, as soon as one begins to think hard about central place theory one realizes that it does not quite hang together as an economic model. ... Christaller suggested the plausibility of a hierarchical structure; he gave no account of how individual actions would produce such a hierarchy ...

In new economic geography, based on a full-fledged general nonlinear market equilibrium approach, Krugman (1991) has developed a core–periphery model,[3] and demonstrated that bifurcation serves as a catalyst to engender agglomeration of population out of uniformly distributed state. Thereafter, new economic geography models sprung up worldwide.[4] The hexagonal patterns in central place theory have yet to be found for core–periphery models as stated[5] by Krugman (1996; P91):

> I have demonstrated the emergence of a regular lattice only for a one-dimensional economy, but I have no doubt that a better mathematician

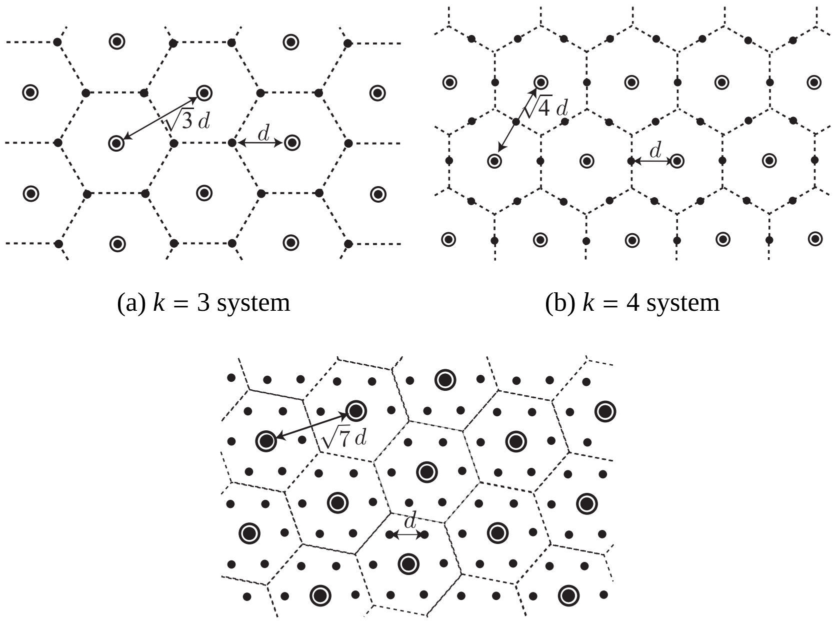
[1] See Clarke and Wilson (1985), Munz and Weidlich (1990) and others for early studies on self-organizing patterns.

[2] For books and reviews for central place theory, see, for example, Lösch (1954), Valavanis (1955), Lloyd and Dicken (1972), Isard (1975), Beavon (1977), King (1984), Dicken and Lloyd (1990), and Allen (2004).

[3] This model expressed the microeconomic underpinning of the spatial economic agglomeration, introduced the Dixit–Stiglitz (1977) model of monopolistic competition into spatial economics, and provided a new framework to explain interactions occurring among increasing returns, transportation costs, and factor mobility.

[4] These models are explained in several books, such as Fujita et al. (1999), Brakman et al., (2001), Fujita and Thisse (2002), Baldwin et al. (2003), Henderson and Thisse (2004), Combes et al. (2008), and Glaeser (2008).

[5] This statement is based on the study of a racetrack economy among a system of places (cf., Fujita et al., 1999). A racetrack economy uses a system of identical places spread uniformly around the circumference of a circle; see, e.g., Krugman (1993,1996), Fujita et al. (1999), Picard and Tabuchi (2009), Akamatsu et al. (2009), Tabuchi and Thisse (2010), and Ikeda et al. (2010).



(a) $k = 3$ system

(b) $k = 4$ system

(c) $k = 7$ system

Figure 1: Three systems predicted by Christaller (the area of a circle indicates the amount of population)

> could show that a system of hexagonal market areas will emerge in two dimensions.

It is the belief of the present authors that core–periphery models themselves have inherent capability to express those hexagonal patterns, but their adequacy has been investigated mainly against two places and sometimes against the racetrack economy with an overly simplified geometry.[6] It is important to note that central place theory is developed for an infinite domain with infinite number of places; however, core–periphery models are developed fundamentally for a finite number of places in a finite domain. Krugman's statement might be interpreted as "a proper mathematical procedure to define an infinite uniform domain could show that a system of hexagonal market areas will emerge in two dimensions."

The objective of this paper is to demonstrate by group-theoretic bifurcation theory the self-organization of hexagonal market areas for core–periphery models in two dimensions. Infinite-periodic-domain approximation that is commonly used in the study of pattern formation (cf., Golubitsky et al., 1988; and Ikeda and Murota, 2010) is employed to express infinite number of places in the framework of core–periphery models; we consider a rhombic domain with periodic boundaries comprising uniformly distributed $n \times n$ places that are connected by roads of the same length forming a regular-triangular mesh. To show model independence of computational results, we employ core–periphery models of two kinds (cf., Section 3 and Appendix A). Hexagonal distributions of population that give

[6]Behrens and Thisse (2007) stated: in multi-regional systems the so-called "three-ness effect" enters the picture and introduces complex feedbacks into the models, which significantly complicates the analysis. Dealing with these spatial interdependencies constitutes one of the main theoretical and empirical challenges NEG and regional economics will surely have to face in the future.

Christaller's three systems in Fig. 1 and a set of nested hexagons are predicted using group-theoretic bifurcation theory[7] and are found by computational bifurcation analysis for those core–periphery models. Although there are bifurcation points of various kinds, those which produce the hexagonal patterns are identified by that theory. Consequently, this paper presents a step toward uniting central place theory and core–periphery models in light of group-theoretic bifurcation theory.

This paper is organized as follows: A system of places that is uniformly spread on an infinite hexagonal lattice in two dimensions is modeled in Section 2. Section 3 introduces core–periphery models and predicts its bifurcation mechanism producing hexagonal distributions by group-theoretic bifurcation theory. Computational bifurcation analysis of $n \times n$ places on the rhombic domain is conducted to find bifurcated patterns that represent hexagonal market areas in Section 4. Details of the core–periphery models are given in Appendix A. Mathematical details of group-theoretic bifurcation analysis are given in Appendices B–D.

2 System of places on a hexagonal lattice

Although an infinite two-dimensional domain is used in the study of the self-organization of hexagonal market areas in central place theory, such domain is incompatible with a naive analysis for the core–periphery models, which are formulated for a finite number of places. As a remedy, we introduce a rhombic domain with periodic boundaries comprising a system of uniformly distributed $n \times n$ places, and prescribes groups expressing the symmetry of this domain. As a spatial configuration of a system of places, we use a hexagonal lattice[8] because it is geometrically consistent with the hexagonal market areas (Lösch, 1954, p.133–134).

2.1 Hexagonal lattice and rhombic domain

Figure 2 portrays the hexagonal lattice, which comprises regular triangles and which covers an infinite two-dimensional domain. A place is allocated at each node of this lattice, expressed by

$$\boldsymbol{p} = n_1\boldsymbol{\ell}_1 + n_2\boldsymbol{\ell}_2, \quad (n_1, n_2 \in \mathbb{Z}),$$

where $\boldsymbol{\ell}_1 = (d, 0)^\top$ and $\boldsymbol{\ell}_2 = (-d/2, d\sqrt{3}/2)^\top$ are oblique basis vectors (d is the length of these vectors); $\mathbb{Z}$ is the set of integers.

To express the infiniteness of the hexagonal lattice in Fig. 2, we consider a rhombic domain that is cut out from the hexagonal lattice and is endowed with periodic boundary conditions: an example of this domain with 2×2 places is

[7]The emergence of hexagons out of uniformity is widely observed for various physical systems. It is explained by group-theoretic bifurcation theory (cf., Buzano and Golubitsky, 1983; Golubitsky et al., 1988; Melbourne, 1999; Dionne and Golubitsky, 1992; Judd and Silber, 2000; Golubitsky and Stewart, 2002; and Ikeda and Murota, 2010). In particular, the hexagonal lattice is employed in the description of convection of fluids and nematic liquid crystals (cf., Peacock et al., 1999; Golubitsky and Stewart, 2002; and Chillingworth and Golubitsky, 2003).

[8]Planar lattices of five kinds exist: rhombic, square, hexagonal, rectangular, and oblique (cf., Golubitsky and Stewart, 2002).

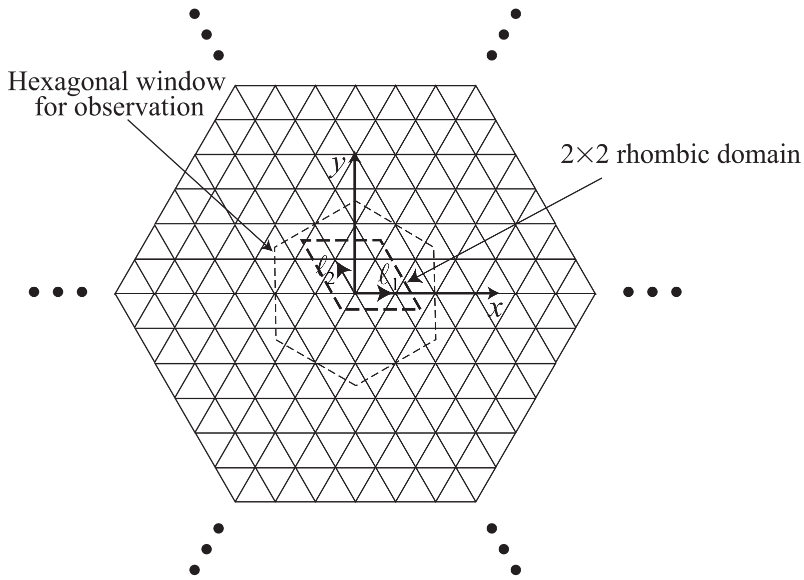
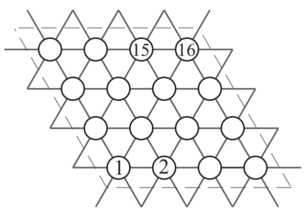
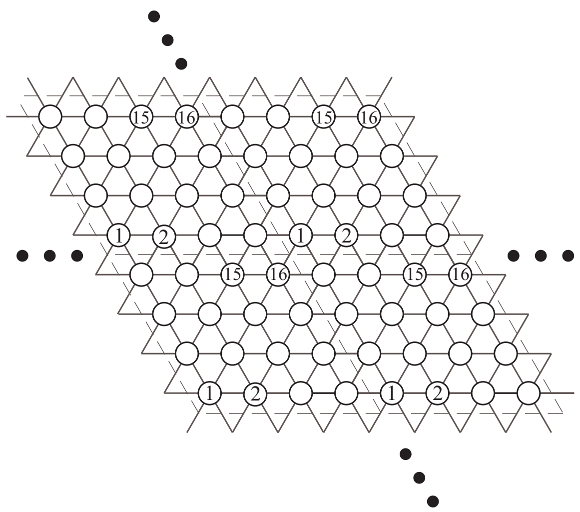


Figure 2: Hexagonal lattice



(a) 4×4 places



(b) Spatially repeated rhombic domains

Figure 3: A system of places in a rhombic domain with periodic boundaries

shown by the dashed lines in this figure. A system of $n \times n$ places in a rhombic domain is modeled as follows:

- Allocate places in an $n \times n$ two-dimensional lattice

$$\boldsymbol{p} = n_1\boldsymbol{\ell}_1 + n_2\boldsymbol{\ell}_2, \quad (n_1, n_2 = 0, 1, \cdots, n-1).$$

- Connect neighboring places by roads (line segments) of the same length d to form regular-triangular meshes.
- Introduce periodic boundaries on the four borders of the domain. This is equivalent to saying that the rhombic domain is repeated spatially and neighboring domains are connected by roads of equal length d (cf., Fig. 3(b)).

2.2 Two-dimensional periodicity and hexagonal distributions

If the population distribution of a system of places (i.e., a subset of nodes) has two-dimensional periodicity, then we can set a pair of independent vectors

$$(\boldsymbol{t}_1, \boldsymbol{t}_2), \tag{1}$$

called the spatial period vectors, such that the system remains invariant under the translations associated with these vectors. The spatial periods (T_1, T_2) are defined as

$$T_i = \|\boldsymbol{t}_i\|, \quad (i = 1, 2).$$

The tilted angle φ between $\boldsymbol{\ell}_1$ and $\boldsymbol{t}_1$ is defined as

$$\sin\varphi = \frac{(\boldsymbol{\ell}_1)^\top \boldsymbol{t}_1}{\|\boldsymbol{t}_1\|}. \tag{2}$$

Although the choice of the vectors $(\boldsymbol{t}_1, \boldsymbol{t}_2)$ is not unique, T_1 and T_2 must be chosen to be as small as possible, and then to choose the smallest non-negative φ.

Among possible doubly-periodic distributions, we specifically examine a hexagonal distribution that is described by

$$\boldsymbol{t}_1 = \alpha\boldsymbol{\ell}_1 + \beta\boldsymbol{\ell}_2, \quad \boldsymbol{t}_2 = -\beta\boldsymbol{\ell}_1 + (\alpha-\beta)\boldsymbol{\ell}_2, \quad (\alpha, \beta \in \mathbb{Z}), \tag{3}$$

for which $T_1 = T_2 (\equiv T)$ is satisfied and the angle between $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ is $2\pi/3$. The associated normalized spatial period is given by

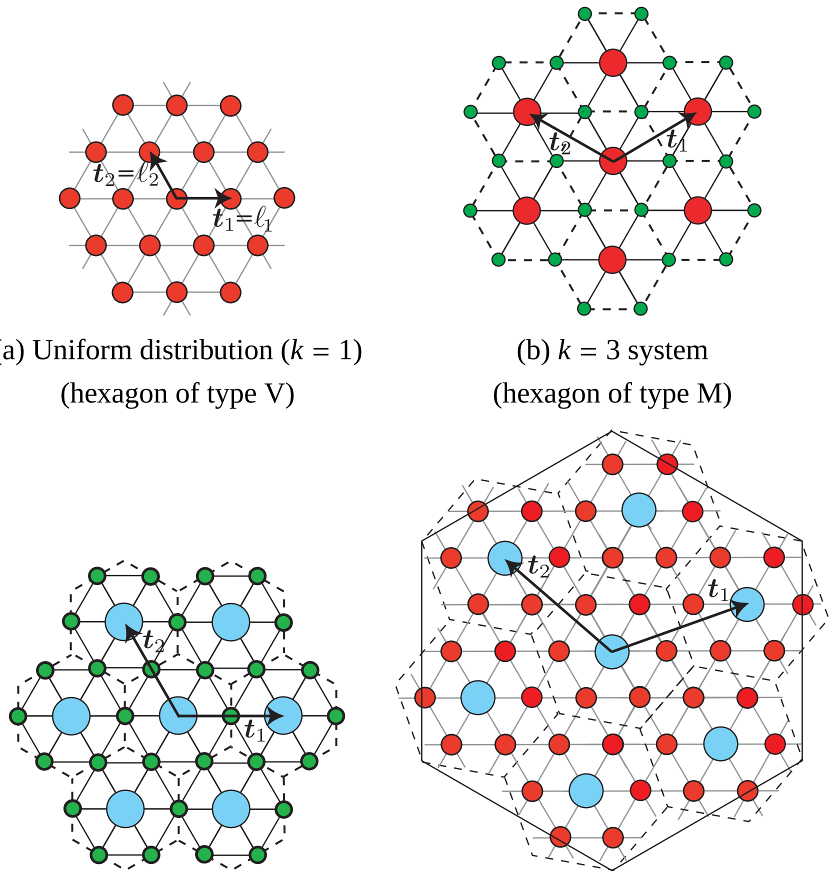
$$T/d = \sqrt{(\alpha - \beta/2)^2 + (\beta\sqrt{3}/2)^2} = \sqrt{\alpha^2 - \alpha\beta + \beta^2}. \tag{4}$$

We consider a positive integer

$$k = \alpha^2 - \alpha\beta + \beta^2,$$

which can take some specific integer values, such as 1, 3, 4, 7, . . . , and rewrite the normalized spatial period in (4) as

$$T/d = \sqrt{k}, \tag{5}$$



(a) Uniform distribution ($k = 1$)
(hexagon of type V)

(b) $k = 3$ system
(hexagon of type M)

(c) $k = 4$ system
(hexagon of type V)

(d) $k = 7$ system
(tilted hexagon)

Figure 4: Hexagonal distributions on the hexagonal lattice

which lies in the range $1 \leq T/d \leq n$ and take some specific values, such as $\sqrt{1}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{7}$, ... We refer to the hexagonal distribution for $k = 1$ as the unifrom distribution (cf., Fig. 4(a)) and those for other k values as $k = 3, 4, 7, \ldots$ systems. The values of (α, β) for these systems are not unique in general but are given, for example, are

$$(\alpha,\beta) = \begin{cases} (1,0): & \text{uniform distribution } (k = 1), \\ (2,1): & k = 3 \text{ system}, \\ (2,0): & k = 4 \text{ system}, \\ (3,1): & k = 7 \text{ system}. \end{cases}$$

Among these systems, we are particularly interested in the three systems associated with $k = 3$, 4, and 7, which correspond to Christaller's $k = 3$, 4, and 7 systems, as depicted in Fig. 4(b)–(d). These three systems are observed in computational bifurcation analysis in Section 4.

With reference to the tilted angle φ defined by (2), we can classify hexagonal distributions into

$$\begin{cases} \text{hexagons of type V}, & \varphi = 0, \\ \text{hexagons of type M}, & \varphi = \pi/6, \\ \text{tilted hexagons}, & \text{otherwise}, \end{cases} \qquad (6)$$

in which "V" signifies that the vertices of the hexagons are located on the x-axis and "M" denotes that midpoints of sides of the hexagons are located on the x-axis. The hexagonal distributions for $k = 3$, 4, and 7 systems can be classified as

$$\begin{cases} k = 3: & \text{hexagon of type M,} \\ k = 4: & \text{hexagon of type V,} \\ k = 7: & \text{tilted hexagon.} \end{cases}$$

2.3 Groups expressing the symmetry

For the study of the agglomeration pattern of population distribution on the rhombic domain, we use group-theoretic bifurcation theory: an established mathematical tool for investigating pattern formation (cf., Subsection 3.3). In this theory, the symmetries of possible bifurcated solutions are determined with resort to the group that labels the symmetry of the system. In this sense, it is the first step of the bifurcation analysis to identify the underlying group.

Symmetry of the $n \times n$ rhombic domain is characterized by invariance with respect to:

- r: counterclockwise rotation about the origin at an angle of $\pi/3$.
- s: reflection $y \mapsto -y$.
- p_1: periodic translation along the $\boldsymbol{\ell}_1$-axis (i.e., the x-axis).
- p_2: periodic translation along the $\boldsymbol{\ell}_2$-axis.

