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Abstract

Let $\mathcal{B}$ be a real hyperplane arrangement which is stable under the action of a Coxeter group W. Then W acts naturally on the set of chambers of $\mathcal{B}$. We assume that $\mathcal{B}$ is disjoint from the Coxeter arrangement $\mathcal{A} = \mathcal{A}(W)$ of W. In this paper, we show that the W-orbits of the set of chambers of $\mathcal{B}$ are in one-to-one correspondence with the chambers of $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ which are contained in an arbitrarily fixed chamber of $\mathcal{A}$. From this fact, we find that the number of W-orbits of the set of chambers of $\mathcal{B}$ is given by the number of chambers of $\mathcal{C}$ divided by the order of W. We will also study the set of chambers of $\mathcal{C}$ which are contained in a chamber b of $\mathcal{B}$. We prove that the cardinality of this set is equal to the order of the isotropy subgroup W_b of b. We illustrate these results with some examples, and solve an open problem in Kamiya, Takemura and Terao [Ranking patterns of unfolding models of codimension one, Adv. in Appl. Math. (2010)] by using our results.

Keywords: all-subset arrangement, braid arrangement, characteristic polynomial, Coxeter arrangement, Coxeter group, finite-field method, mid-hyperplane arrangement, symmetric group.

MSC2010: 20F55, 32S22, 52C35.

1 Introduction

Let $\mathcal{B}$ be a real hyperplane arrangement which is stable under the action of a Coxeter group W. Then W acts naturally on the set $\mathbf{Ch}(\mathcal{B})$ of chambers of $\mathcal{B}$. We want to find the number of W-orbits of $\mathbf{Ch}(\mathcal{B})$. A particular case of this problem was considered in

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the authors' previous paper (Kamiya, Takemura and Terao [9]) and the present paper is motivated by an open problem left in Section 6 of [9]. By the general results of the present paper, we give the affirmative answer to the open problem in Theorem 3.1.

Suppose throughout that $\mathcal{B} \cap \mathcal{A} = \emptyset$, where $\mathcal{A} = \mathcal{A}(W)$ is the Coxeter arrangement of W. In this paper, we will show that the orbit space of $\mathbf{Ch}(\mathcal{B})$ is in one-to-one correspondence with the set of chambers c of $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ which are contained in a, $\{c \in \mathbf{Ch}(\mathcal{C}) \mid c \subseteq a\}$, where $a \in \mathbf{Ch}(\mathcal{A})$ is an arbitrary chamber of $\mathcal{A}$. From this fact, we find that the number of W-orbits of $\mathbf{Ch}(\mathcal{B})$ is given by $|\mathbf{Ch}(\mathcal{C})|/|W|$.

On the other hand, we will also study the set of chambers $c \in \mathbf{Ch}(\mathcal{C})$ which are contained in a chamber $b \in \mathbf{Ch}(\mathcal{B})$ of $\mathcal{B}$, $\{c \in \mathbf{Ch}(\mathcal{C}) \mid c \subseteq b\}$. We will prove that the cardinality of this set is equal to the order of the isotropy subgroup W_b of b. Moreover, we will investigate the structure of W_b.

Kamiya, Takemura and Terao [9] tried to find the number of "inequivalent ranking patterns generated by unfolding models of codimension one" in psychometrics, and obtained an upper bound for this number. It was left open to determine whether this upper bound is actually the exact number. The problem boils down to proving (or disproving) that the orbit space of the chambers of the restricted all-subset arrangement ([9]) $\mathcal{B}$ under the action of the symmetric group $\mathfrak{S}_m$ is in one-to-one correspondence with $\{c \in \mathbf{Ch}(\mathcal{A}(\mathfrak{S}_m) \cup \mathcal{B}) \mid c \subseteq a\}$ for a chamber $a \in \mathbf{Ch}(\mathcal{A}(\mathfrak{S}_m))$ of the braid arrangement $\mathcal{A}(\mathfrak{S}_m)$. The results of the present paper establish the one-to-one correspondence.

