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Modeling of Contagious Credit Events and Risk Analysis of Credit Portfolios

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Abstract

We present a new model of credit events such as rating changes as well as defaults for risk analyses of some portfolio credit derivatives. The framework of our model is based on a so-called top-down approach. To be precise, we firstly pay attention to modeling the point process of each type of credit events in the whole economy with a self-exciting intensity process, and then we characterize the point processes of credit events for the underlying sub-portfolio with some random thinning process specified by the distribution of credit ratings in the sub-portfolio. One of the main features of our model is that the model can capture credit risk contagion simultaneously among several credit portfolios. We show a credit event simulation algorithm based on our model and illustrate as an application of the model some numerical examples of risk analyses of loan portfolios.

keywords Credit risk, Rating change, Self-exciting intensity model, State-dependent, Top-down approach

1 Introduction

This paper presents a new model of the intensities of contagious credit events such as rating changes and defaults. Our modeling framework is based on the top-down approach studied in Giesecke et al. [2] and Nakagawa [4, 5]. That is, we apply self-exciting stochastic processes to model the intensities of credit events in the economy. Then, we apply a random thinning to specify the intensities of the events in a sub-portfolio. Our model enables us to measure risks of several portfolios simultaneously with credit risk contagion.

Giesecke et al. [2] introduced the top-down approach to evaluate the default risk of the portfolio. In the top-down approach, intensity-based models of the loss point process are specified without reference to the portfolio constituents. In addition, the method called random

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thinning decomposes the portfolio loss point process into the sum of its constituent loss point processes. In particular, Giesecke et al. [2] proposed the portfolio default intensity model with self-exciting effect. The word "self-exciting" means that the intensity of next default jumps due to happening of default. A self-exciting default intensity model is one of the models that reflect default contagion.

There are several works on risk analysis and pricing of portfolio credit derivatives within the framework of Giesecke et al. [2]. For instance, Giesecke and Kim [3] analyzed risks of collateralized debt obligations (CDOs) within the framework. They used self-exciting intensity process to model defaults of the underlying credit portfolio of a CDO. In particular, their intensity process has not only self-exciting effect but also state-dependent property. In addition, Giesecke and Kim [3] developed a data generating method to get the default timing data of reference credit portfolio. The data generating method, called acceptance/rejection re-sampling scheme, is based on the random thinning.

Nakagawa [4, 5] proposed a rating change intensity model within the top-down approach, while Giesecke et al. [2] and Giesecke and Kim [3] focused on default intensity model. Nakagawa [4] proposed the self-exciting intensity model for both defaults and rating changes such as up-grades and down-grades. Nakagawa [5] proposed the intensity model that has not only self-exciting property but also mutually exciting property. Moreover, Nakagawa [4, 5] proposed the credit derivative named multi-downgrade protection and mentioned their model is efficient for pricing of the multi-downgrade protection. As Nakagawa [4] noted, rating changes are usually modeled by rating transition intensity matrix, but it is difficult to use the rating transition matrix framework so as to consider dynamic risk dependence in the portfolio. Although, with the model of Nakagawa [4, 5], we can consider dynamic risk dependence in the portfolio without difficulty.

This paper provides a new model of credit events in the whole economy and thinning to measure risk of portfolio credit derivatives. Here, credit events are rating changes (i.e. upgrades and down-grades) and defaults. Our model is an extension of the model proposed by Nakagawa [4]. While the previous works we mentioned above focus on a single credit portfolio, we focus on dealing with several credit portfolios simultaneously. In addition, we propose an event simulation algorithm based on our model. The algorithm can work for other models as well as our intensity model.

We model the credit events in the whole economy with state-dependent type self-exciting intensity processes. Due to modeling economy-wide event with some self-exciting intensity, our model can capture credit risk dependence among several credit portfolios. That is, an event occurrence in one portfolio may influence the probability of the next event in the whole economy, and thus may have some impact on the probability of the next event in other portfolios. Our credit event intensity model is analogous to the default intensity model proposed by Giesecke and Kim [3]. However, it should be noted that our intensity model is different in the respect of introducing jump effect from the default intensity model of Giesecke and Kim [3].

As we consider not only defaults but also rating changes, we can capture changes of credit quality of credit portfolios. We specify the thinning with the distribution of credit quality of portfolios, namely with the relative frequency of credit ratings in the portfolio. As our model treats several portfolios simultaneously, our model is useful for risk analysis of portfolio credit derivatives such as CDOs.

This paper is organized as follows. Section 2 formulates the point process model. Section 3 develops an event simulation algorithm. In section 4, we give some numerical examples on the risk analysis of loan portfolios, and try tentative estimation with historical records on credit events of rated corporate issuers in Japan. Section 5 gives some concluding remarks.

2 Models

In this section, we present our point process model of each type of credit events in the whole economy. In addition, we specify random thinning by the distribution of credit ratings in the sub-portfolios.

2.1 Intensity models for economy-wide events

We will model contagious rating changes and defaults by the point processes with self-exciting intensity processes.

Suppose each firm in the economy is associated with a credit rating. There are $K+1$ ratings and we denote ratings by $1, 2, \cdots, K$ and $K+1$. The order of ratings represents credit quality, that is, the rating $k=1$ represents the best possible credit quality like AAA, $k=K$ represents the worst non-defaulted credit quality and $k=K+1$ represents the state of default. Let S^* denote the set of all rated firms, namely the whole economy. Each rated firm belongs to one of sub-portfolios S_i $(i=1,2,\cdots,I)$. Here we assume $S^* = \bigcup_{i=1}^{I} S_i,\ S_i \cap S_j = \emptyset\ (i \neq j)$.

The uncertainty in the economy is modeled by a filtered complete probability space $(\Omega,\ \mathcal{F},\ \mathbb{P},\ \{\mathcal{F}_t\})$, where $\mathbb{P}$ is the actual probability measure. Here $\{\mathcal{F}_t\}$ is a right-continuous and complete filtration. Let $\ell \in \{1,\ 2,\ \cdots,\ L\}$ denote the types of the credit events. For each ℓ, let $0 < T_1^\ell < T_2^\ell < \cdots$ be an $\{\mathcal{F}_t\}$-adapted point process, that is , an increasing sequence of totally inaccessible $\{\mathcal{F}_t\}$-stopping times. Here we assume $T_n^\ell < \infty$ $a.s.$ $^\forall n \in \mathbb{N}$. We see that $\{T_n^\ell\}_{n\in\mathbb{N}}$ indicates the event times of event type ℓ. In addition, we denote the counting process of event ℓ by $N_t^\ell = \sum_{n\geq 1} \mathbf{1}_{\{T_n^\ell \leq t\}}$. Furthermore, we assume different types of events do not occur at the same time.

