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Abstract

In 2009, McLachlan and Zhang proposed a family of generalized versions of the well-known Camassa–Holm (CH) equation, which they called "modified Camassa–Holm (mCH) equation." The mCH family has one striking feature in common: each of them is equipped with two invariants, "momentum" and "energy," and forms a Hamiltonian equation with respect to the energy. In this paper, we construct three numerical integrators that preserve one or two of the invariants making use of the Hamiltonian structure, and give theoretical analyses of the schemes. We also present several numerical examples, which not only confirm the effectiveness of the schemes but also suggest a new insight that some solutions of the mCH can behave like solitons.

keyword
modified Camassa–Holm equation, conservation, discrete variational derivative method

1 Introduction

In this paper, we consider the numerical integration and behavior of solutions of the modified Camassa–Holm (mCH) equation:

$$m_t + um_x + 2u_x m = 0,\ m = (1 - \partial_x{}^2)^p u, \tag{1}$$

where p is a positive integer and the subscript t (or x, respectively) denotes the differentiation with respect to time variable t (or x).[1] This equation was derived by McLachlan and Zhang [10] as the Euler–Poincaré differential equation on the Bott–Virasoro group with respect to the H^p metric.

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[1] This work is a full version of our recent report [13], with detailed discussions including complete proofs and extra numerical results.

When $p = 1$, (1) reduces to the well-known Camassa–Holm (CH) equation:

$$u_t - u_{xxt} = uu_{xxx} + 2u_x u_{xx} - 3uu_x, \tag{2}$$

which describes shallow water wave. The CH has bi-Hamiltonian structure, is completely-integrable, and has infinitely many conservation laws. Furthermore, it has an interesting feature that it allows strange singular solutions called "peakons," which are peaked solitons; this is in sharp contrast to the classical smooth soliton equations such as the Korteweg-de Vries equation. In order to reveal the rich dynamics of the CH, many numerical studies have been carried out, including the following geometric or structure-preserving integrators: some energy conserving schemes [2, 7, 9, 14, 15] and multisymplectic integrators [1].

In contrast to this, the case $p \geq 2$ has not yet been fully understood, both theoretically and numerically. Here we briefly review some known results, mainly taking the case $p = 2$ as our example. When $p = 2$, the equation expressed in the variable u takes a bit complex form:

$$u_t - 2u_{xxt} + u_{xxxxt} = 2uu_{xxx} + 4u_x u_{xx} - 3uu_x - uu_{xxxxx} - 2u_x u_{xxxx}. \tag{3}$$

On this $p = 2$ case, the global existence of smooth solutions on the circle $\mathbb{S}$ and real line $\mathbb{R}$ were discussed in [10, 17]. In contrast to the CH, only two invariants have been found with the mCH, which read when $p = 2$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int u \mathrm{d}x = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int \tilde{H} \mathrm{d}x = 0, \quad \tilde{H} = \frac{u^2 + 2u_x^2 + u_{xx}^2}{2}. \tag{4}$$

In the present paper, we call them "momentum" and "energy" conservation laws respectively. Also in general p cases, the mCH is equipped with some associated "momentum" and "energy," whose precise forms will be defined in the next section. It is still an open question whether or not there are other invariants, in particular if the mCH is generally completely-integrable. The dynamics of the mCH, for example the possibility of soliton-like solutions, are yet to be understood, except that in [10] it is claimed that an interesting phenomenon "weak blow-up" is numerically suggested (but not confirmed). In this sense, we can say that the study of the mCH has just begun.

In view of this background, in this paper we have the following two objectives. First, we construct numerical integrators that preserve one or two of the invariants, in hope of good numerical schemes well replicating qualitative behavior of the mCH. In a certain area of numerical analysis, it is widely accepted that "structure-preserving" methods, or more specifically "geometric integration" methods generally yield qualitatively better and stable numerical results, and thus are favorable (see, for example, [5, 6]). The construction in this paper is done by fully utilizing the fact that the mCH can be written in Hamiltonian form (see the next section). A similar approach has been already taken by Takeya and one of the present authors for the original CH to find several conservative integrators [14, 15]. Our approach in the present paper is basically a natural extension of theirs, but slightly modified so that it fits more naturally to the mCH case. As a result, we find three conservative integrators: nonlinear and linearly-implicit finite difference schemes preserving both the momentum and energy, and a linearly-implicit finite difference scheme preserving only the energy. The first two reduces to the Takeya–Furihata scheme for the CH when $p = 1$, and the last one is completely new. At this point, the last one with only one invariant might sound obviously inferior and thus not worth consideration; but interestingly enough, this is not true—both theoretically and numerically it has enough virtues to compare well with the first two integrators.

Our second objective is to investigate the dynamics of the mCH, by using the constructed integrators. In particular, in the case of $p = 2$, we numerically show that certain solutions behave like solitons; this support (at least weakly) the view that the mCH may be also a soliton equation.

This paper is organized as follows. In Section 2, some preliminary facts including notation are summarized. In Section 3, the proposed nonlinear scheme is presented, and its properties are discussed. Section 4 is devoted to the linear counterparts. In Section 5, some numerical results are provided. Concluding remarks and comments are given in Section 6.

2 Preliminaries

In this section, our notation and several preliminary results are summarized.

2.1 Notation and Useful Formulas

Numerical solution is denoted by $U_k^{(n)} \simeq u(n\Delta t, k\Delta x)$, where Δx is the space mesh size and Δt the time mesh size. Numerical solution corresponding to m is similarly denoted by $M_k^{(n)}$. In the subsequent sections, we frequently use the abbreviation: $U_k^{(n+\frac{1}{2})} = (U_k^{(n+1)} + U_k^{(n)})/2$, and the vector notation: $\boldsymbol{U}^{(n)} = (U_0^{(n)}, U_1^{(n)}, \ldots, U_{N-1}^{(n)})^\top$ to save space. This also applies to $M_k^{(n)}$.

In the rest of the present paper, we consider the mCH on the circle of length L, which we denote $\mathbb{S}$. In other words, we assume the periodic boundary condition on an interval of length L. We denote the number of the space grid points by N (i.e., $\Delta x = L/N$). In order to treat the periodic boundary condition, we consider $\{U_k^{(n)}\}_{k=-\infty}^{\infty}$, an infinitely long vector, and then its N-dimensional restriction by the discrete periodic boundary condition $U_k^{(n)} = U_{k \bmod N}^{(n)}$ $(\forall k \in \mathbb{Z})$. We denote the latter space by $\mathbb{R}^{(N)}$, and define its inner product by $(\boldsymbol{f}, \boldsymbol{g}) = \sum_{k=0}^{N-1} f_k g_k \Delta x$. It is a natural discretization of $L^2(\mathbb{S})$. The associated 2-norm $\|\cdot\|$ is defined accordingly (we omit the subscript "2"). The discrete Schwarz formula: $|(\boldsymbol{f}, \boldsymbol{g})| \leq \|\boldsymbol{f}\|\,\|\boldsymbol{g}\|$ can be easily checked. We also define $\|\cdot\|_\infty$ by $\|\boldsymbol{U}^{(n)}\|_\infty = \max_{0 \leq k \leq N-1} |U_k^{(n)}|$.

We define δ_k^+ and δ_k^-, and the nth difference operator $\delta_k^{\langle n \rangle}$ (a discretization of the nth differential operator) by

$$
\begin{aligned}
&\delta_k^{\langle 0 \rangle} f_k = f_k, \\
&\delta_k^+ f_k = \frac{f_{k+1} - f_k}{\Delta x}, \ \delta_k^- f_k = \frac{f_k - f_{k-1}}{\Delta x}, \ \delta_k^{\langle 1 \rangle} f_k = \frac{f_{k+1} - f_{k-1}}{2\Delta x}, \\
&\delta_k^{\langle 2 \rangle} f_k = \delta_k^+ \delta_k^- f_k = \frac{f_{k+1} - 2f_k + f_{k+1}}{\Delta x^2}, \\
&\delta_k^{\langle 2l+1 \rangle} = \delta_k^{\langle 1 \rangle} \delta_k^{\langle 2l \rangle}, \ \delta_k^{\langle 2l+2 \rangle} = \delta_k^{\langle 2 \rangle} \delta_k^{\langle 2l \rangle} \ (l \geq 1).
\end{aligned}
$$

When appropriate, we also use the matrix notation, for example, $D^{\langle 1 \rangle} \boldsymbol{f} := (\delta_k^{\langle 1 \rangle} f_0, \delta_k^{\langle 1 \rangle} f_1, \ldots, \delta_k^{\langle 1 \rangle} f_{N-1})^\top$ for $\boldsymbol{f} \in \mathbb{R}^{(N)}$.

We also use the average operator: $\mu_k^{\langle 1 \rangle} f_k = (f_{k+1} + f_{k-1})/2$, and $\mu_k^{\langle 2 \rangle} f_k = (f_{k+1} + 2f_k + f_{k-1})/4$.

As for the difference operators, the following summation-by-parts formulas hold. Their proofs can be found in various studies, including [4, 5].

Lemma 2.1. *For any $\boldsymbol{f}, \boldsymbol{g} \in \mathbb{R}^{(N)}$,*

$$
\sum_{k=0}^{N-1} f_k \delta_k^{\langle 1 \rangle} g_k \Delta x \ = \ -\sum_{k=0}^{N-1} (\delta_k^{\langle 1 \rangle} f_k) g_k \Delta x, \tag{5}
$$

$$\begin{aligned}\sum_{k=0}^{N-1} f_k \delta_k^{\langle 2\rangle} g_k \Delta x &= -\sum_{k=0}^{N-1} \frac{(\delta_k^+ f_k)(\delta_k^+ g_k) + (\delta_k^- f_k)(\delta_k^- g_k)}{2} \Delta x \\ &= \sum_{k=0}^{N-1} (\delta_k^{\langle 2\rangle} f_k) g_k \Delta x. \end{aligned} \tag{6}$$

Obviously they correspond to the integration-by-parts formula. The above formulas also claim the skew-symmetry of (and symmetry of, respectively) $\delta_k^{\langle 1\rangle}$ (and $\delta_k^{\langle 2\rangle}$) with respect to the inner product $(\cdot,\cdot)$. It is convenient to summarize the (skew-)symmetry of various operators in the following corollary, which will be frequently used in the subsequent sections.

Corollary. *With respect to the inner product $(\cdot,\cdot)$, the difference operators $\delta_k^{\langle 2\rangle}$ and $(1 - \delta_k^{\langle 2\rangle})^p$ $(p = 1, 2, \dots)$ are symmetric, and $\delta_k^{\langle 1\rangle}$ and $\delta_k^{\langle 1\rangle}(1 - \delta_k^{\langle 2\rangle})^p$ are skew-symmetric.*

Remark 1. Note that $\sum_{k=0}^{N-1} (\delta_k^+ f_k)(\delta_k^+ g_k)\Delta x = \sum_{k=0}^{N-1} (\delta_k^- f_k)(\delta_k^- g_k)\Delta x$ under the discrete periodic boundary condition, and thus it is not so important to distinguish δ_k^+ and δ_k^- in (6). In what follows, we sometimes just write $\sum_{k=0}^{N-1} f_k \delta_k^{\langle 2\rangle} g_k \Delta x = -\sum_{k=0}^{N-1} (\delta_k^+ f_k)(\delta_k^+ g_k)$ Δx to save space. Similar reductions based on the periodicity will be also done in various calculations. □

2.2 Takeya–Furihata's Approach

In this subsection, Takeya and Furihata's approach for the CH (the mCH with $p = 1$) [14, 15] is briefly summarized.

Their approach is based on the observation that the CH (2) can be written in the Hamiltonian form:

$$m_t = -(m\partial_x + \partial_x m)\frac{\delta \tilde{H}}{\delta m}, \qquad \tilde{H} = \frac{u^2 + u_x^2}{2}, \tag{7}$$

with $m = (1 - \partial_x^2)u$.

Remark 2. Notice that we have used the same symbol $\tilde{H}$ for the different energy functions in (4) and (7). The reason for this, and how we distinguish $\tilde{H}$ and the other energy function H (which will appear later) will be described in the next subsection. □

The symbol $\delta\tilde{H}/\delta m$ denotes the variational derivative of $\tilde{H}$ with respect to m. At first glance, it might seem strange since $\tilde{H}$ is defined with u, but actually it can be calculated in the following way (Takeya–Furihata's approach): we start from an easy one: $\delta\tilde{H}/\delta u = (1 - \partial_x^2)u$, and then utilize the relation $\delta/\delta m = (1 - \partial_x^2)^{-1}(\delta/\delta u)$, which finally yields

$$\frac{\delta \tilde{H}}{\delta m} = u. \tag{8}$$

Then we can see (7) coincides with (1). Here, we would also like to point out that the inverse operator $(1 - \partial_x^2)^{-1}$ is in fact well-defined in functional analytic sense (see, for example, the notice in [9]).

