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A lower bound for the Graver complexity of the incidence matrix of a complete bipartite graph

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Abstract

We give an exponential lower bound for the Graver complexity of the incidence matrix of a complete bipartite graph of arbitrary size. Our result is a generalization of the result by Berstein and Onn [2] for the complete bipartite graph $K_{3,r}$, $r \geq 3$.

Keywords and phrases: algebraic statistics, contingency table, three-way transportation program.

1 Introduction and the main result

The Graver complexity of an integer matrix is currently actively investigated for its importance to integer programming, algebraic statistics and other applications ([2], [4], [5], [1]). In particular, from the universality of the three-way transportation program to general integer programs (De Loera and Onn [3]), the Graver complexity of the incidence matrix of the complete bipartite graph $K_{3,r}$ is particularly important. Berstein and Onn [2] proved that the Graver complexity $g(r)$ for the incidence matrix of $K_{3,r}$, $r \geq 3$, is bounded below as $g(r) = \Omega(2^r)$, where $g(r) \geq 17 \cdot 2^{r-3} - 7$. It is a natural question to generalize this result to the complete bipartite graph $K_{t,r}$ of arbitrary size t, r. We prove that the Graver complexity for $K_{t,r}$ is $\Omega((t-1)^r)$, where $t \geq 4$ is fixed and r diverges to infinity. For proving our result, we employ double induction on r and t starting from the result of [2].

Let $A_{t,r}$ denote the incidence matrix of the complete bipartite graph $K_{t,r}$ and let $g(A_{t,r})$ denote its Graver complexity. Here we state our main theorem. Relevant notations and definitions will be given in the next section.

Theorem 1.1. *The Graver complexity of $A_{t,r}$ for any $4 \leq t \leq r$ is bounded from below as*

$$g(A_{t,r}) \geq (t-1)^{r-t}(b_t + \frac{1}{t-2}) - \frac{1}{t-2},$$

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where

$$b_t = (t-2)! \left(15 + \sum_{i=1}^{t-4} \frac{i+4}{(i+2)!} \right).$$

We give a proof of this theorem in Section 3 after giving necessary definitions and reviewing relevant known results in Section 2. We conclude the paper with some discussion in Section 4.

2 Preliminaries

In this section we summarize our notation and review relevant known results on the Graver complexity following Berstein and Onn [2].

The integer kernel of an $s \times t$ integer matrix A is denoted by $\ker_{\mathbb{Z}}(A) = \{x \in \mathbb{Z}^t \mid Ax = 0\}$. Define a partial order $\sqsubseteq$ on $\mathbb{Z}^t$, which extends the coordinate-wise order $\le$ on $\mathbb{Z}^t_+$, as follows: For two vectors $u, v \in \mathbb{Z}^t$, $u \sqsubseteq v$ if $|u_i| \le |v_i|$ and $u_i v_i \ge 0$ for $i = 1, \ldots, t$. The *Graver basis* $\mathcal{G}(A)$ of A is the finite set of $\sqsubseteq$-minimal elements in the set $\ker_{\mathbb{Z}}(A) \setminus \{0\}$.

For any fixed positive integer h, write an ht-dimensional integer vector $x \in \mathbb{Z}^{ht}$ as $x = (x^1, \ldots, x^h)$ with each block x^i belonging to $\mathbb{Z}^t$. The *type* of $x = (x^1, \ldots, x^h)$ is the number $\text{type}(x) := \#(\{i \mid x^i \neq 0\})$ of nonzero blocks of x. The *h-th Lawrence lifting* of an $s \times t$ matrix A is the following $(t + hs) \times ht$ matrix, with I_t denoting the $t \times t$ identity matrix:

$$A^{(h)} := \begin{pmatrix} A & 0 & 0 & \ldots & 0 \\ 0 & A & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & A \\ I_t & I_t & I_t & \ldots & I_t \end{pmatrix}. \tag{1}$$

The *Graver complexity* of A is defined as

$$g(A) = \sup \left(\{0\} \cup \left\{ \text{type}(x) \;\middle|\; x \in \bigcup_{h \ge 1} \mathcal{G}\left(A^{(h)}\right) \right\} \right). \tag{2}$$

Let $\mathcal{G}(\mathcal{G}(A))$ denote the Graver basis of a matrix whose columns are the elements of $\mathcal{G}(A)$ ordered arbitrarily. The following result shows that the Graver complexity of A is determined by $\mathcal{G}(\mathcal{G}(A))$.

