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**Abstract**

Although quantum hypothesis testing has been recognized as a fundamental theoretical tool, it has not been investigated yet even in the qubit case when the composite hypothesis is adopted. Recently, Hayashi *et al.* [2] have shown optimal hypothesis testing under some invariance conditions when the null hypothesis is a maximally entangled state and the alternative hypothesis is except for the state. They also assume that significance level is set to zero, which seems quite strange in classical statistics. In the present paper, as a first attempt of quantum composite hypothesis testing with an arbitrary significance level, we propose the intuitively reasonable testing of the null hypothesis, which is perfectly mixed state, under composite alternative hypotheses, which is pure states in the qubit case. We show optimality, that is, that the above test is uniformly most powerful unbiased test.

## 1 Introduction

Signal detection is fundamentally important in various fields of physics, astrophysics, particle physics and so on. When microscopic mechanism is relevant with the detection process, we need to deal with the detection scheme by using quantum mechanics [5]. With recent development of precise measurement of the quantum system, theoretical signal detection strategy has been gradually thought of as important.

Quantum hypothesis testing is widely accepted in the field of the quantum information [3, 4]. Basic setting is given in a parallel way to classical hypothesis testing [7] and fundamental results have been shown by many researchers. They mainly focus on the theoretical extension of classical results or fundamental difference between classical results based on measure-theoretical probability theory and quantum analogue and often deal with testing the simple null hypothesis and alternative hypothesis. That is, they assume that the unknown density operator $\rho$ is either $\rho_0$ or $\rho_1$. In this case, it is possible to extend well-known classical results like Neyman-Pearson lemma, Stein's lemma and so on although its proof is technically very hard.

While their result is not satisfactory from a practical viewpoint, unfortunately more reasonable problem of testing hypothesis in the quantum setting has not been investigated yet except for Hayashi *et al.* [2]. They deal with the problem of the hypothesis testing of a maximally entangled state and show the optimal result in the class of LOCC by imposing on some invariance conditions. It is important to show a kind of optimality in the hypothesis testing, but even in the simple cases like the qubit system (two dimensional Hilbert space), it seems so difficult. Indeed, they simplify the original problem in order to obtain a solution. For example, they set significance level $\alpha$ zero, which is usually set $\alpha = 0.01, 0.05$, and their assumption seems quite strange in usual classical statistics.

Thus, as a first attempt to tackle with the composite quantum hypothesis testing, we investigate the qubit system and clarify how the hypothesis testing is rather difficult than classical cases.

We do mainly focus on the practical testing scheme, which is useful in the experimental setup. We present an intuitively reasonable test and show its optimality using the unbiased condition, which is indeed necessary to restrict a class of test. While the quantum detection problem in the Bayesian setting has been well investigated and they need to assume the prior distribution of the possible quantum states, our approach does not need a prior distribution. Optimality result is obtained analytically without any assumption of the prior distribution.

In Section 2, we briefly summarize our problem setting, and introduce some notions and notations. In Section 3, we present main result, a uniformly most powerful(UMP) test in the disturbed qubit model. We also discuss the unbiased condition and compare with invariance condition in Section 4. Finally we give concluding remarks.

## 2 Setting

Now we briefly describe the setting where we consider the optimal quantum hypothesis testing. In the present paper, we consider the qubit system, which is described in the two-dimensional Hilbert space, or $\mathbf{C}^2$ [8]. We have $N$ identical particles $\rho^{\otimes N}$, whose state is unknown but it is either $H_0 : \rho \in \mathcal{S}_0$ (Null hypothesis) or $H_1 : \rho \in \mathcal{S}_1$ (Alternative hypothesis), where $\mathcal{S}_0$ and $\mathcal{S}_1$ are disjoint subsets of the density operators. Assume that the null hypothesis is chosen to be a completely mixed state.

$$\mathcal{S}_0 = \{\rho_{\text{mix}} := \frac{1}{2} I\}$$

The alternative hypothesis is that the unknown state is a pure state:

$$\mathcal{S}_p = \{|\psi\rangle\langle\psi| : \ \|\psi\| = 1\}$$

As we shall see later, $\mathcal{S}_p$ is too large to admit the UMP unbiased test when we consider individual repeated measurements. Thus, we need to restrict the above alternative hypothesis to a reasonable class. For simplicity, we assume that the unknown pure state is in the upper half of the Bloch sphere. To be more specific, we give a parametrization below.