Note that, as usual in this research field, we promise that the "operator" $(m\partial_x + \partial_x m)$ operates to a function f in such a way that $(m\partial_x + \partial_x m)f = mf_x + \partial_x(mf)$. Despite the apparent weirdness, this notation has its own beauty that it clearly shows the skew-symmetry of the block: for any sufficiently smooth L-periodic functions f and g, it holds

$$\int_0^L f(m\partial_x + \partial_x m)g \,\mathrm{d}x = -\int_0^L (m\partial_x + \partial_x m)f \cdot g \,\mathrm{d}x.$$

From the skew-symmetry, the conservation of $\tilde{H}$ is obvious:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \tilde{H}\mathrm{d}x = \int_0^L \frac{\delta \tilde{H}}{\delta m} m_t \mathrm{d}x = -\int_0^L \frac{\delta \tilde{H}}{\delta m}\cdot (m\partial_x + \partial_x m)\frac{\delta \tilde{H}}{\delta m}\mathrm{d}x = 0. \tag{9}$$

The conservation of u is a bit more tricky; to this end, first we note that it is equivalent to prove $(\mathrm{d}/\mathrm{d}t)\int_0^L m\mathrm{d}x = 0$, assuming u (and accordingly m) is sufficiently smooth and the periodic boundary condition. Then we see

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L m\mathrm{d}x &= \int_0^L (m\partial_x + \partial_x m)u\mathrm{d}x = -\int_0^L (mu_x + (mu)_x)\mathrm{d}x \\ &= \int_0^L \partial_x(1-\partial_x^2)^p u\cdot u\mathrm{d}x = 0.\end{aligned} \tag{10}$$

We have used the periodicity of m and u, and the skew-symmetry of $\partial_x(1-\partial_x^2)^p$ (cf. Corollary 2.1). Note that, in this case, we need the concrete form of $\delta\tilde{H}/\delta m = u$ and $m = (1-\partial_x^2)^p u$.

Takeya–Furihata's approach is to follow the discussion above in discrete setting, by utilizing the discrete variational derivative method (see, for example, [5]). We briefly describe the outline. With the discrete energy function:

$$\tilde{H}_k^{(n)} = \frac{2(U_k^{(n)})^2 + (\delta_k^+ U_k^{(n)})^2 + (\delta_k^- U_k^{(n)})^2}{4},$$

one easily find the associated discrete variational derivative:

$$\frac{\delta \tilde{H}}{\delta(\boldsymbol{U}^{(n+1)}, \boldsymbol{U}^{(n)})_k} = (1-\delta_k^{\langle 2\rangle})\left(\frac{U_k^{(n+1)} + U_k^{(n)}}{2}\right).$$

In view of the relation $\delta/\delta u = (1-\partial_x^2)(\delta/\delta m)$, it is straightforward to relate the above discrete variational derivative to that of m:

$$\frac{\delta \tilde{H}}{\delta(\boldsymbol{M}^{(n+1)}, \boldsymbol{M}^{(n)})_k} = (1-\delta_k^{\langle 2\rangle})^{-1}\frac{\delta \tilde{H}}{\delta(\boldsymbol{U}^{(n+1)}, \boldsymbol{U}^{(n)})_k} = \frac{U_k^{(n+1)} + U_k^{(n)}}{2},$$

where $(1-\delta_k^{\langle 2\rangle})^{-1}$ is the inverse operator of $(1-\delta_k^{\langle 2\rangle})$. We note that $(1-\delta_k^{\langle 2\rangle})^{-1}$ is well-defined, since the matrix $(I - D^{\langle 2\rangle})$ is regular (for the proof, see, for example, [11, 14, 15]). The first Takeya–Furihata scheme then reads

$$\frac{M_k^{(n+1)} - M_k^{(n)}}{\Delta t} = -\left(M_k^{(n+\frac{1}{2})}\delta_k^{\langle 1\rangle} + \delta_k^{\langle 1\rangle} M_k^{(n+\frac{1}{2})}\right)\frac{\delta \tilde{H}}{\delta(\boldsymbol{M}^{(n+1)}, \boldsymbol{M}^{(n)})_k},$$

with $M_k^{(n)} = (1-\delta_k^{\langle 2\rangle})U_k^{(n)}$. Notice that by eliminating $M_k^{(n)}$ (or equivalently, $U_k^{(n)}$, whichever), the computation can proceed solely in $U_k^{(n)}$ (or $M_k^{(n)}$, respectively) space. This scheme is conservative in the sense that numerical solutions satisfy

$$\sum_{k=0}^{N-1} U_k^{(n)}\Delta x = \sum_{k=0}^{N-1} U_k^{(0)}\Delta x, \qquad \sum_{k=0}^{N-1} \tilde{H}_k^{(n)}\Delta x = \sum_{k=0}^{N-1} \tilde{H}_k^{(0)}\Delta x,$$

for $n = 1, 2, \ldots$. Notice that this scheme is fully implicit. In fact, recalling the definition: $M_k^{(n+\frac{1}{2})} = (M_k^{(n+1)} + M_k^{(n)})/2$ (defined in Section 2.1), we see that the right hand side include $M_k^{(n+1)}U_k^{(n+1)}$, which makes the scheme nonlinear.

Takeya–Furihata's second scheme, which is linearly implicit, can be obtained in a similar manner from the multi-step discrete energy function:

$$\tilde{H}_k^{(n+\frac{1}{2})} = \frac{2U_k^{(n+1)}U_k^{(n)} + (\delta_k^+ U_k^{(n+1)})(\delta_k^+ U_k^{(n)}) + (\delta_k^- U_k^{(n+1)})(\delta_k^- U_k^{(n)})}{4}, \tag{11}$$

and its associated discrete variational derivative:

$$\frac{\delta \tilde{H}}{\delta(\boldsymbol{U}^{(n+1)}, \boldsymbol{U}^{(n)}, \boldsymbol{U}^{(n-1)})_k} = (1 - \delta_k^{\langle 2 \rangle})U_k^{(n)}.$$

By translating the discrete variational derivative into m, we find

$$\frac{\delta \tilde{H}}{\delta(\boldsymbol{M}^{(n+1)}, \boldsymbol{M}^{(n)}, \boldsymbol{M}^{(n-1)})_k} = (1 - \delta_k^{\langle 2 \rangle})^{-1} \frac{\delta \tilde{H}}{\delta(\boldsymbol{U}^{(n+1)}, \boldsymbol{U}^{(n)}, \boldsymbol{U}^{(n-1)})_k} = U_k^{(n)}.$$

Based on the above discrete quantities, the linearly implicit scheme is represented as

$$\frac{M_k^{(n+1)} - M_k^{(n-1)}}{2\Delta t} = -\left(M_k^{(n)}\delta_k^{\langle 1 \rangle} + \delta_k^{\langle 1 \rangle} M_k^{(n)}\right) \frac{\delta \tilde{H}}{\delta(\boldsymbol{M}^{(n+1)}, \boldsymbol{M}^{(n)}, \boldsymbol{M}^{(n-1)})_k}. \tag{12}$$

This scheme is conservative in the sense that numerical solutions satisfy

$$\sum_{k=0}^{N-1} U_k^{(n)} \Delta x = \sum_{k=0}^{N-1} U_k^{(0)} \Delta x, \qquad \sum_{k=0}^{N-1} \tilde{H}_k^{(n+\frac{1}{2})} \Delta x = \sum_{k=0}^{N-1} \tilde{H}_k^{(\frac{1}{2})} \Delta x,$$

for $n = 1, 2, \ldots$.

Remark 3. Takeya–Furihata [14, 15] considered further two schemes that preserve another invariant of the CH instead of $\tilde{H}$. We omit the description on these schemes, since the invariant is lost in the mCH (with $p \geq 2$), and thus following them does not make any sense. □

2.3 Reformulating the Hamiltonian Form for the mCH

The Takeya–Furihata approach above can be, in principle, extended to the general case $p \geq 2$. For example, we can do exactly the same thing for (3) starting from $\tilde{H} = (u^2 + 2{u_x}^2 + {u_{xx}}^2)/2$. But to handle the general mCH case (with $p \geq 2$), it is more straightforward to start with a slightly different Hamiltonian form:

$$m_t = -(m\partial_x + \partial_x m)\frac{\delta H}{\delta m}, \qquad H = \frac{um}{2}. \tag{13}$$

Notice that the energy function is defined in a different way from above; now the "energy" H is defined with both u and m. Its derivative $\delta H / \delta u$ can be easily found by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L H \mathrm{d}x = \int_0^L \left(\frac{\partial H}{\partial u}u_t + \frac{\partial H}{\partial m}m_t\right)\mathrm{d}x = \int_0^L \left(\frac{m}{2}u_t + \frac{u}{2}m_t\right)\mathrm{d}x = \int_0^L u m_t \mathrm{d}x, \tag{14}$$

from which we see

$$\frac{\delta H}{\delta m} = u.$$

We have repeatedly used the relation $m = (1 - \partial_x^2)^p u$ and its inverse $u = (1 - \partial_x^2)^{-p} m$. The result coincides with the previous case (8), and thus the Hamiltonian form (13) defines exactly the same equation as (7) when $p = 1$.

The situation remains the same for $p \geq 2$. The mCH can be written in two different Hamiltonian forms: one with the energy function $\tilde{H}$ depending only on u, and the other with $H = um/2$. Obviously the latter is more convenient, since $H = um/2$ *does not change its form for any* p, while $\tilde{H}$ changes its form and becomes far complicated as p increases. Actually, for $\tilde{H}$ the next lemma holds (this has not been explicitly written elsewhere, but already implied in [10]).

Lemma 2.2. *For $p = 1, 2, \ldots$, the first Hamiltonian form of the mCH defined with $\tilde{H}$ (that depends only on u) becomes*

$$m_t = -(m\partial_x + \partial_x m)\frac{\delta \tilde{H}}{\delta m}, \qquad \tilde{H} = \frac{1}{2}\sum_{i=0}^{p} {}_pC_i({\partial_x}^i u)^2. \tag{15}$$

Here, for any sufficiently smooth u (and accordingly $m = (1 - \partial_x^2)^p u$), it holds

$$\int_0^L H \mathrm{d}x = \int_0^L \tilde{H} \mathrm{d}x.$$

Proof. Admitting the second form (13), we easily see by expanding $um/2 = u(1 - \partial_x^2)^p u$ and integrating them by parts. □

This lemma tells us that things would get messy, if we stick to the first Hamiltonian form (15), although in the simplest case $p = 1$ (the CH case; see the Hamiltonian form (7)) Takeya–Furihata's choice was fair enough. Due to this observation, in what follows, we employ the second form (13); this enables us to derive conservative schemes in a unified way regardless of p. Interestingly enough (or as expected), the resulting schemes basically coincide with each other whichever Hamiltonian form we start from. We will demonstrate it below.

Finally, at this point, can we completely forget the first form (15) as far as our main concern is on the mCH? The answer is absolutely *no*. An important result immediately follows from this form (again, this has been implied in [10]).

Theorem 2.3. *For any $p \geq 1$, the solution u of the mCH is stable in the sense that $\|u\|_\infty < \infty$.*

Proof. The conservation of $\tilde{H}$ and the Sobolev inequality $\|u\|_\infty \leq c\|u\|_{H^1}$ implies the claim. (Note that the coefficients in $\tilde{H}$ is all positive.) □

Discrete counterparts of this theorem will imply the stability of numerical solutions below.

3 Nonlinear Conservative Scheme

In this section, we propose a nonlinear conservative scheme for general p. Stability and unique solvability of the scheme are discussed. We also show an error analysis for the case $p = 2$.