Proposition 2.1. *[6] The Graver complexity of A satisfies*

$$g(A) = \max\{\|x\|_1 : x \in \mathcal{G}(\mathcal{G}(A))\},$$

where $\|\cdot\|_1$ denotes the 1-norm of a vector.

A *circuit* of an integer matrix A is a nonzero integer vector $x \in \ker_{\mathbb{Z}}(A)$, that has inclusion-minimal support with respect to $\ker_{\mathbb{Z}}(A)$, and whose nonzero entries are relatively prime. Let

$\mathcal{C}(A)$ denote the set of circuits of a matrix A. Then $\mathcal{C}(A) \subseteq \mathcal{G}(A)$ (cf. [7]). An integer relation $h = (h_1, \ldots, h_k)$ on integer vectors $v^1, \ldots, v^k \in \mathbb{Z}^t$

$$0 = h_1 v^1 + \cdots + h_k v^k$$

is *primitive* if $h_1, \ldots, h_k$ are relatively prime positive integers and no $k-1$ of the $\{v^i\}_{i=1}^k$ satisfy any nontrivial linear relation. By $\mathcal{C}(A) \subseteq \mathcal{G}(A)$ and Proposition 2.1 we have the following result.

Proposition 2.2. *[2] Suppose that h is a primitive relation on some set of circuits $\{x^i\}_{i=1}^k$ of an integer matrix A. Then the Graver complexity of A satisfies $g(A) \geq \sum_{i=1}^k h_i$.*

In this paper we consider the Graver complexity of the incidence matrix $A_{t,r}$ for the complete bipartite graph $K_{t,r}$. Let $\mathbf{1}_t = (1, 1, \ldots, 1)$ denote the $1 \times t$ matrix consisting of 1's. Then the r-th Lawrence lifting $A_{t,r} = \mathbf{1}_t^{(r)}$ of $\mathbf{1}_t$ is the incidence matrix of $K_{t,r}$. In algebraic statistics, $A_{t,r}$ is the design matrix specifying the row sums and the column sums of a two-way contingency table. Another Lawrence lifting $(\mathbf{1}_t^{(r)})^{(h)}$ of $\mathbf{1}_t^{(r)}$ is the design matrix for no-three-factor interaction model for $t \times r \times h$ three-way contingency tables ([1], [5]). It is also the coefficient matrix for the three-way transportation program. The Graver complexity $g(A_{t,r}) = g(\mathbf{1}_t^{(r)})$ gives the bound of complexity of the Graver basis for the toric ideal associated with the no-three-factor interaction model for $t \times r \times h$ three-way contingency tables as $h \to \infty$.

We employ below the following notation, where t, r are positive integers. Let

$$V := \{v_1, \ldots, v_t\}, \; U := \{u_1, \ldots, u_r\}. \tag{3}$$

Then $V \oplus U$ and $V \times U$ denote the set of vertices and the set of edges of the complete bipartite graph $K_{t,r}$, respectively. They index the rows and the columns of the incidence matrix $A_{t,r}$ of $K_{t,r}$. Here we explain interpretations of a circuit of $A_{t,r}$ referring to [2]. We interpret each vector $x \in \mathbb{Z}^{V \times U}$ as:

1. an integer valued function on the set of edges $V \times U$;
2. a $t \times r$ matrix with its rows and columns indexed by V and U.

With these interpretations, x is in $\mathcal{C}(A_{t,r})$ if and only if:

1. as a function on $V \times U$ with the following properties: its support is a circuit of $K_{t,r}$, along which its values ± 1 alternate. It can be expressed by the sequence $(v_{i_1}, u_{i_1}, v_{i_2}, u_{i_2}, \ldots, v_{i_l}, u_{i_l})$ of vertices of the circuit of $K_{t,r}$ on which it is supported, with the convention that its value is $+1$ on the first edge (v_{i_1}, u_{i_1}).
2. as a nonzero matrix with the following properties: its elements are $0, \pm 1$, its row sums and columns sums are zeros, and it has an inclusion-minimal support with respect to these properties.

The following example is the base case for our inductive argument for the lower bound of the Graver complexity.