$$\mathcal{S}_1 := \{U(\epsilon_1, \epsilon_2)\rho_1 U^\dagger(\epsilon_1, \epsilon_2), 0 \le \epsilon_1 < 2\pi, 0 \le \epsilon_2 < \frac{\pi}{2}\},$$

where

$$\rho_1 := |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$
$$U(\epsilon_1, \epsilon_2) := e^{-i\frac{\epsilon_1}{2}\sigma_z} e^{-i\frac{\epsilon_2}{2}\sigma_y}.$$

Both $\sigma_y$ and $\sigma_z$ are Pauli matrices. The explicit form of the above density operator is given below. Note that there is a one-to-one correspondence between the Bloch sphere and the parameter space except for $\epsilon_2 = 0$.

Another interpretation of this model is obtained as follows. Suppose that Alice and Bob communicate with each other by transmitting qubits. First, Alice prepare the initial state $\rho = \rho_{\text{mix}}$ or $\rho = |0\rangle\langle 0|$. Then, the transmission channel may cause a kind of state change of unitary type $\sigma \to U\sigma U^\dagger$. In other words, the unknown unitary operator denotes an inevitable disturbance. Assuming that the transmission channel is stable, the disturbance is specified by the finite number of parameter. In such cases, the channel parameter could be estimated by transmitting the known state and analyzing the output state [1].

However, we do not specify the channel parameter because our objective is not to specify the whole unitary channel but to determine whether Alice really sent a pure state to Bob or not. Thus, the output state is either

$$U(\epsilon_1, \epsilon_2)\rho_0 U(\epsilon_1, \epsilon_2)^\dagger = \rho_{\text{mix}}$$

or

$$U(\epsilon_1, \epsilon_2)\rho_1 U(\epsilon_1, \epsilon_2)^\dagger = \frac{1}{2}\begin{pmatrix} 1 + \cos\epsilon_2 & e^{-i\epsilon_1}\sin\epsilon_2 \\ e^{i\epsilon_1}\sin\epsilon_2 & 1 - \cos\epsilon_2 \end{pmatrix}.$$

For convenience, we call this model the *disturbed qubit model.* We use $\rho_{1,\epsilon}$ instead of $U(\epsilon_1, \epsilon_2)\rho_1 U(\epsilon_1, \epsilon_2)^\dagger$ for simplicity.

Although the above disturbed qubit model is easily generalized, in such a generalization, nonessential difficulties arise in various ways. Thus, in the present paper, we only focus on the above simple but meaningful model and hypothesis testing in details.

## 3 UMP unbiased test in the disturbed qubit model

Before we mention main result, we briefly explain the formal definition of hypothesis testing in the quantum setting, see e.g., section 2, Hayashi *et al.* [2] for details. First we fix the significance level $\alpha$, $0 \leq \alpha \leq 1$. A Hermitian operator $T$, $0 \leq T \leq I$ is said to be a *test*. Then, binary decision, like *yes* or *no*, is possible based on the measurement outcome, where a pair of Hermitian matrices $\{T, I-T\}$ is regarded as a POVM. A test $T$ is called a *test with significance level* $\alpha$, or shortly a *test of level-*$\alpha$ if it satisfies

$$\mathrm{Tr}\rho T \leq \alpha, \quad \rho \in \mathcal{S}_0,$$

where the LHS term is called the *first kind of error*. Note that the above condition is equivalent to

$$\sup\{\mathrm{Tr}\rho T : \ \rho \in \mathcal{S}_0\} \leq \alpha.$$

Then, our purpose is to maximize the power function

$$\tilde{\beta}_\rho(T) := \mathrm{Tr}\rho T, \quad \rho \in \mathcal{S}_1,$$

where $\beta_\rho(T) := 1 - \tilde{\beta}_\rho(T) = \mathrm{Tr}\rho(I-T)$ is called the *second kind of error*. When the alternative hypothesis consists of a single point, i.e., $\mathcal{S}_1 = \{\rho'\}$, then a test of level-$\alpha$ is called *most powerful (MP)* if for any other test $T'$ of level-$\alpha$

$$\tilde{\beta}_{\rho'}(T) \geq \tilde{\beta}_{\rho'}(T')$$

holds. Quantum Neyman-Pearson lemma claims that there always exists the MP test. (See, e.g., Helstrom [5] for proof). However, when the alternative hypothesis does not consist of a single point, it becomes more difficult. A test of level-$\alpha$ is called *uniformly most powerful (UMP)* if for any other test $T'$ of level-$\alpha$

$$\tilde{\beta}_\rho(T) \geq \tilde{\beta}_\rho(T'), \quad \rho \in \mathcal{S}_1.$$

Generally, the UMP test does not necessarily exists.