Consequently, the symmetry of the domain is described by the group

$$G = \langle r, s, p_1, p_2 \rangle, \tag{7}$$

where $\langle \cdots \rangle$ denotes a group[9] generated by the elements therein, with the fundamental relations given by

$$\begin{aligned} r^6 &= s^2 = (rs)^2 = p_1{}^n = p_2{}^n = e, \\ rp_1 &= p_1 p_2 r, \quad rp_2 = p_1 r, \quad sp_1 = p_1 s, \quad sp_2 = p_1 p_2 s, \quad p_2 p_1 = p_1 p_2, \end{aligned}$$

where e is the identity element. Each element of G can be represented uniquely in the form of

$$s^l r^m p_1{}^i p_2{}^j, \quad i, j \in \{0, \ldots, n-1\}; \ l \in \{0, 1\}; \ m \in \{0, 1, \ldots, 5\}.$$

The group G contains the dihedral group $\mathrm{D}_6 = \langle r, s \rangle$ and cyclic groups $\mathbb{Z}_n = \langle p_1 \rangle$ and $\mathbb{Z}_n = \langle p_2 \rangle$ as its subgroups. Moreover, it has the structure of semidirect product of D_6 by $\mathbb{Z}_n \times \mathbb{Z}_n$, which is denoted as

$$G = \mathrm{D}_6 \dot{+} (\mathbb{Z}_n \times \mathbb{Z}_n)$$

or $G = \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$.[10]

[9] For more account of group theory, see, e.g., Curtis and Reiner (1962) and Serre (1977).

[10] The former symbol ($\dot{+}$) is used, for example, in Golubitsky et al. (1988).

Among many subgroups of $G = \langle r, s, p_1, p_2 \rangle$, expressing partial symmetries, we search for those containing the element r and the following elements

$$\begin{cases} p_1^2 p_2 \text{ and } p_1^{-1} p_2 & \text{for } k = 3 \text{ system,} \\ p_1^2 \text{ and } p_2^2 & \text{for } k = 4 \text{ system,} \\ p_1^3 p_2 \text{ and } p_1^{-1} p_2^2 & \text{for } k = 7 \text{ system,} \end{cases} \tag{8}$$

which express hexagonal distributions. The correspondence in (8) reveals the compatibility condition on the size n of the rhombic domain for $k = 3$, 4, and 7 systems. For $k = 3$ system we have

$$(p_1^2 p_2) \times (p_1^{-1} p_2)^{-1} = p_1^3,$$

which represents a translation in the direction of the $\boldsymbol{\ell}_1$-axis at the length of $3d$; accordingly, n must be a multiple of 3. For $k = 4$ system, the symmetry of p_1^2 and p_2^2 implies that n is a multiple of 2. For $k = 7$ system we have

$$(p_1^3 p_2)^2 \times (p_1^{-1} p_2^2)^{-1} = p_1^7,$$

from which follows that n is a multiple of 7. Accordingly, we use the size $n = 3$ for $k = 3$ system, $n = 16$ for $k = 4$ system, and $n = 7$ for $k = 7$ system in the computational bifurcation analysis in Section 4.

3 Core–periphery models and bifurcation

In this section, we present multi-regional core–periphery models. The group-equivariance of the governing equation of the system of places for this model is introduced and the mechanism of bifurcation producing hexagonal distributions is studied.

3.1 Core–periphery models

To demonstrate model independence of our results, we employ core–periphery models of two kinds:

i) FO model (Forslid and Ottaviano, 2003) that replaces the production function of Krugman with that of Flam and Helpman (1987).

ii) Pf model (Pflüger, 2004) that replaces, in addition to the production function, the utility function of Krugman with that of the international trade model of Martin and Rogers (1995).

In these models, the economy is composed of K places (labeled $i = 1, \ldots, K$), two factors of production (skilled and unskilled labor), and two sectors (manufacture M and agriculture A). There, H skilled and L unskilled workers consume two final goods: manufactural-sector goods and agricultural-sector goods. Workers supply one unit of each type of labor inelastically; here H is a constant expressing the total number of skilled workers.[11] Skilled workers are mobile across places.

[11]The equality $H = \sum_{i=1}^{K} h_i$ is satisfied by any solution of (9) because $\sum_{i=1}^{K} P_i(\boldsymbol{h}, \tau) = 1$ by (10).

The number of skilled workers in place i is denoted by h_i. Unskilled workers are immobile and equally distributed across all places with the unit density (i.e., $L = 1 \times K$). Hence the population in place i is equal to $h_i + 1$.

Although the details of the models are given in Appendix A, the governing equation of these models is formulated in a standard form of static equilibrium as

$$\boldsymbol{F}(\boldsymbol{h}, \tau) = H\boldsymbol{P}(\boldsymbol{h}) - \boldsymbol{h} = \boldsymbol{0}. \tag{9}$$

Therein $\boldsymbol{h} = (h_i) \in \mathbb{R}^K$ is a K-dimensional vector expressing the population distribution of the skilled workers, $\tau \in \mathbb{R}$ is a (bifurcation) parameter corresponding to the transport parameter, and $\boldsymbol{F}$: $\mathbb{R}^K \times \mathbb{R} \to \mathbb{R}^K$ is a sufficiently smooth nonlinear function in $\boldsymbol{h}$ and τ; $\boldsymbol{P} = (P_i) \in \mathbb{R}^K$ is a K-dimensional vector given by

$$P_i(\boldsymbol{h}, \tau) \equiv \frac{\exp[\theta v_i(\boldsymbol{h}, \tau)]}{\sum_{j=1}^{K} \exp[\theta v_j(\boldsymbol{h}, \tau)]}, \quad i = 1, \ldots, K, \tag{10}$$

where θ is a constant representing the inverse of variance of the idiosyncratic tastes, and $v_i(\boldsymbol{h}, \tau)$ $(i = 1, \ldots, K)$ are nonlinear functions representing the components of an indirect utility function vector $\boldsymbol{v}(\boldsymbol{h}, \tau)$.

3.2 Exploiting symmetry of core–periphery models by group-theoretic bifurcation theory

For investigation of the patterns of the bifurcated solutions, it is crucial to formulate the symmetry that is inherent in the governing equation. In group-theoretic bifurcation theory, the symmetry of the equation for the system of $n \times n$ places on the rhombic domain is described as

$$T(g)\boldsymbol{F}(\boldsymbol{h}, \tau) = \boldsymbol{F}(T(g)\boldsymbol{h}, \tau), \quad g \in G, \tag{11}$$

in terms of an orthogonal matrix representation T of group $G = \langle r, s, p_1, p_2 \rangle$ in (7) on the K-dimensional space $\mathbb{R}^K$. The condition (or property) (11) is called the equivariance of $\boldsymbol{F}(\boldsymbol{h}, \tau)$ to G. The most important consequence of the equivariance (11) is that the symmetries of the whole set of possible bifurcated solutions can be obtained and classified.

In our study of a system of $n \times n$ places in the rhombic domain, each element g of G acts as a permutation among place numbers $(1, \ldots, K)$ for $K = n^2$ and hence each $T(g)$ is a permutation matrix. Then we can show the equivariance (11) to $G = \langle r, s, p_1, p_2 \rangle$ of the core–periphery models presented above as below.

Proof. By expressing the action of $g \in G$ as $g : i \mapsto i^*$ for place numbers i and i^*, we have $v_i(T(g)\boldsymbol{h}, \tau) = v_{i^*}(\boldsymbol{h}, \tau)$ and $P_i(T(g)\boldsymbol{h}, \tau) = P_{i^*}(\boldsymbol{h}, \tau)$ by (10) for any $g \in G$. Therefore, we have

$$\begin{aligned} F_i(T(g)\boldsymbol{h}, \tau) &= HP_i(T(g)\boldsymbol{h}, \tau) - h_{i^*} \\ &= HP_{i^*}(\boldsymbol{h}, \tau) - h_{i^*} \\ &= F_{i^*}(\boldsymbol{h}, \tau). \end{aligned}$$

This proves the equivariance (11). □

According to group-theoretic bifurcation theory the (bifurcation) analysis proceeds as follows. Consider, to be specific, a critical point $(\boldsymbol{h}_{\mathrm{c}}, \tau_{\mathrm{c}})$ of multiplicity $M\,(\geq 1)$, at which the Jacobian matrix of $\boldsymbol{F}$ has M zero eigenvalues.

Using a standard procedure called the *Liapunov–Schmidt reduction with symmetry* (Sattinger, 1979; Golubitsky et al., 1988), the full system of equations

$$\boldsymbol{F}(\boldsymbol{h}, \tau) = \boldsymbol{0} \tag{12}$$

in $\boldsymbol{h} \in \mathbb{R}^K$ (cf., (9)) is reduced, in a neighborhood of $(\boldsymbol{h}_{\mathrm{c}}, \tau_{\mathrm{c}})$, to a system of M equations (called bifurcation equations)

$$\widetilde{\boldsymbol{F}}(\boldsymbol{w}, \widetilde{\tau}) = \boldsymbol{0} \tag{13}$$

in $\boldsymbol{w} \in \mathbb{R}^M$, where $\widetilde{\boldsymbol{F}} : \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}^M$ is a function and $\widetilde{\tau} = \tau - \tau_{\mathrm{c}}$ denotes the increment of τ. In this reduction process the equivariance of the full system, which is formulated in (11), is inherited by the reduced system (13) in the following form:

$$\widetilde{T}(g)\widetilde{\boldsymbol{F}}(\boldsymbol{w}, \widetilde{\tau}) = \widetilde{\boldsymbol{F}}(\widetilde{T}(g)\boldsymbol{w}, \widetilde{\tau}), \qquad g \in G, \tag{14}$$

where $\widetilde{T}$ is the subrepresentation of T on the M-dimensional kernel space of the Jacobian matrix. It is this inheritance of symmetry that plays a key role in determining the symmetry of bifurcating solutions (cf., Appendices B–D).

The reduced equation (13) is to be solved for $\boldsymbol{w}$ as $\boldsymbol{w} = \boldsymbol{w}(\widetilde{\tau})$, which is often possible by virtue of the symmetry of $\widetilde{\boldsymbol{F}}$ described in (14). Since $(\boldsymbol{w}, \widetilde{\tau}) = (\boldsymbol{0}, 0)$ is a singular point of (13), there can be many solutions $\boldsymbol{w} = \boldsymbol{w}(\widetilde{\tau})$ with $\boldsymbol{w}(0) = \boldsymbol{0}$, which gives rise to bifurcation. Each $\boldsymbol{w}$ uniquely determines a solution $\boldsymbol{h}$ of the full system (12).

The symmetry of $\boldsymbol{h}$ is represented by a subgroup of G defined by

$$\Sigma(\boldsymbol{h}; G, T) = \{g \in G \mid T(g)\boldsymbol{h} = \boldsymbol{h}\},$$

called the isotropy subgroup of $\boldsymbol{h}$. The isotropy subgroup $\Sigma(\boldsymbol{h})$ can be computed in terms of the symmetry of the corresponding $\boldsymbol{w}$ as

$$\Sigma(\boldsymbol{h}; G, T) = \Sigma(\boldsymbol{w}; G, \widetilde{T}), \tag{15}$$

where

$$\Sigma(\boldsymbol{w}; G, \widetilde{T}) = \{g \in G \mid \widetilde{T}(g)\boldsymbol{w} = \boldsymbol{w}\}.$$

The relation (15) enables us to determine the symmetry of bifurcated solutions $\boldsymbol{h}$ through the analysis of bifurcation equations in $\boldsymbol{w}$.

In association with repeated bifurcations, one can find a hierarchy of subgroups

$$G = G_0 \to G_1 \to G_2 \to \cdots$$

that characterizes the hierarchical change of symmetries. Here, $\to$ denotes the occurrence of symmetry-breaking bifurcation and G_{i+1} is a subgroup of G_i ($i = 0, 1, \ldots$).

Table 1: Number of irreducible representations of $\mathrm{D}_6 \dot{+} (\mathbb{Z}_n \times \mathbb{Z}_n)$

$n \setminus d$	1	2	3	4	6	12	
	N_1	N_2	N_3	N_4	N_6	N_{12}	$\sum N_d$
$n = 3$	4	4		1	2		11
$n = 7$	4	2			12	1	19
$n = 16$	4	2	4		28	14	52

3.3 Theoretically predicted hexagonal distributions

It is to be noted first that uniformly distributed population of the skilled workers associated with the pre-bifurcation solution is the simplest hexagonal distribution (cf., Remark 3.1 below). Possible bifurcated solutions of the governing equation (9), or (12), representing hexagonal distributions are predicted by group-theoretic bifurcation theory.[12]

The multiplicity M of critical points (i.e., the dimension of the kernel space of the Jacobian matrix of $\boldsymbol{F}$ in (12) at bifurcation points) is generically either 1, 2, 3, 4, 6, or 12, which is a natural consequence of the group-theoretic fact that the dimension d of an irreducible representation of the group G is either $d = 1, 2, 3,$ 4, 6, or 12. For some values of n (treated in Section 4), the numbers N_d of the d-dimensional irreducible representations are listed in Table 1.

In the remainder of this section, we present a possible bifurcation mechanism that can produce hexagonal distributions (cf., Subsection 2.2) of population of skilled workers associated with Christaller's $k = 3$, 4, and 7 systems. Such mechanism is confirmed in Section 4 by the computational bifurcation analysis of the rhombic domain with various sizes n.

Remark 3.1. The governing equation (9) of the system of $n \times n$ places is satisfied by the state of uniformly distributed population of the skilled workers that is expressed by $h_1 = \cdots = h_{n^2} = 1/n^2$. This is the trivial solution that is existent for any value of the transport parameter τ, which serves as the bifurcation parameter. The spatial period vectors in (1) for the uniformly distributed population are given by $(\boldsymbol{t}_1, \boldsymbol{t}_2) = (\boldsymbol{\ell}_1, \boldsymbol{\ell}_2)$ with the shortest spatial period of $T/d = T_1/d = T_2/d = 1$ and with no tilting with $\varphi = 0$. □

3.3.1 $k = 3$ system

When n is a multiple of 3, hexagonal patterns for the $k = 3$ system occur generically as a branch from a double bifurcation point that is associated with the irreducible

[12]The bifurcation from the uniform state of the hexagonal lattice has been investigated using group-theoretic bifurcation theory to show the emergence of hexagonal patterns (cf., Golubitsky and Stewart, 2002 and references therein); bifurcated solutions for the rhombic domain with 2×2 systems of places are investigated extensively and classified (Ikeda and Murota, 2010, Chapter 16).

representation of G given by

$$T(r) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ T(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ T(p_1) = T(p_2) = \begin{pmatrix} \cos 2\pi/3 & -\sin 2\pi/3 \\ \sin 2\pi/3 & \cos 2\pi/3 \end{pmatrix}. \tag{16}$$

This corresponds to one of the four two-dimensional irreducible representations (cf., Table 1). Standard results for bifurcation at a double bifurcation point for the dihedral group symmetry[13] can be adapted, as described in Appendix B. There is a bifurcated path with symmetry

$$\langle r, s, p_1^2 p_2, p_1^{-1} p_2\rangle = \langle r, s\rangle \dot{+} \langle p_1^2 p_2, p_1^{-1} p_2\rangle = \langle r, s\rangle \dot{+} \langle p_1^2 p_2, p_1^3\rangle \simeq \mathrm{D}_6 \dot{+} (\mathbb{Z}_n \times \mathbb{Z}_{n/3}), \tag{17}$$

which expresses the symmetry of the hexagon of type M for the $k = 3$ system in Fig. 4(b); note that $p_1^{-1} p_2 = p_1^2 p_2 \times (p_1^3)^{-1}$.

For example, for $n = 3$, the population distribution $\boldsymbol{h}$ for the $k = 3$ system is given uniquely as

$$\boldsymbol{h} = (a, b, b; b, b, a; b, a, b)^\top, \tag{18}$$

which is invariant to the group in (17), where $(a, b) = (1/9 + 2\delta, 1/9 - \delta)$ with $-1/18 \le \delta \le 1/9$. The population distribution $\boldsymbol{h}$ for $n = 3m$ ($m = 2, 3, \ldots$) can be arrived at by spatially repeating the distribution in (18) for $(a, b) = (1/n^2 + 2\delta, 1/n^2 - \delta)$ with $-1/(2n^2) \le \delta \le 1/n^2$. It is pertinent in the computational bifurcation analysis to know such a special form.

This is a hexagonal distribution with the spatial period vectors

$$(\boldsymbol{t}_1, \boldsymbol{t}_2) = (2\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2, -\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2),$$

which corresponds to $(\alpha, \beta) = (2, 1)$ in (3). The symmetries $p_1^2 p_2$ and $p_1^{-1} p_2$ are apparent from this expression. The spatial period elongates as

$$T/d = 1 \to \sqrt{3}, \tag{19}$$

in agreement with central place theory (cf., (20) for $k = 3$ in Remark 3.2).

Remark 3.2. In central place theory, the spatial period T in (5) can be interpreted as the distance between the first-level centers with the largest population. Christaller's $k = 3$, 4, and 7 systems are hexagonal distributions with the normalized spatial period

$$T/d = \sqrt{k}, \quad (k = 3, 4, 7), \tag{20}$$

which has $\sqrt{k}$-times as large as the spatial period $T/d = 1$ for the state of uniform population. □

3.3.2 $k = 4$ system

When n is a multiple of 2, hexagonal patterns for the $k = 4$ system are predicted using group-theoretic bifurcation analysis to branch from a triple bifurcation point

[13] See, e.g., Sattinger (1979) and Ikeda and Murota (2010, Chapter 8) for analysis of the bifurcation point of this type.

that is associated with the irreducible representation of G given as

$$T(r) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad T(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \tag{21}$$

$$T(p_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T(p_2) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{22}$$

This corresponds to one of the four three-dimensional irreducible representations (cf., Table 1). By a slight extension of the pre-existing result,[14] as worked out in Appendix C, there is a bifurcated solution with the symmetry

$$\langle r, s, p_1^2, p_2^2 \rangle = \langle r, s \rangle \dot{+} \langle p_1^2, p_2^2 \rangle \simeq \mathrm{D}_6 \dot{+} (\mathbb{Z}_{n/2} \times \mathbb{Z}_{n/2}), \tag{23}$$

which expresses the symmetry of the hexagon of type V for the $k = 4$ system in Fig. 4(c).

For example, for $n = 2$, the population distribution $\boldsymbol{h}$ for the $k = 4$ system is given uniquely as

$$\boldsymbol{h} = (a, b; b, b)^\top, \tag{24}$$

which is invariant to the group in (23), where $(a, b) = (1/4 + 3\delta, 1/4 - \delta)$ with $-1/12 \leq \delta \leq 1/4$. The population distribution $\boldsymbol{h}$ for $n = 2m$ ($m = 2, 3, \ldots$) can be arrived at by spatially repeating the distribution in (24) for $(a, b) = (1/n^2 + 3\delta, 1/n^2 - \delta)$ with $-1/(3n^2) \leq \delta \leq 1/n^2$.

