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Masayuki KUMON, Akimichi TAKEMURA and Kei
TAKEUCHI

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DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

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Conformal geometry of statistical manifold with application to sequential estimation

Masayuki Kumon*
Akimichi Takemura †
Kei Takeuchi ‡

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Abstract

We present a geometrical method for analyzing sequential estimating procedures. It is based on the design principle of the second-order efficient sequential estimation provided in Okamoto, Amari and Takeuchi (1991). By introducing a dual conformal curvature quantity, we clarify the conditions for the covariance minimization of sequential estimators. These conditions are further elaborated for the multidimensional curved exponential family. The theoretical results are then numerically examined by using typical statistical models, von Mises-Fisher and hyperboloid models.

Keywords and phrases: Affine connections, Curved exponential family, Hyperboloid distribution, Information geometry, Projective transformation, Riemannian metric, Space of constant curvature, Totally umbilic, von Mises-Fisher distribution.

1 Introduction

Sequential estimation continues observations until the observed sample satisfies a certain prescribed criterion. Its properties have been shown to be superior on the average to those of nonsequential estimation in which the number of observations is fixed a priori. Specifically the developments of higher-order asymptotic theory have suggested that the information loss due to the exponential curvature of the statistical model might be recovered by a sequential estimation procedure which makes use of the ancillary statistic. Such an estimator is expected to have a uniformly better characteristic on the average

*masayuki_kumon@smile.odn.ne.jp

†Graduate School of Information Science and Technology, University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-8656, JAPAN, takemura@stat.t.u-tokyo.ac.jp

‡Emeritus, Graduate School of Economics, University of Tokyo

(see e.g. Sørensen (1986)). Takeuchi and Akahira (1988) formulated this scheme rigorously and analyzed the higher-order efficiency of sequential estimation procedures in the scalar parameter case (see also Akahira and Takeuchi (1989)). They showed that in the sequential case the exponential curvature term in the second-order variance can be eliminated by a second-order efficient estimator, and the maximum likelihood estimator with an appropriate stopping rule gives such a sequential estimator. This also implies that appropriately designed sequential estimators are superior to nonsequential estimators in the asymptotic sense.

Following the work of Takeuchi and Akahira (1988, 1989), Okamoto, Amari and Takeuchi (1991) generalized the results to the multiparameter case by using the geometrical method, and studied characteristics of more general sequential estimation procedures. In the nonsequential case, a statistical manifold is uniformly enlarged by N times when we use N observations, keeping the intrinsic features of the manifold unchanged. In a sequential estimation procedure with a certain stopping rule, the observed sample size N is a random variable depending on the position of a statistical manifold. This causes a nonuniform expansion of the statistical manifold. Such an expansion is called the conformal transformation in geometry, since it changes the scale locally and isotropically but it does not change the shape of a figure (it does not change the orthogonality). The result of Takeuchi and Akahira can be interpreted such that it is possible to reduce the exponential curvature of a statistical manifold to zero by a suitable conformal transformation. The conformal geometry thus is an adequate framework for the analysis of the sequential inferential procedures if we extend the concept of the conformal transformation to the statistical manifold which is the Riemannian manifold with a dual couple of affine connections.

As a sequel to Okamoto, Amari and Takeuchi (1991), this paper investigates the sequential estimating procedures from the information geometrical viewpoint. The novelty of this paper is the introduction of the dual conformal Weyl-Schouten curvature of a statistical manifold, and this quantity will be proved to play a central role when considering the problem of covariance minimization under the sequential estimating procedures. Information geometry was originated with the work of Amari (1985), and it has been establishing a solid status as a mathematical methodology for a variety of statistical sciences (see e.g. Amari et al (1987), Amari and Nagaoka (2000), Kumon (2009, 2010)). In line with these developments, the present paper also intends to provide a starting point for studying the conformal geometry of a statistical manifold itself succeeding to the work of Lauritzen (1987).

The organization of the paper is as follows. The established known results are cited as propositions, and the results obtained in this paper are stated as theorems. In the next section, we prepare some statistical notations and preliminary results which will be relevant in this paper. In Section 3, we formulate a conformal transformation of a statistical manifold, where a set of dual Weyl-Schouten curvature tensors is introduced. Then we elucidate their implications in the structures of statistical manifolds. In this connection, the meaning of conjugate symmetry is also explained, which is the notion first introduced by Lauritzen (1987). In Section 4, the general result in the previous section is used to

write out the structure of multidimensional exponential family. In Section 5, the general result is used to delineate the structure of multidimensional curved exponential family, where the dual Euler-Schouten curvatures are introduced. Then related notions such as totally exponential umbilic and dual quadric hypersurface are shown to involve key elements in studying statistical submanifolds. In Section 6, the geometrical results obtained in the previous sections are applied to the sequential estimation in a multidimensional curved exponential family, where we give a concrete procedure for the covariance minimization. In Section 7, the results in Section 6 are numerically examined by two typical curved exponential families called the von Mises-Fisher model and the hyperboloid model. Section 8 is devoted to some additional discussions and a perspective of future work.

2 Preliminaries

Let us denote by $X(t) = (X_1(t), \dots, X_k(t))^t$ a k -dimensional random process defined on the probability space $[\Omega, \mathcal{F}, P]$ with values in $[E, \mathcal{E}]$, where $E = \mathbb{R}^k$ and \mathcal{E} is the σ -field of all Borel sets in E . The time parameter $t \in T$ runs over all non-negative integers $T = \{0, 1, 2, \dots\}$ or over all non-negative real numbers $T = [0, +\infty)$. Moreover, let $\mathcal{F}_t, t \in T$ denote the σ -field generated by the random vectors $X(s), s \leq t$.

We assume that the probability measure P depends on an unknown parameter $\theta = (\theta^1, \dots, \theta^m)^t \in \Theta$, $P = P_\theta$, where Θ is homeomorphic to \mathbb{R}^m , and we shall consider the case where the following conditions are fulfilled:

- (i) $X(t)$ is continuous in probability, has stationary independent increments and $P_\theta(X(0) = 0) = 1, \forall \theta \in \Theta$.
- (ii) The probability distributions at any time t are dominated by a σ -finite measure μ and have the densities $f(x, t, \theta)$ with respect to μ

$$\frac{dP_\theta}{d\mu}(x, t) = f(x, t, \theta), \quad x \in E, t \in T, \theta \in \Theta. \quad (1)$$

We say that $\{P_\theta \mid \theta \in \Theta\}$ is an m -dimensional full regular minimally represented exponential family (f.r.m. exponential family) when the densities (1) can be written as

$$f_e(x, t, \theta) = \exp\{\theta^i x_i - \psi(\theta)t\}, \quad (2)$$

where $x = (x_1, \dots, x_m)^t \in \mathbb{R}^m$, θ is the natural parameter, and $\eta = \partial\psi(\theta)/\partial\theta$ is the expectation parameter with $\psi(\theta)$ a smooth (infinitely differentiable) convex function of θ . In the right-hand side of (2) and hereafter the Einstein summation convention will be assumed, so that summation will be automatically taken over indices repeated twice in the sense e.g. $\theta^i x_i = \sum_{i=1}^m \theta^i x_i$.

Sequential statistical procedures are characterized by a random sample size, where stopping times are used to stop the observations of the process. We denote by τ an arbitrary stopping time, i.e., a random variable τ defined on Ω with values in $T \cup \{\infty\}$ and possessing the property $\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t, \forall t \in T$. We consider the case to

which the Sudakov lemma applies, where the stopped process $(\tau, X(\tau))$ has the densities $\bar{f}(x, t, \theta)$, and these can be regarded as the same functions in (1) (cf. e.g. Magiera (1974)).

A typical statistical problem is the unbiased estimation of a given m -dimensional real vector valued function $h(\theta) = (h_1(\theta), \dots, h_m(\theta))^t$ of the parameter θ by using observations on the $X(t)$, $t \in T$. An estimating procedure for $h(\theta)$ is defined by a pair $(\tau, Z(\tau, X(\tau)))$, where τ is a stopping variable and $Z(\tau, X(\tau)) = (Z_1(\tau, X(\tau)), \dots, Z_m(\tau, X(\tau)))^t$ is a \mathbb{R}^m -valued function defined on $T \times E$, which is an unbiased estimator of $h(\theta)$, i.e., $E_\theta[Z(\tau, X(\tau))] = h(\theta)$.

Let us look at all estimation procedures satisfying the following regularity conditions:

(i) The $h(\theta)$ gives a smooth one-to-one transformation from Θ to $H = h(\Theta)$ in the sense

$$\text{rank } C_{i\alpha}(\theta) = \text{rank } \frac{\partial h_\alpha}{\partial \theta^i} = m, \quad \forall \theta \in \Theta.$$

(ii) $E_\theta[\tau] < \infty$, $|E_\theta[Z_\alpha(\tau, X(\tau))Z_\beta(\tau, X(\tau))]| < \infty$, $\forall \theta \in \Theta$, $\alpha, \beta = 1, \dots, m$, and the relation $E_\theta[Z(\tau, X(\tau))] = h(\theta)$ can be differentiated with respect to θ under the expectation sign.

Then we have the so-called Cramér-Rao inequality for the covariance matrix of the unbiased estimators.

Proposition 2.1. *If $Z(\tau, X(\tau))$ is an unbiased estimator of $h(\theta)$, then the the covariance matrix of $Z(\tau, X(\tau))$ is bounded below as*

$$E_\theta[(Z(\tau, X(\tau)) - h(\theta))(Z(\tau, X(\tau)) - h(\theta))^t] \geq \bar{G}(h(\theta))^{-1}, \quad (3)$$

where

$$\begin{aligned} \bar{G}(h(\theta)) &= [\bar{g}^{\alpha\beta}(h(\theta))], \quad \bar{g}^{\alpha\beta}(h(\theta)) = E_\theta[\partial^\alpha \bar{l}_\tau \partial^\beta \bar{l}_\tau] = C^{\alpha i} C^{\beta j} \bar{g}_{ij}(\theta), \\ C^{\alpha i}(h) &= \frac{\partial \theta^i}{\partial h_\alpha}, \quad \bar{g}_{ij}(\theta) = E_\theta[\partial_i \bar{l}_\tau \partial_j \bar{l}_\tau], \quad \bar{l}_\tau = \log \bar{f}(X(\tau), \tau, h(\theta)), \quad \partial^\alpha = \frac{\partial}{\partial h_\alpha}, \quad \partial_i = \frac{\partial}{\partial \theta^i}, \end{aligned}$$

and for symmetric matrices A and B , the inequality $A \geq B$ implies that $A - B$ is positive semi-definite. The equality in (3) holds if and only if $Z(\tau, X(\tau))$ can be represented almost everywhere as

$$Z(\tau, X(\tau)) = h(\theta) + \bar{G}(h(\theta))^{-1} \partial^h \bar{l}(X(\tau), \tau, h(\theta)), \quad \partial^h \bar{l} = \left(\frac{\partial \bar{l}}{\partial h_1}, \dots, \frac{\partial \bar{l}}{\partial h_m} \right). \quad (4)$$

The condition (4) is also written as

$$\partial^h \bar{l}(X(\tau), \tau, h(\theta)) = \bar{G}(h(\theta))(Z(\tau, X(\tau)) - h(\theta)),$$

or in component form

$$\partial^\alpha \bar{l}(X(\tau), \tau, h(\theta)) = \bar{g}^{\alpha\beta}(h(\theta))Z_\beta(\tau, X(\tau)) - \bar{g}^{\alpha\beta}(h(\theta))h_\beta(\theta) = k^\alpha(X(\tau), \tau, h(\theta)).$$

The above is a partial differential equation for $\bar{l}(X(\tau), \tau, h(\theta))$, of which integrability condition $\partial^\beta k^\alpha = \partial^\alpha k^\beta$ is

$$[\partial^\beta \bar{g}^{\alpha\gamma}(h(\theta)) - \partial^\alpha \bar{g}^{\beta\gamma}(h(\theta))](Z_\gamma(\tau, X(\tau)) - h_\gamma(\theta)) = 0,$$

and hence the requirement for integrability is

$$\partial^\beta \bar{g}^{\alpha\gamma}(h) = \partial^\alpha \bar{g}^{\beta\gamma}(h) \Leftrightarrow \exists \bar{\phi}(h) \text{ smooth and convex in } h \text{ such that } \bar{g}^{\alpha\beta}(h) = \partial^\alpha \partial^\beta \bar{\phi}(h).$$

In this case, the log likelihood function $\bar{l}(x, t, h(\theta))$ is expressed as

$$\begin{aligned} \bar{l}(x, t, h(\theta)) &= \bar{l}(x, t) + \bar{\xi}^\alpha z_\alpha(t) - \bar{\psi}(\bar{\xi}), \\ \bar{\xi}^\alpha = \bar{\xi}^\alpha(h) &= \partial^\alpha \bar{\phi}(h), \quad \bar{\psi}(\bar{\xi}) = \bar{\xi}^\alpha h_\alpha - \bar{\phi}(h), \end{aligned} \tag{5}$$

which implies that $\{\bar{f}(x, t, \theta)\}$ must be a f.r.m. exponential family with $\bar{\xi}$ and h the natural and the expectation parameters, respectively.