Now let us introduce the unbiased test, which is analogous to the classical counterpart. A test $T$ is called an *unbiased test* with the significance level $\alpha$ if it satisfies

$$\mathrm{Tr}\rho T \leq \alpha, \quad \rho \in \mathcal{S}_0$$

and

$$\tilde{\beta}_\rho(T) = \mathrm{Tr}\rho T \geq \alpha, \quad \rho \in \mathcal{S}_1.$$

The definition is the straightforward extension of unbiasedness in classical hypothesis testing [7]. Intuitively speaking, an unbiased test is not worse than the random decision, where the null hypothesis is rejected with the probability of $\alpha$ without any observation, and thus this condition is very natural. Unbiasedness seems a weak condition but it excludes a certain class of pathological tests.

Unfortunately, in spite of simplification, it is shown to that there is no UMP test for the disturbed qubit model. However, if we restrict the class of hypothesis testing to the unbiased one, then we show that there exists the UMP unbiased test in our setting. It is the same situation as in classical hypothesis testing. For example, in a standard textbook of classical hypothesis testing, they say "*For a large class of problems for which a UMP test does not exist, there does exist a UMP unbiased test*" (Lehmann and Romano [7], section 4.1). Our main result is given below.

**Theorem**
*In the above setting, there exists a UMP unbiased test of level-$\alpha$, where $0 < \alpha < 1$ is an arbitrary significance level.*

Its proof has two steps, first, the unbiased test is restricted to the class of a classical test based on the outcome of the projective measurement of $z$-direction, i.e., $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$. Then, we construct the UMP test for the above class of tests by using the one-sided UMP for classical Bernoulli distribution.

*Proof.*

*Step.1*
From the unbiased condition, $\frac{1}{2}\mathrm{Tr}T \leq \alpha$ and $\tilde{\beta}_{|\psi\rangle\langle\psi|}(T) \geq \alpha$, where $|\psi\rangle$ is equal to $U(\epsilon_1, \epsilon_2)|0\rangle$ up to the phase factor, holds. Then, an unbiased test $T$ must satisfy

$$\tilde{\beta}_{|\psi\rangle\langle\psi|}(T) - \frac{1}{2}\mathrm{Tr}T \geq 0 \tag{1}$$

for an arbitrary $|\psi\rangle\langle\psi| \in \mathcal{S}_1$. As we shall see below, Eq.(1) is restrictive, thus, a test $T$ satisfies Eq.(1) if and only if $T$ is in the following form

$$T = t_0|0\rangle\langle 0| + t_1|1\rangle\langle 1|, \tag{2}$$

for an appropriate pair of $t_0$ and $t_1$ satisfying $0 \leq t_0, t_1 \leq 1$.

In order to prove the above claim, we rewrite $T$

$$T = t_0|e_0\rangle\langle e_0| + t_1|e_1\rangle\langle e_1|,$$

where $t_i, i = 0, 1$ are the eigenvalues and $e_i, i = 0, 1$ are the normalized eigenvectors. Both states $|e_0\rangle\langle e_0|$ and $|e_1\rangle\langle e_1|$ are antipodal points on the Bloch sphere.

If both of them are on the $xy$ plane, which implies $|e_0\rangle\langle e_0| \notin \mathcal{S}_1$ and $|e_1\rangle\langle e_1| \notin \mathcal{S}_1$, then for small arbitrary $\epsilon > 0$, we can choose $|\psi\rangle\langle\psi| \in \mathcal{S}_1$ such that

$$|\langle e_0|\psi\rangle|^2 = 1-\epsilon,$$
$$|\langle e_1|\psi\rangle|^2 = \epsilon.$$

Next, from the unbiased condition we obtain

$$\tilde{\beta}_{|\psi\rangle\langle\psi|}(T) = t_0(1-\epsilon) + t_1\epsilon \geq \alpha$$

Thus, taking limit of $\epsilon \to 0$, we obtain $t_0 \geq \alpha$. Likewise we obtain $t_1 \geq \alpha$. Putting them together with the condition $\mathrm{Tr}\rho_{\mathrm{mix}}T = \frac{t_0+t_1}{2} \geq \alpha$, we obtain $t_0 = t_1 = \alpha$ and $T = \alpha I$, which is in the form (2).