This is a hexagonal distribution with the spatial period vectors

$$(\boldsymbol{t}_1, \boldsymbol{t}_2) = (2\boldsymbol{\ell}_1, 2\boldsymbol{\ell}_2),$$

which corresponds to $(\alpha, \beta) = (2, 0)$ in (3). The symmetries p_1^2 and p_2^2 are apparent from this expression. The spatial period elongates as $T/d = 1 \to \sqrt{4}$, in agreement with central place theory (cf., (20) for $k = 4$ in Remark 3.2 above).

3.3.3 $k = 7$ system

When n is a multiple of 7, hexagonal patterns for $k = 7$ system are predicted to branch by group-theoretic bifurcation analysis for the group $\mathrm{D}_6 \dot{+} (\mathbb{Z}_n \times \mathbb{Z}_n)$ at a bifurcation point of multiplicity 12 associated with a 12-dimensional irreducible representation, as worked out in Appendix D. There is a bifurcated solution with the symmetry

$$\langle r, p_1^3 p_2, p_1^{-1} p_2^2 \rangle = \langle r \rangle \dot{+} \langle p_1^3 p_2, p_1^{-1} p_2^2 \rangle = \langle r \rangle \dot{+} \langle p_1^3 p_2, p_1^7 \rangle \simeq \mathrm{C}_6 \dot{+} (\mathbb{Z}_n \times \mathbb{Z}_{n/7}), \tag{25}$$

which expresses the symmetry of the tilted hexagon for the $k = 7$ system in Fig. 4(d); note that $p_1^{-1} p_2^2 = (p_1^3 p_2)^2 \times (p_1^7)^{-1}$. By (8), this solution is associated with the tilted hexagon for $k = 7$ system, which is also demonstrated in the computational bifurcation analysis for $n = 7$ in Subsection 4.3.

---

[14] This irreducible representation is denoted as $T^{(3,1)}$ in Ikeda and Murota (2010, Chapter 16). The flower mode solution there corresponds to the present solution.

For example, for $n = 7$, the population distribution $\boldsymbol{h}$ for the $k = 7$ system is given uniquely as

$$\begin{aligned} \boldsymbol{h} \ = \ & (a, b, b, b, b, b, b; b, b, b, a, b, b, b; \\ & b, b, b, b, b, b, a; b, b, a, b, b, b, b; \\ & b, b, b, b, b, a, b; b, a, b, b, b, b, b; \\ & b, b, b, b, a, b, b)^\top, \end{aligned} \tag{26}$$

which is invariant to the group in (25), where $(a, b) = (1/49 + 6\delta, 1/49 - \delta)$ with $-1/294 \leq \delta \leq 1/49$. The population distribution $\boldsymbol{h}$ for $n = 7m$ $(m = 2, 3, \ldots)$ can be arrived at by spatially repeating the distribution in (18) for $(a, b) = (1/n^2 + 6\delta, 1/n^2 - \delta)$ with $-1/(6n^2) \leq \delta \leq 1/n^2$.

This is a hexagonal distribution with the spatial period vectors

$$(\boldsymbol{t}_1, \boldsymbol{t}_2) = (3\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2, -\boldsymbol{\ell}_1 + 2\boldsymbol{\ell}_2),$$

which corresponds to $(\alpha, \beta) = (3, 1)$ in (3). The symmetries $p_1^3 p_2$ and $p_1^{-1} p_2^2$ are apparent from this expression. The spatial period elongates as

$$T/d = 1 \to \sqrt{7}, \tag{27}$$

in agreement with central place theory (cf., (20) for $k = 7$ in Remark 3.2 above).

3.3.4 Successive bifurcations producing a set of nested hexagons

Successive bifurcations repeatedly elongate the spatial period T, which starts from the shortest period $T/d = 1$ for the uniform population solution and ends up with the longest spatial period $T/d = n$; a loss of local symmetry is often encountered.

In particular, for $n = 2^m$ (m is a positive integer), there are successive bifurcations associated with a hierarchy of subgroups

$$\begin{aligned} & \mathrm{D}_6 \dot{+} (\mathbb{Z}_n \times \mathbb{Z}_n) \to \mathrm{D}_6 \dot{+} (\mathbb{Z}_{n/2} \times \mathbb{Z}_{n/2}) \to \cdots \\ & \qquad \to \mathrm{D}_6 \dot{+} (\mathbb{Z}_2 \times \mathbb{Z}_2) \to \mathrm{D}_6 \dot{+} (\mathbb{Z}_1 \times \mathbb{Z}_1) = \mathrm{D}_6, \end{aligned} \tag{28}$$

where $\to$ means the occurrence of bifurcation. These successive bifurcations produce a set of nested hexagons (cf., computational analysis in Subsection 4.2). The spatial period doubles successively as

$$T/d = 1 \to 2 \to \cdots \to n/2 \to n, \tag{29}$$

which is called period-doubling bifurcation cascade. The spatial periods that appear in (29) are in agreement with central place theory (cf., (31) in Remark 3.3).

Remark 3.3. The formula (20) in central place theory is extended to a hierarchy of spatial periods (cf., Lösch (1954, p.131))

$$T/d = \sqrt{k},\ \sqrt{k^2},\ \sqrt{k^3},\ \ldots, \quad (k = 3, 4, 7). \tag{30}$$

For example, for $k = 4$ and $n = 2^m$, (30) becomes

$$T/d = 2,\ 2^2,\ 2^3,\ \ldots,\ 2^m = n. \tag{31}$$

□

4 Computationally obtained distribution patterns

In this section, we examine spatial agglomeration patterns of the population of skilled workers among a system of places spread uniformly on a two-dimensional domain. Computational bifurcation analysis is conducted to obtain bifurcated solutions from the uniformly distributed state of population of the skilled workers for a system of $n \times n$ place on the rhombic domain. We respectively use $n = 3$, 16, and 7 to obtain $k = 3$, 4, and 7 systems. As core–periphery models (cf., Subsection 3.1), the FO model and the Pf model are used.

We employ the following parameter values:

- The length d of the road connecting neighboring places is $d = 1/n$.
- The constant expenditure share μ on industrial varieties is $\mu = 0.4$ (cf., Appendix A.1).
- The constant elasticity σ of substitution between any two varieties is $\sigma = 5.0$ (cf., Appendix A.1).
- The inverse θ of variance of the idiosyncratic tastes is $\theta = 1000$ (cf., Appendix A.3).
- The total number H of skilled workers is chosen as $H = 1$, except for the analysis of 16×16 places for the Pf model, in which $H = 16$ is used since no bifurcation is observed for $H = 1$.

4.1 Agglomeration of 3×3 places: $k = 3$ system

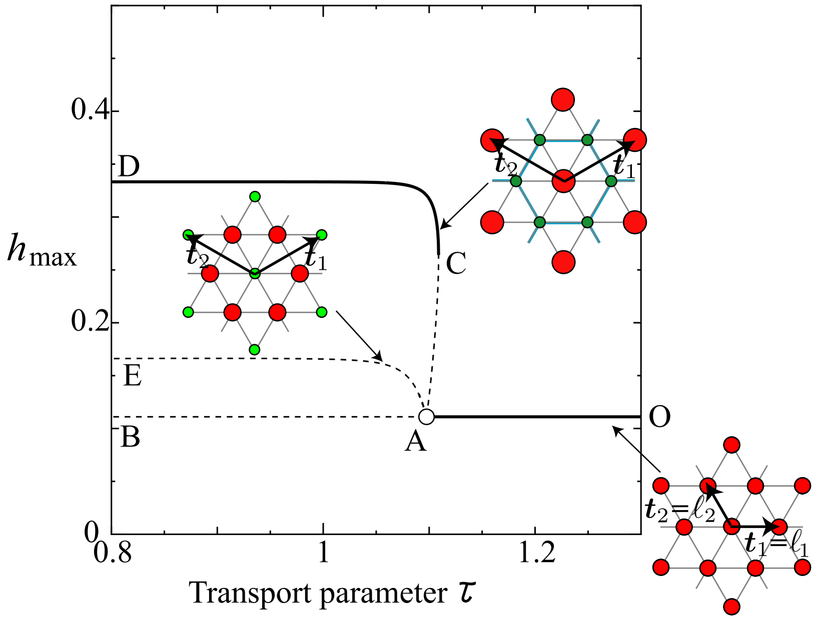
We use a system of 3×3 places on the rhombic domain with $D_6 \dot{+} (\mathbb{Z}_3 \times \mathbb{Z}_3)$-symmetry to explain the basic properties of period $\sqrt{3}$-times bifurcation that produces a population distribution for Christaller's $k = 3$ system.

For the system of 3×3 places on the rhombic domain, bifurcated solutions at a bifurcation point for the uniform population distribution are obtained. Figure 5 depicts equilibrium paths (the maximum population $h_{\max}$ versus the transport parameter τ curves) for the FO and Pf models, where $h_{\max} = \max(h_1, \ldots, h_K)$ ($K = 3 \times 3$). The typical population distribution for each solution is presented on the hexagonal domain[15] in Fig. 5; the area of a circle is proportional to the population of the skilled workers. The two models display quantitatively different but qualitatively identical behaviors: for example, the location of the bifurcation point A is different, but the bifurcated paths have the same symmetry. The following argument, accordingly, is applicable to both models.

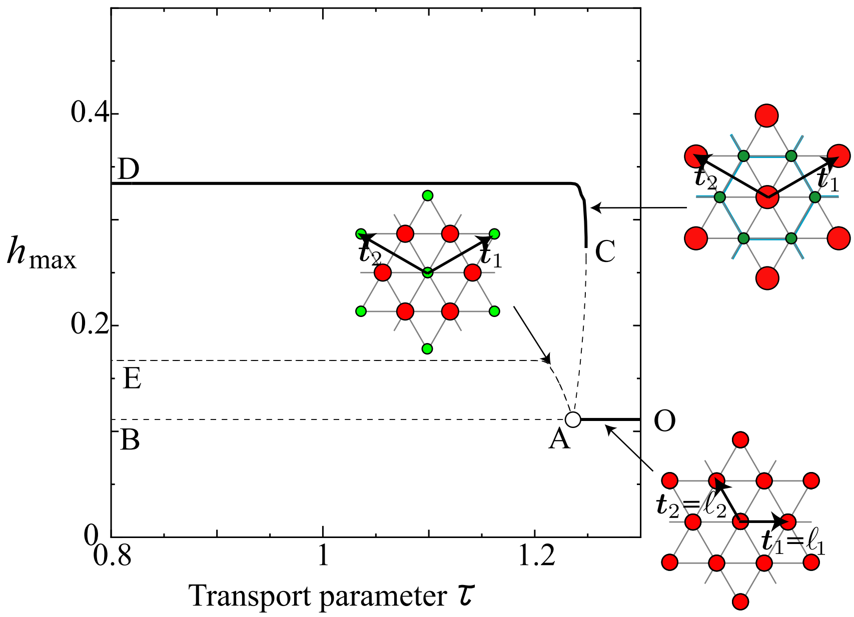
The uniform population solution corresponds to the horizontal path OAB, which is stable during OA (shown by the solid line). The point A on this path corresponds to the double bifurcation point with $\langle r, s, p_1^2 p_2 \rangle$-symmetric[16] bifurcated paths of the form (18) producing a hexagon of type M studied in §3.3.1. These bifurcated paths with the same symmetry have different agglomeration properties:

[15]The hexagonal domain used for this illustration is cut from the infinite domain that is obtained by repeating the 3×3 rhombic domain spatially (cf., the dotted hexagon in Fig. 2).

[16]The group $\langle r, s, p_1^2 p_2, p_1^3 \rangle$ in (17) reduces to $\langle r, s, p_1^2 p_2 \rangle$ because $p_1^3 = e$ for $n = 3$.

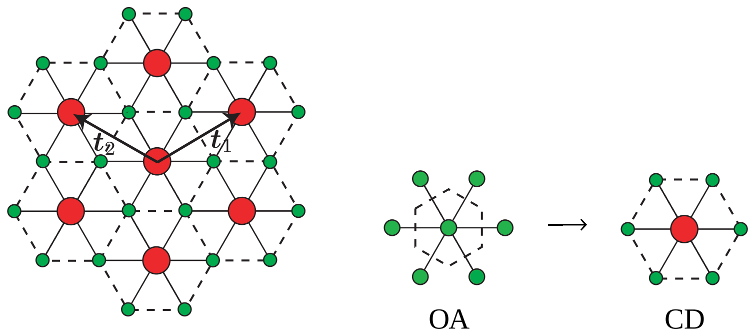


(a) FO model



(b) Pf model

Figure 5: Bifurcation at the first bifurcation point A for a system of 3×3 places and associated population distributions (Solid curve: stable, dashed curve: unstable)



(a) Hexagonal distribution (b) Enlargement of market area

Figure 6: Enlargement of market area and hexagonal distribution observed on the bifurcated path ACD at the first bifurcation point A

- For path ACD, which is stable during CD, the first-level center with the largest population at the center of the hexagonal domain (depicted by the large circle) is surrounded by six regular-hexagonal second-level centers with the second-largest population (depicted by the small circles). The population at the first-level center grows along path CD; at point D, the population at the second-level centers almost disappears.
- For path AE, which is unstable throughout, the second-level center with small population is surrounded by six regular-hexagonal first-level centers. However, the presence of a greater number of first-level centers than the second-level centers is incompatible with an implicit understanding in central place theory.

As depicted in Fig. 5, the spatial period becomes $T/d = \sqrt{3}$ on bifurcated solutions ACD and AE, in agreement with (19) for $k = 3$ system. This is the spatial period $\sqrt{3}$-times bifurcation associated with

$$\begin{array}{rccc}
T/d: & 1 & \rightarrow & \sqrt{3} \\
(\boldsymbol{t}_1, \boldsymbol{t}_2): & (\boldsymbol{\ell}_1, \boldsymbol{\ell}_2) & \rightarrow & (2\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2, -\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2) \\
\text{group}: & \mathrm{D}_6 \dot{+} (\mathbb{Z}_3 \times \mathbb{Z}_3) = \langle r, s, p_1, p_2 \rangle & \rightarrow & \langle r, s, p_1^2 p_2 \rangle \\
\text{path}: & \text{OA} & \rightarrow & \text{ACD}
\end{array}$$

Although the solution curves in Fig. 5 for the FO and Pf models are apparently different, a bifurcated solution with the population distribution for the $k = 3$ system branches at the bifurcation point A in each model, in agreement with the theoretical prediction by group-theoretic bifurcation theory in §3.3.1. This result demonstrates that the emergence of the $k = 3$ system, which is predicted theoretically, is a general phenomenon that is independent of individual models. Such model independence is also demonstrated for $k = 4$ and $k = 7$ systems in Subsections 4.2 and 4.3, respectively.

The hexagonal distribution for the stable bifurcated path CD is repeated spatially to form a distribution depicted in Fig. 6(a), which agrees with the Christaller's market principle for $k = 3$ system:

> In a system of central places developed according to the marketing principle, the great long-distance lines necessarily by-pass places of

considerable importance, ... (cf., Christaller (1966, p.74) and Dicken and Lloyd (1990, Chapter 1)).

For the stable bifurcated path CD, the market area, in the sense of Remark 4.1 below, of the first-level center is the regular hexagon with the radius of 1 depicted at the right of Fig. 6(b) by the dashed lines. The ratio of the number N_1 of the first-level centers to the number N_2 of the second-level centers is equal to

$$N_1 : N_2 = 1 : 2;$$

since each of the six second-level center is shared by three neighboring market areas; in effect, $6/3 = 2$ second-level centers exist in the market area. This is in agreement with the formula (32) for $k = 3$ of central place theory in Remark 4.2 below. Consequently, some commonality exists between the computation for the core–periphery models and the prediction by central place theory, although they are based on different underpinnings. Geometry might be the source of this commonality; these core–periphery models undergo bifurcations that change geometry of population distribution, and central place theory is based on a geometrical approach.

Remark 4.1. For the core–periphery model, the concept of market area is fictitious because the degrees of freedom are allocated only at the nodes of the hexagonal lattice and goods are transported beyond this area. Yet, in this paper, this concept is used for convenience in the description of the progress of agglomeration. Examination of the transportation of goods among places for core–periphery models of various kinds is a topic for future investigation. □

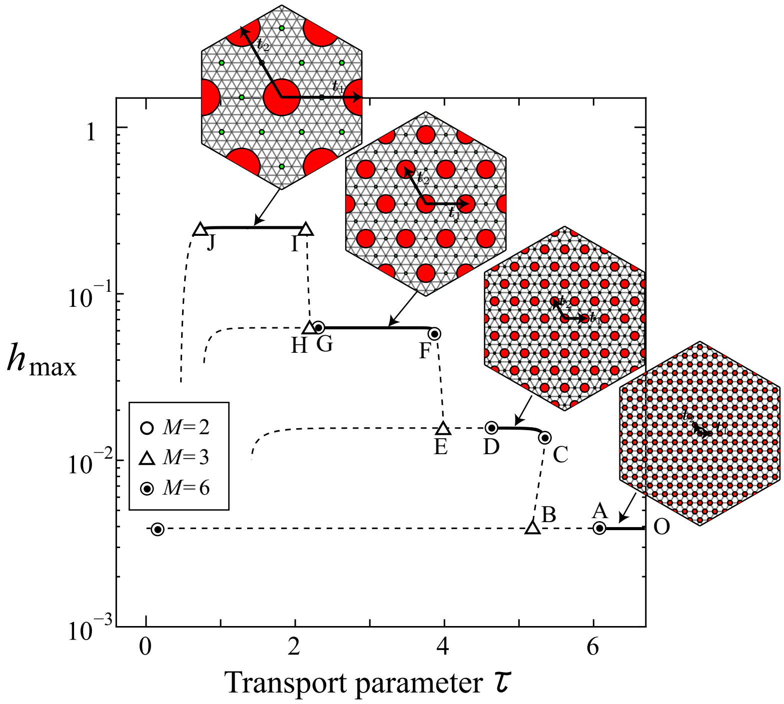
Remark 4.2. A hierarchy of places with different levels exists in the market area governed by the highest-level center. Such a hierarchy is often called, metropolis, city, town, village, and hamlet or A-level center, B-level center, and so on (Dicken and Lloyd, 1990, Chapter 1). The number N_j of the jth-level centers dominated by the highest-order center is expressed as (cf., Dicken and Lloyd, 1990, Chapter 1):

$$N_1 = 1, \quad N_j = k^{j-1} - k^{j-2}, \quad (j = 2, 3, \ldots), \tag{32}$$

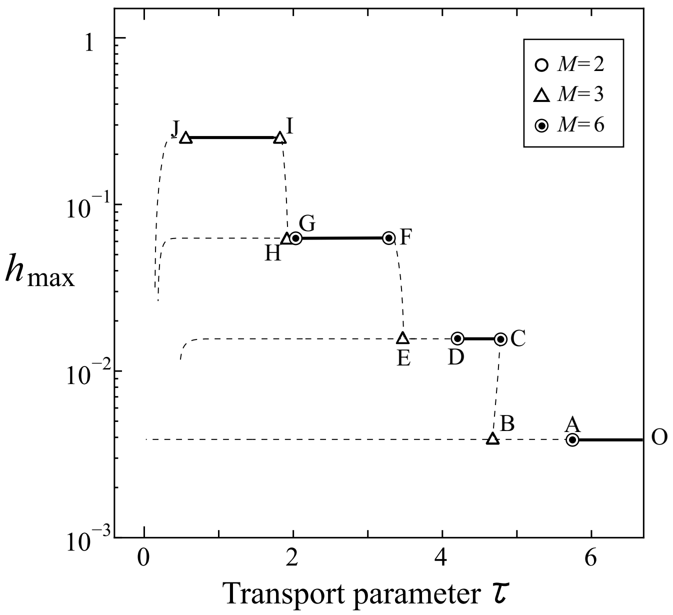
which is applicable to $k = 3$ and 4 systems; its extensibility to other k values is a topic for future. □

4.2 Agglomeration of 16×16 places: $k = 4$ system

In this section, we demonstrate the emergence of the $k = 4$ system for $\mathrm{D}_6 \dot{+} (\mathbb{Z}_{16} \times \mathbb{Z}_{16})$-symmetric 16×16 places on the rhombic domain that undergoes period-doubling bifurcations repeatedly. The agglomeration pattern is shown to display Christaller's $k = 4$ system and the number of different level centers is studied in light of central place theory.