3.1 A Nonlinear Scheme and Its Basic Properties

Let us define a discrete version of $H = um/2$ by

$$H_k^{(n)} := \frac{U_k^{(n)} M_k^{(n)}}{2}, \tag{16}$$

where $M_k^{(n)}$ is related to $U_k^{(n)}$ by $M_k^{(n)} = (1 - \delta_k^{\langle 2 \rangle})^p U_k^{(n)}$. Analogously to the continuous case (14), we have

$$\begin{aligned}
&\frac{1}{\Delta t} \left(\sum_{k=0}^{N-1} H_k^{(n+1)} \Delta x - \sum_{k=0}^{N-1} H_k^{(n)} \Delta x \right) \\
&= \sum_{k=0}^{N-1} \left(\frac{M_k^{(n+\frac{1}{2})}}{2} \frac{U_k^{(n+1)} - U_k^{(n)}}{\Delta t} + \frac{U_k^{(n+\frac{1}{2})}}{2} \frac{M_k^{(n+1)} - M_k^{(n)}}{\Delta t} \right) \Delta x \\
&= \sum_{k=0}^{N-1} U_k^{(n+\frac{1}{2})} \frac{M_k^{(n+1)} - M_k^{(n)}}{\Delta t} \Delta x,
\end{aligned} \tag{17}$$

where again we have used the relation $M_k^{(n)} = (1 - \delta_k^{\langle 2 \rangle})^p U_k^{(n)}$ and its inverse $U_k^{(n)} = (1 - \delta_k^{\langle 2 \rangle})^{-p} M_k^{(n)}$. Then it is natural to define a "discrete variational derivative" that approximate $\delta H / \delta m = u$ by

$$\frac{\delta H}{\delta(\boldsymbol{M}^{(n+1)}, \boldsymbol{M}^{(n)})_k} := U_k^{(n+\frac{1}{2})}.$$

Now we are in a position to define a nonlinear finite difference scheme.

Scheme 1 (Nonlinear conservative scheme)**.** *We define the initial approximate solution by* $U_k^{(0)} = u(0, k\Delta x)$ $(k = 0, 1, \ldots, N-1)$*. Then for* $n = 0, 1, \ldots,$

$$\frac{M_k^{(n+1)} - M_k^{(n)}}{\Delta t} = -(M_k^{(n+\frac{1}{2})} \delta_k^{\langle 1 \rangle} + \delta_k^{\langle 1 \rangle} M_k^{(n+\frac{1}{2})}) \frac{\delta H}{\delta(\boldsymbol{M}^{(n+1)}, \boldsymbol{M}^{(n)})_k} \quad (k = 0, \ldots, N-1). \tag{18}$$

When $p = 1$, Scheme 1 obviously coincides with Takeya–Furihata's nonlinear scheme since

$$\frac{\delta H}{\delta(\boldsymbol{M}^{(n+1)}, \boldsymbol{M}^{(n)})_k} = \frac{\delta \tilde{H}}{\delta(\boldsymbol{M}^{(n+1)}, \boldsymbol{M}^{(n)})_k} = \frac{U_k^{(n+1)} + U_k^{(n)}}{2}.$$

In more general case $p \geq 2$, if we define the discrete version of $\tilde{H}$ by summing (16) by parts as follows, then the scheme should coincide.

Lemma 3.1. *For any* $p \geq 1$*, if we define* $\tilde{H}_k^{(n)}$ *by*

$$\tilde{H}_k^{(n)} = \frac{1}{2} \sum_{i=0}^{\lfloor \frac{p}{2} \rfloor} {}_pC_{2i} (\delta_k^{\langle 2i \rangle} U_k^{(n)})^2 + \frac{1}{4} \sum_{i=0}^{\lfloor \frac{p-1}{2} \rfloor} {}_pC_{2i+1} \left((\delta_k^{\langle 2i \rangle} \delta_k^+ U_k^{(n)})^2 + (\delta_k^{\langle 2i \rangle} \delta_k^- U_k^{(n)})^2 \right),$$

where $\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leq x\}$*, then* $\tilde{H}_k^{(n)}$ *is equivalent to* $H_k^{(n)}$ *in the sense that*

$$\sum_{k=0}^{N-1} H_k^{(n)} \Delta x = \sum_{k=0}^{N-1} \tilde{H}_k^{(n)} \Delta x.$$

Proof. Recall that $M_k^{(n)} = (1-\delta_k^{\langle 2\rangle})^p U_k^{(n)}$. Then

$$
\begin{aligned}
&\sum_{k=0}^{N-1} H_k^{(n)} \Delta x \\
=&\frac{1}{2}\sum_{k=0}^{N-1} U_k^{(n)} (1-\delta_k^{\langle 2\rangle})^p U_k^{(n)} \Delta x \\
=&\frac{1}{2}\sum_{k=0}^{N-1} \left(U_k^{(n)} \sum_{i=0}^{\lfloor \frac{p}{2} \rfloor} {}_pC_{2i} (\delta_k^{\langle 2i\rangle})^2 U_k^{(n)} - U_k^{(n)} \sum_{i=0}^{\lfloor \frac{p-1}{2} \rfloor} {}_pC_{2i+1} \left(\delta_k^{\langle 2\rangle} (\delta_k^{\langle 2i\rangle})^2 U_k^{(n)} \right) \right) \Delta x \\
=&\frac{1}{2}\sum_{k=0}^{N-1} \left(\sum_{i=0}^{\lfloor \frac{p}{2} \rfloor} {}_pC_{2i} (\delta_k^{\langle 2i\rangle} U_k^{(n)})^2 + \frac{1}{2} \sum_{i=0}^{\lfloor \frac{p-1}{2} \rfloor} {}_pC_{2i+1} \left((\delta_k^{\langle 2i\rangle} \delta_k^{+} U_k^{(n)})^2 + (\delta_k^{\langle 2i\rangle} \delta_k^{-} U_k^{(n)})^2 \right) \right) \Delta x \\
=&\sum_{k=0}^{N-1} \tilde{H}_k^{(n)} \Delta x.
\end{aligned}
$$

In the third equality, we have repeatedly used the summation-by-parts formula (6). □

This Lemma corresponds to Lemma 2.2.

Numerical solutions by Scheme 1 conserve two discrete invariants under the discrete periodic boundary condition.

Theorem 3.2 (Scheme 1: discrete conservation laws)**.** *Under the discrete periodic boundary condition, the numerical solution by Scheme 1 conserve the two invariants: for* $n = 1, 2, \ldots,$

$$
\sum_{k=0}^{N-1} U_k^{(n)} \Delta x = \sum_{k=0}^{N-1} U_k^{(0)} \Delta x, \qquad \sum_{k=0}^{N-1} H_k^{(n)} \Delta x = \sum_{k=0}^{N-1} H_k^{(0)} \Delta x.
$$

Proof. We first note that $(M_k^{(n+\frac{1}{2})} \delta_k^{\langle 1\rangle} + \delta_k^{\langle 1\rangle} M_k^{(n+\frac{1}{2})})$ is a skew-symmetric operator: for any $\boldsymbol{f}, \boldsymbol{g} \in \mathbb{R}^{(N)}$, it holds

$$
\begin{aligned}
&\sum_{k=0}^{N-1} f_k \cdot (M_k^{(n+\frac{1}{2})} \delta_k^{\langle 1\rangle} + \delta_k^{\langle 1\rangle} M_k^{(n+\frac{1}{2})}) g_k \Delta x \\
&= -\sum_{k=0}^{N-1} \left((M_k^{(n+\frac{1}{2})} \delta_k^{\langle 1\rangle} + \delta_k^{\langle 1\rangle} M_k^{(n+\frac{1}{2})}) f_k \right) \cdot g_k \Delta x,
\end{aligned} \tag{19}
$$

which is readily seen using (5). Then the proof for $H_k^{(n)}$ goes in the same way as in (9). In fact,

$$
\begin{aligned}
&\frac{1}{\Delta t} \sum_{k=0}^{N-1} (H_k^{(n+1)} - H_k^{(n)}) \Delta x \\
&= \sum_{k=0}^{N-1} \left\{ \frac{\delta H}{\delta(\boldsymbol{M}^{(n+1)}, \boldsymbol{M}^{(n)})_k} \cdot \frac{M_k^{(n+1)} - M_k^{(n)}}{\Delta t} \right\} \Delta x \\
&= \sum_{k=0}^{N-1} \frac{\delta H}{\delta(\boldsymbol{M}^{(n+1)}, \boldsymbol{M}^{(n)})_k} \left\{ -(M_k^{(n+\frac{1}{2})} \delta_k^{\langle 1\rangle} + \delta_k^{\langle 1\rangle} M_k^{(n+\frac{1}{2})}) \frac{\delta H}{\delta(\boldsymbol{M}^{(n+1)}, \boldsymbol{M}^{(n)})_k} \right\} \Delta x
\end{aligned}
$$

$=0.$

Here we have used (17), (18), and (19) in turn.

The first conservation law can be also proved by following the continuous case (10). We first note that it is also sufficient to prove

$$\sum_{k=0}^{N-1} M_k^{(n)}\Delta x = \sum_{k=0}^{N-1} M_k^{(0)}\Delta x, \tag{20}$$

with $\sum_{k=0}^{N-1}(\delta_k^{\langle 2\rangle})^l U_k^{(n)}\Delta x = 0$ $(l=1,2,\ldots)$ in mind, which holds in light of (6) (set $f_k = 1$ and $g_k = U_k^{(n)}$ there). Then the proof of (20) goes, as a discrete analogue of (10),

$$\begin{aligned}
&\frac{1}{\Delta t}\sum_{k=0}^{N-1}(M_k^{(n+1)} - M_k^{(n)})\Delta x\\
&= -\sum_{k=0}^{N-1}(M_k^{(n+\frac{1}{2})}\delta_k^{\langle 1\rangle} + \delta_k^{\langle 1\rangle} M_k^{(n+\frac{1}{2})})U_k^{(n+\frac{1}{2})}\Delta x\\
&= -\sum_{k=0}^{N-1}(M_k^{(n+\frac{1}{2})}\cdot\delta_k^{\langle 1\rangle}U_k^{(n+\frac{1}{2})} + \delta_k^{\langle 1\rangle}(M_k^{(n+\frac{1}{2})}U_k^{(n+\frac{1}{2})}))\Delta x\\
&= \sum_{k=0}^{N-1}\delta_k^{\langle 1\rangle}(1-\delta_k^{\langle 2\rangle})^p U_k^{(n+\frac{1}{2})}\cdot U_k^{(n+\frac{1}{2})}\Delta x\\
&=0.
\end{aligned}$$

We have used (5) with $f_k = 1, g_k = M_k^{(n+\frac{1}{2})}U_k^{(n+\frac{1}{2})}$, and the skew-symmetry of $\delta_k^{\langle 1\rangle}(1-\delta_k^{\langle 2\rangle})^p$ (recall Corollary 2.1). □

Remark 4. As is evident in the proof, the H conservation law is guaranteed, as long as the discretization of $(m\partial_x + \partial_x m)$ is skew-symmetric. Although there are other skew-symmetric discretizations such as $(M_k^{(n)}\delta_k^{\langle 1\rangle} + \delta_k^{\langle 1\rangle}M_k^{(n)})$ or $(M_k^{(n+1)}\delta_k^{\langle 1\rangle} + \delta_k^{\langle 1\rangle}M_k^{(n+1)})$, they are less attractive from the following two reasons: (i) the time symmetry of the resulting scheme would be broken, and consequently the order of accuracy would decrease; (ii) the conservation law (20) would be lost (check the above proof, with the notice after (10) in mind. Careless discretization should destroy the proof). This remark applies also to the linearly implicit schemes below. □

From this conservation laws, stability of numerical solutions by Scheme 1 is guaranteed.

Theorem 3.3 (Stability of Scheme 1)**.** *Scheme 1 is stable in the sense that*

$$\|\boldsymbol{U}^{(n)}\|_\infty < \infty.$$

Proof. First we note the estimates: $\|\boldsymbol{U}^{(n)}\| < \infty, \|D^+\boldsymbol{U}^{(n)}\| < \infty$, which can be easily seen from Lemma 3.1 and Theorem 3.2. Note that $\tilde{H}_k^{(n)}$ can be written as $\tilde{H}_k^{(n)} = c_0(U_k^{(n)})^2 + c_1((\delta_k^+ U_k^{(n)})^2 + (\delta_k^- U_k^{(n)})^2) + c_2$ with some positive constants c_0, c_1, c_2. Then $\|\boldsymbol{U}^{(n)}\|_\infty < \infty$ follows from the discrete Sobolev inequality: for any $\boldsymbol{f} \in \mathbb{R}^{(n)}$, $\|\boldsymbol{f}\|_\infty \le c\|\boldsymbol{f}\|_{H^1}$, where $\|\boldsymbol{f}\|_{H^1} := \|\boldsymbol{f}\|^2 + \|D^+\boldsymbol{f}\|^2$ (see, for example, Furihata–Matsuo [5, Section 3.6]). □

Since Scheme 1 is nonlinear, it requires nonlinear solvers in each time step. The next theorem states that if we set time step Δt adequately based on $M_k^{(n)} = (1-\delta_k^{\langle 2\rangle})^p U_k^{(n)}$, unique existence of the solution is guaranteed.

Theorem 3.4 (unique solvability of Scheme 1)**.** *Let $\boldsymbol{U}^{(n)}$ be given. If Δt satisfies*

$$\Delta t \leq \frac{\Delta x^{3/2}}{6K}, \text{ where } K = \sup_n \|\boldsymbol{M}^{(n)}\|,$$

Scheme 1 has a unique numerical solution $\boldsymbol{U}^{(n+1)}$.