Suppose that the original $\{f(x, t, \theta)\}$ is not a f.r.m. exponential family, then clearly $\{\bar{f}(x, t, \theta)\}$ is not a f.r.m. exponential one, either. Hence we can restrict attention to the case (2), when considering the attainment of the lower bound given by Proposition 1.1. However we should remark that it is only a necessary condition for the attainment of the lower bound. In fact even in the f.r.m. exponential family, only some restricted cases can exactly attain the lower bound due to the problem of the ‘‘overshooting’’ at the efficient stopping times (see Ghosh (1987)).

3 Conformal transformation of statistical manifold

Let $M = \{f(x, 1, \theta) \mid \theta \in \Theta\}$ be an original family of probability densities of unit time, where Θ is homeomorphic to \mathbb{R}^m . The family M can be regarded as a statistical manifold, where the m -dimensional vector parameter θ serves as a coordinate system to specify a point, that is, a density $f(x, 1, \theta) \in M$. The geometry of M is determined by the following two tensor quantities (cf. Amari (1985), Amari and Nagaoka (2000))

$$g_{ij}(\theta) = E_\theta[\partial_i l_1 \partial_j l_1], \quad T_{ijk}(\theta) = E_\theta[\partial_i l_1 \partial_j l_1 \partial_k l_1], \quad l_1 = \log f(x, 1, \theta), \quad \partial_i = \frac{\partial}{\partial \theta^i}.$$

The first is the Fisher information metric and the second is called the skewness tensor. One parameter family of affine connections named the α -connection is defined by

$$\Gamma_{ijk}^{(\alpha)}(\theta) = E_\theta[\partial_i \partial_j l_1 \partial_k l_1] + \frac{1-\alpha}{2} T_{ijk}(\theta),$$

and then the α -Riemann-Christoffel curvature tensor is given by

$$R_{ijkl}^{(\alpha)}(\theta) = \partial_i \Gamma_{jkl}^{(\alpha)} - \partial_j \Gamma_{ikl}^{(\alpha)} + g^{rs} (\Gamma_{ikr}^{(\alpha)} \Gamma_{jsl}^{(\alpha)} - \Gamma_{jkr}^{(\alpha)} \Gamma_{isl}^{(\alpha)}).$$

The α - and the $(-\alpha)$ -connections are mutually dual

$$\partial_i g_{jk} = \Gamma_{ijk}^{(\alpha)} + \Gamma_{ikj}^{(-\alpha)},$$

and the $\pm\alpha$ -RC curvature tensors are in the dual relation

$$R_{ijkl}^{(\alpha)} = -R_{ijlk}^{(-\alpha)}.$$

Let $\bar{M} = \{\bar{f}(x, t, \theta) \mid \theta \in \Theta\}$ be an m -dimensional extended statistical manifold under a sequential statistical procedure. From the Wald identity the metric and the skewness tensors of \bar{M} are given by (see Akahira and Takeuchi (1989))

$$\bar{g}_{ij}(\theta) = \nu g_{ij}, \quad \bar{T}_{ijk}(\theta) = \nu [T_{ijk} + 3g_{(ij}s_k)], \quad (6)$$

$$\nu(\theta) = E_\theta[\tau], \quad s_k(\theta) = \partial_k \log \nu(\theta), \quad 3g_{(ij}s_k) = g_{ij}s_k + g_{jk}s_i + g_{ki}s_j. \quad (7)$$

These relations show that a sequential statistical procedure induces a conformal transformation $M \mapsto \bar{M}$ by the gauge function $\nu(\theta) > 0$. The conformal transformation of a Riemannian manifold implies that the manifold is expanded or contracted isotropically but that an expansion rate depends on each point. Our transformation is a statistical counterpart of this one. A conformal transformation changes the α -connection into

$$\bar{\Gamma}_{ijk}^{(\alpha)} = \nu [\Gamma_{ijk}^{(\alpha)} + \frac{1-\alpha}{2}(g_{ki}s_j + g_{kj}s_i) - \frac{1+\alpha}{2}g_{ij}s_k], \quad (8)$$

$$\bar{\Gamma}_{ij}^{(\alpha)k} = \bar{\Gamma}_{ijl}^{(\alpha)} \bar{g}^{lk} = \Gamma_{ij}^{(\alpha)k} + \frac{1-\alpha}{2}(\delta_i^k s_j + \delta_j^k s_i) - \frac{1+\alpha}{2}g_{ij}s_l g^{lk}, \quad (9)$$

This is obtained by substituting (6) into

$$\bar{\Gamma}_{ijk}^{(\alpha)} = \bar{\Gamma}_{ijk}^{(0)} - \frac{\alpha}{2} \bar{T}_{ijk},$$

and by noting that $\bar{\Gamma}_{ijk}^{(0)}$ is the conformal change of the Riemannian connection $\Gamma_{ijk}^{(0)}$. Then a conformal transformation changes the α -RC curvature tensor into

$$\bar{R}_{ijkl}^{(\alpha)} = \nu [R_{ijkl}^{(\alpha)} - g_{il}s_{jk}^{(\alpha)} + g_{jl}s_{ik}^{(\alpha)} - g_{jk}s_{il}^{(-\alpha)} + g_{ik}s_{jl}^{(-\alpha)}], \quad (10)$$

$$\bar{R}_{ijk}^{(\alpha)l} = \bar{R}_{ijk}^{(\alpha)} \bar{g}^{rl} = R_{ijk}^{(\alpha)l} - \delta_i^l s_{jk}^{(\alpha)} + \delta_j^l s_{ik}^{(\alpha)} - g_{jk}s_{ir}^{(-\alpha)} g^{rl} + g_{ik}s_{jr}^{(-\alpha)} g^{rl}, \quad (11)$$

$$s_{ij}^{(\alpha)} = \frac{1-\alpha}{2} [\nabla_i^{(\alpha)} s_j - \frac{1-\alpha}{2} s_i s_j + \frac{1+\alpha}{4} g_{ij} s_k s_l g^{kl}], \quad \nabla_i^{(\alpha)} s_j = \partial_i s_j - \Gamma_{ij}^{(\alpha)k} s_k. \quad (12)$$

This is obtained by substituting (8) into

$$\bar{R}_{ijkl}^{(\alpha)} = \partial_i \bar{\Gamma}_{jkl}^{(\alpha)} - \partial_j \bar{\Gamma}_{ikl}^{(\alpha)} + \bar{g}^{rs} (\bar{\Gamma}_{ikr}^{(\alpha)} \bar{\Gamma}_{jsl}^{(\alpha)} - \bar{\Gamma}_{jkr}^{(\alpha)} \bar{\Gamma}_{isl}^{(\alpha)}).$$

We note that under a conformal transformation the mutual duality of $\pm\alpha$ -connections is preserved

$$\partial_i \bar{g}_{jk} = \bar{\Gamma}_{ijk}^{(\alpha)} + \bar{\Gamma}_{ikj}^{(-\alpha)}, \quad (13)$$

and also the dual relation of the $\pm\alpha$ -RC curvature tensors is preserved

$$\bar{R}_{ijkl}^{(\alpha)} = -\bar{R}_{ijlk}^{(-\alpha)}. \quad (14)$$

These are confirmed by the direct calculations with (6), (8) and (10).

One of the concerns about the conformal transformation is whether a given manifold can be transformed into a desirable space in some sense. The main objective from the geometrical viewpoint is the flatness or the straightness, and it has been investigated usually in terms of the Riemannian connection. From the statistical viewpoint, the main objective is the flatness or the straightness in terms of the mutually dual ± 1 -connections. Thus we say that a statistical manifold M is *conformally mixture (exponential) flat* when there exists a gauge function $\nu(\theta) > 0$ such that $\bar{R}_{ijk}^{(-1)l} = 0$ ($\bar{R}_{ijk}^{(1)l} = 0$) holds. Note that by (14) M is conformally mixture flat if and only if M is conformally exponential flat.

In view of these observations and also the work of Okamoto (1988), we introduce the set of (-1) -Weyl-Schouten curvature tensors as follows.

Definition 3.1.

$$W_{ijk}^{(-1)l}(\theta) = R_{ijk}^{(-1)l} - \frac{1}{m-1}(\delta_i^l R_{jk}^{(-1)} - \delta_j^l R_{ik}^{(-1)}), \quad (15)$$

$$W_{ijk}^{(-1)}(\theta) = \frac{1}{m-1}(\nabla_i^{(-1)} R_{jk}^{(-1)} - \nabla_j^{(-1)} R_{ik}^{(-1)}), \quad (16)$$

$$W_{ij}^{(-1)}(\theta) = R_{ij}^{(-1)} - R_{ji}^{(-1)}, \quad (17)$$

$$R_{ij}^{(-1)} = R_{lij}^{(-1)}, \quad \nabla_i^{(-1)} R_{jk}^{(-1)} = \partial_i R_{jk}^{(-1)} - \Gamma_{ij}^{(-1)l} R_{lk}^{(-1)} - \Gamma_{ik}^{(-1)l} R_{jl}^{(-1)}. \quad (18)$$

From (11) and (12) we have

$$\begin{aligned} \bar{R}_{ijk}^{(-1)l} &= R_{ijk}^{(-1)l} - \delta_i^l s_{jk}^{(-1)} + \delta_j^l s_{ik}^{(-1)}, \quad s_{jk}^{(-1)} = \nabla_j^{(-1)} s_k - s_j s_k \\ &\Rightarrow \bar{R}_{jk}^{(-1)} = R_{jk}^{(-1)} - (m-1)s_{jk}^{(-1)} \\ &\Rightarrow s_{jk}^{(-1)} = -\frac{1}{m-1}(\bar{R}_{jk}^{(-1)} - R_{jk}^{(-1)}) \\ &\Rightarrow \bar{W}_{ijk}^{(-1)l} = W_{ijk}^{(-1)l}, \quad \bar{W}_{ijk}^{(-1)} = W_{ijk}^{(-1)} + W_{ijk}^{(-1)l} s_l, \quad \bar{W}_{ij}^{(-1)} = W_{ij}^{(-1)}. \end{aligned}$$

For the case $m = 2$ we can also directly check $W_{ijk}^{(-1)l} \equiv 0$, and hence $\bar{W}_{ijk}^{(-1)} = W_{ijk}^{(-1)}$. Then we obtain the following result as to the conditions for the conformal mixture (exponential) flatness.

Theorem 3.1. *A statistical manifold M is conformally mixture flat (or equivalently exponential flat) if and only if*

- (i) $W_{ijk}^{(-1)l} = 0$ when $m = \dim M \geq 3$.
- (ii) $W_{ijk}^{(-1)} = 0$ and $W_{ij}^{(-1)} = 0$ when $m = \dim M = 2$.

For the sake of simplicity we hereafter express the notion such as conformally mixture (or equivalently exponential) flat as conformally $m(e)$ -flat.

Proof. Consider the relation

$$s_{jk}^{(-1)} = \nabla_j^{(-1)} s_k - s_j s_k = -\frac{1}{m-1} (\bar{R}_{jk}^{(-1)} - R_{jk}^{(-1)}).$$

When $\bar{R}_{jk}^{(-1)} = 0$ we note by the integrability condition that

$$\begin{aligned} \exists s_k \text{ such that } \nabla_j^{(-1)} s_k - s_j s_k &= \frac{1}{m-1} R_{jk}^{(-1)} \\ \Leftrightarrow \nabla_i^{(-1)} \nabla_j^{(-1)} s_k - \nabla_j^{(-1)} \nabla_i^{(-1)} s_k &= -R_{ijk}^{(-1)l} s_l \\ \Leftrightarrow W_{ijk}^{(-1)l} s_l + W_{ijk}^{(-1)} &= 0. \end{aligned}$$

We first prove the necessity. Suppose that M is conformally $m(e)$ -flat. Then from $\bar{R}_{ijk}^{(-1)l}, \bar{R}_{jk}^{(-1)} = 0$, when $m \geq 3$ we have $W_{ijk}^{(-1)l} = \bar{W}_{ijk}^{(-1)l} = 0$. When $m = 2$ since $W_{ijk}^{(-1)l} \equiv 0$ we have $W_{ijk}^{(-1)} = \bar{W}_{ijk}^{(-1)} = 0$, and since there exists a log gauge function $s = \log \nu$ we have $W_{ij}^{(-1)} = \bar{W}_{ij}^{(-1)} = 0$.