(a) FO model



(b) Pf model

Figure 7: Bifurcated paths for a system of 16×16 places and associated population distributions in hexagonal windows (Solid curve: stable, dashed curve: unstable; some equilibrium paths are omitted to simplify the figure; M is the multiplicity of the bifurcation point)

4.2.1 Period-doubling bifurcation cascade

The equilibrium paths depicted in Fig. 7 are obtained using the computational bifurcation analysis for the 16×16 places for the FO and Pf models. Again the two models display qualitatively identical behaviors; accordingly, the following argument applies to both models. We plot only the bifurcated paths branching from the triple bifurcation points that produce the hexagonal patterns related to the $k = 4$ system (cf., §3.3.2); bifurcated paths branching at other bifurcation points, such as points A, C, D, F, and G of multiplicity 6, need not be obtained, as it is possible to show that the associated bifurcated patterns are not related to the $k = 4$ system by group-theoretic analysis similar to the one conducted in Appendix C. The information about the symmetries of bifurcated solutions thus is vital in the search for the $k = 4$ system.

From the uniform population solution OA of this system, we found a hierarchy of bifurcated paths:

$$\text{OAB} \rightarrow \text{BCDE} \rightarrow \text{EFGH} \rightarrow \text{HIJ} \tag{33}$$

branching at a series of triple bifurcation points B, E, and H.

These triple bifurcation points are those which are studied theoretically in §3.3.2. The population distribution of these four paths are labeled, respectively, by $D_6\dot{+}(\mathbb{Z}_{16} \times \mathbb{Z}_{16})$, $D_6\dot{+}(\mathbb{Z}_8 \times \mathbb{Z}_8)$, $D_6\dot{+}(\mathbb{Z}_4 \times \mathbb{Z}_4)$, and $D_6\dot{+}(\mathbb{Z}_2 \times \mathbb{Z}_2)$. This is the spatial period-doubling cascade, in which the spatial period T is doubled repeatedly as

$$\begin{array}{rccccccc}
T/d: & 1 & \rightarrow & 2 & \rightarrow & 2^2 & \rightarrow & 2^3 \\
(\boldsymbol{t}_1, \boldsymbol{t}_2): & (\boldsymbol{\ell}_1, \boldsymbol{\ell}_2) & \rightarrow & (2\boldsymbol{\ell}_1, 2\boldsymbol{\ell}_2) & \rightarrow & (2^2\boldsymbol{\ell}_1, 2^2\boldsymbol{\ell}_2) & \rightarrow & (2^3\boldsymbol{\ell}_1, 2^3\boldsymbol{\ell}_2) \\
\text{group}: & D_6\dot{+}(\mathbb{Z}_{16} \times \mathbb{Z}_{16}) & \rightarrow & D_6\dot{+}(\mathbb{Z}_8 \times \mathbb{Z}_8) & \rightarrow & D_6\dot{+}(\mathbb{Z}_4 \times \mathbb{Z}_4) & \rightarrow & D_6\dot{+}(\mathbb{Z}_2 \times \mathbb{Z}_2) \\
\text{path}: & \text{OA} & \rightarrow & \text{CD} & \rightarrow & \text{FG} & \rightarrow & \text{IJ}
\end{array}$$

This hierarchy is in agreement with the theoretically predicted hierarchy (28) and (29) for $n = 16$.

The stable parts OA, CD, FG, and IJ of these paths, which are not continuous but which are existent for most values of the parameter τ, might serve as an economically feasible process of agglomeration. As depicted by the hexagonal windows in Fig. 7, the agglomeration of population progresses in association with this hierarchy of bifurcations in (33), and produces a set of nested hexagons. The hexagonal distributions for these paths are all of type V.

The pattern for CD is Christaller's $k = 4$ system with two-level hierarchy. As depicted in Fig. 7(a), two neighboring first-level centers are connected by a straight road that passes one second-level center. This configuration agrees with Christaller's traffic principle for $k = 4$:

> The traffic principle states that the distribution of central places is most favorable when as many important places as possible lie on one traffic route between two important towns, the route being as straightly and as cheaply as possible (cf., Christaller (1966, p.74) and Dicken and Lloyd (1990, Chapter 1)).

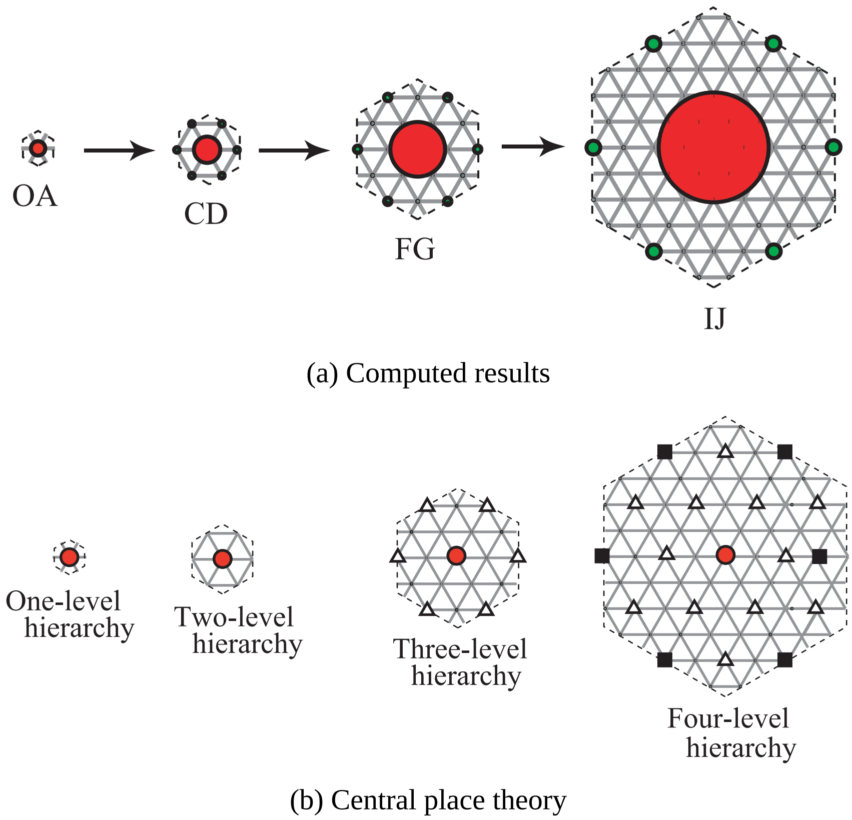


Figure 8: Distribution of different level centers in market areas (In (a), the area of a circle is proportional to the population; in (b), places with the same symbol have the same population, but the sizes of the symbols are not related to the population)

4.2.2 Number of different level centers

We investigate the computed distributions of different level centers in market areas in Fig. 8(a), in comparison with those predicted by central place theory in Fig. 8(b). Although the computation for the core–periphery models and the prediction by central place theory are based on different underpinnings, we can find some commonality.

In central place theory, the ratio of the number N_1 of the first-level centers, the number N_2 of the second-level centers, and so on, is given by the following formula (cf., the recurrence (32) for $k = 4$ in Remark 4.2 above):

$$\begin{cases} \text{One-level hierarchy}: & N_{\mathrm{T}} = 1, \quad N_1 = 1, \\ \text{Two-level hierarchy}: & N_{\mathrm{T}} = 2^2, \quad N_1 : N_2 = 1 : 3, \\ \text{Three-level hierarchy}: & N_{\mathrm{T}} = 4^2, \quad N_1 : N_2 : N_3 = 1 : 3 : 12, \\ \text{Four-level hierarchy}: & N_{\mathrm{T}} = 8^2, \quad N_1 : N_2 : N_3 : N_4 = 1 : 3 : 12 : 48, \end{cases} \tag{34}$$

in which $N_{\mathrm{T}} = \sum_{i=1} N_i$ denotes the total number of places in each market area. As portrayed in Fig. 8(b), for example, for the two-level hierarchy, the first-level center at the center of the market area is surrounded by six second-level centers at the borders of the market area; since each second-level center is shared by two neighboring market areas, there are, in effect, $6/2 = 3$ second-level centers in the market area.

In the computed distributions in Fig. 8(a), with regard to the total number N_{T} of places in each market area for each bifurcated path, we can see a strong correlation with central place theory as follows:

N_{T}	Computed results	Central place theory
1	Uniform solution OA	One-level hierarchy
2^2	Direct bifurcated path CD	Two-level hierarchy
4^2	Secondary bifurcated path FG	Three-level hierarchy
8^2	Tertiary bifurcated path IJ	Four-level hierarchy

The ratio of different level centers for the computed distributions shows agreement with (34) for the uniform population solution OA with single-level hierarchy with $N_1 = 1$, for the direct bifurcated solution CD with two-level hierarchy with $N_1 : N_2 = 1 : 3$, and for the secondary bifurcated solution FG with three-level hierarchy[17] with $N_1 : N_2 : N_3 = 1 : 3 : 12$. See Remark 4.3 below for the bifurcation mechanism to produce such recurrence. Yet such agreement is not observed for the tertiary bifurcated path IJ: we have $N_1 : N_2 : N_3 = 1 : 3 : 60$. It will be a topic for future to investigate the commonality between agglomeration for core–periphery models and that in central place theory so as to extend the horizon of these models.

Remark 4.3. By the period doubling bifurcation cascade, under the assumption that first-level centers are predominantly large, we see that at the onset of each bifurcation one-fourth of the pre-bifurcation first-level centers has increasing population and remains first-level centers, but three-fourths of the pre-bifurcation first-level centers have decreasing population and become second-level centers. This

[17] Although six of the 12 third-level centers have slightly larger population than the other six, they are considered to be identical herein.

preserves the ratio of the number of first-level centers to that of the second-level centers as $N_1 : N_2 = 1 : 3$, in agreement with the ratio observed for the bifurcated solutions OA, CD, and FG. $\square$

4.3 Agglomeration of 7×7 places: $k = 7$ system

We demonstrate the emergence of the $k = 7$ system for $D_6 \dot{+} (\mathbb{Z}_7 \times \mathbb{Z}_7)$-symmetric 7×7 places on the rhombic domain. Figure 9 depicts equilibrium paths of this system for the FO and Pf models, particularly addressing the bifurcated solutions branching at bifurcation points of multiplicity 12. Again the two models display qualitatively identical behaviors; accordingly, the following argument applies to both models. We plot only the bifurcated paths branching from the bifurcation point B of multiplicity 12 that produce the hexagonal patterns related to $k = 7$ system (cf., §3.3.3); bifurcated paths branching at other bifurcation points, such as the point A of multiplicity 6, need not be obtained, as it is possible to show that the associated bifurcated patterns are not related to $k = 7$ system owing to group-theoretic analysis similar to the one conducted in Appendix D.

At bifurcation point B with multiplicity of 12 on the path OABC (for uniform population solution), we found a bifurcated solution BDEF of the form (26) with $\langle r, p_1^3 p_2 \rangle$-symmetry[18] producing the tilted hexagon studied in §3.3.3. The spatial period becomes $T/d = \sqrt{7}$ on the bifurcated solution BDEF, in agreement with (27). This is the spatial period $\sqrt{7}$-times bifurcation associated with

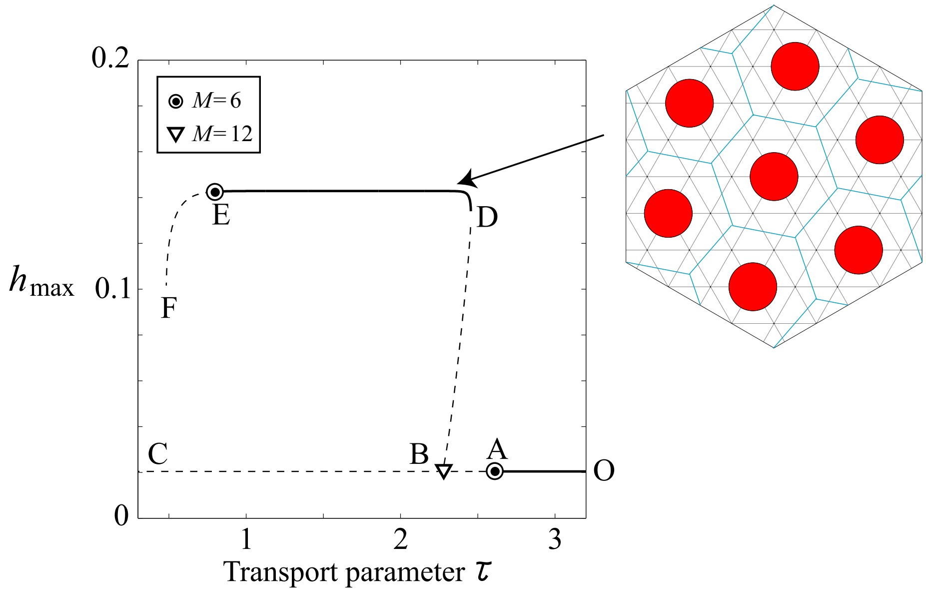
$$\begin{array}{cccc}
T/d: & 1 & \rightarrow & \sqrt{7} \\
(\boldsymbol{t}_1, \boldsymbol{t}_2): & (\boldsymbol{\ell}_1, \boldsymbol{\ell}_2) & \rightarrow & (3\boldsymbol{\ell}_1 - \boldsymbol{\ell}_2, -\boldsymbol{\ell}_1 + 2\boldsymbol{\ell}_2) \\
\text{group}: & D_6 \dot{+} (\mathbb{Z}_7 \times \mathbb{Z}_7) = \langle r, s, p_1, p_2 \rangle & \rightarrow & \langle r, p_1^3 p_2 \rangle \\
\text{path}: & \text{OABC} & \rightarrow & \text{BDEF}
\end{array}$$

Each hexagonal window contains seven market areas, as portrayed in Fig. 9. Each market area contains one first-level center surrounded by six second-level centers with the same population. This agrees with Christaller's administrative principle for $k = 7$ system:

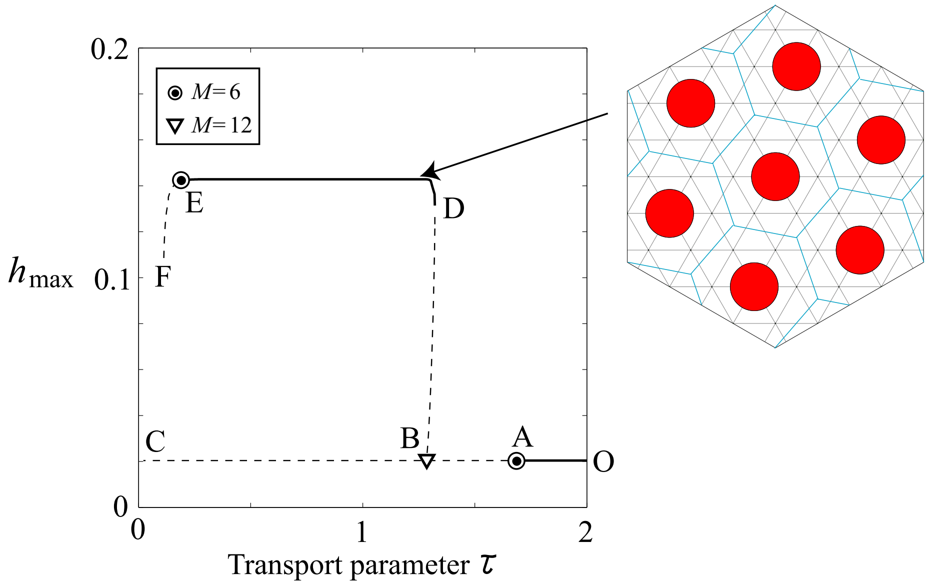
> The ideal of such a spatial community has the nucleus as the capital (a central place of a higher rank), around it, a wreath of satellite places of lesser importance, and toward the edge of the region a thinning population density—and even uninhabited areas (cf., Christaller (1966, p.77)). Lower-order centers are entirely within the hexagon of the higher-order center (cf., Dicken and Lloyd, (1990, Chapter 1)).

The ratio of the number N_1 of the first-level centers to the number N_2 of the second-level centers is equal to $N_1 : N_2 = 1 : 6$. This shows a possible extensibility of the formula (32) of central place theory to $k = 7$.

[18]We have $\langle r, p_1^3 p_2, p_1^{-1} p_2^2 \rangle = \langle r, p_1^3 p_2 \rangle$ for $n = 7$ by $p_1^{-1} p_2^2 = p_1^6 p_2^2 = (p_1^3 p_2)^2$.



(a) FO model



(b) Pf model

Figure 9: Bifurcated paths for a system of 7×7 places and associated population distributions in hexagonal windows (Solid curve: stable, dashed curve: unstable; M is the multiplicity of the bifurcation point)

5 Conclusion

For a two-dimensional system modeled by two different core–periphery models, self-organization of hexagonal population distributions for Christaller's three systems in central place theory is predicted by group-theoretic bifurcation theory, and its existence is verified by computational bifurcation analysis. It demonstrates inherent model-independent capability of the core–periphery models to express those systems provided with pertinent spatial platforms. Moreover, it confirms the prediction by Krugman (1996; P91) of the emergence of a system of hexagonal market areas in two dimensions, thereby paving the way for cross-fertilization between central place theory and new economic geography.