To simplify matters, we focus on rating changes and defaults for the credit events. That is, we set $L=3$ and suppose the event $\ell=1$ indicates up-grade, the event $\ell=2$ indicates down-grade and the event $\ell=3$ indicates default. Furthermore, we assume each defaulted firm vanishes when the default occurs, and the defaulter is not replaced by a new firm.

Suppose each N_t^ℓ has a strictly positive intensity h_t^ℓ such that $\int_0^t h_s^\ell \mathrm{d}s < \infty$ for $^\forall t > 0$ almost surely. Namely, each h_t^ℓ is a $\{\mathcal{F}_t\}$-progressively measurable strictly positive process, and the process

$$N_t^\ell - \int_0^t h_s^\ell \mathrm{d}s$$

is an $\{\mathcal{F}_t\}$-local martingale. To specify h_t^ℓ, let λ_t^ℓ be the self-exciting stochastic process:

$$\mathrm{d}\lambda_t^\ell \quad = \quad \kappa_t^\ell (c_t^\ell - \lambda_t^\ell)\mathrm{d}t + \mathrm{d}J_t^\ell, \tag{1}$$

$$J_t^\ell = \sum_{n\geq 1}(\min(\delta^\ell \lambda^\ell_{T_n^\ell -},\ \gamma^\ell)\mathbf{1}_{\{T_n^\ell \leq t\}}), \tag{2}$$

$$\kappa_t^\ell = \kappa^\ell \lambda^\ell_{T^\ell_{N_t^\ell}},\ c_t^\ell = c^\ell \lambda^\ell_{T^\ell_{N_t^\ell}}. \tag{3}$$

Here, $\lambda^\ell_{t-} = \lim_{s\uparrow t} \lambda^\ell_s$ and the quantities $\kappa^\ell > 0,\ c^\ell \in (0,1),\ \delta^\ell > 0,\ \gamma^\ell \geq 0,\ \lambda^\ell_0 > 0$ are parameters.

With the self-exciting process λ_t^ℓ, we specify intensities of N_t^ℓ , denoted by h_t^ℓ, as follows:

$$h_t^1 := \lambda_t^1 \mathbf{1}_{\{\sum_{k=2}^K X_t^*(k)>0\}}, \tag{4}$$

$$h_t^2 := \lambda_t^2 \mathbf{1}_{\{\sum_{k=1}^{K-1} X_t^*(k)>0\}}, \tag{5}$$

$$h_t^3 := \lambda_t^3 (\sum_{k=1}^K z_k^3 \mathbf{1}_{\{X_t^*(k)>0\}}). \tag{6}$$

Here, $X_t^*(k)$ indicates the number at time t of k-rated firms. z_k^3 are constants satisfying $0 \leq z_k^3 \leq 1$ $(k = 1,\ 2, \cdots ,\ K)$ and $\sum_{k=1}^K z_k^3 = 1$. Also, we suppose $z_1^3 < z_2^3 < \cdots < z_K^3$. Each z_k^3 represents the conditional probability that the rating of a defaulter is k, given that a default occurs. The indicator function in (4) implies that the up-grade intensity become null when the all firms are rated at the highest rating 1. Also the indicator functions in (5) and (6) imply that the down-grade intensity become null when all firms are rated at the lowest rating K and default intensity reduces by $z_k^3\lambda_t^3$ if there are no k-rated firms in the whole economy. In the rest of this paper, we suppose there is enough number of rated firms in each rating category to regard h_t^ℓ are the same as λ_t^ℓ, namely we suppose for any $t \geq 0$, $X_t^*(k) > 0$ $(k = 1,\ 2, \cdots ,\ K)$.

Remark Figure 1 shows a sample path of state-dependent self-exciting intensity process (1) − (3). The intensity processes (1) − (3) are analogous to the default intensity model in Giesecke and Kim [3]. The remarkable characteristics of both intensity processes (1) − (3) and that of Giesecke and Kim [3] are the following. First, the intensity jumps at event times. The jumps of the intensity process generate event correlations. Namely, an event occurrence increases the probability of the next event. This feature facilitates the replication of event clusters. We will mention the details of the jumps later. Second, each event intensity moves deterministically between event times; the intensity jumps at event times. The state-dependent drift involves a reversion level and speed, which are proportional to the intensity at the previous event. Definitely, for $T_n^\ell \leq t < T_{n+1}^\ell$, the behavior of the process is described as

$$\lambda_t^\ell = c^\ell \lambda^\ell_{T_n^\ell} + (1-c^\ell)\lambda^\ell_{T_n^\ell} \exp(-\kappa^\ell \lambda^\ell_{T_n^\ell}(t - T_n^\ell)) \quad T_n^\ell \leq t < T_{n+1}^\ell.$$

Now, we should note the difference between our intensity process and the default intensity process proposed by Giesecke and Kim [3]. Giesecke and Kim [3] introduced the state-dependent self-exciting default intensity process (1), (3) with the jump type of

$$J_t = \sum_{n\geq 1}(\max(\delta\lambda_{T_n -},\ \gamma)\mathbf{1}_{\{T_n \leq t\}}). \tag{7}$$