Proof. The proof bases on the contraction mapping theorem. The case $p = 1$ has been already proved in [14, 15], and the general case $p \geq 2$ is almost trivial extension (see also a similar argument for the Degasperis–Procesi equation [11, 12]). □

3.2 Convergence of Scheme 1 with $p = 2$

In this subsection we consider the convergence of numerical solutions obtained by Scheme 1, limiting ourselves to the case $p = 2$.

We begin by estimating the local truncation error. We define $u_k^{(n)}$ and $m_k^{(n)}$ by $u_k^{(n)} = u(n\Delta t, k\Delta x)$ and $m_k^{(n)} = m(n\Delta t, k\Delta x)$.

Theorem 3.5 (Local truncation error of Scheme 1)**.** *Assume $u(t,\cdot) \in C^7(\mathbb{S})$ and $u(\cdot,x) \in C^3([0,T])$. Then the local truncation error of Scheme 1 is of order $O(\Delta t^2 + \Delta x^2)$.*

Proof. Making use of the Taylor expansion around the point $(t,x) = ((n+\frac{1}{2})\Delta t, k\Delta x)$, we find that

$$\frac{m_k^{(n+1)} - m_k^{(n)}}{\Delta t} = -\left(m_k^{(n+\frac{1}{2})} \delta_k^{\langle 1\rangle} + \delta_k^{\langle 1\rangle} m_k^{(n+\frac{1}{2})} \right) u_k^{(n+\frac{1}{2})} + O(\Delta t^2 + \Delta x^2). \tag{21}$$

□

Now we are in a position to state our main error estimate.

Theorem 3.6 (L^2 and L^∞ error estimates of Scheme 1)**.** *Assume that $u(t,\cdot) \in C^7(\mathbb{S})$ and $u(\cdot,x) \in C^3([0,T])$. Then the numerical solution $\boldsymbol{U}^{(n)}$ by Scheme 1 converges to the exact solution with order $O(\Delta t^2 + \Delta x^2)$ both in $\|\cdot\|_2$ and $\|\cdot\|_\infty$.*

Proof. Let $e_k^{(n)} = U_k^{(n)} - u_k^{(n)}$. From (18) and (21), $e_k^{(n)}$ satisfies

$$\begin{aligned}
&(1 - 2\delta_k^{\langle 2\rangle} + \delta_k^{\langle 4\rangle}) \left(\frac{e_k^{(n+1)} - e_k^{(n)}}{\Delta t} \right) \\
&= - \left\{ M_k^{(n+\frac{1}{2})} \delta_k^{\langle 1\rangle} U_k^{(n+\frac{1}{2})} - m_k^{(n+\frac{1}{2})} \delta_k^{\langle 1\rangle} u_k^{(n+\frac{1}{2})} \right\} \\
&\quad - \left\{ \delta_k^{\langle 1\rangle} (M_k^{(n+\frac{1}{2})} U_k^{(n+\frac{1}{2})}) - \delta_k^{\langle 1\rangle} (m_k^{(n+\frac{1}{2})} u_k^{(n+\frac{1}{2})}) \right\} + O(\Delta t^2 + \Delta x^2) \\
&= - \frac{1}{2} \left\{ (M_k^{(n+\frac{1}{2})} + m_k^{(n+\frac{1}{2})}) \delta_k^{\langle 1\rangle} (U_k^{(n+\frac{1}{2})} - u_k^{(n+\frac{1}{2})}) \right. \\
&\quad \left. + (M_k^{(n+\frac{1}{2})} - m_k^{(n+\frac{1}{2})}) \delta_k^{\langle 1\rangle} (U_k^{(n+\frac{1}{2})} + u_k^{(n+\frac{1}{2})}) \right\} \\
&\quad - \frac{1}{2} \left\{ \delta_k^{\langle 1\rangle} ((M_k^{(n+\frac{1}{2})} + m_k^{(n+\frac{1}{2})})(U_k^{(n+\frac{1}{2})} - u_k^{(n+\frac{1}{2})})) \right.
\end{aligned}$$

$$
\begin{aligned}
&\quad + \delta_k^{\langle 1\rangle}((M_k^{(n+\frac{1}{2})} - m_k^{(n+\frac{1}{2})})(U_k^{(n+\frac{1}{2})} + u_k^{(n+\frac{1}{2})}))\Big\} + O(\Delta t^2 + \Delta x^2) \\
&= -\frac{1}{2}((1-\delta_k^{\langle 2\rangle})^2(U_k^{(n+\frac{1}{2})} + u_k^{(n+\frac{1}{2})}))\delta_k^{\langle 1\rangle} e_k^{(n+\frac{1}{2})} \\
&\quad - \frac{1}{2}((1-\delta_k^{\langle 2\rangle})^2 e_k^{(n+\frac{1}{2})})\delta_k^{\langle 1\rangle}(U_k^{(n+\frac{1}{2})} + u_k^{(n+\frac{1}{2})}) \\
&\quad - \frac{1}{2}\delta_k^{\langle 1\rangle}\left(((1-\delta_k^{\langle 2\rangle})^2(U_k^{(n+\frac{1}{2})} + u_k^{(n+\frac{1}{2})}))e_k^{(n+\frac{1}{2})}\right) \\
&\quad - \frac{1}{2}\delta_k^{\langle 1\rangle}\left(((1-\delta_k^{\langle 2\rangle})^2 e_k^{(n+\frac{1}{2})})(U_k^{(n+\frac{1}{2})} + u_k^{(n+\frac{1}{2})})\right) + O(\Delta t^2 + \Delta x^2) \\
&= A_1 + A_2 + A_3 + A_4 + A_5, \qquad (22)
\end{aligned}
$$

where the last equality defines A_i's as each term of the left hand side. Taking inner product (22) with $\boldsymbol{e}^{(n+\frac{1}{2})}$, we obtain

$$
\begin{aligned}
&\frac{\|\boldsymbol{e}^{(n+1)}\|^2 - \|\boldsymbol{e}^{(n)}\|^2}{2\Delta t} + \frac{\|D^+\boldsymbol{e}^{(n+1)}\|^2 - \|D^+\boldsymbol{e}^{(n)}\|^2}{\Delta t} \\
&\quad + \frac{\|D^{\langle 2\rangle}\boldsymbol{e}^{(n+1)}\|^2 - \|D^{\langle 2\rangle}\boldsymbol{e}^{(n)}\|^2}{2\Delta t} = \left(A_1 + A_2 + A_3 + A_4 + A_5, \boldsymbol{e}^{(n+\frac{1}{2})}\right),
\end{aligned}
$$

where the inner product $(\cdot,\cdot)$ has been introduced in Section 2.1.

In what follows, our main task is to show

$$
\begin{aligned}
&\left(A_1 + A_2 + A_3 + A_4 + A_5, \boldsymbol{e}^{(n+\frac{1}{2})}\right) \\
&\quad \leq c\Big(\|\boldsymbol{e}^{(n+1)}\|^2 + \|\boldsymbol{e}^{(n)}\|^2 + \|D^+\boldsymbol{e}^{(n+1)}\|^2 + \|D^+\boldsymbol{e}^{(n)}\|^2 \\
&\qquad + \|D^{\langle 2\rangle}\boldsymbol{e}^{(n+1)}\|^2 + \|D^{\langle 2\rangle}\boldsymbol{e}^{(n)}\|^2\Big) + \left(O(\Delta t^2 + \Delta x^2)\right)^2, \qquad (23)
\end{aligned}
$$

where hereafter we promise that c is a generic constant independent of the mesh sizes Δt and Δx. As soon as this is shown, we immediately see

$$
\begin{aligned}
&\|\boldsymbol{e}^{(n+1)}\|^2 + \|D^+\boldsymbol{e}^{(n+1)}\|^2 + \|D^{\langle 2\rangle}\boldsymbol{e}^{(n+1)}\|^2 \\
&\quad \leq \left(O(\Delta t^2 + \Delta x^2)\right)^2 + c\Delta t\sum_{l=1}^{n+1}\left(\|\boldsymbol{e}^{(l)}\|^2 + \|D^+\boldsymbol{e}^{(l)}\|^2 + \|D^{\langle 2\rangle}\boldsymbol{e}^{(l)}\|^2\right),
\end{aligned}
$$

and the estimate in turn implies by the discrete Gronwall inequality (see, for example, [5]) $\|\boldsymbol{e}^{(n)}\| + \|D^+\boldsymbol{e}^{(n)}\| + \|D^{\langle 2\rangle}\boldsymbol{e}^{(n)}\| \leq O(\Delta t^2 + \Delta x^2)$. This and the discrete Sobolev lemma yield the final claim regarding $\|\cdot\|_\infty$.

Now, we prove (23). Due to the skew-symmetry of $\delta_k^{\langle 1\rangle}$, obviously $(A_1, \boldsymbol{e}^{(n+\frac{1}{2})}) + (A_3, \boldsymbol{e}^{(n+\frac{1}{2})}) = 0$. Noting $U_k^{(n+\frac{1}{2})} = e_k^{(n+\frac{1}{2})} + u_k^{(n+\frac{1}{2})}$, we also see

$$
\begin{aligned}
&\left(A_2, \boldsymbol{e}^{(n+\frac{1}{2})}\right) + \left(A_4, \boldsymbol{e}^{(n+\frac{1}{2})}\right) \\
&= -\frac{1}{2}\sum_{k=0}^{N-1}(1-\delta_k^{\langle 2\rangle})^2 e_k^{(n+\frac{1}{2})} \cdot \delta_k^{\langle 1\rangle}(e_k^{(n+\frac{1}{2})} + 2u_k^{(n+\frac{1}{2})}) \cdot e_k^{(n+\frac{1}{2})}\Delta x \\
&\quad - \frac{1}{2}\sum_{k=0}^{N-1}\delta_k^{\langle 1\rangle}\left((1-\delta_k^{\langle 2\rangle})^2 e_k^{(n+\frac{1}{2})} \cdot (e_k^{(n+\frac{1}{2})} + 2u_k^{(n+\frac{1}{2})})\right) \cdot e_k^{(n+\frac{1}{2})}\Delta x \\
&= -\frac{1}{2}\sum_{k=0}^{N-1}(1-\delta_k^{\langle 2\rangle})^2 e_k^{(n+\frac{1}{2})} \cdot \delta_k^{\langle 1\rangle} e_k^{(n+\frac{1}{2})} \cdot e_k^{(n+\frac{1}{2})}\Delta x
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2}\sum_{k=0}^{N-1} \delta_k^{\langle 1\rangle}\left((1-\delta_k^{\langle 2\rangle})^2 e_k^{(n+\frac{1}{2})}\cdot e_k^{(n+\frac{1}{2})}\right)\cdot e_k^{(n+\frac{1}{2})}\Delta x\\
&- \sum_{k=0}^{N-1}(1-\delta_k^{\langle 2\rangle})^2 e_k^{(n+\frac{1}{2})}\cdot \delta_k^{\langle 1\rangle}u_k^{(n+\frac{1}{2})}\cdot e_k^{(n+\frac{1}{2})}\Delta x\\
&- \sum_{k=0}^{N-1} \delta_k^{\langle 1\rangle}\left((1-\delta_k^{\langle 2\rangle})^2 e_k^{(n+\frac{1}{2})}\cdot u_k^{(n+\frac{1}{2})}\right)\cdot e_k^{(n+\frac{1}{2})}\Delta x\\
=&- \sum_{k=0}^{N-1}(1-\delta_k^{\langle 2\rangle})^2 e_k^{(n+\frac{1}{2})}\cdot \delta_k^{\langle 1\rangle}u_k^{(n+\frac{1}{2})}\cdot e_k^{(n+\frac{1}{2})}\Delta x\\
&+ \sum_{k=0}^{N-1}(1-\delta_k^{\langle 2\rangle})^2 e_k^{(n+\frac{1}{2})}\cdot u_k^{(n+\frac{1}{2})}\cdot \delta_k^{\langle 1\rangle}e_k^{(n+\frac{1}{2})}\Delta x\\
=&B_1+B_2, \qquad (24)
\end{aligned}
$$

where B_i denotes the ith term of the left hand side. In the third equality, the summation-by-parts formula (5) was repeatedly used.