In central place theory, the three systems are explained based on market, traffic, and administrative principles. In contrast, the present analysis using the core–periphery models based on microeconomic underpinning engenders a hierarchy of different levels of centers without resort to these principles. The results obtained using central place theory must be reconsidered in light of economic geographical modeling to extend the horizon of core–periphery models.

Bifurcations are highlighted as a catalyst to break uniformity to engender the patterns. Group-theoretic bifurcation theory has displayed its usefulness to predict possible agglomeration patterns among a system of places in two dimensions, often associated with successive elongation of spatial periods. Information about symmetries of bifurcated solutions offered by this theory is important in choosing a bifurcation point that produces hexagonal distributions of interest. We computed three hexagons corresponding to the three smallest possible market areas in the sense of Lösch (1954); it will be a topic for future to address other solutions expressing larger hexagons in view of pre-existing results of group-theoretic bifurcation theory.

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A Appendix: Core-Periphery models

Details of the pair of core–periphery models (FO and Pf models) in Section 3 are presented. After presenting basic assumptions, we describe the short-run equilibrium and define the long-run equilibrium and its stability.

A.1 Basic Assumptions

Preferences U over the M- and A-sector goods are identical across individuals, where M signifies manufacture and A stands for agriculture. The utility of an individual in place i is

$$\text{[FO model[19]]} \quad U(C_i^{\mathrm{M}}, C_i^{\mathrm{A}}) = \mu \ln C_i^{\mathrm{M}} + (1-\mu)\ln C_i^{\mathrm{A}} \qquad (0 < \mu < 1), \quad \text{(A.1a)}$$

$$\text{[Pf model]} \quad U(C_i^{\mathrm{M}}, C_i^{\mathrm{A}}) = \mu \ln C_i^{\mathrm{M}} + C_i^{\mathrm{A}} \qquad (\mu > 0), \quad \text{(A.1b)}$$

where μ is the constant expenditure share on industrial varieties, C_i^{A} is the consumption of the A-sector product in place i, and C_i^{M} is the manufacturing aggregate in place i and is defined as

$$C_i^{\mathrm{M}} \equiv \left(\sum_j \int_0^{n_j} q_{ji}(k)^{(\sigma-1)/\sigma} dk \right)^{\sigma/(\sigma-1)},$$

where $q_{ji}(k)$ is the consumption in place i of a variety $k \in [0, n_j]$ produced in place j, n_j is the continuum range of varieties produced in place j, often called the number of available varieties, and $\sigma > 1$ is the constant elasticity of substitution between any two varieties. The budget constraint is given as

$$p_i^{\mathrm{A}} C_i^{\mathrm{A}} + \sum_j \int_0^{n_j} p_{ji}(k) q_{ji}(k) dk = Y_i, \quad \text{(A.2)}$$

where p_i^{A} is the price of A-sector goods in place i, $p_{ji}(k)$ is the price of a variety k in place i produced in place j and Y_i is the income of an individual in place i. The incomes (wages) of the skilled worker and the unskilled worker are represented, respectively, by w_i and w_i^{L}. We denote by K the number of places, and therefore i and j run through 1 to K.

An individual in place i maximizes (A.1) subject to (A.2). This yields the following demand functions:

$$\text{[FO model]} \quad C_i^{\mathrm{A}} = (1-\mu)\frac{Y_i}{p_i^{\mathrm{A}}}, \quad C_i^{\mathrm{M}} = \mu \frac{Y_i}{\rho_i}, \quad q_{ji}(k) = \mu \frac{p_i^{\mathrm{A}} \rho_i^{\sigma-1} Y_i}{p_{ji}(k)^{\sigma}}, \quad \text{(A.3a)}$$

$$\text{[Pf model]} \quad C_i^{\mathrm{A}} = \frac{Y_i}{p_i^{\mathrm{A}}} - \mu, \quad C_i^{\mathrm{M}} = \mu \frac{p_i^{\mathrm{A}}}{\rho_i}, \quad q_{ji}(k) = \mu \frac{p_i^{\mathrm{A}} \rho_i^{\sigma-1}}{p_{ji}(k)^{\sigma}}, \quad \text{(A.3b)}$$

[19] We take logarithms of the Forslid and Ottaviano (2003) type (i.e., Cobb-Douglas-type) utility function to facilitate the analysis. This transformation has no influence on the properties of the model.

where ρ_i denotes the price index of the differentiated product in place i, which is

$$\rho_i = \left(\sum_j \int_0^{n_j} p_{ji}(k)^{1-\sigma} dk \right)^{1/(1-\sigma)}. \tag{A.4}$$

Since the total income and population in place i are $w_i h_i + w_i^{\mathrm{L}}$ and $h_i + 1$, respectively, we have the total demand $Q_{ji}(k)$ in place i for a variety k produced in place j:

$$\text{[FO model]} \quad Q_{ji}(k) = \mu \frac{p_i^{\mathrm{A}} \rho_i^{\sigma-1}}{p_{ji}(k)^{\sigma}} (w_i h_i + w_i^{\mathrm{L}}), \tag{A.5a}$$

$$\text{[Pf model]} \quad Q_{ji}(k) = \mu \frac{p_i^{\mathrm{A}} \rho_i^{\sigma-1}}{p_{ji}(k)^{\sigma}} (h_i + 1). \tag{A.5b}$$

The A-sector is perfectly competitive and produces homogeneous goods under constant returns to scale technology, which requires one unit of unskilled labor in order to produce one unit of output. For simplicity, we assume that the A-sector goods are transported freely between places and that they are chosen as the numèraire. These assumptions mean that, in equilibrium, the wage of an unskilled worker w_i^{L} is equal to the price of A-sector goods in all places (i.e., $p_i^{\mathrm{A}} = w_i^{\mathrm{L}} = 1$ for each $i = 1, \ldots, K$).

The M-sector output is produced under increasing returns to scale technology and Dixit-Stiglitz monopolistic competition. A firm incurs a fixed input requirement of α units of skilled labor and a marginal input requirement of β units of unskilled labor. Given the fixed input requirement α, the skilled labor market clearing implies that, in equilibrium, the number of firms in place i is determined by $n_i = h_i/\alpha$. An M-sector firm located in place i chooses $(p_{ij}(k) \mid j = 1, \ldots, K)$ that maximizes its profit

$$\Pi_i(k) = \sum_j p_{ij}(k) Q_{ij}(k) - (\alpha w_i + \beta x_i(k)),$$

where $x_i(k)$ is the total supply. The transportation costs for M-sector goods are assumed to take the iceberg form.[20] That is, for each unit of M-sector goods transported from place i to place $j \neq i$, only a fraction $1/\phi_{ij} < 1$ arrives. Consequently, the total supply $x_i(k)$ is given as

$$x_i(k) = \sum_j \phi_{ij} Q_{ij}(k). \tag{A.6}$$

To put it concretely, we define the transport cost ϕ_{ij} between the two places i and j as

$$\phi_{ij} = \exp(\tau D_{ij}), \tag{A.7}$$

where τ is the transport parameter and D_{ij} represents the shortest distance between places i and j.

[20] This is a standard term in economics; see, for example, Samuelson (1952).

Since we have a continuum of firms, each firm is negligible in the sense that its action has no impact on the market (i.e., the price indices). Therefore, the first-order condition for profit maximization gives

$$p_{ij}(k) = \frac{\sigma\beta}{\sigma - 1}\phi_{ij}. \tag{A.8}$$

This expression implies that the price of the M-sector product does not depend on variety k, so that $Q_{ij}(k)$ and $x_i(k)$ do not depend on k. Therefore, we describe these variables without the argument k. Substituting (A.8) into (A.4), we have the price index

$$\rho_i = \frac{\sigma\beta}{\sigma - 1}\left(\frac{1}{\alpha}\sum_j h_j d_{ji}\right)^{1/(1-\sigma)}, \tag{A.9}$$

where $d_{ji} = \phi_{ji}^{1-\sigma}$ is a spatial discounting factor between places j and i; from (A.5) and (A.9), d_{ji} is obtained as $(p_{ji}Q_{ji})/(p_{ii}Q_{ii})$, which means that d_{ji} is the ratio of total expenditure in place i for each M-sector product produced in place j to the expenditure for a domestic product.

A.2 Short-run Equilibrium

In the short run, skilled workers are immobile between places, i.e., their spatial distribution ($\boldsymbol{h} = (h_i) \in \mathbb{R}^K$) is assumed to be given. The short-run equilibrium conditions consist of the M-sector goods market clearing condition and the zero-profit condition because of the free entry and exit of firms. The former condition can be written as (A.6). The latter condition requires that the operating profit of a firm is absorbed entirely by the wage bill of its skilled workers:

$$w_i(\boldsymbol{h}, \tau) = \frac{1}{\alpha}\left\{\sum_j p_{ij}Q_{ij}(\boldsymbol{h}, \tau) - \beta x_i(\boldsymbol{h}, \tau)\right\}. \tag{A.10}$$

Substituting (A.5), (A.6), (A.8), and (A.9) into (A.10), we have the short-run equilibrium wage:

$$\text{[FO model]} \quad w_i(\boldsymbol{h}, \tau) = \frac{\mu}{\sigma}\sum_j \frac{d_{ij}}{\Delta_j(\boldsymbol{h}, \tau)}(w_j(\boldsymbol{h}, \tau)h_j + 1), \tag{A.11a}$$

$$\text{[Pf model]} \quad w_i(\boldsymbol{h}, \tau) = \frac{\mu}{\sigma}\sum_j \frac{d_{ij}}{\Delta_j(\boldsymbol{h}, \tau)}(h_j + 1), \tag{A.11b}$$

where $\Delta_j(\boldsymbol{h}, \tau) \equiv \sum_k d_{kj}h_k$ denotes the market size of the M-sector in place j. Consequently, $d_{ij}/\Delta_j(\boldsymbol{h}, \tau)$ defines the market share in place j of each M-sector product produced in place i.

The indirect utility $v_i(\boldsymbol{h}, \tau)$ is obtained by substituting (A.3), (A.9), and (A.11) into (A.1):[21]

$$\text{[FO model]} \quad v_i(\boldsymbol{h}, \tau) = S_i(\boldsymbol{h}, \tau) + \ln[w_i(\boldsymbol{h}, \tau)], \tag{A.12a}$$

$$\text{[Pf model]} \quad v_i(\boldsymbol{h}, \tau) = S_i(\boldsymbol{h}, \tau) + w_i(\boldsymbol{h}, \tau), \tag{A.12b}$$

[21] We ignore the constant terms, which have no influence on the results below.

where

$$S_i(\boldsymbol{h}, \tau) \equiv \mu(\sigma - 1)^{-1} \ln \Delta_i(\boldsymbol{h}, \tau).$$

For convenience in conducting the following analysis, we express the indirect utility function $\boldsymbol{v}(\boldsymbol{h}, \tau)$ in vector form, using the spatial discounting matrix $\boldsymbol{D} = (d_{ij})$, as

$$\text{[FO model]} \quad \boldsymbol{v}(\boldsymbol{h}, \tau) = \boldsymbol{S}(\boldsymbol{h}, \tau) + \ln[\boldsymbol{w}(\boldsymbol{h}, \tau)], \tag{A.13a}$$

$$\boldsymbol{w}(\boldsymbol{h}, \tau) = \frac{\mu}{\sigma}\, [\boldsymbol{I} - \boldsymbol{W}(\boldsymbol{h}, \tau)]^{-1}\, \boldsymbol{w}^{(\mathrm{L})}(\boldsymbol{h}, \tau), \tag{A.13b}$$

$$\text{[Pf model]} \quad \boldsymbol{v}(\boldsymbol{h}, \tau) = \boldsymbol{S}(\boldsymbol{h}, \tau) + \boldsymbol{w}(\boldsymbol{h}, \tau), \tag{A.13c}$$

$$\boldsymbol{w}(\boldsymbol{h}, \tau) = \frac{\mu}{\sigma} \left[\boldsymbol{w}^{(\mathrm{H})}(\boldsymbol{h}, \tau) + \boldsymbol{w}^{(\mathrm{L})}(\boldsymbol{h}, \tau)\right], \tag{A.13d}$$

where

$$\boldsymbol{S}(\boldsymbol{h}, \tau) \equiv [S_1(\boldsymbol{h}, \tau), \ldots, S_K(\boldsymbol{h}, \tau)]^\top, \qquad \ln[\boldsymbol{w}] \equiv [\ln w_1, \ln w_2, \ldots, \ln w_K]^\top,$$

$\boldsymbol{I}$ is a unit matrix, and $\boldsymbol{W}(\boldsymbol{h}, \tau)$, $\boldsymbol{w}^{(\mathrm{H})}$, $\boldsymbol{w}^{(\mathrm{L})}$ and $\boldsymbol{M}$ are defined as

$$\boldsymbol{W} \equiv \frac{\mu}{\sigma} \boldsymbol{M} \mathrm{diag}[\boldsymbol{h}], \quad \boldsymbol{w}^{(\mathrm{H})} \equiv \boldsymbol{M}\boldsymbol{h}, \quad \boldsymbol{w}^{(\mathrm{L})} \equiv \boldsymbol{M}\boldsymbol{1}, \tag{A.14a}$$

$$\boldsymbol{M} \equiv \boldsymbol{D}\boldsymbol{\Delta}^{-1}, \quad \boldsymbol{\Delta} \equiv \mathrm{diag}[\boldsymbol{D}^\top \boldsymbol{h}], \quad \boldsymbol{1} \equiv [1, \ldots, 1]^\top. \tag{A.14b}$$

A.3 Adjustment Process, Long-run Equilibrium and Stability

In the long run, the skilled workers are inter-regionally mobile. They are assumed to be heterogeneous in their preferences for location choice. That is, the indirect utility for an individual s in place i is expressed as

$$v_i^{(s)}(\boldsymbol{h}, \tau) = v_i(\boldsymbol{h}, \tau) + \epsilon_i^{(s)}.$$

In this equation, $\epsilon_i^{(s)}$, which is distributed continuously across individuals, denotes the utility representing the idiosyncratic taste for residential location,

We present the dynamics of the migration of the skilled workers to define the long-run equilibrium and its stability with respect to small perturbations (i.e., local stability). We assume that at each time period t, the opportunity for skilled workers to migrate emerges according to an independent Poisson process with arrival rate λ. That is, for each time interval $[t, t + dt)$, a fraction λdt of skilled workers have the opportunity to migrate. Given an opportunity at time t, each worker chooses the place that provides the highest indirect utility $v_i^{(s)}(\boldsymbol{h}, \tau)$, which depends on the current distribution $\boldsymbol{h} = \boldsymbol{h}(t)$. The fraction of skilled workers who choose place i under distribution $\boldsymbol{h}$ is $P_i(\boldsymbol{v}(\boldsymbol{h}), \tau)$, where

$$P_i(\boldsymbol{v}, \tau) = \Pr[v_i^{(s)} > v_j^{(s)},\ \forall j \neq i].$$

Therefore, we have

$$h_i(t + dt) = (1 - \lambda dt) h_i(t) + \lambda dt H P_i(\boldsymbol{v}(\boldsymbol{h}(t)), \tau).$$

By normalizing the unit of time so that $\lambda = 1$, we obtain the following adjustment process:

$$\dot{\boldsymbol{h}}(t) = \boldsymbol{F}(\boldsymbol{h}(t), \tau) \equiv H\boldsymbol{P}(\boldsymbol{v}(\boldsymbol{h}(t)), \tau) - \boldsymbol{h}(t), \tag{A.15}$$

where $\dot{\boldsymbol{h}}(t)$ denotes the time derivative of $\boldsymbol{h}(t)$, and $\boldsymbol{P}(\boldsymbol{v}(\boldsymbol{h}), \tau) = (P_i(\boldsymbol{v}(\boldsymbol{h}), \tau))$. For the specific functional form of $P_i(\boldsymbol{v}, \tau)$, we use the logit choice function:

$$P_i(\boldsymbol{v}, \tau) \equiv \frac{\exp[\theta v_i]}{\sum_j \exp[\theta v_j]}, \tag{A.16}$$

where $\theta \in (0, \infty)$ is the parameter denoting the inverse of variance of the idiosyncratic tastes. This implies the assumption that the distributions of $(\epsilon_i^{(s)})$'s are Gumbel distributions, which are identical and independent across places (e.g., McFadden, 1974; Anderson et al., 1992). The adjustment process described by (A.15) and (A.16) is the logit dynamics, which has been studied in evolutionary game theory (e.g., Fudenberg and Levine, 1998; Hofbauer and Sandholm, 2007; Sandholm, 2010).

Next, we define the long-run equilibrium and its stability. The long-run equilibrium is a stationary point of the adjustment process of (A.15).

Definition A.1. The long-run equilibrium is defined as the distribution $\boldsymbol{h}^*$ that satisfies

$$\boldsymbol{F}(\boldsymbol{h}^*, \tau) \equiv H\boldsymbol{P}(\boldsymbol{v}(\boldsymbol{h}^*), \tau) - \boldsymbol{h}^* = \boldsymbol{0}. \tag{A.17}$$

The heterogeneous worker case includes the conventional homogeneous worker case. Indeed, when $\theta \to \infty$, the condition given in (A.17) reduces to that for the homogeneous worker case:

$$\begin{cases} V^* - v_i(\boldsymbol{h}^*, \tau) = 0 & \text{if} \quad h_i^* > 0, \\ V^* - v_i(\boldsymbol{h}^*, \tau) \geq 0 & \text{if} \quad h_i^* = 0, \end{cases}$$

where V^* denotes the equilibrium utility.

We restrict our concern to the neighborhood of $\boldsymbol{h}^*$, and define the stability of $\boldsymbol{h}^*$ in the sense of asymptotic stability, the precise definition of which is the following.

Definition A.2. A long-run equilibrium $\boldsymbol{h}^*$ is *asymptotically stable* if, for any $\epsilon > 0$, there is a neighborhood $N(\boldsymbol{h}^*)$ of $\boldsymbol{h}^*$ such that, for every $\boldsymbol{h}_0 \in N(\boldsymbol{h}^*)$, the solution $\boldsymbol{h}(t)$ of (A.15) with an initial value $\boldsymbol{h}(0) \equiv \boldsymbol{h}_0$ satisfies $\|\boldsymbol{h}(t) - \boldsymbol{h}^*\| < \epsilon$ for any time $t \geq 0$, and $\lim_{t\to\infty} \boldsymbol{h}(t) = \boldsymbol{h}^*$. It is *unstable* if equilibrium $\boldsymbol{h}^*$ is not asymptotically stable.