The paper is organized as follows. In Section 2, we verify our main results. Next, in Section 3, we illustrate our general results with four examples. Those examples are mainly taken from the authors' previous studies of unfolding models in psychometrics ([5], [9]). In Section 3, we also solve the open problem of [9] (Theorem 3.1) using our general results in Section 2 applied to one of our examples.

2 Main results

In this section, we state and prove our main results.

Let V be a Euclidean space. Consider a Coxeter group W acting on V. Then the Coxeter arrangement $\mathcal{A} = \mathcal{A}(W)$ is the set of all reflecting hyperplanes of W. Suppose that $\mathcal{B}$ is a hyperplane arrangement which is stable under the natural action of W. We assume $\mathcal{A} \cap \mathcal{B} = \emptyset$ and define

$$\mathcal{C} := \mathcal{A} \cup \mathcal{B}.$$

Let $\mathbf{Ch}(\mathcal{A})$, $\mathbf{Ch}(\mathcal{B})$ and $\mathbf{Ch}(\mathcal{C})$ denote the set of chambers of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, respectively. Define

$$\varphi_{\mathcal{A}} : \mathbf{Ch}(\mathcal{C}) \to \mathbf{Ch}(\mathcal{A}), \quad \varphi_{\mathcal{B}} : \mathbf{Ch}(\mathcal{C}) \to \mathbf{Ch}(\mathcal{B})$$

by

$$\varphi_{\mathcal{A}}(c) := \text{the chamber of } \mathcal{A} \text{ containing } c,$$
$$\varphi_{\mathcal{B}}(c) := \text{the chamber of } \mathcal{B} \text{ containing } c$$

for $c \in \mathbf{Ch}(\mathcal{C})$. Note that the Coxeter group W naturally acts on $\mathbf{Ch}(\mathcal{A})$, $\mathbf{Ch}(\mathcal{B})$ and $\mathbf{Ch}(\mathcal{C})$.

Lemma 2.1. *$\varphi_{\mathcal{A}}$ and $\varphi_{\mathcal{B}}$ are both W-equivariant, i.e.,*

$$\varphi_{\mathcal{A}}(wc) = w(\varphi_{\mathcal{A}}(c)), \quad \varphi_{\mathcal{B}}(wc) = w(\varphi_{\mathcal{B}}(c))$$

for any $w \in W$ and $c \in \mathbf{Ch}(\mathcal{C})$.

The proof is easy and omitted.

The following result is classical (see, e.g., [3, Ch. V, §3. 2. Theorem 1 (iii)]):

Theorem 2.2. *The group W acts on $\mathbf{Ch}(\mathcal{A})$ effectively and transitively. In particular, $|W| = |\mathbf{Ch}(\mathcal{A})|$.*

By Theorem 2.2, we can prove the following lemma.

Lemma 2.3. *The group W acts on $\mathbf{Ch}(\mathcal{C})$ effectively. In particular, each W-orbit of $\mathbf{Ch}(\mathcal{C})$ is of size $|W|$.*

Proof. If $wc = c$ for $w \in W$ and $c \in \mathbf{Ch}(\mathcal{C})$, then we have $\varphi_{\mathcal{A}}(c) = w\varphi_{\mathcal{A}}(c)$, which implies $w = 1$ by Theorem 2.2. □

For $b \in \mathbf{Ch}(\mathcal{B})$, define the isotropy subgroup $W_b := \{w \in W \mid wb = b\}$. Then we have

Lemma 2.4. *For $b \in \mathbf{Ch}(\mathcal{B})$, the group W_b acts on $\varphi_{\mathcal{B}}^{-1}(b)$ effectively and transitively.*

Proof. The effective part follows from Lemma 2.3, so let us prove the transitivity. Let $c_1, c_2 \in \varphi_{\mathcal{B}}^{-1}(b)$. Define

$$\mathcal{A}(c_1, c_2) := \{H \in \mathcal{A} \mid c_1 \text{ and } c_2 \text{ are on different sides of } H\}.$$