Table 1: Circuits for $A_{3,4}$ in a 3×4 matrix form

	u_1	u_2	u_3	u_4	
	0	0	-1	1	v_1
$x^1 = (v_1, u_4, v_3, u_2, v_2, u_3) =$	0	-1	1	0	v_2
	0	1	0	-1	v_3
	-1	1	0	0	v_1
$x^2 = (v_1, u_2, v_3, u_3, v_2, u_1) =$	1	0	-1	0	v_2
	0	-1	1	0	v_3
	0	-1	0	1	v_1
$x^3 = (v_1, u_4, v_2, u_1, v_3, u_2) =$	1	0	0	-1	v_2
	-1	1	0	0	v_3
	-1	0	0	1	v_1
$x^4 = (v_1, u_4, v_2, u_2, v_3, u_1) =$	0	1	0	-1	v_2
	1	-1	0	0	v_3
	1	0	-1	0	v_1
$x^5 = (v_1, u_1, v_2, u_2, v_3, u_3) =$	-1	1	0	0	v_2
	0	-1	1	0	v_3
	0	-1	1	0	v_1
$x^6 = (v_1, u_3, v_2, u_4, v_3, u_2) =$	0	0	-1	1	v_2
	0	1	0	-1	v_3
	0	1	0	-1	v_1
$x^7 = (v_1, u_2, v_2, u_3, v_3, u_4) =$	0	-1	1	0	v_2
	0	0	-1	1	v_3

Example 2.1. *[2] Let $t = 3$ and $r = 4$. Consider seven circuits $\{x^i\}_{i=1}^7$ in Table 1 (written in a 3×4 matrix form) of $A_{3,4} = (1, 1, 1)^{(4)}$. They satisfy a primitive relation*

$$x^1 + 2x^2 + 3x^3 + 3x^4 + 5x^5 + 6x^6 + 7x^7 = 0.$$

Therefore from Proposition 2.2

$$g(A_{3,4}) \geq 1 + 2 + 3 + 3 + 5 + 6 + 7 = 27.$$

3 Proof of the main theorem

In this section we give a proof of our main theorem. Our proof is based on recursive construction of primitive relations for circuits of $A_{t,r}$. We need recursions for t and r, separately. In Lemma 3.1 we give a recursion for t and in Lemma 3.2 we give a recursion for r.

Lemma 3.1. *Let $t \geq 4$. Suppose that there are circuits $\{x^i\}_{i=1}^k$ of $A_{t,t+1} = \mathbf{1}_t^{(t+1)}$ admitting a primitive relation h, where the k-th circuit and the k-th coefficient are*

$$x^k = (v_1, u_2, v_2, u_3, \ldots, v_t, u_{t+1}),$$

$$h_k = 1.$$

Then there are circuits $\{\bar{x}^i\}_{i=1}^{k+t}$ *of* $A_{t+1,t+1} = \mathbf{1}_{t+1}^{(t+1)}$ *admitting a primitive relation* $\bar{h}$*, where the* $(k+t)$*-th circuit and the* $(k+t)$*-th coefficient are*

$$\bar{x}^{k+t} = (v_1, u_1, v_2, u_2, \ldots, v_{t+1}, u_{t+1}),$$
$$\bar{h}_{k+t} = 1.$$

Proof. Using the natural embedding of $K_{t,t+1}$ into $K_{t+1,t+1}$, we can interpret circuits of the former also as circuits of the latter. Put

$$y^i = x^i, \ \forall i = 1, \ldots, k-1,$$

and define

$$y^{k+j-1} = (v_1, u_j, v_{t+1}, u_{j+1}), \ \forall j = 1, \ldots, t,$$
$$y^{k+t} = (v_1, u_2, v_2, u_3, \ldots, v_t, u_{t+1}, v_{t+1}, u_1).$$

Table 2 displays $\{y^{k+j-1}\}_{j=1}^{t+1}$ as matrices, where

$$p = (1, 0, \ldots, 0, -1)^\top \in \mathbb{Z}^{t+1}.$$

Blank entries are zeros. Note that these circuits satisfy $\sum_{j=1}^{t+1} y^{k+j-1} = x^k$.