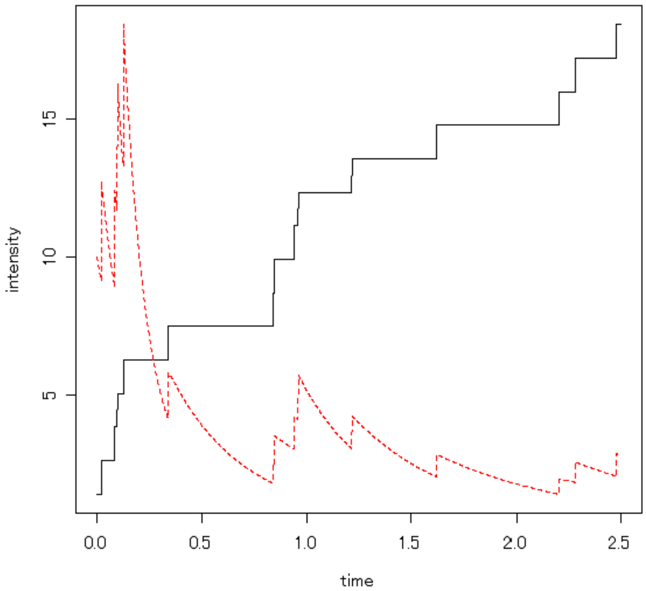

Figure 1: A sample path of the self-exciting process (1) $-$ (3) and cumulative number of the events (the solid line indicates cumulative number of events, the dash line indicates a pass of self-exciting process (1) $-$ (3) with $\kappa = 0.5,\ c = 0.1,\ \delta = 0.4,\ \gamma = 50.0,\ \lambda_0 = 10.0$.)

The difference is that the jump size of our intensity model is bounded, though the jump size of the intensity of Giesecke and Kim [3] is not. We found the estimated intensity model of rating changes with jump part (7) tends to generate unrealistic number of rating changes through simulations. Thus, we propose the intensity model with the limited jump size such as (2).

Figure 2 shows fitted path of the up-grade intensity to the actual data of up-grades. In figure 2, the bar chart displays the number of up-grades in 2004, in Japan, and the broken line graph is the estimated path of the up-grade intensity. Figure 2 shows that the fitted path of the intensity overlaps considerably with the number of the events. This implies the validity of modeling event clusters with the self-exciting process [1].

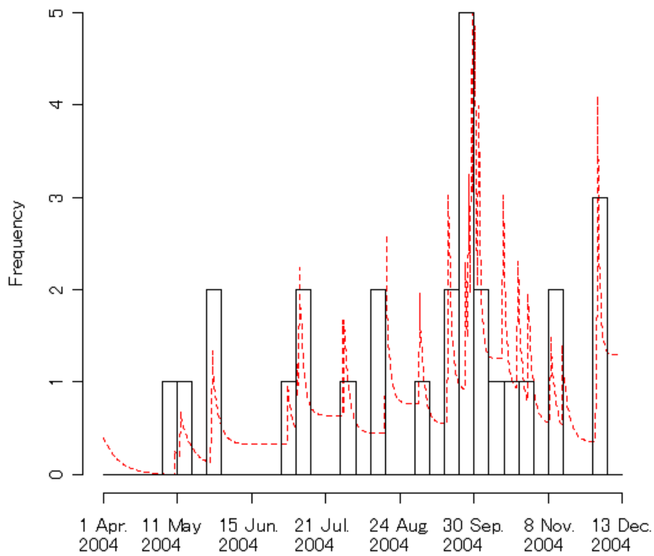

Figure 2: The number of up-grades and estimated up-grade intensity path (the ratings are of R&I, the model parameters in (1) − (3) were estimated with the maximum likelihood method as mentioned later.)

2.2 Thinning process

Next, we decompose the event intensity for the whole economy into intensity for the underlying sub-portfolio with a random thinning based on rating distributions.

We introduce $\{\mathcal{F}_t\}$-adapted processes $\{Z_t^{(i,1)}(k,\, k')\}$, $\{Z_t^{(i,2)}(k,\, k')\}$ and $\{Z_t^{(i,3)}(k)\}$. $Z_t^{(i,1)}(k,\, k')$ represents the conditional probability that the up-grade is the transition from the rating k

[1]Nakagawa [6] find self-exciting effects in rating changes of Japanese firms during April 1998 to September 2009 with mutually exciting intensity model, which is an extension of the self-exciting models.

to the rating $k'(k' = 1, 2, \cdots, k-1)$ of the firm in the portfolio S_i, given that an up-grade occurs in the economy. Similarly, $Z_t^{(i,2)}(k,\ k')$ represents the conditional probability that the down-grade is the transition from the rating k to the $k'(k' = k+1, k+2, \cdots, K)$ of the firm in the portfolio S_i, given that the down-grade occurs in the economy. Also, $Z_t^{(i,3)}(k)$ represents the conditional probability that the defaulter is the k-rated firm in the portfolio S_i, given that the default occurs in the economy.

We assume that $Z_t^{(i,\ell)}(k,\ k')$ $(\ell = 1, 2)$ and $Z_t^{(i,3)}(k)$ are determined by the distribution of ratings in the portfolios. That is, we specify $Z_t^{(i,\ell)}(k,\ k')(\ell = 1, 2)$ and $Z_t^{(i,3)}(k)$ as following [2] :

$$
\begin{aligned}
Z_t^{(i,1)}(k,\ k') &= \frac{X_t^{(i)}(k)}{\sum_{\tilde{k}=(k-k')+1}^{K} X_t^*(\tilde{k})} z^1_{k-k'} \mathbf{1}_{\{\sum_{\tilde{k}=(k-k')+1}^{K} X_t^*(\tilde{k})>0\}}, && (8)\\
Z_t^{(i,2)}(k,\ k') &= \frac{X_t^{(i)}(k)}{\sum_{\tilde{k}=1}^{K-(k'-k)} X_t^*(\tilde{k})} z^2_{k'-k} \mathbf{1}_{\{\sum_{\tilde{k}=1}^{K-(k'-k)} X_t^*(\tilde{k})>0\}}, && (9)\\
Z_t^{(i,3)}(k) &= \frac{X_t^{(i)}(k)}{X_t^*(k)} z^3_k \mathbf{1}_{\{X_t^*(k)>0\}}. && (10)
\end{aligned}
$$