We first estimate B_1. Noticing $(1-\delta_k^{\langle 2\rangle})^2 = 1 - 2\delta_k^{\langle 2\rangle} + \delta_k^{\langle 2\rangle}\delta_k^{\langle 2\rangle}$, and recalling the summation-by-parts formula (6), we obtain

$$
\begin{aligned}
|B_1| \quad \leq \quad & \|D^{\langle 1\rangle}\boldsymbol{u}^{(n+\frac{1}{2})}\|_\infty \|\boldsymbol{e}^{(n+\frac{1}{2})}\|^2 + 2\|D^{\langle 1\rangle}\boldsymbol{u}^{(n+\frac{1}{2})}\|_\infty \|D^{\langle 2\rangle}\boldsymbol{e}^{(n+\frac{1}{2})}\|\,\|\boldsymbol{e}^{(n+\frac{1}{2})}\|\\
&+\left|\sum_{k=0}^{N-1}\delta_k^{\langle 2\rangle}e_k^{(n+\frac{1}{2})}\cdot\delta_k^{\langle 2\rangle}\left(\delta_k^{\langle 1\rangle}u_k^{(n+\frac{1}{2})}\cdot e_k^{(n+\frac{1}{2})}\right)\Delta x\right|.
\end{aligned}
$$

In order to estimate the last term, we need the following identity:

$$
\delta_k^{\langle 2\rangle}(f_k g_k) \quad = \quad \delta_k^{\langle 2\rangle}f_k\cdot\mu_k^{\langle 2\rangle}g_k + 2\delta_k^{\langle 1\rangle}f_k\cdot\delta_k^{\langle 1\rangle}g_k + \mu_k^{\langle 2\rangle}f_k\cdot\delta_k^{\langle 2\rangle}g_k, \qquad (25)
$$

which corresponds to $\partial_x^2(fg) = f_{xx}g + 2f_xg_x + fg_{xx}$. From this we see

$$
\begin{aligned}
&\left|\sum_{k=0}^{N-1}\delta_k^{\langle 2\rangle}e_k^{(n+\frac{1}{2})}\cdot\delta_k^{\langle 2\rangle}\left(\delta_k^{\langle 1\rangle}u_k^{(n+\frac{1}{2})}\cdot e_k^{(n+\frac{1}{2})}\right)\Delta x\right|\\
&\quad\leq \quad \|D^{\langle 3\rangle}\boldsymbol{u}^{(n+\frac{1}{2})}\|_\infty\|D^{\langle 2\rangle}\boldsymbol{e}^{(n+\frac{1}{2})}\|\,\|\boldsymbol{e}^{(n+\frac{1}{2})}\|\\
&\qquad\qquad + 2\|D^{\langle 2\rangle}\boldsymbol{u}^{(n+\frac{1}{2})}\|_\infty\|D^{\langle 2\rangle}\boldsymbol{e}^{(n+\frac{1}{2})}\|\,\|D^{\langle 1\rangle}\boldsymbol{e}^{(n+\frac{1}{2})}\|\\
&\qquad\qquad + \|D^{\langle 1\rangle}\boldsymbol{u}^{(n+\frac{1}{2})}\|_\infty\|D^{\langle 2\rangle}\boldsymbol{e}^{(n+\frac{1}{2})}\|^2.
\end{aligned}
$$

Note that the terms such as $\|D^{\langle 1\rangle}\boldsymbol{u}^{(n+\frac{1}{2})}\|_\infty$ can be bounded independent of the mesh sizes, thanks to the regularity assumption on u. Thus, summing up the estimates above, we obtain

$$
\begin{aligned}
|B_1| \quad \leq \quad & c\Big(\|\boldsymbol{e}^{(n+\frac{1}{2})}\|^2 + \|D^{\langle 2\rangle}\boldsymbol{e}^{(n+\frac{1}{2})}\|\,\|\boldsymbol{e}^{(n+\frac{1}{2})}\| + \|D^{\langle 2\rangle}\boldsymbol{e}^{(n+\frac{1}{2})}\|\,\|D^{\langle 1\rangle}\boldsymbol{e}^{(n+\frac{1}{2})}\|\\
&+\|D^{\langle 2\rangle}\boldsymbol{e}^{(n+\frac{1}{2})}\|^2\Big). \qquad (26)
\end{aligned}
$$

The estimate of B_2 goes in a similar manner.

$$
\begin{aligned}
|B_2| \leq &\|\boldsymbol{u}^{(n+\frac{1}{2})}\|_\infty\|\boldsymbol{e}^{(n+\frac{1}{2})}\|\,\|D^{\langle 1\rangle}\boldsymbol{e}^{(n+\frac{1}{2})}\|\\
&+ 2\|\boldsymbol{u}^{(n+\frac{1}{2})}\|_\infty\|D^{\langle 2\rangle}\boldsymbol{e}^{(n+\frac{1}{2})}\|\,\|D^{\langle 1\rangle}\boldsymbol{e}^{(n+\frac{1}{2})}\|
\end{aligned}
$$

$$+\left|\sum_{k=0}^{N-1}\delta_k^{\langle 2\rangle}e_k^{(n+\frac{1}{2})}\cdot\delta_k^{\langle 2\rangle}\left(u_k^{(n+\frac{1}{2})}\cdot\delta_k^{\langle 1\rangle}e_k^{(n+\frac{1}{2})}\right)\Delta x\right|.$$

The third term can be expanded again with (25) to find

$$\begin{aligned}&\left|\sum_{k=0}^{N-1}\delta_k^{\langle 2\rangle}e_k^{(n+\frac{1}{2})}\cdot\delta_k^{\langle 2\rangle}(u_k^{(n+\frac{1}{2})}\cdot\delta_k^{\langle 1\rangle}e_k^{(n+\frac{1}{2})})\Delta x\right|\\ \leq\ &\|D^{\langle 2\rangle}\boldsymbol{u}^{(n+\frac{1}{2})}\|_\infty\|D^{\langle 2\rangle}\boldsymbol{e}^{(n+\frac{1}{2})}\|\,\|D^{\langle 1\rangle}\boldsymbol{e}^{(n+\frac{1}{2})}\|\\ &+2\|D^{\langle 1\rangle}\boldsymbol{u}^{(n+\frac{1}{2})}\|_\infty\|D^{\langle 2\rangle}\boldsymbol{e}^{(n+\frac{1}{2})}\|^2\\ &+\left|\sum_{k=0}^{N-1}\delta_k^{\langle 2\rangle}e_k^{(n+\frac{1}{2})}\cdot\mu_k^{\langle 2\rangle}u_k^{(n+\frac{1}{2})}\cdot\delta_k^{\langle 3\rangle}e_k^{(n+\frac{1}{2})}\Delta x\right|.\end{aligned}$$

In order to eliminate the (discrete) third derivative $\delta_k^{\langle 3\rangle}e_k^{(n+\frac{1}{2})}$ in the last term (recall our goal (23)), we need a tricky calculation here. With a discrete Leibniz's rule (which can be easily checked): $\delta_k^{\langle 1\rangle}(f_kg_k)=(\delta_k^{\langle 1\rangle}f_k)\cdot\mu_k^{\langle 1\rangle}g_k+\mu_k^{\langle 1\rangle}f_k\cdot(\delta_k^{\langle 1\rangle}g_k)$ in mind, it is straightforward to see

$$\begin{aligned}&\sum_{k=0}^{N-1}\delta_k^{\langle 2\rangle}e_k^{(n+\frac{1}{2})}\cdot u_k^{(n+\frac{1}{2})}\cdot\delta_k^{\langle 3\rangle}e_k^{(n+\frac{1}{2})}\Delta x\\ =&-\sum_{k=0}^{N-1}\delta_k^{\langle 3\rangle}e_k^{(n+\frac{1}{2})}\cdot\mu_k^{\langle 1\rangle}u_k^{(n+\frac{1}{2})}\cdot\delta_k^{\langle 2\rangle}e_k^{(n+\frac{1}{2})}\Delta x\\ &-\sum_{k=0}^{N-1}\delta_k^{\langle 2\rangle}\mu_k^{\langle 1\rangle}e_k^{(n+\frac{1}{2})}\cdot\delta_k^{\langle 1\rangle}u_k^{(n+\frac{1}{2})}\cdot\delta_k^{\langle 2\rangle}e_k^{(n+\frac{1}{2})}\Delta x.\end{aligned}$$

By transposing the first term in the right hand side to the left, we obtain

$$\begin{aligned}&\sum_{k=0}^{N-1}\delta_k^{\langle 2\rangle}e_k^{(n+\frac{1}{2})}\cdot\mu_k^{\langle 2\rangle}u_k^{(n+\frac{1}{2})}\cdot\delta_k^{\langle 3\rangle}e_k^{(n+\frac{1}{2})}\Delta x\\ =&-\frac{1}{2}\sum_{k=0}^{N-1}\delta_k^{\langle 2\rangle}\mu_k^{\langle 1\rangle}e_k^{(n+\frac{1}{2})}\cdot\delta_k^{\langle 1\rangle}u_k^{(n+\frac{1}{2})}\cdot\delta_k^{\langle 2\rangle}e_k^{(n+\frac{1}{2})}\Delta x,\end{aligned}$$

where a trivial identity: $(1+\mu_k^{\langle 1\rangle})f_k=2\mu_k^{\langle 2\rangle}f_k$ that holds for any $\boldsymbol{f}\in\mathbb{R}^{(N)}$ was used. Thus we complete the estimate of B_2 as

$$|B_2|\ \leq\ c\left(\|\boldsymbol{e}^{(n+\frac{1}{2})}\|\,\|D^{\langle 1\rangle}\boldsymbol{e}^{(n+\frac{1}{2})}\|+\|D^{\langle 2\rangle}\boldsymbol{e}^{(n+\frac{1}{2})}\|\,\|D^{\langle 1\rangle}\boldsymbol{e}^{(n+\frac{1}{2})}\|+\|D^{\langle 2\rangle}\boldsymbol{e}^{(n+\frac{1}{2})}\|^2\right).\quad(27)$$

Collecting (24), (26), (27), and having obvious identities $\|D^{\langle 1\rangle}\boldsymbol{e}^{(n)}\|\leq\|D^{+}\boldsymbol{e}^{(n)}\|$ and $\|\boldsymbol{e}^{(n+\frac{1}{2})}\|^2\leq\|\boldsymbol{e}^{(n+1)}\|^2+\|\boldsymbol{e}^{(n)}\|^2$ in mind, we eventually obtain

$$\begin{aligned}&\left(A_2,\boldsymbol{e}^{(n+\frac{1}{2})}\right)+\left(A_4,\boldsymbol{e}^{(n+\frac{1}{2})}\right)\\ &\leq\ c\left(\|\boldsymbol{e}^{(n+1)}\|^2+\|\boldsymbol{e}^{(n)}\|^2+\|D^{+}\boldsymbol{e}^{(n+1)}\|^2+\|D^{+}\boldsymbol{e}^{(n)}\|^2+\|D^{\langle 2\rangle}\boldsymbol{e}^{(n+1)}\|^2+\|D^{\langle 2\rangle}\boldsymbol{e}^{(n)}\|^2\right).\end{aligned}$$

The remaining term A_5 can be easily estimated as

$$\left(A_5,\boldsymbol{e}^{(n+\frac{1}{2})}\right)\leq\left(O(\Delta t^2+\Delta x^2)\right)^2+\|\boldsymbol{e}^{(n+1)}\|^2+\|\boldsymbol{e}^{(n)}\|^2.$$

This completes the proof of (23). □

4 Linear Conservative Schemes

In this section we propose two linear conservative schemes. Similar to Scheme 1 (nonlinear scheme), we first extend Takeya–Furihata's scheme to $p \geq 2$. Then next we show that a slightly different linear conservative scheme can be constructed. The linearization technique utilized throughout this section was first introduced in [8] (see also [3, 5]).

4.1 A Linear Scheme Based on Takeya–Furihata's Scheme

4.1.1 A Linear Conservative Scheme and Its Basic Properties

We here define a multistep discrete energy function by

$$H_k^{(n+\frac{1}{2})} := \frac{U_k^{(n+1)} M_k^{(n)} + U_k^{(n)} M_k^{(n+1)}}{4},$$

from which the associated (multistep) discrete variational derivative can be deduced as

$$\frac{\delta H}{\delta(\boldsymbol{M}^{(n+1)}, \boldsymbol{M}^{(n)}, \boldsymbol{M}^{(n-1)})_k} := U_k^{(n)}.$$

Note that the former approximates $H = um/2$, and the latter $\delta H / \delta m = u$. We omit the detail of the derivation to save space; interested readers may refer the references mentioned above. At the moment, it suffices to realize that the following identity holds similar to (17) (this can be easily checked):

$$\begin{aligned} &\sum_{k=0}^{N-1} H_k^{(n+\frac{1}{2})} \Delta x - \sum_{k=0}^{N-1} H_k^{(n-\frac{1}{2})} \Delta x \\ &\quad = \sum_{k=0}^{N-1} \left\{ \frac{\delta H}{\delta(\boldsymbol{M}^{(n+1)}, \boldsymbol{M}^{(n)}, \boldsymbol{M}^{(n-1)})_k} \frac{M_k^{(n+1)} - M_k^{(n-1)}}{2} \right\} \Delta x. \end{aligned} \tag{28}$$

Then we define the following linear finite difference scheme.