Table 2: Circuits for recursion on t

	u_1	u_2	u_3	$\ldots$	u_t	u_{t+1}
$y^k = (v_1, u_1, v_{t+1}, u_2) =$	p	$-p$				
$y^{k+1} = (v_1, u_2, v_{t+1}, u_3) =$		p	$-p$			
$\vdots$			$\ddots$	$\ddots$		
$y^{k+t-1} = (v_1, u_t, v_{t+1}, u_{t+1}) =$				p	$-p$	
$y^{k+t-1} = (v_1, u_t, v_{t+1}, u_{t+1}) =$					p	$-p$
	-1	1				
		-1	1			
$y^{k+t} = (v_1, u_2, v_2, u_3, \ldots, v_t, u_{t+1}, v_{t+1}, u_1) =$			$\ddots$	$\ddots$		
				-1	1	
					-1	1
	1					-1

Suppose that $\bar{h} \in \mathbb{Z}^{k+t}$ satisfies

$$\bar{h}_i = h_i, \ \forall i = 1, \ldots, k-1,$$
$$\bar{h}_{k+j-1} = h_k, \ \forall j = 1, \ldots, t+1.$$

Then

$$\sum_{i=1}^{k+t} \bar{h}_i y^i = \sum_{i=1}^{k-1} h_i y^i + \sum_{j=1}^{t+1} h_k y^{k+j-1} = \sum_{i=1}^{k-1} h_i x^i + h_k x^k = 0.$$

Therefore $\bar{h}$ is an integer relation of circuits $\{y^i\}_{i=1}^{k+t}$.

Next, we show that $\bar{h}$ is primitive. Suppose that $h' \in \mathbb{Z}^{k+t}$ is a nontrivial relation on the $\{y^i\}_{i=1}^{k+t}$. Without loss of generality we may assume that the $\{h'_i\}_{i=1}^{k+t}$ are relatively prime integers, at least one of which is positive. We look at the row of v_{t+1}. Then it follows that

$$h'_k = h'_{k+1} = \cdots = h'_{k+t}.$$

Therefore

$$0 = \sum_{i=1}^{k+t} h'_i y^i = \sum_{i=1}^{k-1} h'_i y^i + h'_k \sum_{j=1}^{t+1} y^{k+j-1} = \sum_{i=1}^{k-1} h'_i x^i + h'_k x^k.$$

This is an integer relation on $\{x^i\}_{i=1}^{k}$, and because h is primitive,

$$h'_i = h_i, \ \forall i = 1, \ldots, k.$$

Therefore $h' = \bar{h}$ and $\bar{h}$ is primitive.

Now apply to $\{y^i\}_{i=1}^{k+t}$ a permutation of columns so that y^{k+t} becomes $(v_1, u_1, v_2, u_2, \ldots, v_{t+1}, u_{t+1})$. For $i = 1, \ldots, k+t$, let $\bar{x}^i$ be the circuit of $A_{t+1,t+1}$ which is the image of y^i under this permutation. Then $\{\bar{x}^i\}_{i=1}^{k+t}$ also satisfy the primitive relation $\sum_{i=1}^{k+t} \bar{h}_i \bar{x}^i = 0$ with the same coefficients $\bar{h}$. This completes the proof. □

Lemma 3.2. *Let $r \geq t \geq 4$. Suppose that there are circuits $\{x^i\}_{i=1}^{k}$ of $A_{t,r} = \mathbf{1}_t^{(r)}$ admitting a primitive relation h, where the k-th circuit and the k-th coefficient are*

$$x^k = (v_1, u_{r-t+1}, v_2, u_{r-t+2}, \ldots, v_t, u_r),$$
$$h_k = 1.$$

Then there are circuits $\{\bar{x}^i\}_{i=1}^{k+t-1}$ of $A_{t,r+1} = \mathbf{1}_t^{(r+1)}$ admitting primitive relation $\bar{h}$, where the $(k+t-1)$-th circuit is

$$\bar{x}^{k+t-1} = (v_1, u_{r-t+2}, v_2, u_{r-t+3}, \ldots, v_t, u_{r+1})$$

and the elements of $\bar{h}$ are

$$\bar{h}_i = (t-1)h_i, \ \forall i = 1, \ldots, k-1, \ \bar{h}_k = \bar{h}_{k+1} = \cdots = \bar{h}_{k+t-1} = h_k = 1.$$