In dynamic system theory, $\boldsymbol{h}^*$ is asymptotically stable if all the eigenvalues of the Jacobian matrix $\nabla\boldsymbol{F}(\boldsymbol{h}, \tau) \equiv (\partial F_i(\boldsymbol{h}, \tau)/\partial h_j)$ of the adjustment process of (A.15) have negative real parts; otherwise $\boldsymbol{h}^*$ is unstable (see, for example, Hirsch and Smale, 1974). Therefore, the asymptotic stability can be assessed by examining the following Jacobian matrix:

$$\nabla\boldsymbol{F}(\boldsymbol{h}, \tau) = H\boldsymbol{J}(\boldsymbol{v}(\boldsymbol{h}), \tau)\nabla\boldsymbol{v}(\boldsymbol{h}, \tau) - \boldsymbol{I}, \tag{A.18}$$

where $\boldsymbol{J}(\boldsymbol{v}, \tau)$ and $\nabla\boldsymbol{v}(\boldsymbol{h}, \tau)$ are K-by-K matrices, the (i, j) entries of which are, respectively, $\partial P_i(\boldsymbol{v}, \tau)/\partial v_j$ and $\partial v_i(\boldsymbol{h}, \tau)/\partial h_j$. For the logit choice function of (A.16), it is easily verified that the former Jacobian matrix $\boldsymbol{J}(\boldsymbol{v}, \tau)$ is expressed as

$$\boldsymbol{J}(\boldsymbol{v}, \tau) = \theta\{\mathrm{diag}[\boldsymbol{P}(\boldsymbol{v}, \tau)] - \boldsymbol{P}(\boldsymbol{v}, \tau)\boldsymbol{P}(\boldsymbol{v}, \tau)^\top\}. \tag{A.19}$$

The latter Jacobian matrix $\nabla \boldsymbol{v}(\boldsymbol{h}, \tau)$ is given as

$$
\begin{aligned}
&\text{[FO model]} \quad && \nabla \boldsymbol{v}(\boldsymbol{h}, \tau) = \nabla \boldsymbol{S}(\boldsymbol{h}, \tau) + \mathrm{diag}[\boldsymbol{w}(\boldsymbol{h}, \tau)]^{-1} \nabla \boldsymbol{w}(\boldsymbol{h}, \tau), && \text{(A.20a)} \\
& && \nabla \boldsymbol{w}(\boldsymbol{h}, \tau) = \frac{\mu}{\sigma} [\boldsymbol{I} - \boldsymbol{W}(\boldsymbol{h}, \tau)]^{-1} \left\{ \nabla \hat{\boldsymbol{w}}^{(\mathrm{H})}(\boldsymbol{h}, \tau) + \nabla \boldsymbol{w}^{(\mathrm{L})}(\boldsymbol{h}, \tau) \right\}, && \text{(A.20b)} \\
&\text{[Pf model]} \quad && \nabla \boldsymbol{v}(\boldsymbol{h}, \tau) = \nabla \boldsymbol{S}(\boldsymbol{h}, \tau) + \frac{\mu}{\sigma} \left\{ \nabla \boldsymbol{w}^{(\mathrm{H})}(\boldsymbol{h}, \tau) + \nabla \boldsymbol{w}^{(\mathrm{L})}(\boldsymbol{h}, \tau) \right\}, && \text{(A.20c)}
\end{aligned}
$$

where the matrices $\nabla \boldsymbol{S}(\boldsymbol{h}, \tau)$, $\nabla \hat{\boldsymbol{w}}^{(\mathrm{H})}(\boldsymbol{h}, \tau)$, $\nabla \boldsymbol{w}^{(\mathrm{H})}(\boldsymbol{h}, \tau)$ and $\nabla \boldsymbol{w}^{(\mathrm{L})}(\boldsymbol{h}, \tau)$ are obtained as

$$\nabla \boldsymbol{S}(\boldsymbol{h}, \tau) = \mu(\sigma - 1)^{-1} \boldsymbol{M}^\top, \tag{A.21}$$

$$\nabla \hat{\boldsymbol{w}}^{(\mathrm{H})}(\boldsymbol{h}, \tau) = \boldsymbol{M}\mathrm{diag}[\boldsymbol{w}(\boldsymbol{h}, \tau)] - \boldsymbol{M}\mathrm{diag}[\boldsymbol{w}(\boldsymbol{h}, \tau)]\mathrm{diag}[\boldsymbol{h}]\boldsymbol{M}^\top, \tag{A.22}$$

$$\nabla \boldsymbol{w}^{(\mathrm{H})}(\boldsymbol{h}, \tau) = \boldsymbol{M} - \boldsymbol{M}\mathrm{diag}[\boldsymbol{h}]\boldsymbol{M}^\top, \tag{A.23}$$

$$\nabla \boldsymbol{w}^{(\mathrm{L})}(\boldsymbol{h}, \tau) = -\boldsymbol{M}\boldsymbol{M}^\top. \tag{A.24}$$

B Bifurcated solutions at group-theoretic bifurcation point of multiplicity 2 for $D_6\dot{+}(\mathbb{Z}_n \times \mathbb{Z}_n)$

We derive bifurcated solutions at a group-theoretic bifurcation point of multiplicity 2 for $D_6\dot{+}(\mathbb{Z}_n \times \mathbb{Z}_n)$ that are given in Subsection 3.3. We assume that n is divisible by 3.

B.1 Irreducible representations

A two-dimensional irreducible representation of the group $G = \langle r, s, p_1, p_2\rangle \simeq D_6\dot{+}(\mathbb{Z}_n \times \mathbb{Z}_n)$ is given by (16) as

$$T(r) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ T(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ T(p_1) = T(p_2) = \begin{pmatrix} \cos 2\pi/3 & -\sin 2\pi/3 \\ \sin 2\pi/3 & \cos 2\pi/3 \end{pmatrix}. \tag{B.1}$$

The action given in (B.1) on two-dimensional vectors, say, (w_1, w_2), can be expressed for complex variables $z = w_1 + \mathrm{i}w_2$ as

$$r: \quad z \mapsto \bar{z}, \tag{B.2}$$

$$s: \quad z \mapsto z, \tag{B.3}$$

$$p_1, p_2: \quad z \mapsto \omega z, \tag{B.4}$$

where $\omega = \exp(\mathrm{i}2\pi/3)$.

B.2 Equivariance of bifurcation equation

The bifurcation equation for the group-theoretic bifurcation point of multiplicity 2 is a two-dimensional equation over $\mathbb{R}$. This equation can be expressed as a two-dimensional complex-valued equation in complex variables as

$$F(z, \bar{z}, \tau) = \overline{F(z, \bar{z}, \tau)} = 0, \tag{B.5}$$

where $(z, \bar{z}, \tau) = (0, 0, 0)$ is assumed to correspond to the bifurcation point. We often omit τ in the subsequent derivation.

Since the group $D_6\dot{+}(\mathbb{Z}_n \times \mathbb{Z}_n)$ is generated by the four elements r, s, p_1, p_2, the equivariance of the bifurcation equation to the group $D_6\dot{+}(\mathbb{Z}_n \times \mathbb{Z}_n)$ is identical to the equivariance to the action of these four elements. Therefore, the equivariance condition (14) of the bifurcation equation (B.5) can be written as

$$r: \quad \overline{F(z, \bar{z})} = F(\bar{z}, z), \tag{B.6}$$

$$s: \quad F(z, \bar{z}) = F(z, \bar{z}), \tag{B.7}$$

$$p_1, p_2: \quad \omega F(z, \bar{z}) = F(\omega z, \overline{\omega z}). \tag{B.8}$$

We expand F as

$$F(z, \bar{z}, \tau) = \sum_{a=0} \sum_{b=0} A_{ab}(\tau) z^a \bar{z}^b. \tag{B.9}$$

The equivariance condition (B.6) with respect to r gives $A_{ab} = \overline{A_{ab}}$, i.e., A_{ab} are real. For $A_{ab} \neq 0$ in (B.9), condition (B.8) gives

$$a - b - 1 = 3p, \quad p \in \mathbb{Z}.$$

Using this relation in (B.9), we obtain the bifurcation equation (B.5) as

$$\begin{aligned} F(z,\bar{z},\tau) &= \sum_{a=0} A_{a+1,a}(\tau)\, z^{a+1}\bar{z}^{a} \\ &\quad + \sum_{p=1}\sum_{a=0} [A_{a+1+3p,a}(\tau)\, z^{a+1+3p}\bar{z}^{a} + A_{a,a-1+3p}\,(\tau)\, z^{a}\bar{z}^{a-1+3p}\,] = 0. \end{aligned} \tag{B.10}$$

Therein, $A_{a+1,a}(\tau)$, $A_{a+1+3p,a}(\tau)$, and $A_{a,a-1+3p}\,(\tau)$ are real and generically distinct from zero (because no reason exists for the disappearance of these terms).

Because $(z,\bar{z},\tau) = (0,0,0)$ corresponds to the double critical point, we have

$$A_{00}(0) = 0, \quad A_{10}(0) = 0, \quad A_{01}(0) = 0. \tag{B.11}$$

Therefore, we have

$$A_{10}(\tau) \approx A\tau \tag{B.12}$$

for some constant A, which is generically nonzero.

B.3 Bifurcated solutions

The equation (B.10) has the trivial solution $z = 0$ since each term in (B.10) vanishes if $z = \bar{z} = 0$. The nontrivial solution of (B.10) is determined from $F/z = 0$. If we put

$$\hat{F}(\rho,\theta,\tau) = \frac{F(\rho\exp(\mathrm{i}\theta), \rho\exp(-\mathrm{i}\theta), \tau)}{\rho\exp(\mathrm{i}\theta)} \quad \left(= \frac{F}{z}\right)$$

using the polar coordinates $z = w_1 + \mathrm{i}w_2 = \rho\exp(\mathrm{i}\theta)$ $(\rho \geq 0)$, then we have

$$\begin{aligned} \mathrm{Re}(\hat{F}) &= \sum_{a=0} A_{a+1,a}(\tau)\rho^{2a} \\ &\quad + \sum_{p=1}\sum_{a=0} [A_{a+1+3p,a}(\tau)\rho^{2a+3p} + A_{a,a-1+3p}\,(\tau)\,\rho^{2(a-1)+3p}\,]\cos(3p\theta), \\ \mathrm{Im}(\hat{F}) &= \sum_{p=1}\sum_{a=0} [A_{a+1+3p,a}(\tau)\rho^{2a+3p} - A_{a,a-1+3p}\,(\tau)\,\rho^{2(a-1)+3p}\,]\sin(3p\theta). \end{aligned}$$

Then the nontrivial solution of (B.10) is determined from $\mathrm{Re}(\hat{F}) = \mathrm{Im}(\hat{F}) = 0$.

Equation $\mathrm{Im}(\hat{F}) = 0$ is satisfied by

$$\theta = -\pi\frac{k-1}{3}, \quad (k = 1,\ldots,6).$$

for which $\sin(3p\theta) = \sin(-p(k-1)\pi) = 0$ and

$$\cos(3p\theta) = \cos(-p(k-1)\pi) = (-1)^{p(k-1)}. \tag{B.13}$$

By (B.13), $\mathrm{Re}(\hat{F}) = 0$ is given as

$$\begin{aligned} &\sum_{a=0} A_{a+1,a}(\tau)\rho^{2a} \\ &+ \sum_{p=1}\sum_{a=0} (-1)^{p(k-1)}\,[A_{a+1+3p,a}(\tau)\rho^{2a+3p} + A_{a,a-1+3p}\,(\tau)\,\rho^{2(a-1)+3p}\,] = 0. \end{aligned} \tag{B.14}$$

By $A_{10}(\tau) \approx A\tau$ in (B.12), the leading part of this equation is

$$A\tau + (-1)^k A_{02}(0)\rho = 0,$$

where $A_{02}(0) \neq 0$ (generically). Consequently, a solution of the form $\rho = \mathrm{O}(\tau)$ exists, which we set

$$\rho = \begin{cases} A\tau/A_{02}(0), & (k = 1, 3, 5), \\ -A\tau/A_{02}(0), & (k = 2, 4, 6). \end{cases}$$

From the actions (B.2)–(B.4), we can determine the symmetry of the solutions $z = \rho$ for $k = 1$ and $z = -\rho$ for $k = 4$ as

$$\Sigma(z) = \langle r, s, p_1^2 p_2, p_1^{-1} p_2 \rangle,$$

which simplifies to $\langle r, s, p_1^2 p_2 \rangle$ for $n = 3$. Since other solutions are obtainable from z as $T(p_1)z$ or $T(p_1^2)z$, the symmetry of the other solutions is

$$p_1 \cdot \Sigma(z) \cdot p_1^{-1}, \quad p_1^2 \cdot \Sigma(z) \cdot p_1^{-2}.$$

C Bifurcated solutions at group-theoretic bifurcation point of multiplicity 3 for $\mathrm{D}_6\dot{+}(\mathbb{Z}_n \times \mathbb{Z}_n)$

We derive bifurcated solutions at a group-theoretic bifurcation point of multiplicity 3 for $\mathrm{D}_6\dot{+}(\mathbb{Z}_n \times \mathbb{Z}_n)$ that are given in Subsection 3.3. We assume that n is divisible by 2.

C.1 Irreducible representations

A three-dimensional irreducible representation[22] of the group $G = \langle r, s, p_1, p_2 \rangle \simeq \mathrm{D}_6\dot{+}(\mathbb{Z}_n \times \mathbb{Z}_n)$ is given by (21) and (22) as

$$T(r) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad T(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \tag{C.1}$$

$$T(p_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T(p_2) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{C.2}$$

For this irreducible representation, the action of $G = \langle r, s, p_1, p_2 \rangle$ on $\boldsymbol{w} = (w_1, w_2, w_3)^\top$ is given as

$$\begin{array}{rlll} r: & w_1 \mapsto w_2, & w_2 \mapsto w_3, & w_3 \mapsto w_1, \\ s: & w_1 \mapsto w_1, & w_2 \mapsto w_3, & w_3 \mapsto w_2, \\ p_1: & w_1 \mapsto w_1, & w_2 \mapsto -w_2, & w_3 \mapsto -w_3, \\ p_2: & w_1 \mapsto -w_1, & w_2 \mapsto w_2, & w_3 \mapsto -w_3. \end{array}$$

C.2 Equivariance of bifurcation equation

The bifurcation equation for the group-theoretic bifurcation point of multiplicity 3 is a three-dimensional equation over $\mathbb{R}$. This equation can be expressed as

$$F_1(w_1, w_2, w_3, \tau) = F_2(w_1, w_2, w_3, \tau) = F_3(w_1, w_2, w_3, \tau) = 0. \tag{C.3}$$

It is assumed that $(w_1, w_2, w_3, \tau) = (0, 0, 0, 0)$ corresponds to the triple bifurcation point. We often omit τ in the subsequent derivation.

Since the group $\mathrm{D}_6\dot{+}(\mathbb{Z}_n \times \mathbb{Z}_n)$ is generated by the four elements r, s, p_1, p_2, the equivariance of the bifurcation equation to the group $\mathrm{D}_6\dot{+}(\mathbb{Z}_n \times \mathbb{Z}_n)$ is identical to the equivariance to the action of these four elements. Therefore, the equivariance condition (14) of the bifurcation equation (C.3) can be written as

$$\begin{aligned} F_1(w_1, w_2, w_3) &= F_1(w_1, -w_2, -w_3), && \text{(C.4)} \\ -F_1(w_1, w_2, w_3) &= F_1(-w_1, w_2, -w_3), && \text{(C.5)} \\ F_1(w_1, w_2, w_3) &= F_1(w_1, w_3, w_2), && \text{(C.6)} \\ F_2(w_1, w_2, w_3) &= F_1(w_2, w_3, w_1), && \text{(C.7)} \\ F_3(w_1, w_2, w_3) &= F_1(w_3, w_1, w_2). && \text{(C.8)} \end{aligned}$$

[22] This irreducible representation corresponds to that denoted as (3, 1) in Ikeda and Murota (2010).

We expand F_1 as

$$F_1(w_1, w_2, w_3, \tau) = \sum_{a=0} \sum_{b=0} \sum_{c=0} A_{abc}(\tau) {w_1}^a {w_2}^b {w_3}^c. \tag{C.9}$$

For the nonzero terms in (C.9), conditions (C.4) and (C.5) give

$$(-1)^{b+c} = (-1)^{a+c-1} = 1,$$

which means that $(a, b, c) = (\text{odd}, \text{even}, \text{even})$ or (even, odd, odd). Therefore, F_1 reduces to

$$\begin{aligned} F_1(w_1, w_2, w_3, \tau) &= \sum_{a:\text{odd}\geq 1} \sum_{b:\text{even}\geq 0} \sum_{c:\text{even}\geq 0} A_{abc}(\tau)\, {w_1}^a {w_2}^b {w_3}^c \\ &\quad + \sum_{a:\text{even}\geq 0} \sum_{b:\text{odd}\geq 1} \sum_{c:\text{odd}\geq 1} A_{abc}(\tau)\, {w_1}^a {w_2}^b {w_3}^c \\ &= w_1 \sum_{a=0} \sum_{b=0} \sum_{c=0} A_{2a+1,2b,2c}(\tau)\, {w_1}^{2a} {w_2}^{2b} {w_3}^{2c} \\ &\quad + w_2 w_3 \sum_{a=0} \sum_{b=0} \sum_{c=0} A_{2a,2b+1,2c+1}(\tau)\, {w_1}^{2a} {w_2}^{2b} {w_3}^{2c}. \end{aligned} \tag{C.10}$$

The condition (C.6) gives

$$A_{abc}(\tau) = A_{acb}(\tau).$$

The expressions in (C.7) and (C.8) with the forms of F_1 in (C.10) give F_2 and F_3 as

$$\begin{aligned} F_2(w_1, w_2, w_3, \tau) &= w_2 \sum_{a=0} \sum_{b=0} \sum_{c=0} A_{2a+1,2b,2c}(\tau)\, {w_2}^{2a} {w_3}^{2b} {w_1}^{2c} \\ &\quad + w_3 w_1 \sum_{a=0} \sum_{b=0} \sum_{c=0} A_{2a,2b+1,2c+1}(\tau)\, {w_2}^{2a} {w_3}^{2b} {w_1}^{2c}, \end{aligned} \tag{C.11}$$

$$\begin{aligned} F_3(w_1, w_2, w_3, \tau) &= w_3 \sum_{a=0} \sum_{b=0} \sum_{c=0} A_{2a+1,2b,2c}(\tau)\, {w_3}^{2a} {w_1}^{2b} {w_2}^{2c} \\ &\quad + w_1 w_2 \sum_{a=0} \sum_{b=0} \sum_{c=0} A_{2a,2b+1,2c+1}(\tau)\, {w_3}^{2a} {w_1}^{2b} {w_2}^{2c}. \end{aligned} \tag{C.12}$$

Because $(w_1, w_2, w_3, \tau) = (0, 0, 0, 0)$ corresponds to the triple critical point and the Jacobian matrix

$$(\partial F_i / \partial w_j \mid i, j = 1, 2, 3)$$

at this point is equal to $A_{100}(0) I_3$, we have

$$A_{100}(0) = 0.$$

Therefore, we have

$$A_{100}(\tau) \approx A\tau \tag{C.13}$$

for some constant A, which is generically nonzero.