Scheme 2 (Linear conservative scheme)**.** *We define the initial approximate solution by* $U_k^{(0)} = u(0, k\Delta x)$ $(k = 0, 1, \ldots, N-1)$*. Then for* $n = 1, 2, \ldots,$

$$\frac{M_k^{(n+1)} - M_k^{(n-1)}}{2\Delta t} = -(M_k^{(n)} \delta_k^{\langle 1 \rangle} + \delta_k^{\langle 1 \rangle} M_k^{(n)}) \frac{\delta H}{\delta(\boldsymbol{M}^{(n+1)}, \boldsymbol{M}^{(n)}, \boldsymbol{M}^{(n-1)})_k} \quad (k = 0, \ldots, N-1). \tag{29}$$

Obviously (29) corresponds to (13). When $p = 1$, Scheme 2 coincides with Takeya–Furihata's linear conservative scheme; in fact, (29) coincides with (12), and furthermore, the above defined multistep energy function $H_k^{(n+\frac{1}{2})}$ is equivalent to Takeya–Furihata's $\tilde{H}_k^{(n+\frac{1}{2})}$ defined in (11) as the next lemma shows.

Lemma 4.1. *When* $p = 1$*,* $\tilde{H}_k^{(n+\frac{1}{2})}$ *is equivalent to* $H_k^{(n+\frac{1}{2})}$ *in the following sense: for* $n = 0, 1, 2, \ldots,$

$$\sum_{k=0}^{N-1} H_k^{(n+\frac{1}{2})} \Delta x = \sum_{k=0}^{N-1} \tilde{H}_k^{(n+\frac{1}{2})} \Delta x.$$

Proof. Recall that $M_k^{(n)} = (1-\delta_k^{\langle 2\rangle})U_k^{(n)}$ when $p=1$. Then,

$$\begin{aligned}
&\sum_{k=0}^{N-1} H_k^{(n+\frac{1}{2})}\Delta x \\
&= \sum_{k=0}^{N-1} \frac{U_k^{(n+1)}M_k^{(n)} + U_k^{(n)}M_k^{(n+1)}}{4}\Delta x \\
&= \sum_{k=0}^{N-1} \frac{U_k^{(n+1)}(U_k^{(n)} - \delta_k^{\langle 2\rangle}U_k^{(n)}) + U_k^{(n)}(U_k^{(n+1)} - \delta_k^{\langle 2\rangle}U_k^{(n+1)})}{4}\Delta x \\
&= \sum_{k=0}^{N-1} \frac{2(U_k^{(n+1)})(U_k^{(n)}) + (\delta_k^{+}U_k^{(n+1)})(\delta_k^{+}U_k^{(n)}) + (\delta_k^{-}U_k^{(n+1)})(\delta_k^{-}U_k^{(n)})}{4}\Delta x.
\end{aligned}$$

The third equality is by the summation-by-parts formula (6). □

Similar to Lemma 3.1, we can do the same argument for $p \geq 2$. But since a concrete form of $\tilde{H}_k^{(n+\frac{1}{2})}$ is too cumbersome, and the stability is not guaranteed based on this argument as will be noted soon, we omit it.

Although Scheme 2 might seem explicit at first glance (observe that the right hand side does not include the term on the time step $(n+1)$), but the need of the transformation $\boldsymbol{M}^{(n+1)} \mapsto \boldsymbol{U}^{(n+1)}$ forces the scheme to be substantially linearly "implicit."

We note that since Scheme 2 is a multistep scheme, we need not only the initial value $\boldsymbol{U}^{(0)}$ but also a starting value $\boldsymbol{U}^{(1)}$. If we choose Scheme 1 for the first step, we get the following conservation laws. Note that also other schemes can be used if Δt is chosen appropriately small such that the invariants are kept with enough accuracy.

Theorem 4.2 (Scheme 2: discrete conservation laws)**.** *Assume that Scheme 1 is chosen for the first step in Scheme 2. Then under the discrete periodic boundary condition, the numerical solution by Scheme 2 conserve the two invariants: for $n = 1, 2, \ldots$,*

$$\sum_{k=0}^{N-1} U_k^{(n)}\Delta x = \sum_{k=0}^{N-1} U_k^{(0)}\Delta x, \quad \sum_{k=0}^{N-1} H_k^{(n+\frac{1}{2})}\Delta x = \sum_{k=0}^{N-1} H_k^{(\frac{1}{2})}\Delta x.$$

Proof. The second conservation law is immediate by the same argument as the nonlinear case (Theorem 3.2); note (28) and recall (19). For the first conservation law, it is again immediate to see $\frac{1}{2\Delta t}\sum_{k=0}^{N-1}(M_k^{(n+1)} - M_k^{(n-1)})\Delta x = 0$ by the same argument as in Theorem 3.2, which in turn implies $\sum_{k=0}^{N-1} U_k^{(n+2)}\Delta x = \sum_{k=0}^{N-1} U_k^{(n)}\Delta x$ $(n = 0, 1, 2, \ldots)$. Since in the scheme we demanded the first time step to be conservative: $\sum_{k=0}^{N-1} U_k^{(1)}\Delta x = \sum_{k=0}^{N-1} U_k^{(0)}\Delta x$, $\sum_{k=0}^{N-1} U_k^{(n)}\Delta x = \sum_{k=0}^{N-1} U_k^{(0)}\Delta x$ $(n = 0, 1, 2, \ldots)$ holds for every n. □

Moreover the unique existence of numerical solutions is obvious.

Theorem 4.3 (Unique solvability of Scheme 2)**.** *For $n \geq 2$, Scheme 2 has a unique numerical solution $\boldsymbol{U}^{(n+1)}$ independent of prescribed $\Delta t, \Delta x$.*

Proof. Scheme 2 can be rewritten as follows:

$$(I - D^{\langle 2\rangle})^p \boldsymbol{U}^{(n+1)} = \boldsymbol{F}(\boldsymbol{U}^{(n)}, \boldsymbol{U}^{(n-1)}),$$

where $\boldsymbol{F}$ denotes the remaining terms with $\boldsymbol{U}^{(n)}, \boldsymbol{U}^{(n-1)}$. Since the coefficient matrix of the left hand side is a constant nonsingular matrix $(I - D^{\langle 2\rangle})^p$, the unique existence of $\boldsymbol{U}^{(n+1)}$ follows. □

Although Scheme 2 has two discrete invariants, the sup norm stability is not theoretically guaranteed. Observe that the discrete multistep energy function expressed in $\boldsymbol{U}^{(n)}$ in Lemma 4.1 fails to suppress the possible blowup of $\|D^+\boldsymbol{U}^{(n)}\|$; in fact, some subsequence of it can blowup while the products of two consecutive values remain bounded. We will demonstrate it numerically in Section 5.

Remark 5. We would like to point out that the meaning of the "linearization" here is slightly different from those in the above mentioned literature, where main effort has been devoted to decompose a highly nonlinear energy function (say, u^4) to a quadratic one (say, $(U_k^{(n+1)})^2(U_k^{(n)})^2$). In the present case, H is already quadratic, and thus $\delta H/\delta m$ is linear by itself. What makes the equation (and accordingly the scheme) nonlinear is the combination of the linear variational derivative and the variable m included in the operator $(m\partial_x + \partial_x m)$. Thus the game here is: Find appropriate discretizations of the variational derivative and the operator $(m\partial_x + \partial_x m)$ such that only in one of them, at most, the unknown variable $U_k^{(n+1)}$ (or equivalently, $M_k^{(n+1)}$) appears. $\square$

4.1.2 Convergence of Scheme 2 with $p = 2$

In this subsection we present an error estimate of the numerical solutions obtained by Scheme 2, limiting ourselves to the case $p = 2$. Since the argument is essentially the same as in the nonlinear case, we omit the proof and show only the result.

Theorem 4.4 (L^∞ error estimate of Scheme 2)**.** *Assume that* $u(t,\cdot) \in C^7(\mathbb{S})$ *and* $u(\cdot, x) \in C^3([0,T])$. *Then the numerical solution* $\boldsymbol{U}^{(n)}$ *by Scheme 2 converges to the exact solution* $u(n\Delta t, \cdot)$ *with order* $O(\Delta t^2 + \Delta x^2)$ *by both in* $\|\cdot\|$ *and* $\|\cdot\|_\infty$.

4.2 A New Stable Linear Conservative Scheme

4.2.1 A New Linear Conservative Scheme and Its Basic Properties

By slightly modifying the Takeya–Furihata approach, we can construct another linear conservative scheme, which is more stable than Scheme 2.

Let us start with the discrete energy function:

$$H_k^{(n+\frac{1}{2})} := \frac{U_k^{(n+1)}M_k^{(n+1)} + U_k^{(n)}M_k^{(n)}}{2},$$

from which we obtain the multistep discrete variational derivative:

$$\frac{\delta H}{\delta(\boldsymbol{M}^{(n+1)}, \boldsymbol{M}^{(n)}, \boldsymbol{M}^{(n-1)})_k} := \frac{U_k^{(n+1)} + U_k^{(n-1)}}{2}.$$

This satisfies the following key equality:

$$\begin{aligned}\sum_{k=0}^{N-1} H_k^{(n+\frac{1}{2})}\Delta x - \sum_{k=0}^{N-1} H_k^{(n-\frac{1}{2})}\Delta x \\ = \sum_{k=0}^{N-1}\left\{\frac{\delta H}{\delta(\boldsymbol{M}^{(n+1)}, \boldsymbol{M}^{(n)}, \boldsymbol{M}^{(n-1)})_k}\frac{M_k^{(n+1)} - M_k^{(n-1)}}{2}\right\}\Delta x.\end{aligned}$$

We then define the following linear finite difference scheme.

Scheme 3 (A stable linear conservative scheme)**.** *We define the initial approximate solution by* $U_k^{(0)} = u(0, k\Delta x)$ $(k = 0, 1, \dots, N-1)$*. Then for* $n = 1, 2, \dots,$

$$\frac{M_k^{(n+1)} - M_k^{(n-1)}}{2\Delta t} = -(M_k^{(n)}\delta_k^{\langle 1\rangle} + \delta_k^{\langle 1\rangle}M_k^{(n)})\frac{\delta H}{\delta(\boldsymbol{M}^{(n+1)}, \boldsymbol{M}^{(n)}, \boldsymbol{M}^{(n-1)})_k} \quad (k = 0, \dots, N-1).$$

We can do a similar argument to Lemma 3.1 and Lemma 4.1. Here we show only the cases $p = 1, 2$.

Lemma 4.5. *If we define* $\tilde{H}_k^{(n+\frac{1}{2})}$ *by*

$$\tilde{H}_k^{(n+\frac{1}{2})} = \frac{(U_k^{(n+1)})^2 + (U_k^{(n)})^2 + (\delta_k^+ U_k^{(n+1)})^2 + (\delta_k^+ U_k^{(n)})^2}{4}$$

for $p = 1$*, and*

$$\tilde{H}_k^{(n+\frac{1}{2})} = \frac{(U_k^{(n+1)})^2 + (U_k^{(n)})^2 + 2(\delta_k^+ U_k^{(n+1)})^2 + 2(\delta_k^+ U_k^{(n)})^2}{4} + \frac{(\delta_k^{\langle 2\rangle} U_k^{(n+1)})^2 + (\delta_k^{\langle 2\rangle} U_k^{(n)})^2}{4}$$

for $p = 2$*, then* $\tilde{H}_k^{(n+\frac{1}{2})}$ *is equivalent to* $H_k^{(n+\frac{1}{2})}$ *in the following sense*

$$\sum_{k=0}^{N-1} H_k^{(n+\frac{1}{2})}\Delta x = \sum_{k=0}^{N-1} \tilde{H}_k^{(n+\frac{1}{2})}\Delta x.$$

Proof. Easy by using appropriate summation-by-parts formulas. □

Scheme 3 is also a linear scheme, but has only one discrete invariant.

Theorem 4.6 (Scheme 3: discrete conservation laws)**.** *Under the discrete periodic boundary condition, the numerical solution by Scheme 3 conserve the invariant*:

$$\sum_{k=0}^{N-1} H_k^{(n+\frac{1}{2})}\Delta x = \sum_{k=0}^{N-1} H_k^{(\frac{1}{2})}\Delta x, \qquad n = 1, 2, \dots.$$

Proof. Similar to Scheme 1 and 2. □

The next theorem states that unique existence of the solution is guaranteed if Δt is sufficiently small.