Proof. Using the natural embedding of $K_{t,r}$ into $K_{t,r+1}$, we can interpret circuits of the former also as circuits of the latter. Put

$$y^i = x^i, \ \forall i = 1, \ldots, k-1,$$

Table 3: Circuits for recursion on r

	$\dots$	u_{r-t+1}	u_{r-t+2}	u_{r-t+3}	$\dots$	u_r	u_{r+1}
$y^k = (v_1, u_{r+1}, v_2, u_{r-t+2}, \dots, v_{t-1}, u_{r-1}, v_t, u_r) =$	$\dots$	0	q^2	q^3	$\dots$	q^t	q^1
$y^{k+1} = (v_1, u_{r-t+1}, v_2, u_{r+1}, \dots, v_{t-1}, u_{r-1}, v_t, u_r) =$	$\dots$	q^1	0	q^3	$\dots$	q^t	q^2
$\vdots$				$\vdots$			
$y^{k+t-2} = (v_1, u_{r-t+1}, v_2, u_{r-t+2}, \dots, v_{t-1}, u_{r+1}, v_t, u_r) =$	$\dots$	q^1	q^2	q^3	$\dots$	q^t	q^{t-1}
$y^{k+t-1} = (v_1, u_{r-t+1}, v_2, u_{r-t+2}, \dots, v_{t-1}, u_{r-1}, v_t, u_{r+1}) =$	$\dots$	q^1	q^2	q^3	$\dots$	0	q^t

and for all $j = 1, \dots, t$, let y^{k+j-1} denote vectors obtained by changing vertex u_{r-j+1} of x^k to u_{r+1}. Table 3 displays these circuits as matrices. Here for each $i = 1, \dots, t$, $q^i \in \mathbb{Z}^t$ denotes a vector satisfying

$$q^i_i = 1,\ q^i_{i+1} = -1,$$

and the rest are zeros. Here we identify $t+1$ with 1.

Notice that

$$\sum_{j=1}^{t} y^{k+j-1} = (t-1)x^k.$$

Define

$$\bar{h}_i = (t-1)h_i,\ \forall i = 1, \dots, k-1,$$
$$\bar{h}_{k+j-1} = h_k = 1,\ \forall j = 1, \dots, t+1.$$

Then

$$\sum_{i=1}^{k+t-1} \bar{h}_i y^i = \sum_{i=1}^{k-1} (t-1)h_i y^i + \sum_{j=1}^{t} h_k y^{k+j-1} = \sum_{i=1}^{k-1} h_i x^i + h_k x^k = 0.$$

Therefore $\bar{h}$ is an integer relation on circuits $\{y^i\}_{i=1}^{k+t-1}$.

Next we show that $\bar{h}$ is primitive. Suppose that $h' \in \mathbb{Z}^{k+t-1}$ is a nontrivial relation on the $\{y^i\}_{i=1}^{k+t-1}$. Without loss of generality we may assume that $\{h'_i\}_{i=1}^{k+t-1}$ are relatively prime integers, at least one of which is positive. Consider the column of u_{r+1}. Then

$$h'_k = h'_{k+1} = \cdots = h'_{k+t-1}.$$

Therefore

$$0 = \sum_{i=1}^{k+t-1} h'_i y^i = \sum_{i=1}^{k-1} h'_i y^i + h'_k \sum_{j=1}^{t} y^{k+j-1} = \sum_{i=1}^{k-1} h'_i x^i + (t-1)h'_k x^k.$$

This is an integer relation on $\{x^i\}_{i=1}^{k}$. Therefore there exists $\alpha \in \mathbb{Z}$ such that

$$h'_i = \alpha h_i,\ \forall i = 1, \dots, k-1, \tag{4}$$

$$(t-1)h'_k = \alpha h_k = \alpha. \tag{5}$$

Since $h_i > 0$ for all i and there is an i such that $h'_i > 0$, equations (4) and (5) imply $\alpha > 0$. Therefore (4) and (5) imply that $h'_i > 0$ for all i. Hence $\bar{h}$ is primitive.