From (C.10), (C.11), and (C.12), the system of bifurcation equation is expressed as

$$
\begin{aligned}
F_1(w_1, w_2, w_3, \tau) &= w_1 \sum_{a=0} \sum_{b=0} \sum_{c=0} A_{2a+1,2b,2c}(\tau)\, {w_1}^{2a} {w_2}^{2b} {w_3}^{2c} \\
&\quad + w_2 w_3 \sum_{a=0} \sum_{b=0} \sum_{c=0} A_{2a,2b+1,2c+1}(\tau)\, {w_1}^{2a} {w_2}^{2b} {w_3}^{2c} = 0, \qquad \text{(C.14)} \\
F_2(w_1, w_2, w_3, \tau) &= w_2 \sum_{a=0} \sum_{b=0} \sum_{c=0} A_{2a+1,2b,2c}(\tau)\, {w_2}^{2a} {w_3}^{2b} {w_1}^{2c} \\
&\quad + w_3 w_1 \sum_{a=0} \sum_{b=0} \sum_{c=0} A_{2a,2b+1,2c+1}(\tau)\, {w_2}^{2a} {w_3}^{2b} {w_1}^{2c} = 0, \qquad \text{(C.15)} \\
F_3(w_1, w_2, w_3, \tau) &= w_3 \sum_{a=0} \sum_{b=0} \sum_{c=0} A_{2a+1,2b,2c}(\tau)\, {w_3}^{2a} {w_1}^{2b} {w_2}^{2c} \\
&\quad + w_1 w_2 \sum_{a=0} \sum_{b=0} \sum_{c=0} A_{2a,2b+1,2c+1}(\tau)\, {w_3}^{2a} {w_1}^{2b} {w_2}^{2c} = 0. \qquad \text{(C.16)}
\end{aligned}
$$

C.3 Bifurcated solutions

For the bifurcation equations (C.14)–(C.16) above, we seek solutions with $|w_1| = |w_2| = |w_3|$, which have $\langle p_1^2, p_2^2 \rangle$- or higher symmetry by (C.2). Those bifurcated solutions are relevant for our purpose since they possess symmetry corresponding to Christaller's $k = 4$ system.

With $|w_1| = |w_2| = |w_3|$, the bifurcation equations in (C.14)–(C.16) become identical and read as

$$
\begin{aligned}
&\sum_{a=0} \sum_{b=0} \sum_{c=0} A_{2a+1,2b,2c}(\tau)\, {w_1}^{2(a+b+c)} \\
&\qquad + \alpha |w_1| \sum_{a=0} \sum_{b=0} \sum_{c=0} A_{2a,2b+1,2c+1}(\tau)\, {w_1}^{2(a+b+c)} = 0, \qquad \text{(C.17)}
\end{aligned}
$$

where $\alpha = \mathrm{sign}(w_1 w_2 w_3)$. By (C.13), the leading part of (C.17) is

$$
A\tau + \alpha A_{011}(0)|w_1| = 0,
$$

where $A_{011}(0) \neq 0$ (generically). Consequently, a solution of the form $w_1 = \mathrm{O}(\tau)$ to the equation (C.17) exists, which we set

$$
w_1 = \begin{cases} \Phi_1(\tau) & \text{for } \alpha = 1, \\ \Phi_2(\tau) & \text{for } \alpha = -1. \end{cases}
$$

Four bifurcated paths—eight half-branches—exist, which are associated with

$$
\begin{aligned}
&(w_1, w_2, w_3) \\
&= (\Phi_1(\tau), \Phi_1(\tau), \Phi_1(\tau)), \\
&\quad (-\Phi_1(\tau), -\Phi_1(\tau), \Phi_1(\tau)),\ (-\Phi_1(\tau), \Phi_1(\tau), -\Phi_1(\tau)),\ (\Phi_1(\tau), -\Phi_1(\tau), -\Phi_1(\tau)), \\
&\quad (-\Phi_2(\tau), -\Phi_2(\tau), -\Phi_2(\tau)), \\
&\quad (\Phi_2(\tau), \Phi_2(\tau), -\Phi_2(\tau)),\ (\Phi_2(\tau), -\Phi_2(\tau), \Phi_2(\tau)),\ (-\Phi_2(\tau), \Phi_2(\tau), \Phi_2(\tau)).
\end{aligned}
$$

For symmetry of the solutions, we have

$$
\Sigma(\boldsymbol{w}) = \langle r, s, p_1^2, p_2^2 \rangle
$$

for $\boldsymbol{w} = \pm(\Phi_i(\tau), \Phi_i(\tau), \Phi_i(\tau))^\top$ $(i = 1, 2)$. Since other solutions are obtainable from $\boldsymbol{w}$ as $T(p_1)\boldsymbol{w}$, $T(p_2)\boldsymbol{w}$, or $T(p_1p_2)\boldsymbol{w}$, the symmetry of the other solutions is obtained as

$$p_1 \cdot \Sigma(\boldsymbol{w}) \cdot p_1^{-1}, \quad p_2 \cdot \Sigma(\boldsymbol{w}) \cdot p_2^{-1}, \quad p_1p_2 \cdot \Sigma(\boldsymbol{w}) \cdot (p_1p_2)^{-1}.$$

By starting with the assumption $|w_1| = |w_2| = |w_3|$, we have not excluded the possibility of solutions of other types with $w_1w_2w_3 \neq 0$.

D Bifurcated solutions at group-theoretic bifurcation point of multiplicity 12 for $\mathrm{D}_6\dot{+}(\mathbb{Z}_n \times \mathbb{Z}_n)$

We derive bifurcated solutions at a group-theoretic bifurcation point of multiplicity 12 for $\mathrm{D}_6\dot{+}(\mathbb{Z}_n \times \mathbb{Z}_n)$ that are given in Subsection 3.3.

D.1 Irreducible representations

The group $\mathrm{D}_6\dot{+}(\mathbb{Z}_n \times \mathbb{Z}_n)$, with $n \geq 6$, has 12-dimensional irreducible representations. We can designate them by (k, ℓ) with

$$1 \leq \ell \leq k-1, \quad 2k + \ell \leq n-1, \tag{D.1}$$

where the irreducible representation (k, ℓ) is defined as

$$T^{(k,\ell)}(r) = \left(\begin{array}{ccc|ccc} & & S & & & \\ S & & & & & \\ & S & & & & \\ \hline & & & & S & \\ & & & & & S \\ & & & S & & \end{array} \right), \quad T^{(k,\ell)}(s) = \left(\begin{array}{ccc|ccc} & & & I & & \\ & & & & I & \\ & & & & & I \\ \hline I & & & & & \\ & I & & & & \\ & & I & & & \end{array} \right), \tag{D.2}$$

$$T^{(k,\ell)}(p_1) = \left(\begin{array}{ccc|ccc} R^k & & & & & \\ & R^\ell & & & & \\ & & R^{-k-\ell} & & & \\ \hline & & & R^k & & \\ & & & & R^\ell & \\ & & & & & R^{-k-\ell} \end{array} \right), T^{(k,\ell)}(p_2) = \left(\begin{array}{ccc|ccc} R^\ell & & & & & \\ & R^{-k-\ell} & & & & \\ & & R^k & & & \\ \hline & & & R^{-k-\ell} & & \\ & & & & R^k & \\ & & & & & R^\ell \end{array} \right) \tag{D.3}$$

with

$$R = \begin{pmatrix} \cos 2\pi/n & -\sin 2\pi/n \\ \sin 2\pi/n & \cos 2\pi/n \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The action given in (D.2) and (D.3) on 12-dimensional vectors, say, $(w_1, \cdots, w_{12})$, can be expressed for complex variables $z_j = w_{2j-1} + \mathrm{i}w_{2j}$ $(j = 1, \ldots, 6)$ as

$$r: \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ \overline{z_1} \\ \overline{z_2} \\ \overline{z_3} \\ \overline{z_4} \\ \overline{z_5} \\ \overline{z_6} \end{pmatrix} \mapsto \begin{pmatrix} \overline{z_3} \\ \overline{z_1} \\ \overline{z_2} \\ \overline{z_5} \\ \overline{z_6} \\ \overline{z_4} \\ z_3 \\ z_1 \\ z_2 \\ z_5 \\ z_6 \\ z_4 \end{pmatrix}, \quad s: \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ \overline{z_1} \\ \overline{z_2} \\ \overline{z_3} \\ \overline{z_4} \\ \overline{z_5} \\ \overline{z_6} \end{pmatrix} \mapsto \begin{pmatrix} z_4 \\ z_5 \\ z_6 \\ z_1 \\ z_2 \\ z_3 \\ \overline{z_4} \\ \overline{z_5} \\ \overline{z_6} \\ \overline{z_1} \\ \overline{z_2} \\ \overline{z_3} \end{pmatrix}, \tag{D.4}$$

$$
p_1 : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ \overline{z_1} \\ \overline{z_2} \\ \overline{z_3} \\ \overline{z_4} \\ \overline{z_5} \\ \overline{z_6} \end{pmatrix} \mapsto \begin{pmatrix} \omega^{k}\, z_1 \\ \omega^{\ell}\, z_2 \\ \omega^{-k-\ell}\, z_3 \\ \omega^{k}\, z_4 \\ \omega^{\ell}\, z_5 \\ \omega^{-k-\ell}\, z_6 \\ \omega^{-k}\, \overline{z_1} \\ \omega^{-\ell}\, \overline{z_2} \\ \omega^{k+\ell}\, \overline{z_3} \\ \omega^{-k}\, \overline{z_4} \\ \omega^{-\ell}\, \overline{z_5} \\ \omega^{k+\ell}\, \overline{z_6} \end{pmatrix}, \quad p_2 : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ \overline{z_1} \\ \overline{z_2} \\ \overline{z_3} \\ \overline{z_4} \\ \overline{z_5} \\ \overline{z_6} \end{pmatrix} \mapsto \begin{pmatrix} \omega^{\ell}\, z_1 \\ \omega^{-k-\ell}\, z_2 \\ \omega^{k}\, z_3 \\ \omega^{-k-\ell}\, z_4 \\ \omega^{k}\, z_5 \\ \omega^{\ell}\, z_6 \\ \omega^{-\ell}\, \overline{z_1} \\ \omega^{k+\ell}\, \overline{z_2} \\ \omega^{-k}\, \overline{z_3} \\ \omega^{k+\ell}\, \overline{z_4} \\ \omega^{-k}\, \overline{z_5} \\ \omega^{-\ell}\, \overline{z_6} \end{pmatrix}, \tag{D.5}
$$

where $\omega = \exp(\mathrm{i}2\pi/n)$.

D.2 Equivariance of bifurcation equation

We consider a 12-dimensional bifurcation equation associated with the irreducible representation (k, ℓ) of the group $\mathrm{D}_6\dot{+}(\mathbb{Z}_n \times \mathbb{Z}_n)$. Our main concern lies in the case where n is a multiple of 7, i.e., $n = 7m$ for an integer $m \geq 1$, and the irreducible representation is $(k, \ell) = (2m, m)$. We treat general n and (k, ℓ) to the greatest degree possible.

The bifurcation equation for the group-theoretic bifurcation point of multiplicity 12 is a 12-dimensional equation over $\mathbb{R}$. This equation can be expressed as a 6-dimensional complex-valued equation in complex variables as

$$
F_i(z_1, \ldots, z_6, \overline{z_1}, \ldots, \overline{z_6}, \tau) = \overline{F_i(z_1, \ldots, z_6, \overline{z_1}, \ldots, \overline{z_6}, \tau)} = 0, \qquad i = 1, \ldots, 6, \tag{D.6}
$$

where

$$
(z_1, \ldots, z_6, \overline{z_1}, \ldots, \overline{z_6}, \tau) = (0, \ldots, 0)
$$

is assumed to correspond to the bifurcation point. We often omit τ in the subsequent derivation.

Since the group $\mathrm{D}_6\dot{+}(\mathbb{Z}_n \times \mathbb{Z}_n)$ is generated by the four elements r, s, p_1, p_2, the equivariance of the bifurcation equation to the group $\mathrm{D}_6\dot{+}(\mathbb{Z}_n \times \mathbb{Z}_n)$ is identical to the equivariance to the action of these four elements. Therefore, the equivariance condition of the bifurcation equation (D.6) can be written as

$$
\begin{aligned}
r:\quad \overline{F_3(z_1,\cdots,\overline{z_6})} &= F_1(\overline{z_3},\overline{z_1},\overline{z_2},\overline{z_5},\overline{z_6},\overline{z_4},z_3,z_1,z_2,z_5,z_6,z_4), && \text{(D.7)}\\
\overline{F_1(z_1,\cdots,\overline{z_6})} &= F_2(\overline{z_3},\overline{z_1},\overline{z_2},\overline{z_5},\overline{z_6},\overline{z_4},z_3,z_1,z_2,z_5,z_6,z_4), && \text{(D.8)}\\
\overline{F_2(z_1,\cdots,\overline{z_6})} &= F_3(\overline{z_3},\overline{z_1},\overline{z_2},\overline{z_5},\overline{z_6},\overline{z_4},z_3,z_1,z_2,z_5,z_6,z_4), && \text{(D.9)}\\
\overline{F_5(z_1,\cdots,\overline{z_6})} &= F_4(\overline{z_3},\overline{z_1},\overline{z_2},\overline{z_5},\overline{z_6},\overline{z_4},z_3,z_1,z_2,z_5,z_6,z_4), && \text{(D.10)}\\
\overline{F_6(z_1,\cdots,\overline{z_6})} &= F_5(\overline{z_3},\overline{z_1},\overline{z_2},\overline{z_5},\overline{z_6},\overline{z_4},z_3,z_1,z_2,z_5,z_6,z_4), && \text{(D.11)}\\
\overline{F_4(z_1,\cdots,\overline{z_6})} &= F_6(\overline{z_3},\overline{z_1},\overline{z_2},\overline{z_5},\overline{z_6},\overline{z_4},z_3,z_1,z_2,z_5,z_6,z_4), && \text{(D.12)}\\
F_3(z_1,\cdots,\overline{z_6}) &= \overline{F_1(\overline{z_3},\overline{z_1},\overline{z_2},\overline{z_5},\overline{z_6},\overline{z_4},z_3,z_1,z_2,z_5,z_6,z_4)}, && \text{(D.13)}\\
F_1(z_1,\cdots,\overline{z_6}) &= \overline{F_2(\overline{z_3},\overline{z_1},\overline{z_2},\overline{z_5},\overline{z_6},\overline{z_4},z_3,z_1,z_2,z_5,z_6,z_4)}, && \text{(D.14)}\\
F_2(z_1,\cdots,\overline{z_6}) &= \overline{F_3(\overline{z_3},\overline{z_1},\overline{z_2},\overline{z_5},\overline{z_6},\overline{z_4},z_3,z_1,z_2,z_5,z_6,z_4)}, && \text{(D.15)}\\
F_5(z_1,\cdots,\overline{z_6}) &= \overline{F_4(\overline{z_3},\overline{z_1},\overline{z_2},\overline{z_5},\overline{z_6},\overline{z_4},z_3,z_1,z_2,z_5,z_6,z_4)}, && \text{(D.16)}\\
F_6(z_1,\cdots,\overline{z_6}) &= \overline{F_5(\overline{z_3},\overline{z_1},\overline{z_2},\overline{z_5},\overline{z_6},\overline{z_4},z_3,z_1,z_2,z_5,z_6,z_4)}, && \text{(D.17)}\\
F_4(z_1,\cdots,\overline{z_6}) &= \overline{F_6(\overline{z_3},\overline{z_1},\overline{z_2},\overline{z_5},\overline{z_6},\overline{z_4},z_3,z_1,z_2,z_5,z_6,z_4)}; && \text{(D.18)}
\end{aligned}
$$

$$
\begin{aligned}
s:\quad F_{i+3}(z_1,\cdots,\overline{z_6}) &= F_i(z_4,z_5,z_6,z_1,z_2,z_3,\overline{z_4},\overline{z_5},\overline{z_6},\overline{z_1},\overline{z_2},\overline{z_3}),\\
F_i(z_1,\cdots,\overline{z_6}) &= F_{i+3}(z_4,z_5,z_6,z_1,z_2,z_3,\overline{z_4},\overline{z_5},\overline{z_6},\overline{z_1},\overline{z_2},\overline{z_3}),\\
\overline{F_{i+3}(z_1,\cdots,\overline{z_6})} &= \overline{F_i(z_4,z_5,z_6,z_1,z_2,z_3,\overline{z_4},\overline{z_5},\overline{z_6},\overline{z_1},\overline{z_2},\overline{z_3})},\\
\overline{F_i(z_1,\cdots,\overline{z_6})} &= \overline{F_{i+3}(z_4,z_5,z_6,z_1,z_2,z_3,\overline{z_4},\overline{z_5},\overline{z_6},\overline{z_1},\overline{z_2},\overline{z_3})},\\
&\qquad\qquad i=1,2,3; \qquad \text{(D.19)}
\end{aligned}
$$

$$
\begin{aligned}
p_j:\quad \omega_{ji}F_i(z_1,\cdots,\overline{z_6}) &= F_i(\omega_{j1}z_1,\ldots,\omega_{j6}z_6,\overline{\omega_{j1}z_1},\ldots,\overline{\omega_{j6}z_6}),\\
&\qquad j=1,2;\ i=1,\ldots,6, \qquad \text{(D.20)}
\end{aligned}
$$

where

$$
\begin{aligned}
(\omega_{11},\ldots,\omega_{16}) &= (\omega^k,\omega^\ell,\omega^{-k-\ell},\omega^k,\omega^\ell,\omega^{-k-\ell}),\\
(\omega_{21},\ldots,\omega_{26}) &= (\omega^\ell,\omega^{-k-\ell},\omega^k,\omega^{-k-\ell},\omega^k,\omega^\ell).
\end{aligned}
$$

We expand F_1 as

$$
\begin{aligned}
F_1(z_1,\cdots,\overline{z_6}) &= \sum_{a=0}\sum_{b=0}\sum_{c=0}\sum_{d=0}\sum_{e=0}\sum_{g=0}\sum_{h=0}\sum_{i=0}\sum_{j=0}\sum_{s=0}\sum_{t=0}\sum_{u=0}\\
&\quad A_{abcdeghijstu}(\tau)z_1^a z_2^b z_3^c z_4^d z_5^e z_6^g \overline{z_1}^h \overline{z_2}^i \overline{z_3}^j \overline{z_4}^s \overline{z_5}^t \overline{z_6}^u. \qquad \text{(D.21)}
\end{aligned}
$$