We next prove the sufficiency when $m \geq 3$. From the Bianchi's second identity (cf. Schouten (1954), p.147)

$$\begin{aligned} \nabla_l^{(-1)} R_{ijk}^{(-1)l} + \nabla_j^{(-1)} R_{lik}^{(-1)l} + \nabla_i^{(-1)} R_{jlk}^{(-1)l} &= 0 \\ \Rightarrow \nabla_l^{(-1)} R_{ijk}^{(-1)l} &= \nabla_i^{(-1)} R_{jk}^{(-1)} - \nabla_j^{(-1)} R_{ik}^{(-1)}, \end{aligned}$$

and from $W_{ijk}^{(-1)l} = 0$ we have

$$\nabla_l^{(-1)} W_{ijk}^{(-1)l} = \nabla_l^{(-1)} R_{ijk}^{(-1)l} - \frac{1}{m-1} (\nabla_i^{(-1)} R_{jk}^{(-1)} - \nabla_j^{(-1)} R_{ik}^{(-1)}) = (m-2) W_{ijk}^{(-1)} = 0,$$

so that $W_{ijk}^{(-1)} = 0$. Then as noted before, there exists a covariant vector field s_k such that $\nabla_j^{(-1)} s_k - s_j s_k = \frac{1}{m-1} R_{jk}^{(-1)}$. From the Bianchi's first identity (cf. Schouten (1954), p.144)

$$R_{ijk}^{(\pm 1)i} + R_{kij}^{(\pm 1)i} + R_{jki}^{(\pm 1)i} = 0 \Rightarrow R_{jk}^{(\pm 1)} - R_{kj}^{(\pm 1)} + R_{jkil}^{(\pm 1)} g^{li} = 0,$$

and from the duality of $R_{ijkl}^{(\pm 1)}$

$$R_{ijkl}^{(1)} + R_{ijkl}^{(-1)} = -R_{ijlk}^{(1)} - R_{ijlk}^{(-1)},$$

we have

$$R_{jk}^{(1)} - R_{kj}^{(1)} + R_{jk}^{(-1)} - R_{kj}^{(-1)} = 0.$$

On the other hand, from $W_{ijk}^{(-1)l} = 0$ we have

$$\begin{aligned} R_{jilk}^{(1)} &= R_{ijkl}^{(-1)} = \frac{1}{m-1}(g_{il}R_{jk}^{(-1)} - g_{jl}R_{ik}^{(-1)}) \\ \Rightarrow R_{il}^{(1)} &= R_{jilk}^{(1)}g^{jk} = \frac{1}{m-1}(g_{il}R^{(-1)} - R_{il}^{(-1)}), \quad R^{(-1)} = R_{jk}^{(-1)}g^{jk}. \end{aligned}$$

By combining these two relations we obtain

$$\frac{m-2}{m-1}(R_{jk}^{(-1)} - R_{kj}^{(-1)}) = 0 \Rightarrow R_{jk}^{(-1)} - R_{kj}^{(-1)} = 0 \Rightarrow \partial_j s_k - \partial_k s_j = 0,$$

and hence there exists a log gauge function $s = \log \nu$ such that $s_k = \partial_k s$.

Finally we prove the sufficiency when $m = 2$. From $W_{ijk}^{(-1)l} \equiv 0$, $W_{ijk}^{(-1)} = 0$, as noted above, there exists a covariant vector field s_k such that $\nabla_j^{(-1)} s_k - s_j s_k = \frac{1}{m-1} R_{jk}^{(-1)}$. Then from

$$W_{jk}^{(-1)} = R_{jk}^{(-1)} - R_{kj}^{(-1)} = 0 \Rightarrow \partial_j s_k - \partial_k s_j = 0,$$

there exists a log gauge function $s = \log \nu$ such that $s_k = \partial_k s$.

This completes the proof of the theorem. \square

We further investigate the implications of Theorem 3.1. Suppose that the (-1) -RC curvature tensor of a statistical manifold M is expressed as

$$R_{ijkl}^{(-1)} = \lambda(g_{jk}g_{il} - g_{ik}g_{jl}), \quad (19)$$

where λ is constant on M . In this case we have

$$R_{ijkl}^{(-1)} = -R_{ijlk}^{(-1)} = R_{ijkl}^{(1)} = R_{jilk}^{(-1)}. \quad (20)$$

A statistical manifold M satisfying (19) is said to be a *space of constant mixture (exponential) curvature*, and M satisfying (20) is said to be *conjugate mixture (exponential) symmetric*. The conjugate $m(e)$ -symmetry is the special notion of the conjugate $\pm\alpha$ -symmetry introduced by Lauritzen (1987). By definition we know

M is a space of constant $m(e)$ -curvature $\Rightarrow M$ is conjugate $m(e)$ -symmetric.

The connections among these notions are summarized in the following theorem.

Theorem 3.2. *For the conformal $m(e)$ -flatness of a statistical manifold M , the following relations hold.*

- (i) *M is conjugate $m(e)$ -symmetric and is conformally $m(e)$ -flat if and only if M is a space of constant $m(e)$ -curvature.*
- (ii) *A f.r.m. exponential family M_e is always conformally $m(e)$ -flat.*

Proof. We first prove the sufficiency of (i). Suppose that M is a space of constant $m(e)$ -curvature. From

$$R_{ijkl}^{(-1)} = \lambda(g_{jk}g_{il} - g_{ik}g_{jl}),$$

we have

$$R_{jk}^{(-1)} = R_{ijkl}^{(-1)} g^{il} = (m-1)\lambda g_{jk}.$$

Then we obtain

$$\begin{aligned} W_{ijkl}^{(-1)} &= W_{ijk}^{(-1)r} g_{rl} = R_{ijkl}^{(-1)} - \frac{1}{m-1}(g_{il}R_{jk}^{(-1)} - g_{jl}R_{ik}^{(-1)}) = 0, \\ W_{ijk}^{(-1)} &= \lambda(\nabla_i^{(-1)}g_{jk} - \nabla_j^{(-1)}g_{ik}) = 0 \quad (\text{since } \nabla_i^{(\alpha)}g_{jk} = \alpha T_{ijk}), \\ W_{ij}^{(-1)} &= (m-1)\lambda(g_{ij} - g_{ji}) = 0. \end{aligned}$$

We next prove the necessity of (i). Suppose that M is conjugate $m(e)$ -symmetric and is conformally $m(e)$ -flat. When $m \geq 3$, we have

$$\begin{aligned} R_{jilk}^{(-1)} &= R_{ijkl}^{(-1)} = \frac{1}{m-1}(g_{il}R_{jk}^{(-1)} - g_{jl}R_{ik}^{(-1)}), \\ R_{il}^{(-1)} &= R_{jilk}^{(-1)} g^{jk} = \frac{1}{m-1}(g_{il}R^{(-1)} - R_{il}^{(-1)}) \\ \Rightarrow R_{il}^{(-1)} &= \frac{R^{(-1)}}{m} g_{il} \\ \Rightarrow R_{ijkl}^{(-1)} &= \frac{R^{(-1)}}{m(m-1)}(g_{jk}g_{il} - g_{ik}g_{jl}) \\ \Rightarrow R_{ijk}^{(-1)l} &= \rho(g_{jk}\delta_i^l - g_{ik}\delta_j^l), \quad \rho = \frac{R^{(-1)}}{m(m-1)}. \end{aligned}$$

By substituting this expression into the Bianchi's second identity, we have

$$\begin{aligned} \nabla_r^{(-1)}R_{ijk}^{(-1)l} + \nabla_j^{(-1)}R_{rik}^{(-1)l} + \nabla_i^{(-1)}R_{jrk}^{(-1)l} &= 0 \\ \Rightarrow \nabla_r^{(-1)}\rho(g_{jk}\delta_i^l - g_{ik}\delta_j^l) + \nabla_j^{(-1)}\rho(g_{ik}\delta_r^l - g_{rk}\delta_i^l) + \nabla_i^{(-1)}\rho(g_{rk}\delta_j^l - g_{jk}\delta_r^l) &= 0 \\ (\text{since } \nabla_i^{(-1)}g_{jk} = -T_{ijk} \text{ is symmetric in } i, j, k) & \\ \Rightarrow (m-1)(m-2)\nabla_r^{(-1)}\rho = (m-1)(m-2)\partial_r\rho = 0, & \end{aligned}$$

so that ρ is constant on M . When $m = 2$ we have

$$\begin{aligned} \nabla_i^{(-1)}R_{jk}^{(-1)} &= \nabla_j^{(-1)}R_{ik}^{(-1)}, \quad R_{jk}^{(-1)} = \frac{R^{(-1)}}{2}g_{jk} \\ \Rightarrow \partial_i R^{(-1)}g_{jk} &= \partial_j R^{(-1)}g_{ik} \quad (\text{since } \nabla_i^{(-1)}g_{jk} = -T_{ijk}) \\ \Rightarrow \partial_i R^{(-1)} &= 0, \end{aligned}$$

and again $R^{(-1)}$ is constant on M . This completes the proof of (i).

Since M_e is a space of zero $m(e)$ -curvature $R_{ijkl}^{(\pm 1)} = 0$, (ii) is obtained from (i). \square

Figure 1 illustrates the relations among several notions in Theorem 3.2.

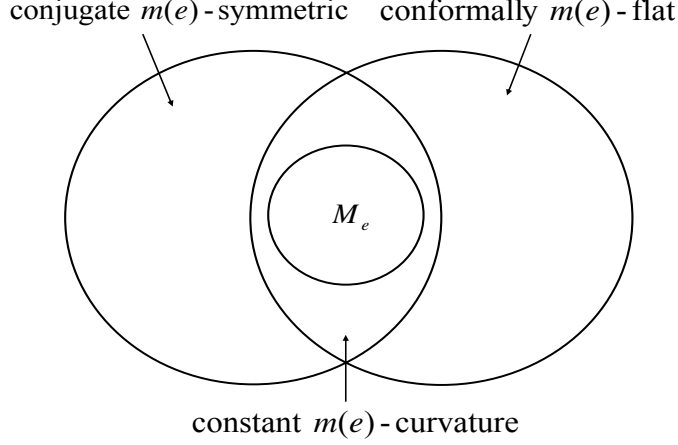


Figure 1: Relations among several notions on M

4 Conformal geometry of exponential family

Based on Theorem 3.2 (ii), we seek a concrete conformal transformation $M_e \mapsto \bar{M}_e$ such that \bar{M}_e is ± 1 -flat. When M_e is a f.r.m. exponential family, it is ± 1 -flat, i.e., $R_{ijkl}^{(\pm 1)} = 0$, in which the natural parameter θ and the expectation parameter η provide the ± 1 -affine coordinate systems of M_e in the sense (cf. Amari (1985), Amari and Nagaoka (2000))

$$\Gamma_{ijk}^{(1)}(\theta) = E_\theta[\partial_i \partial_j l_1 \partial_k l_1] = 0, \quad \partial_i = \frac{\partial}{\partial \theta^i},$$

$$\Gamma^{(-1)ijk}(\eta) = E_\eta[(\partial^i \partial^j l_1 + \partial^i l_1 \partial^j l_1) \partial^k l_1] = 0, \quad \partial^i = \frac{\partial}{\partial \eta_i},$$

and there exist two potential functions $\psi(\theta)$ and $\phi(\eta)$ such that

$$\theta^i = \partial^i \phi(\eta), \quad \eta_i = \partial_i \psi(\theta), \quad g_{ij}(\theta) = \partial_i \partial_j \psi(\theta), \quad g^{ij}(\eta) = \partial^i \partial^j \phi(\eta), \quad \psi(\theta) + \phi(\eta) - \theta^i \eta_i = 0.$$

By the formula (8) a conformal transformation $M_e \mapsto \bar{M}_e$ with gauge function $\nu(\eta) > 0$ changes $\Gamma^{(-1)ijk}(\eta)$ into

$$\bar{\Gamma}^{(-1)ijk}(\eta) = \nu[g^{ki} s^j + g^{kj} s^i], \quad g^{ki} = E_\eta[\partial^k l_1 \partial^i l_1], \quad s^j = \partial^j \log \nu.$$

We consider a coordinate transformation from η to h which will provide a (-1) -affine coordinate system of \bar{M}_e . The (-1) -connection $\bar{\Gamma}^{(-1)ijk}(\eta)$ transforms to

$$\begin{aligned} \bar{\Gamma}^{(-1)\alpha\beta\gamma}(h) &= B_i^\alpha B_j^\beta B_k^\gamma \bar{\Gamma}^{(-1)ijk}(\eta) + \bar{g}^{jk} B_k^\gamma \partial^\alpha B_j^\beta, \quad B_i^\alpha = \frac{\partial \eta_i}{\partial h_\alpha}, \quad \partial^\alpha = \frac{\partial}{\partial h_\alpha}, \\ &= \nu B_k^\gamma [B_i^\alpha B_j^\beta (g^{ki} s^j + g^{kj} s^i) + g^{jk} \partial^\alpha B_j^\beta], \end{aligned}$$

and hence

$$\begin{aligned}
\bar{M}_e \text{ is } m\text{-flat} &\Leftrightarrow \bar{\Gamma}^{(-1)\alpha\beta\gamma}(h) = 0 \\
&\Leftrightarrow \exists h, \nu \text{ such that } B_i^\alpha B_j^\beta (g^{ki} s^j + g^{kj} s^i) + g^{jk} \partial^\alpha B_j^\beta = 0 \\
&\Leftrightarrow \exists h, \nu \text{ such that } \partial^i C_\alpha^j - s^i C_\alpha^j - s^j C_\alpha^i = 0, \quad s^{(-1)ij} = \partial^i s^j - s^i s^j = 0, \quad C_\alpha^i = \frac{\partial h_\alpha}{\partial \eta_i},
\end{aligned}$$

where $s^{(-1)ij} = 0$ is the integrability condition of the first equation on the right-hand side. We can solve the above two partial differential equations for $s(\eta) = \log \nu(\eta)$ and $h_\alpha(\eta)$ as shown in the following theorem.