Now apply to $\{y^i\}_{i=1}^{k+t-1}$ a permutation of columns so that y^{k+t-1} becomes $(v_1, u_{r-t+2}, v_2, u_{r-t+3}, \dots, v_t, u_{r+1})$. For $i = 1, \dots, k+t-1$, let $\bar{x}^i$ be the circuit of $A_{t,r+1}$ which is the image of y^i under this permutation. Then $\{\bar{x}^i\}_{i=1}^{k+t-1}$ also satisfy the primitive relation $\sum_{i=1}^{k+t-1} \bar{h}_i \bar{x}^i = 0$ with the same coefficients $\bar{h}$. This completes the proof. □

We are now ready to prove Theorem 1.1. In the proof we use the following notation. Let $\mathscr{A}(\{x^i\}_{i=1}^k) = \{\bar{x}^i\}_{i=1}^{k+t}$ and $\mathscr{B}(h) = \bar{h} = (\bar{h}_1, \dots, \bar{h}_{k+t-1}, 1)$ denote circuits of $A_{t+1,t+1}$ and the primitive relation which are obtained by the operation of Lemma 3.1 to circuits $\{x^i\}_{i=1}^k$ of $A_{t,t+1}$ and the primitive relation h. Note that $\|\mathscr{B}(h)\|_1 = \|h\|_1 + t$. Furthermore let $\mathscr{A}'(\{x^i\}_{i=1}^k) = \{\bar{x}^i\}_{i=1}^{k+t-1}$ and $\mathscr{B}'(h) = \bar{h} = (\bar{h}_1, \dots, \bar{h}_{k+t-2}, 1)$ denote circuits of $A_{t,r+1}$ and the primitive relation which are obtained by the operation of Lemma 3.2 to circuits $\{x^i\}_{i=1}^k$ of $A_{t,r}$ and the primitive relation h. Note that $\|\mathscr{B}'(h)\|_1 = (t-1)(\|h\|_1 - 1) + t$.

Our proof uses induction on t, r. We will construct a primitive relation $h^{(t\times r)}$ on circuits $\mathscr{X}_{(t\times r)}$ of $A_{t,r}$ by induction. Therefore we obtain $g(A_{t,r}) \geq \|h^{(t\times r)}\|_1$. Our induction is illustrated in Figure 1. There, a down arrow corresponds to the operation of Lemma 3.1, and a right arrow corresponds to the operation of Lemma 3.2.

$$\begin{array}{ccccccc}
\|h^{(3\times4)}\|_1 & \to & \|h^{(3\times5)}\|_1 & \to & \|h^{(3\times6)}\|_1 & \to & \cdots \\
\downarrow & & & & & & \\
\|h^{(4\times4)}\|_1 & \to & \|h^{(4\times5)}\|_1 & \to & \|h^{(4\times6)}\|_1 & \to & \cdots \\
 & & \downarrow & & & & \\
 & & \|h^{(5\times5)}\|_1 & \to & \|h^{(5\times6)}\|_1 & \to & \cdots \\
 & & & & \downarrow & & \\
 & & & & \vdots & &
\end{array}$$

Figure 1: Induction on t, r

Proof of Theorem 1.1. By induction on t we will prove that for all $t \geq 4$ there exist $k(t) = t^2 - 2t + 2$ circuits $\mathscr{X}_{(t\times t)} = \{x^i_{(t\times t)}\}_{i=1}^{k(t)} \subset \mathcal{C}(A_{t,t})$ and the primitive relation $h^{(t\times t)}$ such that

$$x^{k(t)}_{(t\times t)} = (v_1, u_1, v_2, u_2, \dots, v_t, u_t),$$

$$\sum_{i=1}^{k(t)} h_i^{(t\times t)} x^i = 0,$$

$$h^{(t\times t)}_{k(t)} = 1,$$

$$\|h^{(t\times t)}\|_1 = (t-2)! \left(15 + \sum_{i=1}^{t-4} \frac{i+4}{(i+2)!}\right).$$

Exchange x^1 and x^7 of circuits of Example 2.1 and apply to the circuits a permutation of vertices so that

$$x^7 = (v_1, u_2, v_2, u_3, v_3, u_4).$$

Let $\mathscr{X}_{(3\times 4)} = \{x^i_{(3\times 4)}\}_{i=1}^7$ be the image of $\{x^i\}_{i=1}^7$ under this permutation. The primitive relation $h^{(3\times 4)}$ on these circuits satisfy

$$h^{(3\times 4)} = (7, 2, 3, 3, 5, 6, 1).$$

Notice that $h^{(3\times 4)}_7 = 1$ holds.