Here, $X_t^{(i)}(k)$ denotes the number at time t of k-rated firms in the portfolio S_i. As we have already mentioned, $X_t^*(k)$ denotes the number at time t of k-rated firms in the whole economy. $z^\ell_m(\ell = 1, 2,\ m = 1,\ 2,\ \cdots,\ K-1)$ denote conditional probability that the rating change is the m-step rating change, given that the rating change occurs. $z^\ell_m(\ell = 1, 2,\ m = 1,\ 2,\ \cdots,\ K-1)$ are constants and satisfies $0 \le z^\ell_m \le 1$ $(\ell = 1,\ 2)$ and $\sum_{m=1}^{K-1} z^\ell_m = 1$. We also suppose $z^\ell_1 > z^\ell_2 > \ldots > z^\ell_{K-1}$ $(\ell = 1,\ 2)$. Recall that each z^3_k represents the conditional probability that the rating of a defaulter is k, given that a default occurs and z^3_k are constants satisfying $0 \le z^3_k \le 1$ $(k = 1,\ 2, \cdots,\ K), \sum_{k=1}^{K} z^3_k = 1$ and $z^3_1 < z^3_2 < \cdots < z^3_K$. In (8) − (10), the quotients are taken to be 0 when the denominator vanishes. $Z_t^{(i,\ell)}(k)$ satisfy the following properties: for each ℓ, (1) $Z_t^{(i,\ell)}(k)$ takes values in the unit interval $[0, 1]$, (2) $\sum_{i,k} Z_t^{(i,\ell)}(k) = 1$. We give the specific models of $z^\ell_m(\ell = 1, 2,\ m = 1,\ 2,\ \cdots,\ K-1)$ and z^3_k in the sub section 4.2.

Let $N_t^{(i,1)}(k,\ k')$ be the counting process of the up-grades to k'-rated of k-rated firms in portfolio S_i. Also, let $N_t^{(i,2)}(k,\ k')$ be the counting process of the down-grades to k'-rated of

[2] From proposition in Giesecke et al. [2], $Z_t^{(i,\ell)}(k,\ k')$ $(\ell = 1, 2)$, are described definitely as

$$
\begin{aligned}
&Z_t^{(i,\ell)}(k,\ k') = \lim_{\varepsilon \to 0} Z_t^{(i,\ell)}(k,\ k', \varepsilon),\\
&Z_t^{(i,\ell)}(k,\ k', \varepsilon)\\
&\quad = \sum_n \frac{P[\{T_n^\ell \in \tau^\ell(S_i)\} \cap \{T_n^\ell \in \tau^\ell(k,\ k')\} \cap \{T_n^\ell \le t+\varepsilon\} \mid \mathcal{F}_t]}{P[T_n^\ell \le t+\varepsilon \mid \mathcal{F}_t]} \mathbf{1}_{\{T_{n-1}^\ell < t \le T_n^\ell\}}.
\end{aligned}
$$

Here $\tau^\ell(S_i)$ denotes the set of event times of event ℓ in the portfolio S_i. $\tau^\ell(k,\ k')$ $(\ell = 1,\ 2)$ denotes the set of the time that rating changes from k to k', $\ell = 1$ indicates the rating change is up-grade and $\ell = 2$ the rating change is indicates down-grade. Also, suppose $\varepsilon > 0$ and the quotients on the right are taken to be 0 when the denominator vanishes. $Z_t^{(i,3)}(k)$ is described in the same way.

k-rated firms in portfolio S_i and $N_t^{(i,3)}(k)$ be the counting process of the defaults of k-rated firms in portfolio S_i [3]. With $Z_t^{(i,1)}(k,\ k'), Z_t^{(i,2)}(k,\ k')$ and $Z_t^{(i,3)}(k)$, we obtain the intensities of $N_t^{(i,1)}(k,\ k')$, $N_t^{(i,2)}(k,\ k')$ and $N_t^{(i,3)}(k)$ as following:

$$\begin{aligned}\lambda_t^{(i,1)}(k,\ k') &= Z_t^{(i,1)}(k,\ k')\lambda_t^1, \\ \lambda_t^{(i,2)}(k,\ k') &= Z_t^{(i,2)}(k,\ k')\lambda_t^2, \\ \lambda_t^{(i,3)}(k) &= Z_t^{(i,3)}(k)\lambda_t^3.\end{aligned}$$

3 Event time Simulation

This section shows a main points of event simulation algorithm based on the model introduced in the section 2. We should note that our algorithm can work for other models as well as our intensity model. The step 2 and 3 are based on the point process simulation method proposed by Ogata [7]. In the step 4, we determine details of the event time by the thinning processes. The details of the algorithm are in Appendix A.

Algorithm: generating event times over $[0, H]$

1. **[Set values of parameters and initialize simulation settings]**
   - Set parameters of the event intensity models and the thinning models. Set $S = 0$ which indicate the present time.
2. **[Generate a candidate event time** T**]**
   - Set $\Lambda = \sum_{\ell=1}^{3} \lambda_S^\ell$. Draw $\mathcal{E} \sim \exp(\Lambda)$.
   - Set $T = S + \mathcal{E}$. If $T > H$, stop.
3. **[Decide to accept or to reject the candidate event time** T**]**
   - Accept T as the event time of event ℓ with probability λ_T^ℓ/Λ. If T is rejected, go to 5.
4. **[Thinning]**
   - For the accepted event time of type ℓ, decide the details of the event with probability of (8),(9),(10). Update $X_T^{(i)}(k)$, N_T^ℓ and $T_{N_T^\ell}^\ell$.
5. Set $S = T$ and go to step 2.

[3] These counting processes are given by

$$\begin{aligned}N_t^{(i,\ell)}(k,\ k') &= \sum_{n\geq 1} \mathbf{1}_{\{T_n^\ell \leq t\}\cap\{T_n^\ell \in \tau^\ell(S_i)\}\cap\{T_n^\ell \in \tau^\ell(k,\ k')\}}\ (\ell = 1, 2), \\ N_t^{(i,3)}(k) &= \sum_{n\geq 1} \mathbf{1}_{\{T_n^3 \leq t\}\cap\{T_n^3 \in \tau^3(S_i)\}\cap\{T_n^3 \in \tau^3(k)\}}.\end{aligned}$$

Here, $\tau^3(k)$ denotes the default times of k-rated firms.

4 Numerical example

In this section, we show some numerical examples on risk analysis of loan portfolios. In addition, we try estimating the model parameters from the actual data on rating changes in Japan.

4.1 Estimation of intensity model

In this subsection, we mention about the parameter estimation methods for the intensity processes (1) – (3) and the thinning processes (8) – (10) respectively. In addition, we present some tentative estimation results with actual data of rating transitions in Japan.