Now, without loss of generality, we take $|e_0\rangle\langle e_0| \in \mathcal{S}_1$. When $|\psi\rangle\langle\psi| = |e_0\rangle\langle e_0|$, from Eq.(1) we obtain

$$\tilde{\beta}_{|e_0\rangle\langle e_0|}(T) - \frac{1}{2}\mathrm{Tr}T = \frac{1}{2}(t_0 - t_1) \geq 0.$$

If $t_0 = t_1$, then our claim holds. If $t_0 > t_1$, then we can choose $|\psi\rangle\langle\psi| \in \mathcal{S}_1$ such that $|\langle e_1|\psi\rangle|^2 > |\langle e_0|\psi\rangle|^2$ unless $|e_0\rangle\langle e_0| = |0\rangle\langle 0|$. Since $|\langle e_0|\psi\rangle|^2 + |\langle e_1|\psi\rangle|^2 = 1$,

$$|\langle e_1|\psi\rangle|^2 > \frac{1}{2} > |\langle e_0|\psi\rangle|^2$$

holds. Thus, we obtain

$$\begin{aligned}
&\tilde{\beta}_{|\psi\rangle\langle\psi|}(T) - \frac{1}{2}\mathrm{Tr}T \\
&= t_0|\langle e_0|\psi\rangle|^2 + t_1|\langle e_1|\psi\rangle|^2 - \frac{1}{2}(t_0+t_1) \\
&= -(t_0 - t_1)\left(|\langle e_1|\psi\rangle|^2 - \frac{1}{2}\right) < 0.
\end{aligned}$$

The unbiased condition does not hold unless $|e_0\rangle\langle e_0| = |0\rangle\langle 0|$. Therefore, $|e_0\rangle\langle e_0| = |0\rangle\langle 0|$ and $|e_1\rangle\langle e_1| = |1\rangle\langle 1|$ holds.

*Step.2*

We only consider an unbiased test $T$ in the form (2). Now we consider the pinching map $\kappa$ using PVM $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$, Since $\kappa(T) = T$ and $\mathrm{Tr}\kappa(A)B = \mathrm{Tr}\kappa(A)\kappa(B)$ for two Hermitian operators $A$ and $B$, both kinds of error is rewritten by

$$\mathrm{Tr}\rho_{\mathrm{mix}}T = \mathrm{Tr}\kappa(\rho_{\mathrm{mix}})T,$$
$$\tilde{\beta}_{\rho_{1,\epsilon}}(T) = \mathrm{Tr}\rho_{1,\epsilon}T = \mathrm{Tr}\kappa(\rho_{1,\epsilon})T,$$

where

$$\kappa(\rho_{1,\epsilon}) = \frac{1}{2}\begin{pmatrix} 1+\cos\epsilon_2 & 0 \\ 0 & 1-\cos\epsilon_2 \end{pmatrix}.$$

Now seeking the optimal test among the unbiased tests is reduced to seeking the optimal test in the following classical hypothesis testing in the Bernoulli model (e.g., coin toss). Suppose that we have a coin whose head comes with the probability $p$. Null hypothesis is $H_{0,cl} : p = \frac{1}{2}$ and alternative hypothesis is $H_{1,cl} : p = \frac{1}{2}(1 - \cos \epsilon_2), 0 \leq \epsilon_2 < \frac{\pi}{2}$. Since we do not know the parameter $\epsilon_2$, the latter is rewritten as $H_{1,cl} : 0 \leq p < \frac{1}{2}$. This is a one-sided hypothesis testing and there exists the UMP test [7]. *Q.E.D.*

According to theorem, the optimal strategy is as follows. To be specific, we fix some parameter, say, $\alpha = 0.01$ and $N = 20$. First we perform a measurement based on PVM $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ for each particle. Then, we count on the number of 1, say, $x$. If $x = 0, 1, 2, 3$ and 4, then reject the null hypothesis. If $x = 5$, then reject with the probability of $\gamma = 0.276$. Otherwise, we accept the null hypothesis. For other parameter, testing is performed in the same manner.