Theorem 4.7 (Unique solvability of Scheme 3)**.** *At the time step n, if the prescribed time mesh size* $\Delta t > 0$ *is sufficiently small, then Scheme 3 has a unique numerical solution* $\boldsymbol{U}^{(n+1)}$.

Proof. Scheme 3 can be rewritten as follows:

$$\left[(I - D^{\langle 2\rangle})^p + \Delta t\left(M^{(n)}D^{\langle 1\rangle} + D^{\langle 1\rangle}M^{(n)}\right)\right]\boldsymbol{U}^{(n+1)} = \boldsymbol{G}(\boldsymbol{U}^{(n)}, \boldsymbol{U}^{(n-1)}).$$

where $M^{(n)} = \mathrm{diag}(M_0^{(n)}, \ldots, M_{N-1}^{(n)})$, and G denotes the remaining terms with $\boldsymbol{U}^{(n)}, \boldsymbol{U}^{(n-1)}$. Let us consider the coefficient matrix of the left hand side. Since $(I - D^{\langle 2\rangle})^p$ is nonsingular, the coefficient matrix should be nonsingular for sufficiently small Δt by the continuity of determinant. (See a similar argument for the Degasperis–Procesi equation [11, 12]). □

Above two theorems indicate that Scheme 3 has only one invariant and the solvability is not guaranteed when Δt is large. Despite these disadvantages, this scheme still deserves serious consideration, since it has an important preferable feature that the stability is assured as far as the numerical solution exists.

Theorem 4.8 (Stability of Scheme 3)**.** *Scheme 3 is stable in the sense that*

$$\|\boldsymbol{U}^{(n)}\|_\infty < \infty.$$

Proof. Similarly to Scheme 1, we can prove it by the combination of Lemma 4.5, Theorem 4.6 and the discrete Sobolev lemma. □

Remark 6. We here hope to point out that Scheme 3 is a generic scheme for all $p \geq 1$, which in particular includes the CH case $p = 1$. That is, a new stable linearly implicit scheme is proposed also for the CH in the present paper. □

4.2.2 Convergence of Scheme 3 with $p = 2$

In this subsection we present an error estimate of the numerical solutions obtained by Scheme 3, limiting ourselves to the case $p = 2$. Since the argument is essentially the same as in the nonlinear case, we omit the proof and show only the result.

Theorem 4.9 (L^∞ error estimate of Scheme 3)**.** *Assume that $u(t, \cdot) \in C^7(\mathbb{S})$ and $u(\cdot, x) \in C^3([0, T])$. Then the numerical solution $\boldsymbol{U}^{(n)}$ by Scheme 3 converges to the exact solution $u(n\Delta t, \cdot)$ with order $O(\Delta t^2 + \Delta x^2)$ by both in $\|\cdot\|$ and $\|\cdot\|_\infty$.*

5 Numerical Examples with Soliton-Like Solutions

In this section we present some numerical examples for the mCH (1). Throughout this section, we consider the case $p = 2$.

In the CH ($p = 1$), the singular soliton solutions, the "peakons," can be obtained by formally setting $m = \delta(x)$ (the Dirac delta function). In view of the strong similarity between the CH and the mCH, a natural expectation would be that also in the mCH the delta function behaves as a soliton. In fact, this has been already suggested by Zhang in his Ph.D. thesis [18] for a slightly different generalized version of the CH with $m = (1 - \partial_x + {\partial_x}^2)u$, where he wrote that the delta function is formally a "soliton." But whether it is a stable solution or not has not been confirmed both theoretically and numerically.

Taking these into account, we here call the delta function a "soliton-like" solution. For the moment, let us consider the mCH on the whole real line. By formally integrating $m = c\delta(x - ct)$ where c is a generic constant, we obtain

$$u(t, x) = \frac{c}{4}(1 + |x - ct|)\mathrm{e}^{-|x-ct|}. \tag{30}$$

If we denote the fundamental solution of the inertia operator $(1-{\partial_x}^2)^2$ by G, (30) is equivalent to $u(t,x)=cG(x-ct)$. The concrete form of G is given in [10] for all p; for example, $G=\dfrac{1}{4}(1+|x|)\mathrm{e}^{-|x|}$ when $p=2$. The discussion gets cumbersome as soon as we consider the circle $\mathbb{S}$ case. In order to avoid the complication, in what follows we simply truncate the (some combinations of the) soliton-like solutions for our initial data.

Below we present two groups of numerical examples. The first group is mainly for the check of the proposed conservative schemes: Scheme 1, 2 and 3, picking some "soliton-like" or "multi-soliton like" solutions as our target. For comparison, we also test the standard Heun Scheme (which is also a second-order scheme) based on the ODE:

$$\frac{\mathrm{d}}{\mathrm{d}t}M_k=-(M_k\delta_k^{\langle 1\rangle}+\delta_k^{\langle 1\rangle}M_k)U_k \qquad (k=0,\ldots,N-1).$$

Through these examples, it will be shown that the (multi-)soliton like solutions in fact behave like (multi-)solitons. In the second group, more challenging initial data will be considered to investigate the dynamics of the (multi-)soliton like solutions.

All the computations were done in the computation environment: CPU Xeon(3.00GHz), 16GB memory, Linux OS. We used MATLAB (R2007b), where nonlinear equations were solved by "fsolve" with tolerance $TolFun=10^{-10}$.

5.1 Check of the Conservative Schemes

We compare Scheme 1, 2, 3 and the Heun Scheme for a "soliton-like" solution. The parameters were set to $t\in[0,50], x\in[-15,15], \Delta t=0.1, \Delta x=0.1$, and the initial value was to $u(0,x)=(1+|x|)\mathrm{e}^{-|x|}$.

Fig. 1 and Fig. 2 show the evolution of the discrete invariants. More specifically, for each scheme the following discrete quantities are plotted: for Scheme 1 (the nonlinear scheme), the discrete momentum and energy noted in Theorem 3.2; for Scheme 2 (the first linearly implicit scheme), those in Theorem 4.2; for Scheme 3 (the second linearly implicit scheme), those in Theorem 4.6; and finally for the Heun scheme, the same ones as the one-step nonlinear scheme.

Fig. 1 indicates that three schemes except for Scheme 3 preserve $\sum U_k^{(n)}\Delta x$. Scheme 3 is not conservative in this sense, as theoretically suggested, but the deviation is under 0.1% which is fair enough. Fig. 2 shows that the Heun scheme dose not conserve H, and it shows strong deviation of 20% or more.

Fig. 3 shows the numerical solutions. Compared to the conservative schemes, we observe that the numerical solution by the Heun scheme is unstable (see around $(t,x)=(50,10)$). Thus proposed schemes (Scheme 1, 2 and 3) are better as expected in the sense of the qualitative behavior of solutions. We also would like to comment that the "soliton-like" solution in fact behave like a soliton.

Next we compare the computation time. Table 1 shows the computation times of Scheme 1, 2, 3 and the Heun scheme required for 20 time steps for various Δx. We set other parameters and the initial value to those employed in the above example. In contrast to Scheme 2 and 3, Scheme 1 costs much more when the space mesh size Δx is small. We also observe that although Heun scheme is linear, it costs more than Scheme 2; this is mainly caused by the fact that two simultaneous equations should be solved in each time step, while Scheme 2 requires only one. Scheme 3 is slightly expensive than Scheme 2, since the coefficient matrix should be updated at every time step, while it remains the same in Scheme 2 (recall the proofs of Theorem 4.3 and Theorem 4.7).

We compare the numerical stability of Scheme 2 and 3 in detail. As noted in Section 4, the theoretical sup norm stability of Scheme 2 is not certified. Here we set the space

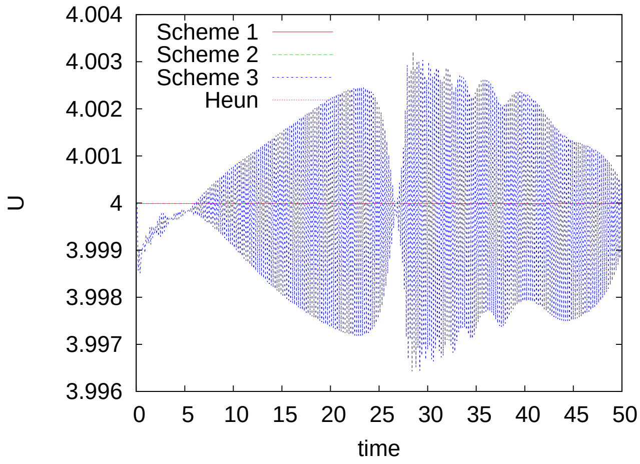

Figure 1: Evolution of summation of the discrete momentum $U_k^{(n)}$.

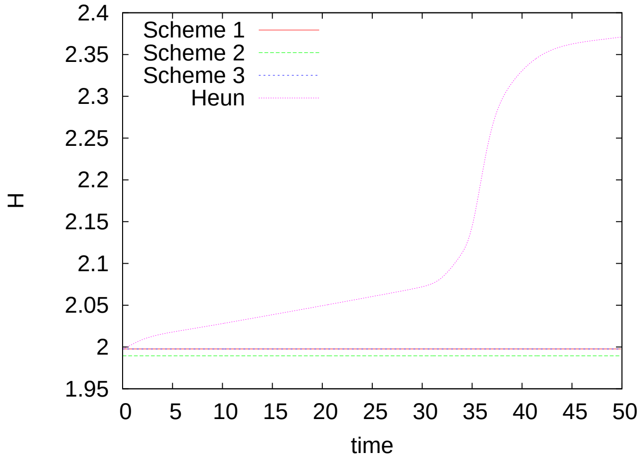

Figure 2: Evolution of summation of the discrete energy $H_k^{(n)}$.

Table 1: Computation time in Scheme 1, 2, 3 and the Heun scheme.

Δx	$30/2^5$	$30/2^6$	$30/2^7$	$30/2^8$	$30/2^9$
Scheme 1	0.322828s	0.481097s	1.501954s	7.292101s	80.65813s
Scheme 2	0.003134s	0.004419s	0.010809s	0.039865s	0.232313s
Scheme 3	0.003694s	0.007731s	0.030569s	0.190028s	1.535142s
Heun Scheme	0.005427s	0.008645s	0.030920s	0.115230s	0.633522s

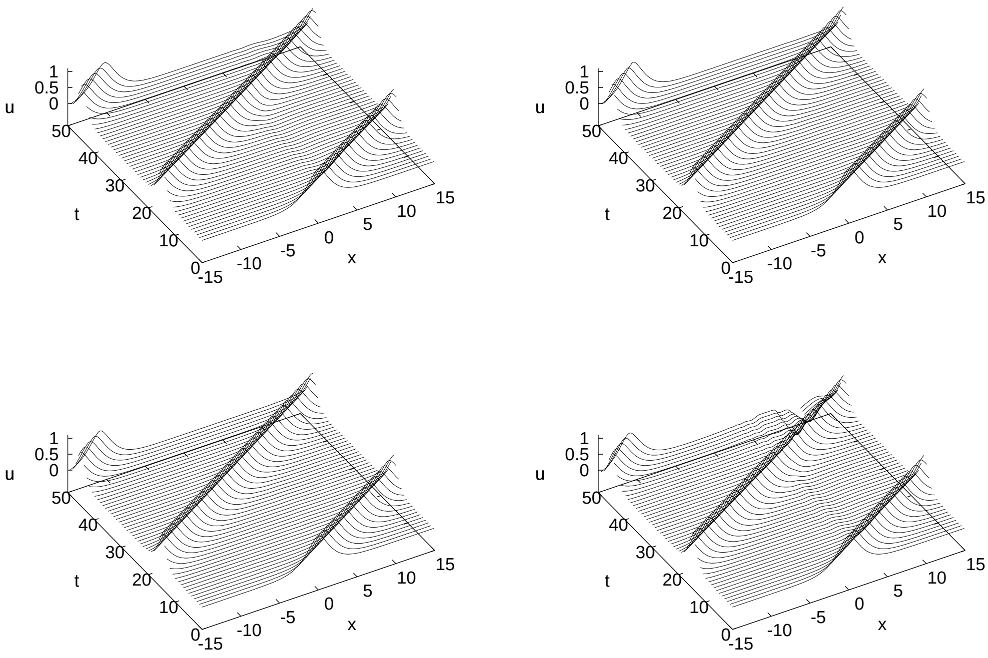

Figure 3: The numerical solutions obtained by (top left) Scheme 1, (top right) Scheme 2, (bottom left) Scheme 3, and (bottom right) the Heun scheme with $\Delta t = 0.1$ and $\Delta x = 0.1$.

mesh size $\Delta x = 0.1$, and tried four time mesh sizes: $\Delta t = 10/99 \simeq 0.101$, $\Delta t = 0.1$, $\Delta t = 10/150 \simeq 0.067$ and $\Delta t = 0.05$. Initial value was set to $u(0, x) = 1.1(1 + |x|)\mathrm{e}^{-|x|}$. Fig. 4 and Fig. 5 are the numerical solutions by Scheme 2 and 3 at $t = 10$. For relatively fine time meshes: $\Delta t = 10/150 \simeq 0.067$ and $\Delta t = 0.05$, the numerical solutions by Scheme 2 and 3 are both fine. But as Δt gets large, the results by Scheme 2 become slightly worse. In fact, undesirable oscillation appear in the result of Scheme 2 around the peak. On the other hand, Scheme 3 holds on even for the large Δt's (although it is true that the shape of the wave gets worse as Δt increases). These observations agree with the analysis in Section 4.