Theorem 4.1. *When a statistical manifold M_e is an m -dimensional f.r.m. exponential family with (-1) -affine coordinate system η , it is conformally $m(e)$ -flat by the gauge function $\nu(\eta) > 0$ and the new (-1) -affine coordinate system h given by*

$$\nu(\eta) = \frac{1}{|c^0 + c^i \eta_i|}, \quad h_\alpha = \nu(\eta)(d_\alpha + D_\alpha^i \eta_i), \quad (21)$$

where $c^0, c^i, d_\alpha, D_\alpha^i$ ($i, \alpha = 1, \dots, m$) are constants, and rank $D_\alpha^i = m$.

The new 1-affine coordinate system ξ , two potential functions $\bar{\psi}(\xi)$ and $\bar{\phi}(h)$ are respectively given as

$$\xi^\alpha = \partial^\alpha \bar{\phi}(h), \quad h_\alpha = \partial_\alpha \bar{\psi}(\xi), \quad \bar{\phi}(h) = \nu(\eta) \phi(\eta), \quad \bar{\psi}(\xi) + \bar{\phi}(h) - \xi^\alpha h_\alpha = 0, \quad (22)$$

$$\bar{g}_{\alpha\beta}(\xi) = \partial_\alpha \partial_\beta \bar{\psi}(\xi), \quad \bar{g}^{\alpha\beta}(h) = \partial^\alpha \partial^\beta \bar{\phi}(h). \quad (23)$$

Proof. We first prove (21). By putting $s = -\log r$ we have

$$\partial^i s^j - s^i s^j = 0 \Leftrightarrow \partial^i \partial^j r = 0 \Leftrightarrow r = c^0 + c^i \eta_i \Leftrightarrow \nu = \frac{1}{r} = \frac{1}{c^0 + c^i \eta_i},$$

and by putting $h_\alpha = e^s y_\alpha$ we have

$$\partial^i C_\alpha^j - s^i C_\alpha^j - s^j C_\alpha^i = 0 \Leftrightarrow \partial^i \partial^j y_\alpha = 0 \Leftrightarrow y_\alpha = d_\alpha + D_\alpha^i \eta_i \Leftrightarrow h_\alpha = \frac{d_\alpha + D_\alpha^i \eta_i}{c^0 + c^i \eta_i}.$$

We next prove (22), (23). By the direct calculation we can confirm

$$\bar{g}^{\alpha\beta}(h) = \partial^\alpha \partial^\beta \bar{\phi}(h), \quad \bar{\phi}(h) = \nu(\eta) \phi(\eta),$$

and then the others are immediately obtained.

This completes the proof of the theorem. \square

We remark that $\nu(\eta)$ and $h(\eta)$ in (21) cover the general solution and these are the same as those given in Winkler and Franz (1979), which were derived from the statistical considerations of the efficient sequential estimators attaining the Cramér-Rao bound.

5 Conformal geometry of curved exponential family

We first introduce a curved exponential family. A family of probability densities $M_c = \{f_c(x, t, u) \mid u \in U\}$ parameterized by an m -dimensional vector parameter $u = (u^1, \dots, u^m)^t$ is said to be an (n, m) -curved exponential family when it is smoothly imbedded in an n -dimensional f.r.m. exponential family $M_e = \{f_e(x, t, \theta) \mid \theta \in \Theta\}$ in the sense

$$f_c(x, t, u) = f_e(x, t, \theta(u)) = \exp\{\theta^i(u)x_i - \psi(\theta(u))t\}, \quad (24)$$

where U is homeomorphic to $\mathbb{R}^m (m < n)$ and $\theta(u) = (\theta^1(u), \dots, \theta^n(u))^t$ is a smooth function of u having a full rank Jacobian matrix. We use indices i, j, k and so on to denote quantities in terms of the coordinate system θ or η of M_e , and indices a, b, c and so on to denote quantities in terms of the coordinate system u of M_c .

For analyzing the geometrical properties of M_c imbedded in M_e , it is convenient to introduce a new coordinate system $w = (u, v)$ of M_e in the following manner. We attach to each point $u \in M_c$ an $(n - m)$ -dimensional smooth submanifold $A(u)$ of M_e which transverses M_c at $\theta(u)$ or equivalently at $\eta(u)$. We assume that the family $A = \{A(u) \mid u \in M_c\}$ fills up at least a neighborhood of M_c in M_e , that is, A is a foliation of the tubular neighborhood of M_c in M_e . Such an $A(u)$ is called an ancillary submanifold rigging u , and A is called an ancillary family rigging M_c .

We introduce a coordinate system $v = (v^{m+1}, \dots, v^n)$ to each $A(u)$ such that the origin $v = 0$ is at the intersection of $A(u)$ and M_c . Then the combined system

$$w = (w^\alpha) = (u^a, v^\kappa), \quad \alpha = 1, \dots, n, \quad a = 1, \dots, m, \quad \kappa = m + 1, \dots, n$$

gives a new local coordinate system of M_e . We use indices α, β, γ and so on for quantities related to the coordinate system w , and indices κ, λ, μ and so on for quantities related to the coordinate system v .

The basic tensors of M_e are written as

$$g_{\alpha\beta}(w) = g_{ij}(\theta)B_\alpha^i B_\beta^j, \quad T_{\alpha\beta\gamma}(w) = T_{ijk}(\theta)B_\alpha^i B_\beta^j B_\gamma^k, \quad B_\alpha^i = \frac{\partial \theta^i}{\partial w^\alpha},$$

and the α -connection is given by

$$\begin{aligned} \Gamma_{\beta\gamma\delta}^{(\alpha)}(w) &= \Gamma_{ijk}^{(\alpha)}(\theta)B_\beta^i B_\gamma^j B_\delta^k + g_{ij}(\theta)B_\delta^i \partial_\beta B_\gamma^j = \frac{1 - \alpha}{2} T_{\beta\gamma\delta} + (\partial_\beta B_\gamma^j) B_{\delta j} \\ &= \Gamma^{(\alpha)ijk}(\eta)B_{\beta i} B_{\gamma j} B_{\delta k} + g^{ij}(\eta)B_{\delta i} \partial_\beta B_{\gamma j} = -\frac{1 + \alpha}{2} T_{\beta\gamma\delta} + (\partial_\beta B_{\gamma j}) B_\delta^j, \\ B_{\beta i} &= \frac{\partial \eta_i}{\partial w^\beta} = g_{ij} B_\beta^j, \end{aligned}$$

in the w -coordinate system. When we evaluate a quantity $q(u, v)$ on M_c , i.e., at $v = 0$, we often denote it by $q(u)$ instead of by $q(u, 0)$ for brevity's sake. The metric tensors of M_c and $A(u)$ are given by

$$g_{ab}(u) = B_a^i B_b^j g_{ij} = B_{ai} B_{bj} g^{ij}, \quad g_{\kappa\lambda}(u) = B_\kappa^i B_\lambda^j g_{ij} = B_{\kappa i} B_{\lambda j} g^{ij},$$

and then indices can be lowered or uppered by using these metric tensors or their inverses $g^{ab}(u), g^{\kappa\lambda}(u)$. The ± 1 -connections of M_c are given by

$$\Gamma_{abc}^{(1)}(u) = (\partial_a B_b^j) B_{cj}, \quad \Gamma_{abc}^{(-1)}(u) = (\partial_a B_{bj}) B_c^j.$$

We call $A = \{A(u) \mid u \in U\}$ an orthogonal ancillary family when $g_{a\kappa}(u) = 0, \forall u \in U$, and we assume this property in the following. The mixed parts $\Gamma_{ab\kappa}^{(\pm 1)}(u)$ play central roles in the evaluation of statistical inferences, which are defined as follows.

Definition 5.1.

$$H_{ab\kappa}^{(1)}(u) = \Gamma_{ab\kappa}^{(1)}(u) = (\partial_a B_b^j) B_{\kappa j}, \quad H_{ab\kappa}^{(-1)}(u) = \Gamma_{ab\kappa}^{(-1)}(u) = (\partial_a B_{bj}) B_{\kappa}^j,$$

and we call $H_{ab\kappa}^{(\pm 1)}$ the ± 1 -Euler-Schouten curvature tensors of M_c .

The ± 1 -RC curvature tensors and the \pm -ES curvature tensors of M_c are connected by the equations of Gauss (cf. Schouten (1954), p.266)

$$\begin{aligned} R_{abcd}^{(\pm 1)}(u) &= R_{ijkl}^{(\pm 1)} B_a^i B_b^j B_c^k B_d^l + (H_{ad\kappa}^{(\mp 1)} H_{bc\lambda}^{(\pm 1)} - H_{bd\kappa}^{(\mp 1)} H_{ac\lambda}^{(\pm 1)}) g^{\kappa\lambda} \\ &= (H_{ad\kappa}^{(\mp 1)} H_{bc\lambda}^{(\pm 1)} - H_{bd\kappa}^{(\mp 1)} H_{ac\lambda}^{(\pm 1)}) g^{\kappa\lambda}. \end{aligned} \quad (25)$$

Suppose that the ± 1 -ES curvature tensors of M_c are related as

$$H_{ab\kappa}^{(-1)}(u) = \epsilon H_{ab\kappa}^{(1)}(u), \quad (26)$$

where $\epsilon (\neq 0)$ is a constant. In this case we have

$$R_{abcd}^{(-1)}(u) = R_{abcd}^{(1)}(u) = \epsilon (H_{ad\kappa}^{(1)} H_{bc\lambda}^{(1)} - H_{bd\kappa}^{(1)} H_{ac\lambda}^{(1)}) g^{\kappa\lambda},$$

so that M_c is conjugate $m(e)$ -symmetric. Thus we say that M_c satisfying (26) is *ES conjugate $m(e)$ -symmetric*. Suppose further that the 1-ES curvature tensor of M_c is written as

$$H_{ab\kappa}^{(1)}(u) = H_{\kappa}^{(1)} g_{ab}(u), \quad H_{\kappa}^{(1)}(u) = \frac{1}{m} H_{ab\kappa}^{(1)} g^{ab}, \quad \forall u \in M_c, \quad (27)$$

where $H_{\kappa}^{(1)}$ is called the mean 1-ES curvature of M_c , and M_c satisfying (27) is said to be *totally exponential umbilic (e-umbilic)*.

The implications of these notions are summarized in the following manner.

Theorem 5.1. *For an (n, m) -curved exponential family M_c , the following relation holds.*

Let $m \geq 3$ or $n = m + 1$, and suppose that M_c is ES conjugate $m(e)$ -symmetric and totally e-umbilic. Then M_c is a space of constant $m(e)$ -curvature.

Proof. Suppose that M_c is ES conjugate $m(e)$ -symmetric and totally e -umbilic. Then from (26) and (27) we have

$$R_{abcd}^{(-1)}(u) = R_{abcd}^{(1)}(u) = \epsilon H^{(1)2}(g_{ad}g_{bc} - g_{ac}g_{bd}), \quad H^{(1)2}(u) = H_\kappa^{(1)}H_\lambda^{(1)}g^{\kappa\lambda}.$$

When $m \geq 3$, as noted in the proof of Theorem 3.2, $\epsilon H^{(1)2}$ is constant on M_c . When $n = m + 1$, from the equation of Codazzi (cf. Schouten (1954), p.266)

$$\begin{aligned} 0 &= R_{ijk}^{(1)l} B_a^i B_b^j B_c^k B_l^\kappa = \nabla_a^{(1)} H_{bc}^{(1)\kappa} - \nabla_b^{(1)} H_{ac}^{(1)\kappa}, \quad B_l^\kappa = g_{li} g^{\kappa\lambda} B_\lambda^i \\ &\Rightarrow \epsilon(\nabla_a^{(1)} H^{(1)\kappa}) g_{bc} - \epsilon(\nabla_b^{(1)} H^{(1)\kappa}) g_{ac} = 0 \quad (\text{since } \nabla_a^{(1)} g_{bc} = T_{abc}) \\ &\Rightarrow \epsilon(m-1) \nabla_a^{(1)} H^{(1)\kappa} = \epsilon(m-1) \partial_a H^{(1)\kappa} = 0 \quad (\text{since } H^{(1)\kappa} \text{ is a scalar on } M_c, \kappa = m+1), \end{aligned}$$

and without loss of generality we can set $g_{\kappa\kappa}(u) = 1$, so that $\epsilon H^{(1)2} = \epsilon H^{(1)\kappa} H^{(1)\kappa} g_{\kappa\kappa}$ is again constant on M_c . \square

We further deal with the case of $n = m + 1$. Suppose that M_c satisfies the equations

$$B_\kappa^i(u) = k_0(\theta^i(u) - \theta_0^i), \quad B_{\kappa i}(u) = l_0(\eta_i(u) - \eta_i^0), \quad g_{\kappa\kappa}(u) = 1, \quad \kappa = m+1, \quad (28)$$

where k_0, l_0 are non-zero constants and θ_0^i, η_i^0 are constant vectors. In this case M_c is expressed as

$$(\theta^i(u) - \theta_0^i)(\eta_i(u) - \eta_i^0) = \frac{1}{k_0 l_0},$$

and we call M_c satisfying (28) a *dual quadric hypersurface*. In Section 7 it will be shown that the von Mises-Fisher model and the hyperboloid model are the examples of the dual quadric hypersurface. The meaning of this hypersurface is described in the following theorem.