## 4 Discussion

In the above problem, since $\mathcal{S}_1$ has a very simple form, hypothesis testing of $H_0$ against $H_1$ is rotationally invariant in the direction of $z$-axis. If we impose the invariance condition instead of the unbiasedness, we obtain the same class. However, generally speaking, the invariance condition is so strong that it often causes too restricted class of tests. On the other hand, the unbiasedness is rather weak and there could exist an optimal test in a more complicated model.

In the disturbed qubit model, if there are two points $\psi$ and $\psi'$ such that $|\langle e_0|\psi\rangle|^2 < |\langle e_1|\psi\rangle|^2$ and $|\langle e_0|\psi'\rangle|^2 > |\langle e_1|\psi'\rangle|^2$ hold, the unbiased condition fails. If we introduce the Bures distance, $d_F$ [8, 4], which is defined by

$$d_F(|\psi\rangle, |\phi\rangle) := 1 - |\langle\psi|\phi\rangle|^2,$$

we obtain a more geometrical interpretation of the unbiased condition. We introduce a semisphere whose point is strictly closer to $|e_0\rangle$ than $|e_1\rangle$, i.e.,

$$\mathcal{M} := \{|\psi\rangle\langle\psi| : \; d_F(|e_0\rangle, |\psi\rangle) < d_F(|e_1\rangle, |\psi\rangle) \; \}.$$

Then, $\mathcal{S}_1$ is totally covered with $\mathcal{M}$ if and only if $T = t_0|e_0\rangle\langle e_0| + t_1|e_1\rangle\langle e_1|$ with an appropriate choice of $t_0$ and $t_1$ satisfies the unbiased condition. Otherwise, noncommutative unbiased tests appear. In particular, the full model of pure states, $\mathcal{S}_p$ is not covered with $\mathcal{M}$ and thus there is no unbiased test. Then, what we can do is at most only to guess randomly according to the significance level $\alpha$. Further analysis from geometrical perspective is left for future study.

**Example**
We restrict the alternative hypothesis to a bit complicated form, which breaks rotational invariance.

$$\mathcal{S}_2 = \{\rho_{1,\epsilon} : 0 \le \epsilon_1 < 2\pi,\ \frac{\pi}{3}\cos^2\epsilon_1 \le \epsilon_2 < \frac{\pi}{2}\}$$

Still, it is shown that there exists the UMP unbiased test by the similar argument to theorem.

In the present paper, we assume that the unitary channel is stable. Thus, we consider the unknown parameter $\epsilon_1, \epsilon_2$ fixed. One may think what happens if the channel parameter is also random. Then, we have to adopt the Bayesian scheme. We assume that the distribution of the unknown parameter is given by $\pi(\epsilon_1, \epsilon_2)$, where $\int \pi(\epsilon_1, \epsilon_2)\mathrm{d}\epsilon_1\mathrm{d}\epsilon_2 = 1$. Suppose that we use the repeated measurement. We define

$$\begin{aligned}\mathcal{S}_\pi &:= \{\rho_\pi\},\\ \rho_\pi &:= \int \rho_{1,\epsilon}\pi(\epsilon_1, \epsilon_2)\mathrm{d}\epsilon_1\mathrm{d}\epsilon_2\end{aligned}$$

Then, this problem is equivalent to the hypothesis testing of the simple null hypothesis $\mathcal{S}_0$ under the simple alternative hypothesis $\mathcal{S}_\pi$ instead of $\mathcal{S}_1$, which is easily solved due to quantum Neyman-Pearson lemma and the UMP test is given.

# 5 Concluding Remarks

We have obtained the UMP unbiased test in the disturbed qubit model. Since Helstrom's work, composite hypothesis testing in the quantum setting has not been investigated even in the qubit model in spite of its importance in a practical application.

There are much of problems remaining. For example, collective measurements may enhance the performance of testing, that is, $\beta$ is improved for fixed $\alpha$. It is also interesting to consider a nonunitary channel (TPCP map) in the disturbed qubit model and an extension to higher dimensional Hilbert spaces. Due to the unbiasedness condition, which seems weaker than invariance, it would be possible to give the UMP unbiased test for the null hypothesis of maximally entangled states and Gaussian states under composite hypotheses.

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