Let $\mathscr{X}_{(4\times 4)} = \mathscr{A}(\mathscr{X}_{(3\times 4)})$ and $h^{(4\times 4)} = \mathscr{B}(h^{(3\times 4)})$ denote the image of $\mathscr{X}_{(3\times 4)}$ and $h^{(3\times 4)}$ under the operation of Lemma 3.1. Then we have $h^{(4\times 4)} = (7, 2, 3, 3, 5, 6, 1, 1, 1, 1) \in \mathbb{Z}^{10}$ and

$$\begin{aligned} x^{10}_{(4\times 4)} &= (v_1, u_1, v_2, u_2, v_3, u_3, v_4, u_4), \\ h^{(4\times 4)}_{10} &= 1, \\ \|h^{(4\times 4)}\|_1 &= \|h^{(3\times 4)}\|_1 + 3 = 30. \end{aligned}$$

Therefore we have verified the initial condition at $t = 4$ for the induction.

Suppose now that the result holds for $t \geq 4$. Let $\mathscr{X}_{(t\times(t+1))} = \mathscr{A}'(\mathscr{X}_{(t\times t)})$ and $h^{(t\times(t+1))} = \mathscr{B}'(h^{(t\times t)})$ denote the image of $\mathscr{X}_{(t\times t)}$ and $h^{(t\times t)} \in \mathbb{Z}^{k(t)}$ under the operation of Lemma 3.2.

$$\begin{aligned} x^{k(t)+t-1}_{(t\times(t+1))} &= (v_1, u_2, v_2, u_3, \ldots, v_t, u_{t+1}), \\ h^{(t\times(t+1))}_{k(t)+t-1} &= 1, \\ \|h^{(t\times(t+1))}\|_1 &= (t-1)(\|h^{(t\times t)}\|_1 - 1) + t \end{aligned}$$

follows from Lemma 3.2. Now let $\mathscr{X}_{((t+1)\times(t+1))} = \mathscr{A}(\mathscr{X}_{(t\times(t+1))})$ and $h^{((t+1)\times(t+1))} = \mathscr{B}(h^{(t\times(t+1))})$ denote the image of $\mathscr{X}_{(t\times(t+1))}$ and $h^{(t\times(t+1))}$ under the operation of Lemma 3.1. Then

$$\begin{aligned} x^{k(t)+2t-1}_{((t+1)\times(t+1))} &= (v_1, u_1, v_2, u_2, \ldots, v_{t+1}, u_{t+1}), \\ h^{((t+1)\times(t+1))}_{k(t)+2t-1} &= 1, \\ \|h^{((t+1)\times(t+1))}\|_1 &= (t-1)(\|h^{(t\times t)}\|_1 - 1) + 2t \\ &= (t-1)!\left(15 + \sum_{i=1}^{t-4} \frac{i+4}{(i+2)!}\right) + t + 1 \\ &= ((t+1)-2)!\left(15 + \sum_{i=1}^{(t+1)-4} \frac{i+4}{(i+2)!}\right) \end{aligned}$$

follows from Lemma 3.1. Here $k(t+1) = k(t) + 2t - 1$ and $k(4) = 10$ imply $k(t) = t^2 - 2t + 2$. Therefore the result holds for $t + 1$. Henceforth, let $b_t = \|h^{(t\times t)}\|_1$.

We fix $t \geq 4$ arbitrarily. We prove by induction on r that, for all $r \geq t$, there are circuits $\mathscr{X}_{(t\times r)} = \{x^i_{(t\times r)}\}_{i=1}^{k(t)} \subset \mathcal{C}(A_{t,r})$ and the primitive relation $h^{(t\times r)}$ such that

$$\begin{aligned} x^{k(t)}_{(t\times r)} &= (v_1, u_{r-t+1}, v_2, u_{r-t+2}, \ldots, v_t, u_r), \\ \sum_{i=1}^{k(t)} h_i^{(t\times r)} x^i_{(t\times r)} &= 0, \\ h^{(t\times r)}_{k(t)} &= 1, \\ \|h^{(t\times r)}\|_1 &= (t-1)^{r-t}\left(b_t + \frac{1}{t-2}\right) - \frac{1}{t-2}. \end{aligned}$$

The initial condition of the induction, at $r = t$, follows from $\|h^{(t\times t)}\|_1 = b_t$.