We estimate the parameter of the intensity model $(\kappa^\ell, c^\ell, \delta^\ell, \gamma^\ell, \lambda_0^\ell)$ from the data that consists of the sequence of event data. Let us recall that we suppose that there are enough numbers of firms to identify the self-exciting process λ_t^ℓ with the intensity of the corresponding type of events, namely $h_t^\ell = \lambda_t^\ell$. Note that the data we used for the estimation satisfies the assumption.

For the purpose, we apply the maximum likelihood method executed in Giesecke and Kim [3]. Suppose that we have event times data of event type ℓ, $0 < T_1^\ell < T_2^\ell < \ldots < T_N^\ell (\leq H)$. Then the log-likelihood function of the intensity of the event ℓ is following:

$$\sum_{n=1}^{N} \log \lambda^\ell_{T_n^\ell -} - \int_0^H \lambda_s^\ell \mathrm{d}s. \tag{11}$$

We specify the parameters that maximize (11).

For testing validity of estimated model to the data, we apply the Kolmogorov-Smirnov test, that Azizpour and Giesecke [1] and Nakagawa [6] executed as following. First, we transform the event times $\{T_n^\ell\}_{n=1}^N$ into A_n^ℓ by

$$A_n^\ell := \int_0^{T_n^\ell} \lambda_s^\ell \mathrm{d}s.$$

We execute the Kolmogorov-Smirnov test using the fact that $\{A_n^\ell\}_{n=1}^N$ will be jump times of the standard Poisson process in the case of $\{T_n^\ell\}_{n=1}^N$ are generated by λ_t^ℓ. Hence, the null hypothesis is that $\{A_{n+1}^\ell - A_n^\ell\}_{n=1}^{N-1}$ are independent and exponentially-distributed with parameter 1.

The data for parameter estimation are the records on rating changes of Japanese firms. The ratings are announced by Rating and Investment Information, Inc. (R&I). R&I is one of the largest rating agencies in Japan. Since there are no defaults in the data, we treated the rating below BBB− as quasi-default. Then, rating considered are AAA, AA+, AA, AA−, A+, A, A−, BBB+, BBB, BBB−, Default, that is $K = 10$.

We used the rating change records from April 1, 2004 to April 1, 2009 for estimating intensity models. During the period (April 1, 2004 — April 1, 2009), 347 up-grades, 125 down-grades and 18 quasi-defaults were observed. Excluding no-business days, we transformed calendar times April 1, 2004, April 1, 2005, $\cdots$ to $t = 0, 1, \cdots$. There are a lot of events are in the same day, so we slide the event times with uniform random number so as to make every event times different.

We execute the maximum likelihood estimation of the parameters with the free statistical software package `R`. In particular, we use the intrinsic function "`optim`" to maximize the objective function. We execute the maximization for 30 sets of initial values, and finally choose the estimates that maximize the objective function among the initial value sets. In addition, we execute Kolmogorov-Smirnov test with `R`, using the intrinsic function "`ks.test`".

Table 1 shows the estimation result of the intensity processes. As all the obtained p-values are over 0.05, the estimated models are not rejected with the 5% significant level.

Table 1: Estimated parameters of the intensity processes

	κ	c	δ	γ	λ_0	p-value
up-grade ($\ell = 1$)	1.745	0.350	1.2(fixed)	90.804	26.486	0.062
down-grade ($\ell = 2$)	1.643	0.281	1.2(fixed)	168.839	82.676	0.063
default ($\ell = 3$)	3.450	0.503	1.2(fixed)	23.384	1.181	0.974

4.2 Estimation of thinning process models

We model z_m^1, z_m^2 and z_k^3 in (8), (9) and (10) as follows:

$$\begin{aligned} z_m^\ell &= \frac{\exp(a^\ell(K-m))}{\sum_m \exp(a^\ell(K-m))} \quad (\ell = 1, 2), \\ z_k^3 &= \frac{\exp(a^3 k)}{\sum_k \exp(a^3 k)}. \end{aligned}$$

Here, we suppose $a^\ell \geq 0$ ($\ell = 1, 2, 3$). To estimate z_m^1, z_m^2 and z_k^3, we prepare variables $\tilde{z}_m^1, \tilde{z}_m^2$ and $\tilde{z}_k^3$. $\tilde{z}_m^1$ describes the ratio of m-step up-grades to all up-grades obtained from data, $\tilde{z}_m^2$ describes the ratio of m-step down-grades to all down-grades obtained from data. Also, $\tilde{z}_k^3$ describes ratio of defaults from rating k to all defaults obtained from data.

We get estimate values by minimizing $\sum_m (z_m^\ell - \hat{z}_m^\ell)^2$ ($\ell = 1, 2$), $\sum_k (z_k^3 - \hat{z}_k^3)^2$. That is, we obtain a^ℓ ($\ell = 1, 2, 3$) that minimize $\sum_m (z_m^\ell - \hat{z}_m^\ell)^2$, $\sum_k (z_k^3 - \hat{z}_k^3)^2$, then we calculate estimation values $\hat{z}_m^1$, $\hat{z}_m^2$, $\hat{z}_k^3$.

We used the records from April 1, 1998 to April 1, 2009 on rating changes for estimating thinning process models. Table 2 and 3 shows the historical ratios $\tilde{z}_m^1, \tilde{z}_m^2$ and $\tilde{z}_k^3$ obtained from the data and our estimation results [4] .

4.3 Numerical example on risk analysis of loan portfolio

In this subsection, we show some results of numerical examples on a risk analysis of loan portfolios. The purpose of our numerical experiments is just to show some features of our model and to illustrate what we can with our model on a risk analysis of credit portfolios.

[4] We obtained $\hat{a}^1 = 2.327$, $\hat{a}^2 = 1.979$, $\hat{a}^3 = 1.238$.