Let us prove that there exists $w \in W$ such that $wc_1 = c_2$ by an induction on $|\mathcal{A}(c_1, c_2)|$. When $|\mathcal{A}(c_1, c_2)| = 0$, we have $\mathcal{A}(c_1, c_2) = \emptyset$ and $c_1 = c_2$. Thus we may choose $w = 1$. If $\mathcal{A}(c_1, c_2)$ is non-empty, then there exists $H_1 \in \mathcal{A}(c_1, c_2)$ such that H_1 contains a wall of c_1. Let s_1 denote the reflection with respect to H_1. Then

$$\mathcal{A}(s_1c_1, c_2) = \mathcal{A}(c_1, c_2) \setminus \{H_1\}.$$

By the induction assumption, there exists $w_1 \in W$ with $w_1s_1c_1 = c_2$. Set $w := w_1s_1$. Then $wc_1 = c_2$ and $c_2 = (wc_1) \cap c_2 \subseteq (wb) \cap b$, which implies that $(wb) \cap b$ is not empty. Thus $wb = b$ and $w \in W_b$. □

The following lemma states that the W-orbits of $\mathbf{Ch}(\mathcal{C})$ and those of $\mathbf{Ch}(\mathcal{B})$ are in one-to-one correspondence.

Lemma 2.5. *The map $\varphi_{\mathcal{B}} : \mathbf{Ch}(\mathcal{C}) \to \mathbf{Ch}(\mathcal{B})$ induces a bijection from the set of W-orbits of $\mathbf{Ch}(\mathcal{C})$ to the set of W-orbits of $\mathbf{Ch}(\mathcal{B})$.*

Proof. For $b \in \mathbf{Ch}(\mathcal{B})$ and $c \in \mathbf{Ch}(\mathcal{C})$, we denote the W-orbit of b and the W-orbit of c by $\mathcal{O}(b)$ and by $\mathcal{O}(c)$, respectively. It is easy to see that

$$\varphi_{\mathcal{B}}(\mathcal{O}(c)) = \mathcal{O}(\varphi_{\mathcal{B}}(c)), \quad c \in \mathbf{Ch}(\mathcal{C}),$$

by Lemma 2.1. Thus $\varphi_{\mathcal{B}}$ induces a map from the set of W-orbits of $\mathbf{Ch}(\mathcal{C})$ to the set of W-orbits of $\mathbf{Ch}(\mathcal{B})$. We will show the map is bijective.

Surjectivity: Let $\mathcal{O}(b)$ be an arbitrary orbit of $\mathbf{Ch}(\mathcal{B})$ with a representative point $b \in \mathbf{Ch}(\mathcal{B})$. Take an arbitrary $c \in \varphi_{\mathcal{B}}^{-1}(b)$. Then

$$\varphi_{\mathcal{B}}(\mathcal{O}(c)) = \mathcal{O}(\varphi_{\mathcal{B}}(c)) = \mathcal{O}(b),$$

which shows the surjectivity.

Injectivity: Suppose $\varphi_{\mathcal{B}}(\mathcal{O}(c_1)) = \varphi_{\mathcal{B}}(\mathcal{O}(c_2))$ ($c_1, c_2 \in \mathbf{Ch}(\mathcal{C})$). Set $b_i := \varphi_{\mathcal{B}}(c_i)$ for $i = 1, 2$. We have

$$\mathcal{O}(b_1) = \mathcal{O}(\varphi_{\mathcal{B}}(c_1)) = \varphi_{\mathcal{B}}(\mathcal{O}(c_1)) = \varphi_{\mathcal{B}}(\mathcal{O}(c_2)) = \mathcal{O}(\varphi_{\mathcal{B}}(c_2)) = \mathcal{O}(b_2),$$

so we can pick $w \in W$ such that $wb_2 = b_1$. Then

$$\varphi_{\mathcal{B}}(wc_2) = w(\varphi_{\mathcal{B}}(c_2)) = wb_2 = b_1.$$

Therefore, both c_1 and wc_2 lie in $\varphi_{\mathcal{B}}^{-1}(b_1)$. By Lemma 2.4, we have $\mathcal{O}(c_1) = \mathcal{O}(wc_2) = \mathcal{O}(c_2)$. $\square$

We are now in a position to state the main results of this paper.