Theorem 5.2. *For an $(m+1, m)$ -curved exponential family M_c , the following two conditions are equivalent.*

- (i) M_c is a dual quadric hypersurface.
- (ii) M_c is ES conjugate $m(e)$ -symmetric and totally e -umbilic with constant $m(e)$ -curvature $k_0 l_0$, and $T_{\alpha\kappa\kappa}(u) = 0$ on M_c .

Proof. We first prove (i) \Rightarrow (ii). By the definition (28) we have

$$\begin{aligned} \partial_a B_\kappa^i(u) &= \Gamma_{a\kappa}^{(1)b} B_{bi} + \Gamma_{a\kappa}^{(1)\kappa} B_{\kappa i} = k_0 B_a^i(u), \quad \partial_a B_{\kappa i}(u) = \Gamma_{a\kappa}^{(-1)b} B_{bi} + \Gamma_{a\kappa}^{(-1)\kappa} B_{\kappa i} = l_0 B_{ai}(u) \\ &\Rightarrow \Gamma_{a\kappa b}^{(1)}(u) = k_0 g_{ab}(u), \quad \Gamma_{a\kappa\kappa}^{(1)}(u) = 0, \quad \Gamma_{a\kappa b}^{(-1)}(u) = l_0 g_{ab}(u), \quad \Gamma_{a\kappa\kappa}^{(-1)}(u) = 0. \end{aligned}$$

On the other hand

$$0 = \partial_b g_{a\kappa}(u) = H_{ab\kappa}^{(-1)} + \Gamma_{b\kappa a}^{(1)} = H_{ab\kappa}^{(1)} + \Gamma_{b\kappa a}^{(-1)},$$

and hence

$$\begin{aligned} H_{ab\kappa}^{(-1)}(u) &= -k_0 g_{ab}(u) = \frac{k_0}{l_0} H_{ab\kappa}^{(1)}(u), & H_{ab\kappa}^{(1)}(u) &= -l_0 g_{ab}(u), & T_{a\kappa\kappa}(u) &= \Gamma_{a\kappa\kappa}^{(-1)} - \Gamma_{a\kappa\kappa}^{(1)} = 0 \\ \Rightarrow R_{abcd}^{(\pm 1)}(u) &= k_0 l_0 (g_{ad} g_{bc} - g_{ac} g_{bd}). \end{aligned}$$

We next prove (ii) \Rightarrow (i). By the definitions (26) and (27) we have

$$\begin{aligned} R_{abcd}^{(\pm 1)}(u) &= \epsilon H^{(1)2} (g_{ad} g_{bc} - g_{ac} g_{bd}) \\ \Rightarrow \epsilon H^{(1)2} &= \epsilon H_{\kappa}^{(1)} H_{\kappa}^{(1)} g^{\kappa\kappa} = k_0 l_0, & g^{\kappa\kappa}(u) &= 1 \\ \Rightarrow H_{ab\kappa}^{(1)}(u) &= H_{\kappa}^{(1)} g_{ab}(u) = -\sqrt{|k_0 l_0 / \epsilon|} g_{ab}(u), & H_{ab\kappa}^{(-1)}(u) &= -\sqrt{|\epsilon k_0 l_0|} g_{ab}(u). \end{aligned}$$

On the other hand

$$\begin{aligned} \Gamma_{b\kappa a}^{(1)}(u) &= -H_{ab\kappa}^{(-1)}(u), & \Gamma_{b\kappa a}^{(-1)}(u) &= -H_{ab\kappa}^{(1)}(u), \\ 0 = \partial_a g_{\kappa\kappa}(u) &= \Gamma_{a\kappa\kappa}^{(1)}(u) + \Gamma_{a\kappa\kappa}^{(-1)}(u), & T_{a\kappa\kappa}(u) &= \Gamma_{a\kappa\kappa}^{(-1)} - \Gamma_{a\kappa\kappa}^{(1)} = 0, \end{aligned}$$

and hence

$$\begin{aligned} \Gamma_{a\kappa\kappa}^{(1)}(u) &= 0, & \Gamma_{a\kappa\kappa}^{(-1)}(u) &= 0 \\ \Rightarrow \partial_a B_{\kappa}^i(u) &= \Gamma_{a\kappa}^{(1)b} B_b^i(u) + \Gamma_{a\kappa}^{(1)\kappa} B_{\kappa}^i(u) = \sqrt{|\epsilon k_0 l_0|} B_a^i(u) \\ \partial_a B_{\kappa i}(u) &= \Gamma_{a\kappa}^{(-1)b} B_{bi}(u) + \Gamma_{a\kappa}^{(-1)\kappa} B_{\kappa i}(u) = \sqrt{|k_0 l_0 / \epsilon|} B_{ai}(u) \\ \Rightarrow \partial_a (B_{\kappa}^i(u) - \sqrt{|\epsilon k_0 l_0|} \theta^i(u)) &= 0, & \partial_a (B_{\kappa i}(u) - \sqrt{|k_0 l_0 / \epsilon|} \eta_i(u)) &= 0 \\ \Rightarrow B_{\kappa}^i(u) &= \sqrt{|\epsilon k_0 l_0|} (\theta^i(u) - \theta_0^i), & B_{\kappa i}(u) &= \sqrt{|k_0 l_0 / \epsilon|} (\eta_i(u) - \eta_i^0). \end{aligned}$$

This completes the proof of the theorem. \square

Figure 2 illustrates the relations among several notions in Theorems 5.1 and 5.2.

From Theorems 3.2 and 5.2, the dual quadric hypersurface is conformally $m(e)$ -flat, and we obtain the following result as to its dual structure.

Theorem 5.3. *When M_c is an m -dimensional dual quadric hypersurface, it is conformally $m(e)$ -flat by the log gauge function $s(u) = \log \nu(u)$ and the (-1) -affine coordinate system $\bar{u}^{\bar{a}}$, $\bar{a} = 1, \dots, m$, satisfying*

$$\partial_a s_b - \Gamma_{ab}^{(-1)c} s_c - s_a s_b = k_0 l_0 g_{ab}, \quad \bar{u}^{\bar{a}} = \nu(u) D^{\bar{a}i} (\eta_i(u) - \eta_i^0), \quad (29)$$

where $\eta_i^0, D^{\bar{a}i}$ ($\bar{a} = 1, \dots, m$, $i = 1, \dots, m+1$) are constants and $\text{rank } D^{\bar{a}i} = m$.

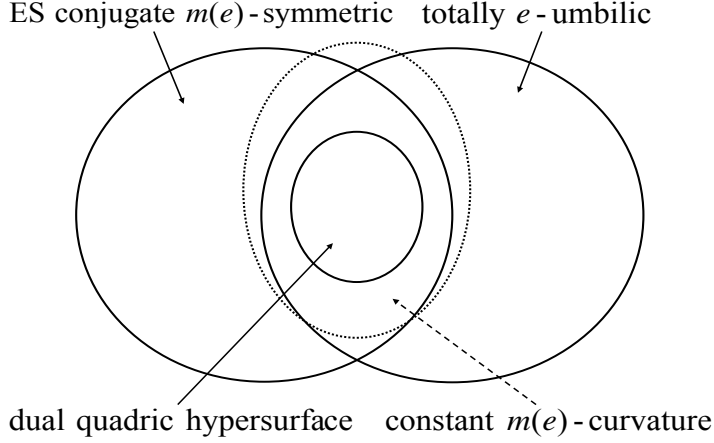


Figure 2: Relations among several notions on M_c

The 1-affine coordinate system $\bar{v}_{\bar{a}}$, two potential functions $\bar{\psi}(\bar{v})$ and $\bar{\phi}(\bar{u})$ of \bar{M}_c are respectively given as

$$\bar{v}_{\bar{a}} = \partial_{\bar{a}} \bar{\phi}(\bar{u}), \quad \bar{u}^{\bar{a}} = \partial^{\bar{a}} \bar{\psi}(\bar{v}), \quad \bar{\phi}(\bar{u}) = \frac{\nu(u)}{k_0 l_0}, \quad \bar{\psi}(\bar{v}) + \bar{\phi}(\bar{u}) - \bar{v}_{\bar{a}} \bar{u}^{\bar{a}} = 0, \quad (30)$$

$$g^{\bar{a}\bar{b}}(\bar{v}) = \partial^{\bar{a}} \partial^{\bar{b}} \bar{\psi}(\bar{v}), \quad g_{\bar{a}\bar{b}}(\bar{u}) = \partial_{\bar{a}} \partial_{\bar{b}} \bar{\phi}(\bar{u}). \quad (31)$$

Proof. We first prove (29). As noted in the proof of Theorem 3.1, the partial differential equation for $s(u) = \log \nu(u)$ is

$$\partial_a s_b - \Gamma_{ab}^{(-1)c} s_c - s_a s_b = \frac{1}{m-1} R_{ab}^{(\pm 1)}.$$

When M_c is an m -dimensional dual quadric hypersurface, from

$$R_{abcd}^{(\pm 1)} = k_0 l_0 (g_{ad} g_{bc} - g_{ac} g_{bd}) \Rightarrow R_{bc}^{(\pm 1)} = R_{abcd}^{(\pm 1)} g^{ad} = k_0 l_0 (m-1) g_{bc},$$

we have the first relation of (29).

The partial differential equation for $\bar{u} = \bar{u}(u)$ is given as

$$\begin{aligned} \bar{\Gamma}_{\bar{a}\bar{b}\bar{c}}^{(-1)}(\bar{u}) &= B_{\bar{a}}^a B_{\bar{b}}^b B_{\bar{c}}^c \bar{\Gamma}_{abc}^{(-1)}(u) + \bar{g}_{bc} B_{\bar{c}}^c \partial_{\bar{a}} B_{\bar{b}}^b = 0 \\ \Leftrightarrow B_{\bar{c}}^c [B_{\bar{a}}^a B_{\bar{b}}^b \bar{\Gamma}_{abc}^{(-1)} + g_{bc} \partial_{\bar{a}} B_{\bar{b}}^b] &= 0, \quad \bar{\Gamma}_{abc}^{(-1)} = \nu [\Gamma_{abc}^{(-1)} + g_{ac} s_b + g_{bc} s_a] \\ \Leftrightarrow \partial_b C_a^{\bar{a}} - \bar{\Gamma}_{ba}^{(-1)c} C_c^{\bar{a}} &= 0, \quad C_a^{\bar{a}} = \frac{\partial \bar{u}^{\bar{a}}}{\partial u^a}, \quad \bar{\Gamma}_{ba}^{(-1)c} = \Gamma_{ba}^{(-1)c} + \delta_a^c s_b + \delta_b^c s_a. \end{aligned}$$

When M_c is a dual quadric hypersurface, we can directly show that the above is satisfied by the second relation of (29).

We next prove (30), (31). By the direct calculation we can confirm

$$g_{\bar{a}\bar{b}}(\bar{u}) = \partial_{\bar{a}}\partial_{\bar{b}}\bar{\phi}(\bar{u}), \quad \bar{\phi}(\bar{u}) = \frac{\nu(u)}{k_0 l_0},$$

and then the others are immediately obtained.

This completes the proof of the theorem. \square

We next consider a conformal transformation $M_e \mapsto \bar{M}_e$ by the gauge function $\nu(w) > 0$. As shown by (8), the α -connection in terms of the w -coordinate system is changed into

$$\bar{\Gamma}_{\beta\gamma\delta}^{(\alpha)} = \nu[\Gamma_{\beta\gamma\delta}^{(\alpha)} + \frac{1-\alpha}{2}(g_{\delta\beta}s_\gamma + g_{\delta\gamma}s_\beta) - \frac{1+\alpha}{2}g_{\beta\gamma}s_\delta].$$

Then we can express the change of quantities related to \bar{M}_c , that is, the (-1) -connection of M_c , the 1-ES curvature of M_c and the (-1) -ES curvature of $A(u)$ are respectively changed into

$$\bar{\Gamma}_{abc}^{(-1)} = \nu[\Gamma_{abc}^{(-1)} + g_{ca}s_b + g_{cb}s_a], \quad \bar{H}_{ab\kappa}^{(1)} = \nu[H_{ab\kappa}^{(1)} - g_{ab}s_\kappa], \quad \bar{H}_{\kappa\lambda a}^{(-1)} = \nu H_{\kappa\lambda a}^{(-1)}. \quad (32)$$

It is also seen that

$$\bar{K}_{ab\kappa}^{(1)} = \nu K_{ab\kappa}^{(1)}, \quad K_{ab\kappa}^{(1)}(u) = H_{ab\kappa}^{(1)} - g_{ab}H_\kappa^{(1)}, \quad (33)$$

and we call $K_{ab\kappa}^{(1)}$ the conformal 1-ES curvature tensor.