Table 2: Estimated parameters of the thinning processes (8), (9)

	m	1	2	3	4	5	6	7	8	9
$\tilde{z}^1_m$	Historical	0.902	0.0732	0.0244	0	0	0	0	0	0
	Model	0.902	0.0881	0.00860	8.40 $\times 10^{-4}$	8.29 $\times 10^{-5}$	9.00 $\times 10^{-6}$	1.78 $\times 10^{-6}$	1.08 $\times 10^{-6}$	1.01 $\times 10^{-6}$
$\tilde{z}^2_m$	Historical	0.862	0.129	0.00460	0.00460	0	0	0	0	0
	Model	0.862	0.119	0.0165	0.00228	3.15 $\times 10^{-4}$	4.44 $\times 10^{-5}$	7.00 $\times 10^{-6}$	1.82 $\times 10^{-6}$	1.11 $\times 10^{-6}$

Table 3: Estimated parameters of the thinning processes (10)

	k	1	2	3	4	5	6	7	8	9	10
$\tilde{z}^3_k$	Historical	0	0	0	0	0	0	0	0.0300	0.260	0.710
	Model	1.01 $\times 10^{-5}$	3.53 $\times 10^{-5}$	1.22 $\times 10^{-4}$	4.22 $\times 10^{-4}$	1.46 $\times 10^{-3}$	5.02 $\times 10^{-3}$	1.73 $\times 10^{-2}$	0.0597	0.206	0.710

The rating distributions of loan portfolios in our numerical examples are described in table 4. Table 4 shows there are 100 corporate bonds for each rating. The portfolio 1 is high credit quality with many high rating loans, the portfolio 3 is low credit quality with many low rating loans, and the portfolio 2 is average credit quality. We set each loan has common maturity, loan amount and recovery rate. We set recovery rate as 40%. For setting the model parameters, we used the estimated parameter values in table1, 2 and 3.

Table 4: Rating distributions of the loan portfolios

rating	1	2	3	4	5	6	7	8	9	10
portfolio 1	15	15	15	15	15	5	5	5	5	5
portfolio 2	10	10	10	10	10	10	10	10	10	10
portfolio 3	5	5	5	5	5	15	15	15	15	15
residual	10	10	10	10	10	10	10	10	10	10

We execute Monte-Carlo simulation with 1 million scenarios and obtained loss distributions of each portfolio. The results of our numerical experiments are the following. First, we compared the loss distribution from our model and that of the Poisson model (constant intensity model). Here, we set the average loss obtained from both models are the same. Figure 3 shows the one year economy-wide loss distributions obtained by our model and that of Poisson model. Table 5 shows the risk measures of one year economy-wide loss distribution obtained by our model and the Poisson model [5] . Figure 3 and table 5 indicates that the tail part of loss distribution from self-exciting model is fatter than that from Poisson model. This is the intuitively obvious result from the self-exciting property. That is, a default occurrence in one portfolio may influence the probability of the next default in the whole economy, and thus may have impacts on the probability of the next default in other portfolios.

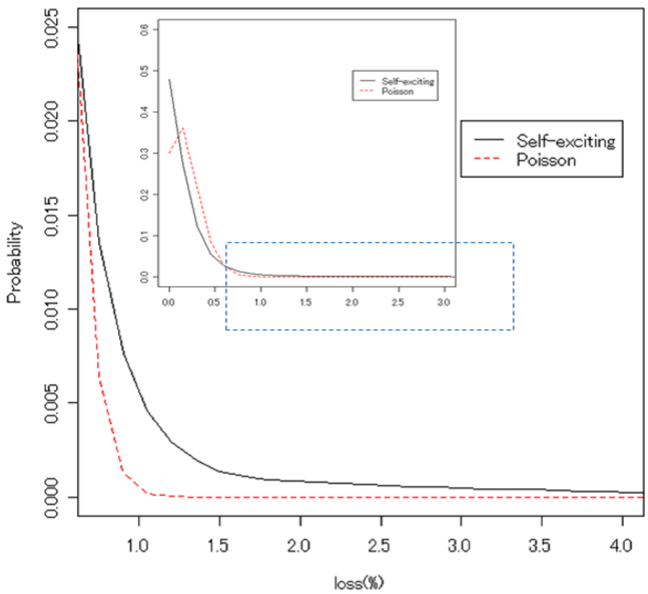

Figure 3: Loss distribution of economy

Table 5: Risk measures of one year economy-wide loss

	Self-exciting	Poisson
Average	0.18%	0.18%
Max	8.25%	1.50%
95%VaR	0.60%	0.45%
95%ES	1.32%	0.58%
99%VaR	1.80%	0.60%
99%ES	2.97%	0.74%

Next, we derive the loss distribution of each portfolio. Figure 4 shows the one year loss rate distribution of each portfolio. Table 6 shows risk measures on one year loss of each portfolio.

[5] We obtained α%VaR by the "$\alpha\%\times$ (Number of samples)"-th value of ordered loss sample values.

Figure 4 and table 6 indicates that as the credit quality of portfolio become lower, the loss of the portfolio increase. This is because the default intensity of a low credit quality portfolio tends to become higher than that of a high credit quality portfolio, with our thinning processes models.