Since $(z_1,z_2,z_3,z_4,z_5,z_6,\overline{z_1},\overline{z_2},\overline{z_3},\overline{z_4},\overline{z_5},\overline{z_6},\tau) = (0,0,0,0,0,0,0,0,0,0,0,0,0)$ corresponds to the bifurcation point of multiplicity 12, we have

$$
\begin{aligned}
&A_{000000000000}(0) = 0, && \text{(D.22)}\\
&A_{100000000000}(0) = A_{010000000000}(0) = \cdots = A_{000000000001}(0). && \text{(D.23)}
\end{aligned}
$$

The equivariance conditions (D.7)–(D.9) with respect to r give

$$
\begin{aligned}
F_1(z_1,\cdots,\overline{z_6}) &= \overline{F_2(\overline{z_3},\overline{z_1},\overline{z_2},\overline{z_5},\overline{z_6},\overline{z_4},z_3,z_1,z_2,z_5,z_6,z_4)}\\
&= F_3(z_2,z_3,z_1,z_6,z_4,z_5,\overline{z_2},\overline{z_3},\overline{z_1},\overline{z_6},\overline{z_4},\overline{z_5})\\
&= \overline{F_1(\overline{z_1},\overline{z_2},\overline{z_3},\overline{z_4},\overline{z_5},\overline{z_6},z_1,z_2,z_3,z_4,z_5,z_6)},
\end{aligned}
$$

from which we see that $A_{ab\cdots tu}$ are real. Then $F_2, \cdots, F_6$ are obtained from the equivariance conditions (D.7)–(D.18) and (D.19) with respect to r and s as

$$
\begin{aligned}
F_2(z_1,\cdots,\overline{z_6}) &= F_1(z_2,z_3,z_1,z_6,z_4,z_5,\overline{z_2},\overline{z_3},\overline{z_1},\overline{z_6},\overline{z_4},\overline{z_5}), && \text{(D.24)}\\
F_3(z_1,\cdots,\overline{z_6}) &= F_1(z_3,z_1,z_2,z_5,z_6,z_4,\overline{z_3},\overline{z_1},\overline{z_2},\overline{z_5},\overline{z_6},\overline{z_4}), && \text{(D.25)}\\
F_4(z_1,\cdots,\overline{z_6}) &= F_1(z_4,z_5,z_6,z_1,z_2,z_3,\overline{z_4},\overline{z_5},\overline{z_6},\overline{z_1},\overline{z_2},\overline{z_3}), && \text{(D.26)}\\
F_5(z_1,\cdots,\overline{z_6}) &= F_1(z_6,z_4,z_5,z_2,z_3,z_1,\overline{z_6},\overline{z_4},\overline{z_5},\overline{z_2},\overline{z_3},\overline{z_1}), && \text{(D.27)}\\
F_6(z_1,\cdots,\overline{z_6}) &= F_1(z_5,z_6,z_4,z_3,z_1,z_2,\overline{z_5},\overline{z_6},\overline{z_4},\overline{z_3},\overline{z_1},\overline{z_2}). && \text{(D.28)}
\end{aligned}
$$

For the index $(a,b,\dots,t,u)$ of a nonvanishing coefficient $A_{ab\cdots tu}$, the equivariance conditions (D.20) with respect to p_1 and p_2 yield

$$
\begin{aligned}
&k(a-h)+\ell(b-i)-(k+\ell)(c-j)+k(d-s)+\ell(e-t)-(k+\ell)(g-u)\\
&= k+np', \quad p' \in \mathbb{Z}, && \text{(D.29)}\\
&\ell(a-h)-(k+\ell)(b-i)+k(c-j)-(k+\ell)(d-s)+k(e-t)+\ell(g-u)\\
&= \ell+nq', \quad q' \in \mathbb{Z}. && \text{(D.30)}
\end{aligned}
$$

In what follows, we assume that n is a multiple of 7, i.e., $n = 7m$ for an integer $m \geq 1$, and the irreducible representation is $(k,\ell) = (2m,m)$. The condition (D.1) is met by $(k,\ell) = (2m,m)$. Then (D.29) and (D.30) above reduce to

$$2(a-h)+(b-i)-3(c-j)+2(d-s)+(e-t)-3(g-u) = 2+7p', \quad p' \in \mathbb{Z}, \tag{D.31}$$

$$(a-h)-3(b-i)+2(c-j)-3(d-s)+2(e-t)+(g-u) = 1+7q', \quad q' \in \mathbb{Z}, \tag{D.32}$$

which are equivalent to

$$
\begin{aligned}
\text{(D.31)}\times 3+\text{(D.32)}\times 2: &\quad (a-h)-3(b-i)+2(c-j) = 1+7p, \quad p \in \mathbb{Z}, && \text{(D.33)}\\
\text{(D.31)}-\text{(D.32)}\times 2: &\quad (d-s)-3(e-t)+2(g-u) = 7q, \quad q \in \mathbb{Z}. && \text{(D.34)}
\end{aligned}
$$

Accordingly, we define

$$
\begin{aligned}
P &= \{(a,b,c,h,i,j) \mid (1,-3,2)\cdot(a-h,b-i,c-j) \equiv 1 \mod 7\}, && \text{(D.35)}\\
Q &= \{(d,e,g,s,t,u) \mid (1,-3,2)\cdot(d-s,e-t,g-u) \equiv 0 \mod 7\}, && \text{(D.36)}
\end{aligned}
$$

where "·" denotes the inner product of vectors. It is noteworthy that

$$(0,0,0,0,0,0) \notin P. \tag{D.37}$$

Use of (D.33) and (D.34) in (D.21) yields

$$F_1(z_1,\cdots,\overline{z_6}) = \hat{F}(z_1,\cdots,\overline{z_6}), \tag{D.38}$$

in which

$$\hat{F}(z_1,\cdots,\overline{z_6}) = \sum_P \sum_Q A_{abcdeghijstu}(\tau) z_1^a z_2^b z_3^c z_4^d z_5^e z_6^g \overline{z_1}^h \overline{z_2}^i \overline{z_3}^j \overline{z_4}^s \overline{z_5}^t \overline{z_6}^u, \tag{D.39}$$

where the summation is taken over all $(a, b, c, h, i, j) \in P$ and $(d, e, g, s, t, u) \in Q$. Use of (D.38) in (D.24)–(D.28) then gives the bifurcation equation

$$
\begin{aligned}
F_1(z_1, \cdots, \overline{z_6}) &= \hat{F}(z_1, z_2, z_3, z_4, z_5, z_6, \overline{z_1}, \overline{z_2}, \overline{z_3}, \overline{z_4}, \overline{z_5}, \overline{z_6}) = 0, && \text{(D.40)}\\
F_2(z_1, \cdots, \overline{z_6}) &= \hat{F}(z_2, z_3, z_1, z_6, z_4, z_5, \overline{z_2}, \overline{z_3}, \overline{z_1}, \overline{z_6}, \overline{z_4}, \overline{z_5}) = 0, && \text{(D.41)}\\
F_3(z_1, \cdots, \overline{z_6}) &= \hat{F}(z_3, z_1, z_2, z_5, z_6, z_4, \overline{z_3}, \overline{z_1}, \overline{z_2}, \overline{z_5}, \overline{z_6}, \overline{z_4}) = 0, && \text{(D.42)}\\
F_4(z_1, \cdots, \overline{z_6}) &= \hat{F}(z_4, z_5, z_6, z_1, z_2, z_3, \overline{z_4}, \overline{z_5}, \overline{z_6}, \overline{z_1}, \overline{z_2}, \overline{z_3}) = 0, && \text{(D.43)}\\
F_5(z_1, \cdots, \overline{z_6}) &= \hat{F}(z_6, z_4, z_5, z_2, z_3, z_1, \overline{z_6}, \overline{z_4}, \overline{z_5}, \overline{z_2}, \overline{z_3}, \overline{z_1}) = 0, && \text{(D.44)}\\
F_6(z_1, \cdots, \overline{z_6}) &= \hat{F}(z_5, z_6, z_4, z_3, z_1, z_2, \overline{z_5}, \overline{z_6}, \overline{z_4}, \overline{z_3}, \overline{z_1}, \overline{z_2}) = 0. && \text{(D.45)}
\end{aligned}
$$

D.3 Bifurcated solutions

For the bifurcation equation (D.40)–(D.45) above, we show the presence of bifurcated solutions such that

$$|z_1| = |z_2| = |z_3|, \quad z_4 = z_5 = z_6 = 0. \tag{D.46}$$

Such solutions have $\langle p_1^3 p_2, p_1^{-1} p_2^2\rangle$- or higher symmetry by (D.5). As their conjugate solutions, there also exist bifurcated solutions with

$$z_1 = z_2 = z_3 = 0, \quad |z_4| = |z_5| = |z_6|, \tag{D.47}$$

which have $\langle p_1 p_2^3, p_1^2 p_2^{-1}\rangle$- or higher symmetry. Although we do not exclude the possibility of other bifurcated solutions, those bifurcated solutions are sufficient for our purpose since they possess the symmetry that corresponds to Christaller's $k = 7$ system.

We first search for solutions of the form $|z_1| = |z_2| = |z_3|$ and $z_4 = z_5 = z_6 = 0$ in (D.46). Such solutions satisfy $F_4 = F_5 = F_6 = 0$ since, by (D.37), we have $(a, b, c, h, i, j) \neq (0, 0, 0, 0, 0, 0)$ in the expression (D.39) for $\hat{F}$, which implies, by (D.43)–(D.45), that each term of F_4, F_5, and F_6 contains z_4, z_5, z_6, $\overline{z_4}$, $\overline{z_5}$, or $\overline{z_6}$.

To find the solutions for $F_1 = F_2 = F_3 = 0$, we set

$$z_j = \rho \exp(\mathrm{i}\theta_j), \quad (j = 1, 2, 3).$$

Then from (D.40)–(D.42) with (D.39), we obtain

$$
\begin{aligned}
F_1 &= \sum_P A_{abc000hij000}(\tau) z_1^a z_2^b z_3^c \overline{z_1}^h \overline{z_2}^i \overline{z_3}^j \\
&= \sum_P A_{abc000hij000}(\tau) \rho^{a+b+c+h+i+j} \exp \mathrm{i}[(\theta_1, \theta_2, \theta_3) \cdot (a-h, b-i, c-j)], \\
F_2 &= \sum_P A_{abc000hij000}(\tau) z_2^a z_3^b z_1^c \overline{z_2}^h \overline{z_3}^i \overline{z_1}^j \\
&= \sum_P A_{abc000hij000}(\tau) \rho^{a+b+c+h+i+j} \exp \mathrm{i}[(\theta_2, \theta_3, \theta_1) \cdot (a-h, b-i, c-j)], \\
F_3 &= \sum_P A_{abc000hij000}(\tau) z_3^a z_1^b z_2^c \overline{z_3}^h \overline{z_1}^i \overline{z_2}^j \\
&= \sum_P A_{abc000hij000}(\tau) \rho^{a+b+c+h+i+j} \exp \mathrm{i}[(\theta_3, \theta_1, \theta_2) \cdot (a-h, b-i, c-j)].
\end{aligned}
$$

If we choose

$$(\theta_1, \theta_2, \theta_3) = \frac{2\pi k}{7}(1, -3, 2), \qquad (k = 0, 1, \ldots, 6), \tag{D.48}$$

then we have

$$(\theta_1, \theta_2, \theta_3)\cdot(a-h, b-i, c-j) = \frac{2\pi k}{7}(1, -3, 2)\cdot(a-h, b-i, c-j) \equiv \frac{2\pi k}{7} = \theta_1 \mod 2\pi$$

by (D.35). Therefore,

$$F_1 = \rho \exp(\mathrm{i}\theta_1) \sum_{(a,b,c,h,i,j)\in P} A_{abc000hij000}(\tau)\rho^{a+b+c+h+i+j-1}.$$

For F_2 we note

$$(-3) \times (1, -3, 2) = (-3, 9, -6) \equiv (-3, 2, 1) \mod 7$$

to obtain

$$\begin{aligned}
&(\theta_2, \theta_3, \theta_1) \cdot (a-h, b-i, c-j) \\
&= \frac{2\pi k}{7} \times (-3, 2, 1) \cdot (a-h, b-i, c-j) \\
&\equiv \frac{2\pi k}{7} \times (-3) \times [(1, -3, 2) \cdot (a-h, b-i, c-j)] \\
&\equiv \frac{2\pi k}{7} \times (-3) = \theta_2 \mod 2\pi.
\end{aligned}$$

Therefore,

$$F_2 = \rho \exp(\mathrm{i}\theta_2) \sum_{(a,b,c,h,i,j)\in P} A_{abc000hij000}(\tau)\rho^{a+b+c+h+i+j-1}.$$

Similarly, for F_3 we note

$$2 \times (1, -3, 2) = (2, -6, 4) \equiv (2, 1, -3) \mod 7$$

to obtain

$$\begin{aligned}
&(\theta_3, \theta_1, \theta_2) \cdot (a-h, b-i, c-j) \\
&= \frac{2\pi k}{7} \times (2, 1, -3) \cdot (a-h, b-i, c-j) \\
&\equiv \frac{2\pi k}{7} \times 2 \times [(1, -3, 2) \cdot (a-h, b-i, c-j)] \\
&\equiv \frac{2\pi k}{7} \times 2 = \theta_3 \mod 2\pi.
\end{aligned}$$

Hence

$$F_3 = \rho \exp(\mathrm{i}\theta_3) \sum_{(a,b,c,h,i,j)\in P} A_{abc000hij000}(\tau)\rho^{a+b+c+h+i+j-1}.$$

Therefore,

$$\frac{F_1}{\rho \exp(\mathrm{i}\theta_1)} = \frac{F_2}{\rho \exp(\mathrm{i}\theta_2)} = \frac{F_3}{\rho \exp(\mathrm{i}\theta_3)} = \sum_{(a,b,c,h,i,j)\in P} A_{abc000hij000}(\tau)\rho^{a+b+c+h+i+j-1},$$

and the related bifurcated solution curve is determined from

$$\sum_{(a,b,c,h,i,j)\in P} A_{abc000hij000}(\tau)\rho^{a+b+c+h+i+j-1} = 0. \tag{D.49}$$

The leading terms of (D.49) are given as

$$A'_{100000000000}(0)\tau + A_{020000000000}(0)\rho = 0, \tag{D.50}$$

where $A'_{100000000000}(0)$ means the derivative of $A_{100000000000}(\tau)$ with respect to τ, evaluated at $\tau = 0$. The expression (D.50) can be derived from the following observations for $(a, b, c, h, i, j) \in P$ (cf. (D.33), (D.35)):

$$\begin{aligned} &a + b + c + h + i + j \geq 1 && \forall(a, b, c, h, i, j) \in P, \\ &a + b + c + h + i + j = 1 \quad\Longleftrightarrow && (a, b, c, h, i, j) = (1, 0, 0, 0, 0, 0), \\ &a + b + c + h + i + j = 2 \quad\Longleftrightarrow && (a, b, c, h, i, j) = (0, 2, 0, 0, 0, 0), \end{aligned}$$

combined with (D.23). Both $A'_{100000000000}(0)$ and $A_{020000000000}(0)$ are generically distinct from zero. Therefore, the equation (D.50) has a solution of the form $\rho = c\tau$ for some $c \neq 0$, which shows the generic existence of bifurcated solutions for all $(\theta_1, \theta_2, \theta_3)$ in (D.48).

To reveal the symmetry of the bifurcated solutions, we first consider the case of $(\theta_1, \theta_2, \theta_3) = (0, 0, 0)$ in (D.48). Then $z_1 = z_2 = z_3 = \rho \in \mathbb{R}$, whereas $z_4 = z_5 = z_6 = 0$. This solution, denoted $\mathrm{z}^{(0)}$, is invariant to the action of r by (D.2). Then the isotropy subgroup $\Sigma(\mathrm{z}^{(0)})$ representing the symmetry of this solution contains $\langle r\rangle$. By (D.5) with $(k, \ell) = (2m, m)$ and $n = 7m$, this solution has additional symmetry of the form $p_1^{\alpha} p_2^{\beta}$ if and only if (α,β) satisfies

$$2\alpha + \beta \equiv 0, \quad \alpha - 3\beta \equiv 0, \quad -3\alpha + 2\beta \equiv 0 \mod 7,$$

which condition is equivalent to

$$(\alpha,\beta) = p(3, 1) + q(-1, 2), \qquad p, q \in \mathbb{Z}.$$

This shows $\Sigma(z^{(0)}) \supseteq \langle p_1^3 p_2, p_1^{-1} p_2^2\rangle$. It therefore follows that $\Sigma(z^{(0)}) \supseteq \langle r, p_1^3 p_2, p_1^{-1} p_2^2\rangle$, where it can be verified that the inclusion is in fact equality, i.e.,

$$\Sigma(\mathrm{z}^{(0)}) = \langle r, p_1^3 p_2, p_1^{-1} p_2^2\rangle. \tag{D.51}$$

Let $z^{(k)}$ denote the solution corresponding to k in (D.48), where $1 \leq k \leq 6$. We can see from

$$(1, -3, 2) \equiv (2, 1, -3) - (1, -3, 2) \mod 7$$

and (D.5) that $z^{(k)}$ is obtained from $\mathrm{z}^{(0)}$ by the transformation with $(p_1 p_2^{-1})^k$, which we designate as $z^{(k)} = (p_1 p_2^{-1})^k \cdot \mathrm{z}^{(0)}$. Then the isotropy subgroup of $z^{(k)}$ is a conjugate subgroup of that of $\mathrm{z}^{(0)}$, i.e.,

$$\Sigma(\mathrm{z}^{(k)}) = (p_1 p_2^{-1})^k \cdot \Sigma(\mathrm{z}^{(0)}) \cdot (p_1 p_2^{-1})^{-k}.$$

This means, in particular, that the solutions $z^{(k)}$ for $k \geq 1$ are fundamentally (or geometrically) equivalent to $z^{(0)}$.

A bifurcated solution of the form of (D.47), with $z_1 = z_2 = z_3 = 0$ and $|z_4| = |z_5| = |z_6|$, can be obtained from $z^{(0)}$ by transforming $z^{(0)}$ with s. The isotropy group representing the symmetry of this solution $s \cdot z^{(0)}$ is obtained as

$$s \cdot \langle r, p_1^3 p_2, p_1^{-1} p_2^2 \rangle \cdot s^{-1} = \langle r^{-1}, p_1^2 p_2^{-1}, p_1^{-3} p_2^{-2} \rangle = \langle r, p_1^2 p_2^{-1}, p_1 p_2^3 \rangle. \quad \text{(D.52)}$$

It is noted, however, such conjugate solutions should be identified from a geometrical point of view.