Note that the change of the (-1) -connection

$$\bar{\Gamma}_{\alpha\beta}^{(-1)\gamma} = \Gamma_{\alpha\beta}^{(-1)\gamma} + \delta_\alpha^\gamma s_\beta + \delta_\beta^\gamma s_\alpha$$

induces the projective transformation at the same time, which implies that the mixture geodesic is preserved under the transformation (cf. Schouten (1954), p.287). The effect of constant $m(e)$ -curvature is given in the following theorem.

Theorem 5.4. *Suppose that a curved exponential family M_c is a space of constant $m(e)$ -curvature. Then there exists a conformal transformation $M_e \mapsto \bar{M}_e$ and a coordinate system $\bar{u} = (\bar{u}^{\bar{a}})$, $\bar{a} = 1, \dots, m$, of \bar{M}_c such that the followings hold.*

- (i) $\bar{\Gamma}_{\bar{a}\bar{b}\bar{c}}^{(-1)}(\bar{u}) = 0, \quad \forall \bar{u} \in \bar{M}_c.$
- (ii) *If M_c is totally e -umbilic, then $\bar{H}_{\bar{a}\bar{b}\kappa}^{(1)}(\bar{u}) = 0, \quad \forall \bar{u} \in \bar{M}_c.$*

Proof. We first prove (i). When M_c is a space of constant $m(e)$ -curvature, from Theorem 3.2, M_c is conformally $m(e)$ -flat, so that we have

$$\bar{R}_{\bar{a}\bar{b}\bar{c}\bar{d}}^{(-1)}(u) = 0 \Leftrightarrow \exists \nu(u) > 0, \quad \bar{u} = (\bar{u}^{\bar{a}}) \text{ such that } \bar{\Gamma}_{\bar{a}\bar{b}\bar{c}}^{(-1)}(\bar{u}) = 0, \quad \forall \bar{u} \in \bar{M}_c.$$

We next prove (ii). Let us take $s_\kappa(u) = H_\kappa^{(1)}(u)$ (see Okamoto, Amari and Takeuchi (1991)). Then for the totally e -umbilic M_c , from (27), (32) and (33) we have

$$\bar{H}_{\bar{a}\bar{b}\kappa}^{(1)}(u) = \bar{K}_{\bar{a}\bar{b}\kappa}^{(1)}(u) = \nu K_{\bar{a}\bar{b}\kappa}^{(1)}(u) = 0, \quad \forall u \in \bar{M}_c \Rightarrow \bar{H}_{\bar{a}\bar{b}\kappa}^{(1)}(\bar{u}) = 0, \quad \forall \bar{u} \in \bar{M}_c. \quad \square$$

6 Sequential estimation in curved exponential family

We consider sequential estimations in an (n, m) -curved exponential family M_c . Let $K > 0$ be a parameter controlling the average sample size, and let $\nu(\eta) > 0$ ($\nu(w) > 0$) be a smooth gauge function defined on M_e in the η -(w -)coordinate system.

We denote by $\bar{X}_t = X(t)/t$ the sample mean up to time t . It has the same value in the η -coordinate system, and its value in the w -coordinate system is denoted by $\hat{w}_t = (\hat{u}_t, \hat{v}_t) = \eta^{-1}(\bar{X}_t)$. The random stopping time τ is assumed to satisfy (see Okamoto, Amari and Takeuchi (1991))

$$\begin{aligned} \tau &= K\nu(\hat{w}_\tau) + c(\hat{u}_\tau) + \varepsilon, & c(u) &= -\frac{1}{2}(\partial_\alpha s_\beta - \Gamma_{\alpha\beta}^{(-1)\gamma} s_\gamma - s_\alpha s_\beta)g^{\alpha\beta}, \\ \varepsilon &= O_p(1), \quad E_u[\varepsilon] = o(1), & E_u[\tau] &= K\nu(u), \quad V_u[\tau] = O(K). \end{aligned}$$

The term c is due to the bias of \hat{w}_τ from the true $w = (u, 0)$, which is obtained by the requirement $E_u[\tau] = K\nu(u)$. The term ε includes a rounding error and the ‘‘overshooting’’ at the stopping time τ .

We cite the established results concerning the asymptotics of sequential estimators of u from Okamoto, Amari and Takeuchi (1991).

Proposition 6.1. *For a consistent sequential estimator \hat{u} of u , the following relations hold.*

(i) *The estimator \hat{u} is first-order efficient, that is, $\sqrt{K\nu}(\hat{u} - u) \rightarrow N(0, g^{ab}(u))$ as $K \rightarrow \infty$, if and only if $A = \{A(u)\}$ is an orthogonal ancillary family.*

(ii) *The bias-corrected estimator \hat{u}^* of \hat{u} is given by*

$$\hat{u}^{*a} = \hat{u}^a + \frac{1}{2K\nu} \Gamma_{\alpha\beta}^{(-1)a} g^{\alpha\beta}(\hat{u}), \quad \Gamma_{\alpha\beta}^{(-1)a} = \Gamma_{\alpha\beta}^{(-1)a} + \delta_\alpha^a s_\beta + \delta_\beta^a s_\alpha. \quad (34)$$

(iii) *The asymptotic covariance of \hat{u}^* is given by*

$$E[K\nu(\hat{u}^{*a} - u^a)(\hat{u}^{*b} - u^b)] = g^{ab} + \frac{1}{K\nu} \left\{ \frac{1}{2}(\Gamma_{M_c}^{(-1)})^{2ab} + (H_{M_c}^{(1)})^{2ab} + \frac{1}{2}(H_A^{(-1)})^{2ab} \right\} + O(K^{-2}), \quad (35)$$

where

$$\begin{aligned} (\Gamma_{M_c}^{(-1)})^{2ab} &= (\Gamma_{M_c}^{(-1)})_{cd}^2 g^{ac} g^{bd}, & (\Gamma_{M_c}^{(-1)})_{ab} &= \Gamma_{cda}^{(-1)} \Gamma_{efb}^{(-1)} g^{ce} g^{df}, & \Gamma_{abc}^{(-1)} &= \Gamma_{abc}^{(-1)} + g_{ca} s_b + g_{cb} s_a, \\ (H_{M_c}^{(1)})^{2ab} &= (H_{M_c}^{(1)})_{cd}^2 g^{ac} g^{bd}, & (H_{M_c}^{(1)})_{ab} &= H_{ac\kappa}^{(1)} H_{bd\lambda}^{(1)} g^{c\kappa} g^{d\lambda}, & H_{ab\kappa}^{(1)} &= H_{ab\kappa}^{(1)} - g_{ab} s_\kappa, \\ (H_A^{(-1)})^{2ab} &= (H_A^{(-1)})_{cd}^2 g^{ac} g^{bd}, & (H_A^{(-1)})_{ab} &= H_{\kappa\lambda a}^{(-1)} H_{\mu\nu b}^{(-1)} g^{\kappa\mu} g^{\lambda\nu}. \end{aligned}$$

Based on Theorem 5.4 we obtain the following result for the possibility of covariance minimization.

Theorem 6.1. *Suppose that a curved exponential family M_c is a space of constant $m(e)$ -curvature and is totally e -umblic. Then there exists a conformal transformation $M_e \mapsto \bar{M}_e$ and a coordinate system $\bar{u} = (\bar{u}^{\bar{a}})$, $\bar{a} = 1, \dots, m$, of \bar{M}_e such that the following holds for the maximum likelihood estimator $\hat{u}_{mle}^{\bar{a}}$ of $\bar{u}^{\bar{a}}$ without bias-correction:*

$$E[K\nu(\hat{u}_{mle}^{\bar{a}} - \bar{u}^{\bar{a}})(\hat{u}_{mle}^{\bar{b}} - \bar{u}^{\bar{b}})] = g^{\bar{a}\bar{b}} + O(K^{-2}). \quad (36)$$

When M_c itself is a f.r.m. exponential family, (36) holds by (21) given in Theorem 4.1. When M_c is a dual quadric hypersurface, (36) holds by (29) given in Theorem 5.3.

Proof. Since $H_{\kappa\lambda a}^{(-1)} = 0$ holds for the maximum likelihood estimator (m.l.e.), from (34) and Theorem 5.4, we have for the bias of the m.l.e.

$$b_{mle}^{\bar{a}} = -\Gamma_{\alpha\beta}^{(-1)\bar{a}} g^{\alpha\beta} = -\Gamma_{\bar{b}\bar{c}}^{(-1)\bar{a}} g^{\bar{b}\bar{c}} - \Gamma_{\kappa\lambda}^{(-1)\bar{a}} g^{\kappa\lambda} = -\Gamma_{\bar{b}\bar{c}}^{(-1)\bar{a}} g^{\bar{b}\bar{c}} - H_{\kappa\lambda}^{(-1)\bar{a}} g^{\kappa\lambda} = 0,$$

and the expression (36) is derived.

When M_c itself is a f.r.m. exponential family with expectation parameter u , the partial differential equations for $s(u) = \log \nu(u)$ and for $\bar{u} = \bar{u}(u)$ are

$$\partial_a s_b - s_a s_b = 0, \quad \partial_b C_a^{\bar{a}} - s_b C_a^{\bar{a}} - s_a C_b^{\bar{a}} = 0, \quad C_a^{\bar{a}} = \frac{\partial \bar{u}^{\bar{a}}}{\partial u^a},$$

as noted in Theorem 4.1. When M_c is a dual quadric hypersurface, the partial differential equations for $s(u) = \log \nu(u)$ and for $\bar{u} = \bar{u}(u)$ are

$$\partial_a s_b - \Gamma_{ab}^{(-1)c} s_c - s_a s_b = k_0 l_0 g_{ab}, \quad \partial_b C_a^{\bar{a}} - (\Gamma_{ba}^{(-1)c} + \delta_a^c s_b + \delta_b^c s_a) C_c^{\bar{a}} = 0,$$

as noted in Theorem 5.3.

This completes the proof of the theorem. \square

7 Examples

7.1 von Mises-Fisher model

This is an $(m+1, m)$ -curved exponential family, of which density functions with respect to the invariant measure on the m -dimensional unit sphere under rotational transformations are given by (cf. Barndorff-Nielsen et al (1989), p.76)

$$\begin{aligned} f_c(x, 1, u) &= \exp\{\theta(u) \cdot x - \psi(\theta(u))\}, \quad \theta \cdot x = \theta^1 x_1 + \theta^2 x_2 + \dots + \theta^{m+1} x_{m+1}, \\ \theta &= r\xi = (r\xi^i), \quad \xi \in S^m = \{\xi \in \mathbb{R}^{m+1} \mid \xi \cdot \xi = 1\}, \quad x = (x_i) \in S^m, \\ \psi(\theta) &= -\log a_m(r), \quad 1/a_m(r) = (2\pi)^{(m+1)/2} r^{(1-m)/2} I_{(m-1)/2}(r), \quad r > 0, \end{aligned}$$

where $I_{(m-1)/2}(r)$ is the modified Bessel function of the first kind and of order $(m-1)/2$. We assume that the concentration parameter r is assumed to be a given positive constant.

The parametric representations $\theta = \theta(u)$ and $\eta = \eta(u)$ are given by

$$\begin{aligned} \theta^1(u) &= r \cos u^1 & \eta_1(u) &= r^\dagger \cos u^1 \\ \theta^2(u) &= r \sin u^1 \cos u^2 & \eta_2(u) &= r^\dagger \sin u^1 \cos u^2 \\ \theta^3(u) &= r \sin u^1 \sin u^2 \cos u^3 & \eta_3(u) &= r^\dagger \sin u^1 \sin u^2 \cos u^3 \\ &\dots & &\dots \\ \theta^{m+1}(u) &= r \sin u^1 \sin u^2 \dots \sin u^{m-1} \sin u^m & \eta_{m+1}(u) &= r^\dagger \sin u^1 \sin u^2 \dots \sin u^{m-1} \sin u^m, \end{aligned}$$

where $0 \leq u^1, \dots, u^{m-1} \leq \pi$, $0 \leq u^m < 2\pi$ and $r^\dagger = -d \log a_m(r)/dr = I_{(m+1)/2}(r)/I_{(m-1)/2}(r)$. Note that $E[x] = r^\dagger \xi$, and r^\dagger is a strictly increasing function of r which maps $(0, \infty)$ onto $(0, 1)$.