Suppose now that the result holds for some $r \geq t$. Let $\mathscr{X}_{(t\times(r+1))} = \mathscr{A}'(\mathscr{X}_{(t\times r)})$ and $h^{(t\times(r+1))} = \mathscr{B}'(h^{(t\times r)})$ denote the image of $\mathscr{X}_{(t\times r)}$ and $h^{(t\times r)}$ under the operation of Lemma 3.2. Then

$$\begin{aligned} x^{k(t)+t-1}_{(t\times(r+1))} &= (v_1, u_{r-t+2}, v_2, u_{r-t+3}, \ldots, v_t, u_{r+1}), \\ h^{(t\times(r+1))}_{k(t)+t-1} &= 1, \\ \|h^{(t\times(r+1))}\|_1 &= (t-1)(\|h^{(t\times r)}\|_1 - 1) + t \\ &= (t-1)\left((t-1)^{r-t}\left(b_t + \frac{1}{t-2}\right) - \frac{1}{t-2} - 1\right) + t \\ &= (t-1)^{r+1-t}\left(b_t + \frac{1}{t-2}\right) - \frac{1}{t-2} \end{aligned}$$

follows from Lemma 3.2. Therefore the result holds for $r+1$ and

$$g(A_{t,r}) \geq (t-1)^{r-t}(b_t + \frac{1}{t-2}) - \frac{1}{t-2}$$

follows from Lemma 2.2. □

4 Discussion

In this paper we provided a lower bound in Theorem 1.1 by the induction on t, r. Here we discuss some ideas for improving our lower bound.

Look at Figure 1 again. On the step $\|h^{(3\times 4)}\|_1 \to \|h^{(4\times 4)}\|_1$, we can construct a larger primitive relation than the relation constructed in the proof.

Example 4.1. *Let $\{x^i\}_{i=1}^7$ denote the circuits in Example 2.1. Using the natural embedding of $K_{3,4}$ into $K_{4,4}$, let*

$$\bar{x}^i = x^i, \ \forall i = 1, \ldots, 6.$$

and for $i = 7, \ldots, 10$, we define $\bar{x}^i$ as shown in Table 4. Then $\{\bar{x}^i\}_{i=1}^{10}$ are circuits of $\ker_{\mathbb{Z}}(\mathbf{1}_4^{(4)})$

Table 4: Circuits of $A_{4,4}$

	u_1	u_2	u_3	u_4	
	0	1	0	−1	v_1
$\bar{x}^7 = (v_1, u_2, v_4, u_1, v_3, u_4) =$	0	0	0	0	v_2
	−1	0	0	1	v_3
	1	−1	0	0	v_4
	0	0	0	0	v_1
$\bar{x}^8 = (v_2, u_3, v_4, u_2) =$	0	−1	1	0	v_2
	0	0	0	0	v_3
	0	1	−1	0	v_4
	0	0	0	0	v_1
$\bar{x}^9 = (v_3, u_4, v_4, u_3) =$	0	0	0	0	v_2
	0	0	−1	1	v_3
	0	0	1	−1	v_4
	0	1	0	−1	v_1
$\bar{x}^{10} = (v_1, u_2, v_2, u_3, v_3, u_1, v_4, u_4) =$	0	−1	1	0	v_2
	1	0	−1	0	v_3
	−1	0	0	1	v_4

and $\bar{h} = (2, 4, 6, 6, 10, 12, 7, 7, 7, 7)$ *is its primitive relation. Then, by Proposition 2.2,*

$$g(\mathbf{1}_4^{(4)}) \geq 2 + 4 + 6 + 6 + 10 + 12 + 7 + 7 + 7 + 7 = 68. \tag{6}$$

Equation (6) is sharper than the evaluation in Theorem 1.1.

We could start induction from circuits and its primitive relation in Example 4.1. Then we obtain a sharper evaluation for some small t, r. However, unfortunately it turns out that, if we start from Example 4.1 then on the step $\|h^{(8\times r)}\|_1 \to \|h^{(8\times(r+1))}\|_1$, we can not obtain an exponential lower bound. Therefore we did not use Example 4.1 in the proof of Theorem 1.1. However this example suggests that there may be some other better initial set of circuits for our induction.

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