5.2 Dynamics of the Soliton-Like Solutions

We now turn our attention to the "soliton-like" solutions. In what follows, we use Scheme 3 for numerical experiments, which is well balanced in both efficiency and stability.

In order to see if the solutions actually interact like solitons, we considered the initial value $u(0, x) = (1 + |x + 5|)\mathrm{e}^{-|x+5|} + \dfrac{1}{2}(1 + |x - 5|)\mathrm{e}^{-|x-5|}$. The parameters were set as follows: $t \in [0, 100], x \in [-20, 20], \Delta x = 0.1$ and $\Delta t = 0.1$. The result is shown in Fig. 6, which seems to support our view that the solution behaves like a two-soliton solution; the taller wave takes over the shorter one. The right graph in Fig. 6 is a contour graph of the solution. It clearly shows the typical soliton behavior.

Finally, we try some more general initial data. In many soliton equations such as the Korteweg–de Vries equation and the Camassa–Holm equation, it is often observed that a big wave collapses into a series of solitons. For example, Matsuo–Yamaguchi observed in the CH [9] that a triangle shaped initial data soon splits into a series of peakons.

Fig. 7 shows the numerical results with the parameters: $t \in [0, 250], x \in [0, 200], \Delta t =$

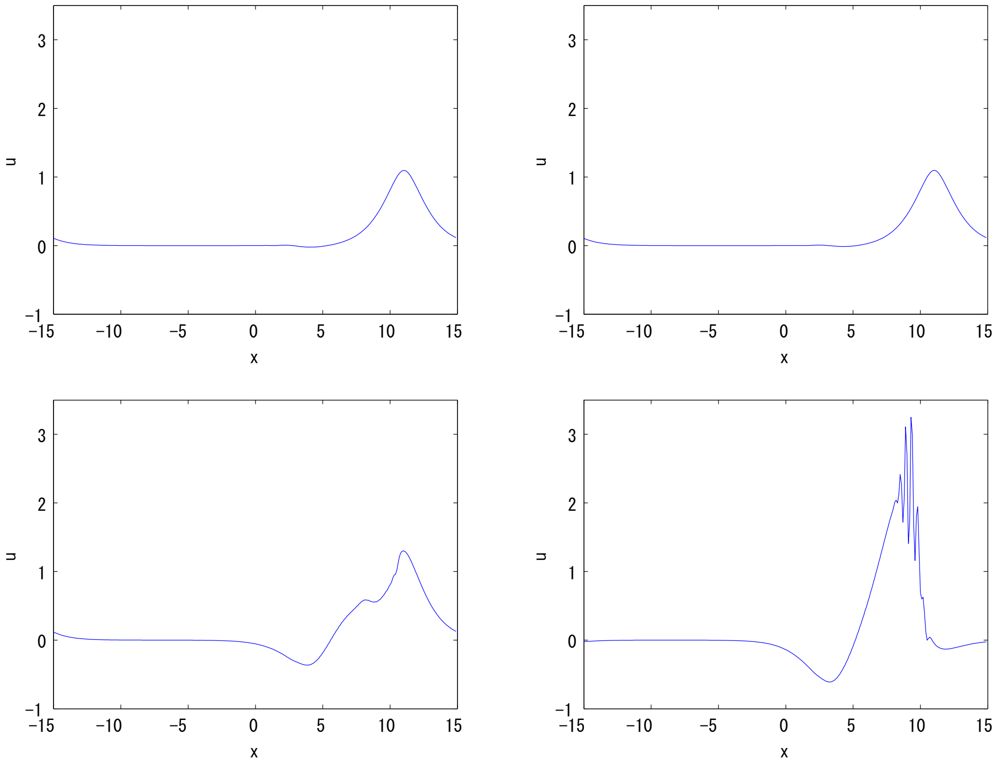

Figure 4: The numerical solution of “soliton-like” solution obtained by Scheme 2: (top left) $\Delta t = 0.05$, (top right) $\Delta t = 0.067$, (bottom left) $\Delta t = 0.1$, (bottom right) $\Delta t = 0.101$.

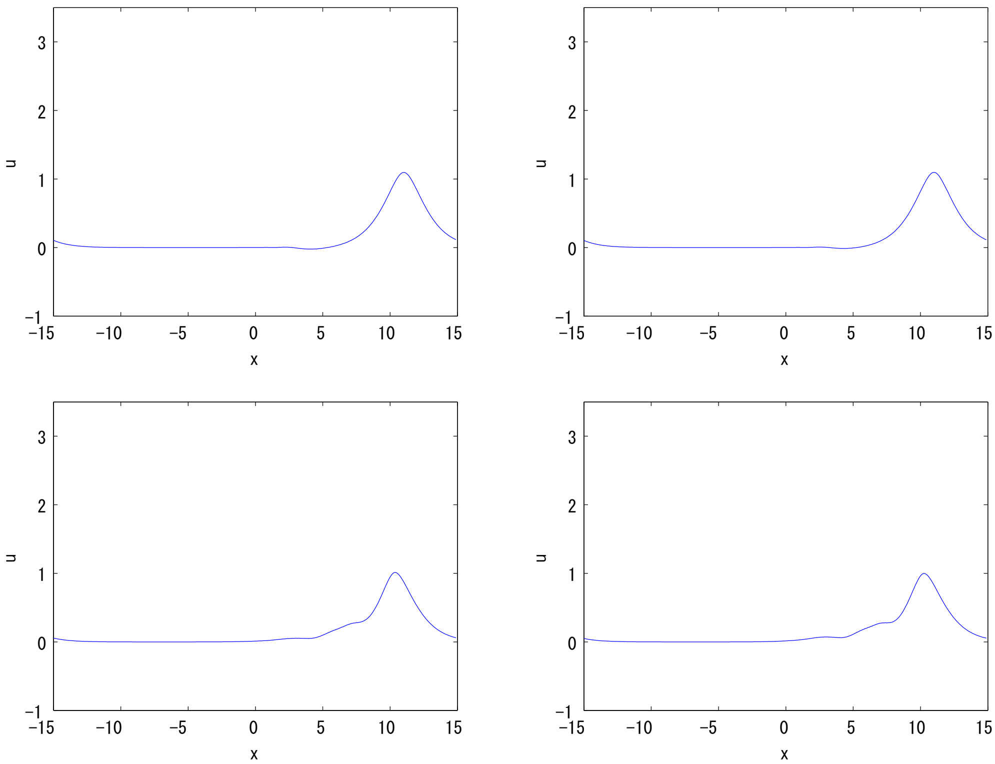

Figure 5: The numerical solution of “soliton-like” solution obtained by Scheme 3: (top left) $\Delta t = 0.05$, (top right) $\Delta t = 0.067$, (bottom left) $\Delta t = 0.1$, (bottom right) $\Delta t = 0.101$.

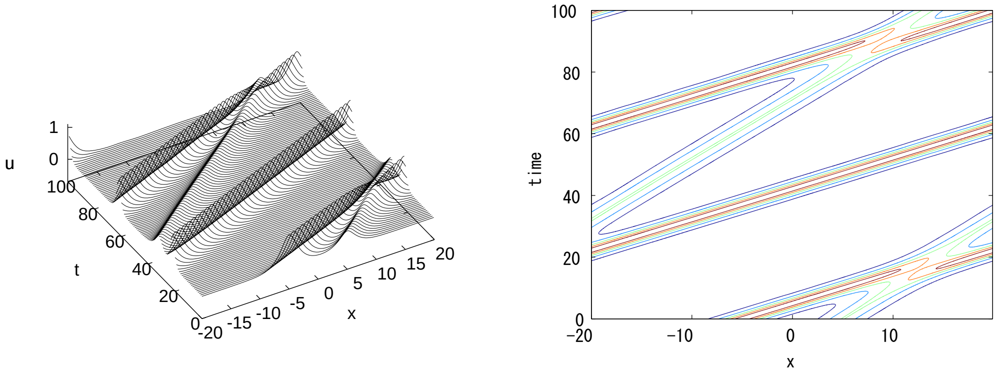

Figure 6: The numerical solution of “2-soliton like” solution obtained by Scheme 3.

$0.01, \Delta x = 0.2$ starting from the initial data:

$$u(0, x) = \mathrm{sech}^2(0.1(x - 50)).$$

The result suggests the same phenomenon can happen also in the mCH.

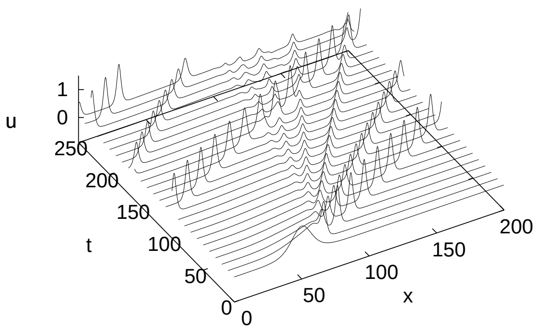

Figure 7: Generation of soliton-like solutions.

Fig. 8 shows the result starting from more challenging non-smooth initial data:

$$u(0, x) = \begin{cases} x - 10.05, & x \in [10.05, 20.05), \\ -x + 30.05, & x \in [20.05, 30.05), \\ 0, & \text{otherwise}. \end{cases}$$

The parameters were set as follows: $t \in [0, 15], x \in [0, 200], \Delta t = 0.01, \Delta x = 0.2$. It shows the same splitting occur for the non-smooth initial data.

6 Concluding Remarks

We proposed two finite difference schemes (Scheme 1 and 2) for the mCH equation preserving the two associate invariants, and one (Scheme 3) conserving only one invariant. The key idea there is to reformulate the mCH equation, so that the scheme derivation can be done in a unified way for all p (the index in the mCH family). We also proved the stability of Scheme 1 and 3, and the unique solvability and the convergence of all proposed schemes.

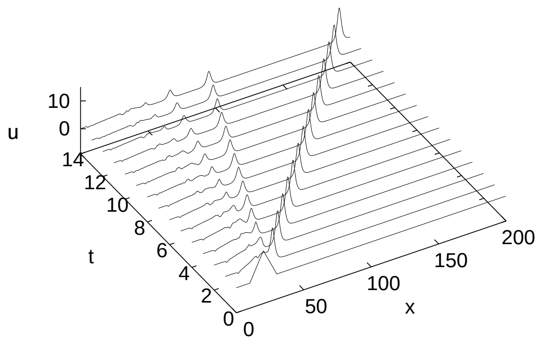

Figure 8: Generation of soliton-like solutions with the triangle shaped initial data.

Although our error estimate was limited to the case $p = 2$, we feel it may be extended for the case $p \geq 3$. We considered a "soliton-like" solution for the mCH, and numerical examples on the "soliton-like" solutions indicate that these solutions in fact behave like solitons; in particular, we observed the splitting of soliton-like solutions, which is often observed in wide range of soliton equations.

Left issues to be investigated include the followings. Extension to the non-uniform grids is an interesting topic; there the mapping technique developed by Yaguchi et al. [16] would be worth trying. The finite element approach for the CH by Matsuo–Yamaguchi [9] does not automatically carry to the mCH case, since the higher derivatives in the energy function cause serious problems in formulating conservative weak forms. We are now investigating a way for circumventing this difficulty, and the result will be reported in the near future elsewhere.

Finally, as noted in the introduction, the study of the mCH has just started, and many open problems still remain. Does the mCH admit other invariants? Or more aggressively, is the mCH completely-integrable? As for the dynamical aspects, although in the present study we could find a "soliton-like" solution in analogy with the standard CH, it is not clear at all whether or not the entire dynamics can be understood in a similar way. The answer should be negative, at least partly, since it has been shown in [10] that the blow-up in the sense of "wave-breaking" should not occur in the mCH ($p \geq 2$). Thus much more effort should be devoted in this topic, and there we believe that the presented conservative schemes serve as effective numerical tools.

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