From these representations the tangent vectors $B_a^i(u)$ and $B_{ai}(u)$ can be calculated, and then the unit normal vectors $B_\kappa^i(u)$ and $B_{\kappa i}(u)$ ($\kappa = m+1$) are derived from the relations $B_\kappa^i(u)B_{ai}(u) = 0$ and $B_{\kappa i}(u)B_a^i(u) = 0$ as follows.

$$\begin{aligned} B_\kappa^1(u) &= \cos u^1 & B_{\kappa 1}(u) &= \cos u^1 \\ B_\kappa^2(u) &= \sin u^1 \cos u^2 & B_{\kappa 2}(u) &= \sin u^1 \cos u^2 \\ B_\kappa^3(u) &= \sin u^1 \sin u^2 \cos u^3 & B_{\kappa 3}(u) &= \sin u^1 \sin u^2 \cos u^3 \\ &\dots & &\dots \\ B_\kappa^{m+1}(u) &= \sin u^1 \sin u^2 \dots \sin u^{m-1} \sin u^m & B_{\kappa m+1}(u) &= \sin u^1 \sin u^2 \dots \sin u^{m-1} \sin u^m. \end{aligned}$$

The above expressions show that

$$B_\kappa^i(u) = \frac{1}{r} \theta^i(u), \quad B_{\kappa i}(u) = \frac{1}{r^\dagger} \eta_i(u),$$

and so this model is a dual quadric hypersurface. From Theorem 5.2 we also see that this model is ES conjugate $m(e)$ -symmetric and totally e -umblic with constant $m(e)$ -curvature $1/(rr^\dagger) > 0$. The related geometrical quantities are given below.

$$\begin{aligned} g_{ab}(u) &= \delta_{ab} r r^\dagger \prod_{c=1}^a \sin^2 u^{c-1}, \quad \sin^2 u^0 = 1, \\ H_{ab\kappa}^{(1)}(u) &= -\frac{1}{r^\dagger} g_{ab}(u), \quad H_{ab\kappa}^{(-1)}(u) = -\frac{1}{r} g_{ab}(u) = \frac{r^\dagger}{r} H_{ab\kappa}^{(1)}(u), \\ R_{abba}^{(\pm 1)}(u) &= \frac{1}{r r^\dagger} g_{aa}(u) g_{bb}(u), \quad a \neq b, \quad R_{ab}^{(\pm 1)}(u) = \frac{m-1}{r r^\dagger} g_{ab}(u). \end{aligned}$$

From Theorem 3.2 (i) this M_c is conformally $m(e)$ -flat, so that there exist a gauge function $\nu(u) > 0$ and a (-1) -affine coordinate system $\bar{u} = (\bar{u}^{\bar{a}})$ such that $\bar{\Gamma}_{\bar{a}\bar{b}\bar{c}}^{(-1)}(\bar{u}) = 0$, $\forall \bar{u} \in \bar{M}_c$. As given by (29), the partial differential equation for $s(u) = \log \nu(u)$ is

$$\partial_a s_b - \Gamma_{ab}^{(-1)c} s_c - s_a s_b = \frac{1}{r r^\dagger} g_{ab},$$

of which one solution is

$$\nu(u) = \frac{1}{\prod_{a=1}^m |\sin u^a|},$$

and then $\bar{u}^{\bar{a}} = \nu(u) D^{\bar{a}i} \eta_i(u)$.

7.2 Hyperboloid model

This is an $(m + 1, m)$ -curved exponential family, of which density functions with respect to the invariant measure on the m -dimensional unit hyperboloid under hyperbolic transformations are given by (cf. Barndorff-Nielsen et al. (1989), p.104)

$$\begin{aligned} f_c(x, 1, u) &= \exp\{\theta(u) \cdot x - \psi(\theta(u))\}, \quad \theta^1 = -r\xi^1, \quad \theta^i = r\xi^i, \quad i = 2, \dots, m + 1, \\ \xi &= (\xi^i) \in H^m = \{\xi \in \mathbb{R}^{m+1} \mid \xi * \xi = 1, \xi^1 > 0\}, \quad x = (x_i) \in H^m, \\ \xi * \xi &= (\xi^1)^2 - (\xi^2)^2 - \dots - (\xi^{m+1})^2, \\ \psi(\theta) &= -\log a_m(r), \quad 1/a_m(r) = 2(2\pi)^{(m-1)/2} r^{(1-m)/2} K_{(m-1)/2}(r), \quad r > 0, \end{aligned}$$

where $K_{(m-1)/2}(r)$ is the modified Bessel function of the third kind and of order $(m-1)/2$. We assume that the concentration parameter r is assumed to be a given positive constant. The parametric representations $\theta = \theta(u)$ and $\eta = \eta(u)$ are given by

$$\begin{aligned} \theta^1(u) &= -r \cosh u^1 & \eta_1(u) &= r^\dagger \cosh u^1 \\ \theta^2(u) &= r \sinh u^1 \cos u^2 & \eta_2(u) &= r^\dagger \sinh u^1 \cos u^2 \\ \theta^3(u) &= r \sinh u^1 \sin u^2 \cos u^3 & \eta_3(u) &= r^\dagger \sinh u^1 \sin u^2 \cos u^3 \\ &\dots & &\dots \\ \theta^{m+1}(u) &= r \sinh u^1 \sin u^2 \dots \sin u^{m-1} \sin u^m & \eta_{m+1}(u) &= r^\dagger \sinh u^1 \sin u^2 \dots \sin u^{m-1} \sin u^m, \end{aligned}$$

where $u^1 \in \mathbb{R}$, $0 \leq u^2, \dots, u^{m-1} \leq \pi$, $0 \leq u^m < 2\pi$ and $r^\dagger = d \log a_m(r) / dr = K_{(m+1)/2}(r) / K_{(m-1)/2}(r)$. Note that $E[x] = r^\dagger \xi$, and r^\dagger is a strictly decreasing function of r which maps $(0, \infty)$ onto $(1, \infty)$.

From these representations the tangent vectors $B_a^i(u)$ and $B_{ai}(u)$ can be calculated, and then the unit normal vectors $B_\kappa^i(u)$ and $B_{\kappa i}(u)$ ($\kappa = m + 1$) are derived from the relations $B_\kappa^i(u) B_{ai}(u) = 0$ and $B_{\kappa i}(u) B_a^i(u) = 0$ as follows.

$$\begin{aligned} B_\kappa^1(u) &= \cosh u^1 & B_{\kappa 1}(u) &= \cosh u^1 \\ B_\kappa^2(u) &= -\sinh u^1 \cos u^2 & B_{\kappa 2}(u) &= \sinh u^1 \cos u^2 \\ B_\kappa^3(u) &= -\sinh u^1 \sin u^2 \cos u^3 & B_{\kappa 3}(u) &= \sinh u^1 \sin u^2 \cos u^3 \\ &\dots & &\dots \\ B_\kappa^{m+1}(u) &= -\sinh u^1 \sin u^2 \dots \sin u^{m-1} \sin u^m & B_{\kappa \ m+1}(u) &= \sinh u^1 \sin u^2 \dots \sin u^{m-1} \sin u^m. \end{aligned}$$

The above expressions show that

$$B_\kappa^i(u) = -\frac{1}{r} \theta^i(u), \quad B_{\kappa i}(u) = \frac{1}{r^\dagger} \eta_i(u),$$

and again this model is a dual quadric hypersurface. From Theorem 5.2 we also see that this model is ES conjugate $m(e)$ -symmetric and totally e -umblic with constant $m(e)$ -

curvature $-1/(rr^\dagger) < 0$. The related geometrical quantities are given below.

$$g_{11}(u) = rr^\dagger, \quad g_{ab}(u) = \delta_{ab} rr^\dagger \sinh^2 u^1 \prod_{c=2}^a \sin^2 u^{c-1}, \quad \sin^2 u^1 = 1, \quad a = 2, \dots, m,$$

$$H_{ab\kappa}^{(1)}(u) = -\frac{1}{r^\dagger} g_{ab}(u), \quad H_{ab\kappa}^{(-1)}(u) = \frac{1}{r} g_{ab}(u) = -\frac{r^\dagger}{r} H_{ab\kappa}^{(1)}(u),$$

$$R_{abba}^{(\pm 1)}(u) = -\frac{1}{rr^\dagger} g_{aa}(u) g_{bb}(u), \quad a \neq b, \quad R_{ab}^{(\pm 1)}(u) = -\frac{m-1}{rr^\dagger} g_{ab}(u).$$

From Theorem 3.2 (i) this M_c is conformally $m(e)$ -flat, so that there exist a gauge function $\nu(u) > 0$ and a (-1) -affine coordinate system $\bar{u} = (\bar{u}^{\bar{a}})$ such that $\bar{\Gamma}_{\bar{a}\bar{b}\bar{c}}^{(-1)}(\bar{u}) = 0$, $\forall \bar{u} \in \bar{M}_c$. As given by (29), the partial differential equation for $s(u) = \log \nu(u)$ is

$$\partial_a s_b - \Gamma_{ab}^{(-1)c} s_c - s_a s_b = -\frac{1}{rr^\dagger} g_{ab},$$

of which one solution is

$$\nu(u) = \frac{1}{|\sinh u^1| \prod_{a=2}^m |\sin u^a|},$$

and then $\bar{u}^{\bar{a}} = \nu(u) D^{\bar{a}i} \eta_i(u)$.

7.3 Numerical results

We examine our theoretical results numerically by using the von Mises-Fisher and the hyperboloid models. We take 10 kinds of number N (nonsequential case) and K (sequential case) of observations, and for each N or K , we generate 500 random simulated data. Then the empirical means of covariances $E_{emp}[(\hat{u}_{mle}^{*a} - u_0^a)(\hat{u}_{mle}^{*b} - u_0^b)]$ (nonsequential case) and $E_{emp}[(\hat{u}_{mle}^{\bar{a}} - \bar{u}_0^{\bar{a}})(\hat{u}_{mle}^{\bar{b}} - \bar{u}_0^{\bar{b}})]$ (sequential case) of the m.l.e. over this 500 sample size are used for evaluation, where u_0^a and $\bar{u}_0^{\bar{a}}$ denote the true values of u^a and $\bar{u}^{\bar{a}}$. The stopping times τ for the sequential estimations are determined by (see Okamoto, Amari and Takeuchi (1991))

$$\tau = \inf \left\{ t \mid -\frac{1}{m} \partial_a \partial_b l(x, t, \hat{u}_{mle}) g^{ab}(\hat{u}_{mle}) \geq K \nu(\hat{u}_{mle}) + c \right\},$$

$$c = -\frac{1}{2} \left(\frac{m}{rr^\dagger} - \frac{1}{r^{\dagger 2}} \right) : \text{ von Mises-Fisher} \quad c = -\frac{1}{2} \left(-\frac{m}{rr^\dagger} - \frac{1}{r^{\dagger 2}} \right) : \text{ hyperboloid.}$$

As for the von Mises-Fisher model, numerical results are based on the following set of values

$$m = 2, \quad r = 0.25, \quad (u_0^1, u_0^2) = (\pi/6, \pi/3),$$

$$D^{\bar{a}i} = \delta^{\bar{a}i}, \text{ i.e., } \bar{u}^1 = \nu(u) \eta_1(u), \quad \bar{u}^2 = \nu(u) \eta_2(u), \quad \nu(u) = 1/(|\sin u^1| |\sin u^2|),$$

and for the hyperboloid model, numerical results are based on the following set of values

$$m = 2, \quad r = 0.1, \quad (u_0^1, u_0^2) = (0.1, \pi/3),$$

$$D^{\bar{a}i} = \delta^{\bar{a}i}/100, \text{ i.e., } \bar{u}^1 = \nu(u)\eta_1(u)/100, \quad \bar{u}^2 = \nu(u)\eta_2(u)/100, \quad \nu(u) = 1/(|\sinh u^1| |\sin u^2|).$$

Figures 3-8 show the von Mises-Fisher model, and Figures 9-14 show the hyperboloid model. The notations in the figures indicate the following quantities.

· nonsequential case

$$OCOVab = NE_{emp}[(\hat{u}_{mle}^{*a} - u_0^a)(\hat{u}_{mle}^{*b} - u_0^b)], \quad a, b = 1, 2$$

$$OCRBab = g^{ab}(u_0),$$

$$OALBab = g^{ab}(u_0) + \frac{1}{N} \left\{ \frac{1}{2} (\Gamma_{M_c}^{(-1)}(u_0))^{2ab} + (H_{M_c}^{(1)}(u_0))^{2ab} \right\}$$

· sequential case

$$CCOV\bar{a}\bar{b} = E_{emp}(\tau)E_{emp}[(\hat{u}_{mle}^{\bar{a}} - \bar{u}_0^{\bar{a}})(\hat{u}_{mle}^{\bar{b}} - \bar{u}_0^{\bar{b}})], \quad \bar{a}, \bar{b} = 1, 2$$

$$CCRB\bar{a}\bar{b} = g^{\bar{a}\bar{b}}(\bar{u}_0),$$

$$MST = E_{emp}(\tau) : \text{ empirical mean of } \tau$$

$$SDST = \sqrt{V_{emp}(\tau)} : \text{ empirical standard deviation of } \tau.$$

We see that in the nonsequential case $OCOVab$ approach to the asymptotic lower bound $OALBab$ exhibiting the differential geometrical loss $OALBab - OCRBab$, and in the sequential case $CCOV\bar{a}\bar{b}$ nearly attain the Cramér-Rao lower bound $CCRB\bar{a}\bar{b}$ as if the model were a f.r.m. exponential family. Figures 8, 14 confirm that the assumptions $MST = O(K)$, $SDST = O(\sqrt{K})$ are satisfied in each model.