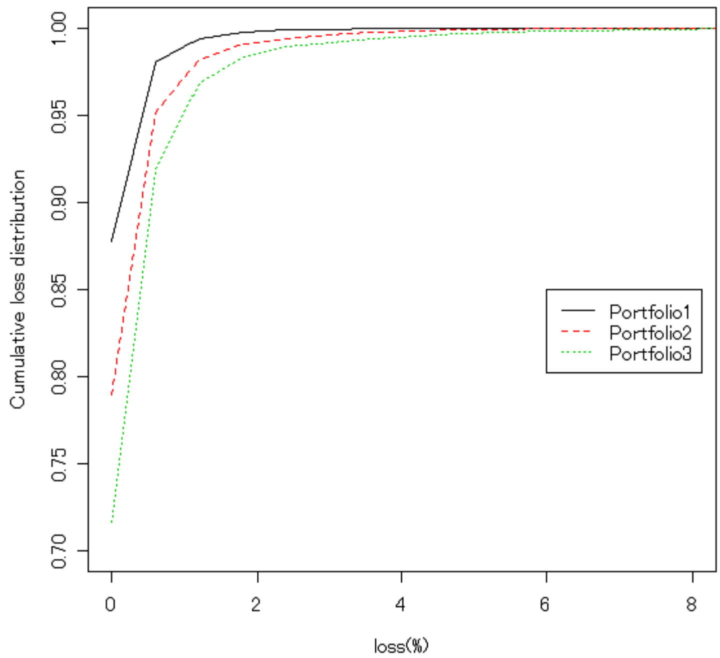

Figure 4: One year loss of each portfolio

Next, we see the most important feature of our model, that is the risk contagion among portfolios. As our model treats several portfolios simultaneously, we obtain conditional loss distribution of each portfolio, on the condition of alternative portfolio loss. Here, we consider the loss of each portfolio conditioned on the first one year loss of portfolio 2. Table 7 shows the distribution of numbers of defaults in portfolio 2 in the first one year . Table 8 shows risk measures of portfolio 1 obtained from conditional loss distribution on the condition of loss of portfolio 2. Table 8 indicates that as loss of portfolio 2 increases, the loss of other portfolios also increases. This means that we can analyze credit risk contagion with our model. We should remark on that risk contagion among the portfolios exists though the portfolios are disjoint. In addition, table 8 shows the risk measures of portfolio 1 obtained from conditional loss distribution on the condition of loss of portfolio 2 with Poisson model. Table 8 indicates that the conditional loss of portfolio 1 are the same and we are not able to capture risk contagion with constant intensities.

We saw the risk contagion among first one year loss of the portfolios. Now, we see the risk

Table 6: Risk measures of each portfolios.

	One year loss		
	Portfolio 1	Portfolio 2	Portfolio 3
Average	0.09%	0.18%	0.27%
Max	6.00%	9.60%	15.00%
95%VaR	0.60%	0.60%	1.20%
95%ES	0.95%	1.68%	2.23%
99%VaR	1.20%	1.80%	3.00%
99%ES	1.79%	3.25%	4.66%

Table 7: First one year loss distribution of portfolio 2

Numbers of defaults	Frequency
0	78.92%
1	16.22%
2	3.12%
$3, 4, \cdots$	1.74%

Table 8: First one year conditional risk measure portfolio 1

Numbers of defaults in one year in Portfolio 2		One year loss					
		Portfolio 1					
		Average	Max	95%VaR	95%ES	99%VaR	99%ES
Self-exciting	0	0.06%	3.60%	0.60%	0.70%	0.60%	1.08%
	1	0.12%	3.60%	0.60%	0.94%	1.20%	1.45%
	2	0.24%	4.80%	1.20%	1.52%	1.80%	2.20%
	$3, 4, \cdots$	0.93%	6.00%	3.00%	3.30%	3.60%	3.99%
Poisson		0.09%	3.00%	0.60%	0.72%	0.60%	1.21%

contagion from first one year loss to the second one year loss. Table 9 shows risk measures of second one year loss obtained from conditional loss distribution of each portfolio on the condition of first one year loss of portfolio 2. Table 9 indicates as first one year loss of portfolio 2 increases, the second one year loss of each portfolio increases. This means that we can analyze credit risk contagion from the past to the future. In addition, table 9 shows the risk measures of portfolio 1 obtained from conditional loss distribution on the condition of loss of portfolio 2 with the Poisson model. Table 9 indicates that we are not able to capture risk contagion with constant intensity.

Table 9: Expected loss of second one year, conditioned on first one year loss of portfolio 2.

Numbers of defaults in first one year in Port.2		Expected loss of second one year			
		Economy	Port.1	Port.2	Port.3
Self-exciting	0	0.20%	0.10%	0.20%	0.30%
	1	0.57%	0.29%	0.57%	0.86%
	2	1.56%	0.78%	1.55%	2.37%
	$3, 4, \cdots$	3.72%	1.85%	3.56%	5.74%
Poisson		0.18%	0.09%	0.18%	0.27%

Next, we see the sensitivity of losses to the event intensity parameters. Table 10 shows sensitivity of expected loss of portfolio 3 to the default intensity parameters. Here, the estimated value of parameter in table 1 is the benchmark. Direction of shift in table 10 are intuitive: decreasing of κ^3,increasing of c^3 and δ^3 and γ^3 makes default intensity higher, and that tends to cause more losses. Table 10 indicates that the model is sensitive to the parameters except the intensity jump size boundary γ^3. Though decline speed of default intensity is depend on both of κ^3 and c^3, table 10 indicates c^3 has more impact on default risks than that of κ^3 has.

Table 10: Parameter sensitivity of expected loss of portfolio 3

Shift of parameters	Shift of expected loss				
	κ^3	c^3	δ^3	γ^3	λ_0^3
-10%	$+7.56\%$	-15.99%	-10.24%	-0.83%	-12.11%
-5%	$+3.00\%$	-8.89%	-5.19%	-0.38%	-6.23%
-1%	$+0.61\%$	-1.92%	-1.25%	-0.18%	-1.87%
$+1\%$	-0.35%	$+1.77\%$	$+1.35\%$	$+0.13\%$	$+1.33\%$
$+5\%$	-2.33%	$+10.69\%$	$+5.85\%$	$+0.38\%$	$+6.33\%$
$+10\%$	-5.98%	$+23.33\%$	$+11.92\%$	$+0.74\%$	$+12.52\%$

At the end of our numerical examples, we consider the situation that the defaulter of each loan portfolio are replaced by loans of the residual portfolio [6] . This is easily realized with

[6] We replaced defaulter by a loan of residuals which has the same rating of the defaulter in the initial time.

our simulation algorithm. Table 11 shows risk measures of second one year losses without the replacing and losses with the replacing defaulter. As Table 11 indicates, the replacing makes the possibility of default in the portfolio 1,2,3 larger and possibility of default from residual portfolio smaller.

Table 11: Risk measures of second one year loss with replacing

		Second one year loss				
		Economy	Portfolio 1	Portfolio 2	Portfolio 3	residual
Average	no Replacing	0.36%	0.18%	0.36%	0.55%	0.36%
	Replacing	0.37%	0.21%	0.42%	0.63%	0.20%
Max	no Replacing	8.33%	6.67%	11.13%	14.55%	12.00%
	Replacing	8.90%	9.00%	13.75%	19.15%	6.00%
95%VaR	no Replacing	1.97%	1.20%	1.84%	3.03%	1.84%
	Replacing	1.97%	1.20%	2.40%	3.60%	1.20%
95%ES	no Replacing	3.84%	2.06%	3.96%	5.90%	3.96%
	Replacing	3.85%	2.51%	4.91%	7.34%	1.62%
99%VaR	no Replacing	4.89%	3.00%	5.11%	7.74%	5.16%
	Replacing	4.91%	3.60%	6.60%	9.70%	1.84%
99%ES	no Replacing	5.59%	3.41%	6.19%	9.02%	6.21%
	Replacing	5.62%	4.19%	7.77%	11.38%	2.52%

5 Concluding remarks

We presented a new model of credit events based on the top-down approach. Here we were interested in modeling rating changes (up-grades and down-grades) and defaults in the whole economy. Our model consists of the intensities of some credit events that are specified by the self-exciting processes with a state-dependent property. The model is similar to that of Giesecke and Kim [3], but different from them since we suppose the jump sizes of our intensity model have an upper limit.