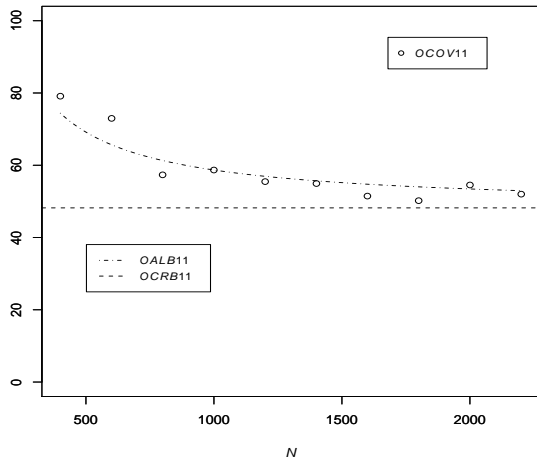


Figure 3: $OCOV11$ von Mises-Fisher

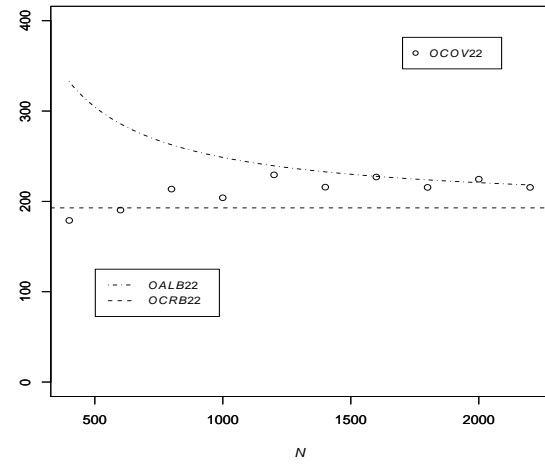


Figure 4: $OCOV22$ von Mises-Fisher

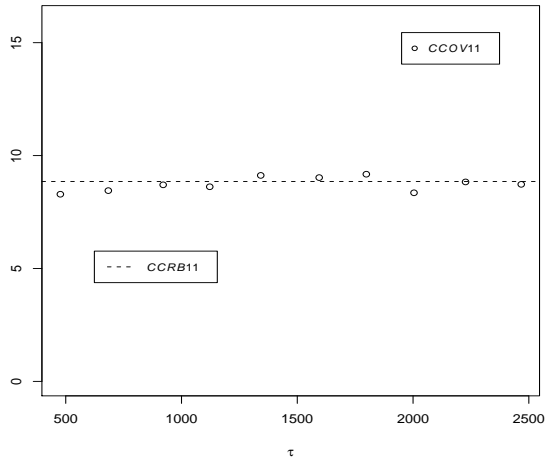


Figure 5: $CCOV11$ von Mises-Fisher

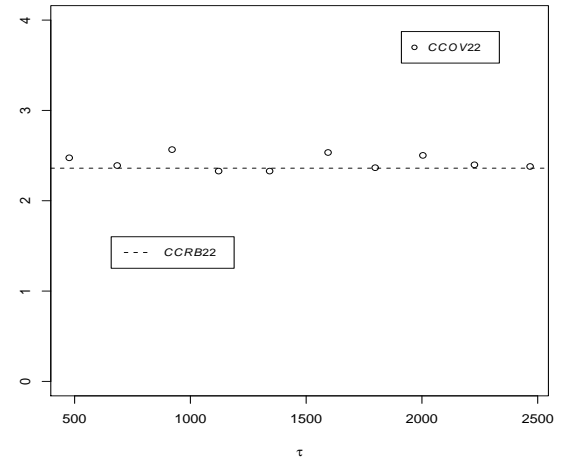


Figure 6: $CCOV22$ von Mises-Fisher

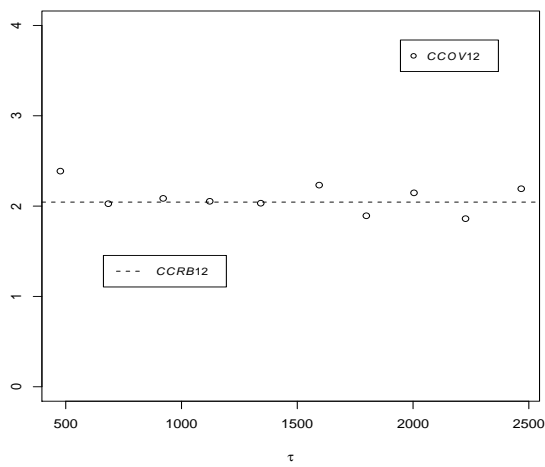


Figure 7: $CCOV12$ von Mises-Fisher

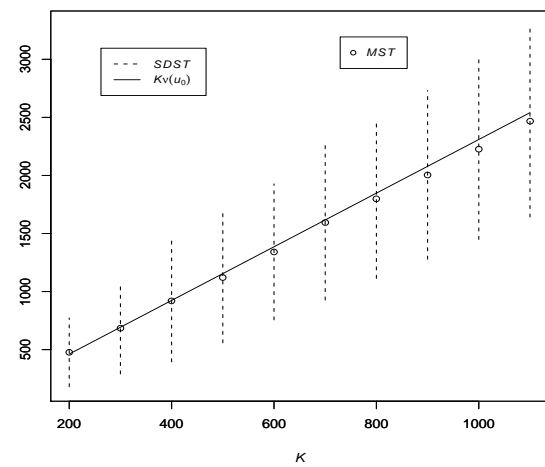


Figure 8: MST $SDST$ von Mises-Fisher

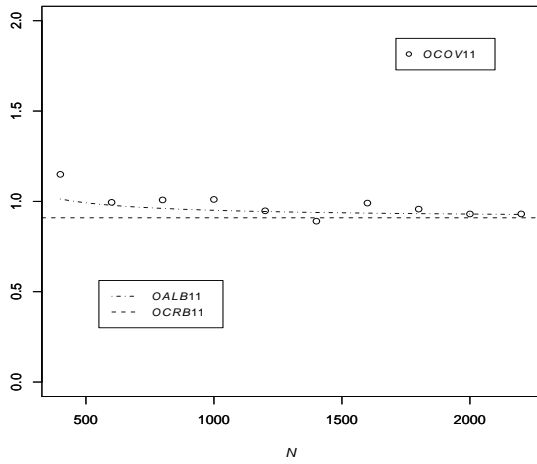


Figure 9: *OCOV11* hyperboloid

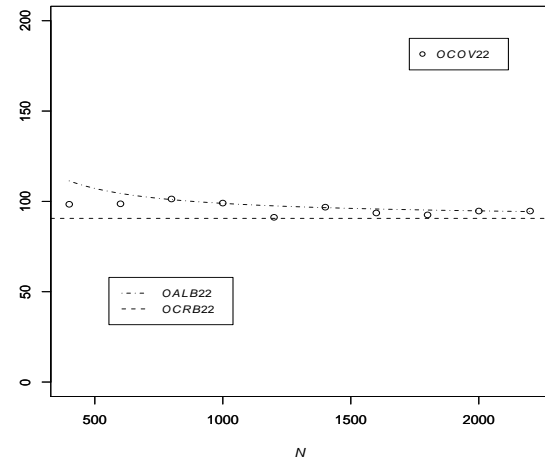


Figure 10: *OCOV22* hyperboloid

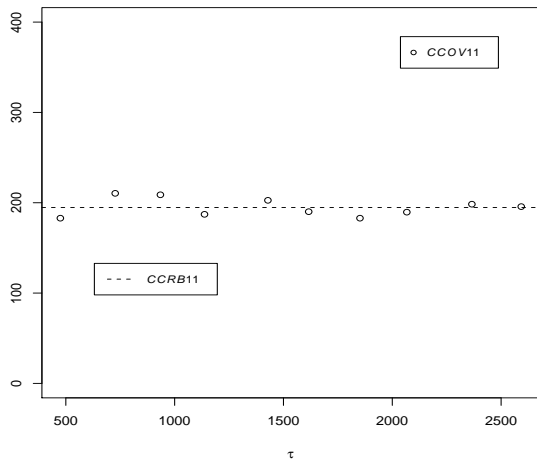


Figure 11: *CCOV11* hyperboloid

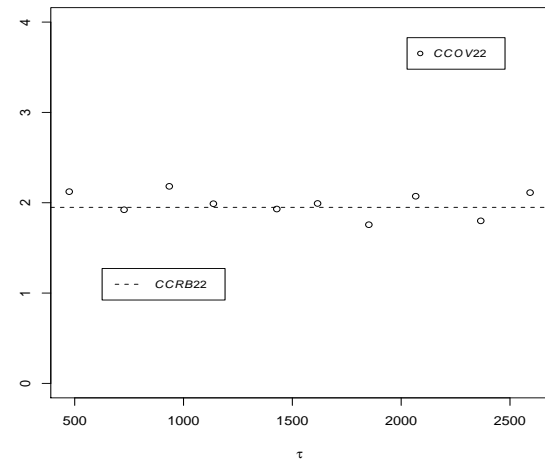


Figure 12: *CCOV22* hyperboloid

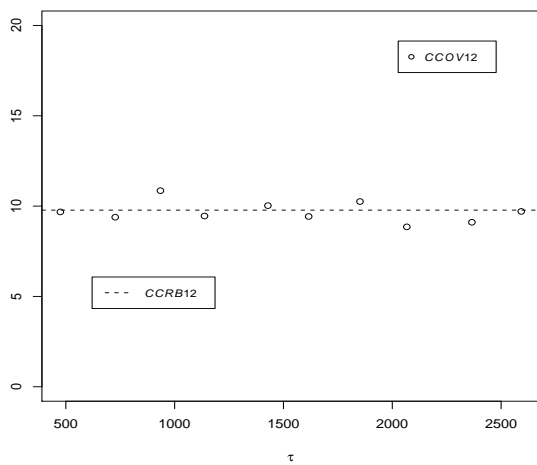


Figure 13: *CCOV12* hyperboloid

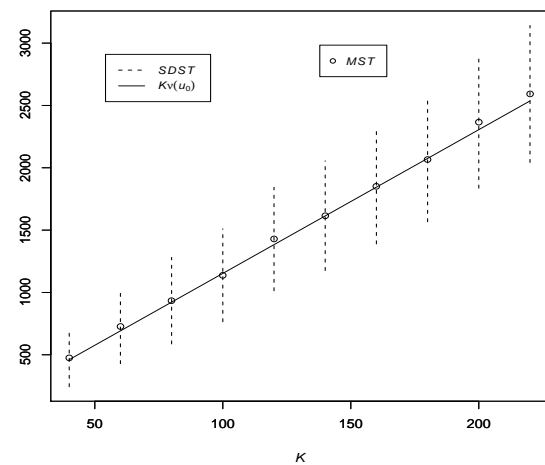


Figure 14: *MST SDST* hyperboloid

8 Discussion

We have analyzed sequential estimation procedures in terms of the conformal geometry of statistical manifolds. We have also constructed a concrete procedure for the covariance minimization in a multidimensional curved exponential family M_c . The method is divided into two separate stages: one is to choose a stopping rule which is effective for reducing the 1-ES curvature $H_{M_c}^{(1)}$ and the other is to choose a gauge function $\nu(u)$ on M_c effective for reducing the (-1) -connection $\Gamma_{M_c}^{(-1)}$. Another typical choice of $\nu(u)$ is the one effective for the covariance stabilization, as suggested in Okamoto, Amari and Takeuchi (1991). These choices contradict each other in general multidimensional cases, and this fact reflects the difference between the ordinary Riemannian geometry and the mutually dual geometry as exhibited in several geometrical notions introduced in this paper.

The present method is also applicable to investigating sequential testing procedures. The geometrical theory of higher-order asymptotics of testing hypothesis in nonsequential case was developed by Kumon and Amari (1983) and Amari (1985). The main results are summarized as follows.

The power function $P_T(t)$ of a test T is expanded as

$$P_T(t) = P_{T1}(t) + P_{T2}(t)/\sqrt{N} + P_{T3}(t)/N + O(N^{-3/2}),$$

where N denotes the number of observations, and t/\sqrt{N} indicates the geodesic distance between the null hypothesis and the point in the alternative hypothesis.

(i) The first-order power function $P_{T1}(t)$ and the second-order power function $P_{T2}(t)$ are maximized uniformly in t if and only if the ancillary family (boundaries of the critical region) associated with a test T is asymptotically an orthogonal family.

(ii) The third-order power loss function $\Delta P_{T3}(t) = \sup_T P_{T3}(t) - P_{T3}(t)$ is expressed as the weighted sum of two kinds of the square of the 1-ES curvatures $H_{M_c}^{(1)}$, the square of the (-1) -ES mixture curvature $H_T^{(-1)}$ of the associated ancillary family, and also the square of the (-1) -mixture connection $\Gamma_{M_c}^{(-1)}$ (when there are unknown nuisance parameters).

Based on these nonsequential results, we can utilize the conformal geometry to the analysis and the construction of most powerful sequential tests. Specifically when a statistical manifold is a f.r.m. exponential family or a dual quadric hypersurface, it is expected that one can design sequential tests without any power loss. This is a subject which will be treated in a future work.

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