In addition, we specify the random thinning processes in terms of the distribution of credit ratings in the sub-portfolios. Such a specification of the random thinning can ensure that the credit risk of one sub-portfolio increases as the whole credit quality in the portfolio becomes worse. Also, our model can capture credit risk contagion among several portfolios so that it is possible to evaluate the portfolios simultaneously.

In addition, we present the credit event simulation algorithm based on the model. As we illustrate some numerical examples on risk analysis of loan portfolios, our model is tractable and will work effectively to analyze risks of portfolio credit derivatives such as CDOs.

If there are no loans satisfy the condition, we randomly choose the loan for replacing.

Appendix A Simulation Algorithm

Algorithm: Generating event times over $[0, H]$ for a horizon H and the model in the section 2

1. **[Set values of the parameters and initialize simulation settings]**
   - Set $(\kappa^\ell,\ c^\ell,\ \delta^\ell,\ \gamma^\ell,\ \lambda_0^\ell)$ $(\ell = 1,\ 2,\ 3)$, $z_m^\ell(\ell = 1, 2,\ m = 1,\ 2,\ \cdots,\ K-1)$ and z_k $(k = 1,\ 2,\ \cdots,\ K)$.
   - Initialize $S = 0$, which indicate the present time. Set the time horizon $H(> 0)$.
   - Set $X_T^{(i)}(k) = X_0^{(i)}(k)$ $(i = 1,\ 2,\ \cdots,\ I,\quad k = 1,\ 2,\ \cdots,\ K)$, the number of k-rated firms in the portfolio S_i.
   - Initialize the cumulative event number $N^\ell = 0$ $(\ell = 1,\ 2,\ 3)$.
   - Initialize the last event time $T_{N^\ell}^\ell = 0$ $(\ell = 1,\ 2,\ 3)$.

2. **[Generate a candidate event time T]**
   - Set $\Lambda = \sum_{\ell=1}^{3} \lambda_S^\ell$. Draw $\mathcal{E} \sim \exp(\Lambda)$.
   - Set $T = S + \mathcal{E}$. If $T > H$, stop.

3. **[Decide to accept or to reject the candidate event time T]**
   - For $\ell = 1,\ 2,\ 3$, evaluate $\lambda_T^\ell = c^\ell \lambda_{T_{N^\ell}^\ell}^\ell + (\lambda_S^\ell - c^\ell \lambda_{T_{N^\ell}^\ell}^j) \exp(-\kappa^\ell \lambda_{T_{N^\ell}^\ell}^\ell (T - S))$.
   - Draw $u_1 \sim U(0, 1)$.
   - If $u_1 < \lambda_T^1/\Lambda$ and $\sum_{k=2}^{K} \sum_{i=1}^{I} X_T^{(i)}(k) > 0$,
     then set $\ell = 1$. (Accept T as a event time of the event type 1)
     Else if $u_1 < (\lambda_T^1 + \lambda_T^2)/\Lambda$ and $\sum_{k=1}^{K-1} \sum_{i=1}^{I} X_T^{(i)}(k) > 0$,
     then set $\ell = 2$. (Accept T as a event time of the event type 2)
     Else if $u_1 < (\lambda_T^1 + \lambda_T^2 + \lambda_T^3)/\Lambda$ and $\sum_{k=1}^{K} \sum_{i=1}^{I} X_T^{(i)}(k) > 0$,
     then set $\ell = 3$. (Accept T as a event time of the event type 3)
     Else, go to 6. (Reject T)

4. **[Thinning]**
   - Draw $u_2 \sim U(0, 1)$.
   - Evaluate $Z_T^{(i,\ell)}(k,\ k')$, $Z_T^{(i,3)}(k)(i = 1,\ 2,\ \cdots,\ I,\quad k = 1,\ 2,\ \cdots,\ K)$ with employing (8),(9),(10).
   - Set $Z = 0$.
     If$(\ell = 1)\{$
     For$(k' = 1$ to $K - 1)\{$

For($i = 1$ to I){
    For($k = k' + 1$ to K){
      $Z = Z + Z_T^{(i,\ell)}(k,\ k')$.
      If($u_2 < Z$){
        · $X_T^{(i)}(k) = X_T^{(i)}(k) - 1$
        and $X_T^{(i)}(k') = X_T^{(i)}(k') + 1$.
        · Go to 6.
      }
    }
  }
}
}

Else If($\ell = 2$){
  For($k' = 2$ to K){
    For($i = 1$ to I){
      For($k = 1$ to $k' - 1$){
        $Z = Z + Z_T^{(i,\ell)}(k,\ k')$.
        If ($u_2 < Z$){
          · $X_T^{(i)}(k) = X_T^{(i)}(k) - 1$
          and $X_T^{(i)}(k') = X_T^{(i)}(k') + 1$.
          · Go to 6.
        }
      }
    }
  }
}

Else If ($\ell = 3$){
  For ($i = 1$ to I){
    For ($k = 1$ to K){
      $Z = Z + Z_T^{(i,\ell)}(k)$.
      If($u_2 < Z$){
        · $X_T^{(i)}(k) = X_T^{(i)}(k) - 1$.
        · Go to 6.
      }
    }
  }
}

5. **[Updating]**
   For the event ℓ accepted in the step 3,
   - $N^\ell = N^\ell + 1$, (Update cumulative number of event ℓ)

- $T^\ell_{N^\ell} = T$, (Update the last event ℓ time)
- $\lambda^\ell_T = \lambda^\ell_T + \min(\delta^\ell \lambda^\ell_T, \gamma^\ell)$. (Update the intensity λ^ℓ)

6. Set $S = T$ and go to step 2.

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