Theorem 2.6. *The cardinalities of $\varphi_{\mathcal{A}}^{-1}(a)$, $\varphi_{\mathcal{B}}^{-1}(b)$ for $a \in \mathbf{Ch}(\mathcal{A})$, $b \in \mathbf{Ch}(\mathcal{B})$ are given as follows:*

1. *For $a \in \mathbf{Ch}(\mathcal{A})$, we have*

$$\begin{aligned} |\varphi_{\mathcal{A}}^{-1}(a)| = \frac{|\mathbf{Ch}(\mathcal{C})|}{|\mathbf{Ch}(\mathcal{A})|} = \frac{|\mathbf{Ch}(\mathcal{C})|}{|W|} &= |\{W\textit{-orbits of } \mathbf{Ch}(\mathcal{C})\}| \\ &= |\{W\textit{-orbits of } \mathbf{Ch}(\mathcal{B})\}|. \end{aligned}$$

2. *For $b \in \mathbf{Ch}(\mathcal{B})$, we have $|\varphi_{\mathcal{B}}^{-1}(b)| = |W_b|$.*

Proof. Part 2 follows from Lemma 2.4, so we will prove Part 1. Since the map $\varphi_{\mathcal{A}} : \mathbf{Ch}(\mathcal{C}) \to \mathbf{Ch}(\mathcal{A})$ is W-equivariant (Lemma 2.1), we have for each $w \in W$ a bijection

$$\varphi_{\mathcal{A}}^{-1}(a) \to \varphi_{\mathcal{A}}^{-1}(wa)$$

sending $c \in \varphi_{\mathcal{A}}^{-1}(a)$ to wc. Thus every fiber of $\varphi_{\mathcal{A}}$ has the same cardinality because W acts transitively on $\mathbf{Ch}(\mathcal{A})$ (Theorem 2.2). The cardinality is equal to

$$\frac{|\mathbf{Ch}(\mathcal{C})|}{|\mathbf{Ch}(\mathcal{A})|} = \frac{|\mathbf{Ch}(\mathcal{C})|}{|W|}.$$

By Lemma 2.3, we have

$$|\{W\text{-orbits of } \mathbf{Ch}(\mathcal{C})\}| = \frac{|\mathbf{Ch}(\mathcal{C})|}{|W|}.$$

Finally, Lemma 2.5 proves the last equality. $\square$

Next, let $x \in V \setminus \bigcup_{H \in \mathcal{B}} H$. Let $b \in \mathbf{Ch}(\mathcal{B})$ denote the unique chamber that contains x. Define the average $z(x)$ of x over the action of W_b:

$$z(x) = \frac{1}{|W_b|} \sum_{w \in W_b} wx.$$

Then it is easily seen that $z(x)$ lies in b because of the convexity of b, and that the map z is W-equivariant. Concerning the structure of W_b, we obtain

Proposition 2.7. *The following statements hold true:*

1. *For any $b \in \mathbf{Ch}(\mathcal{B})$, the set $\{z(x) \mid x \in b\}$ is equal to the set of all W_b-invariant points of b.*

2. *For any $x \in V \setminus \bigcup_{H \in \mathcal{B}} H$, the isotropy subgroup $W_{z(x)}$ of $z(x)$ is equal to W_b, where $b \in \mathbf{Ch}(\mathcal{B})$ is the unique chamber that contains x. In particular, $W_{z(x)}$ depends only on the chamber $b \in \mathbf{Ch}(\mathcal{B})$ containing x.*

Proof.

1. By the linearity of the action of W on V, the average $z(x) \in b$ of $x \in b$ is W_b-invariant: $wz(x) = z(x),\ w \in W_b$. Conversely, the average of any W_b-invariant point of b is the point itself.

2. Assume $w \in W_{z(x)}$. Then $z(x) = wz(x) \in b \cap wb$, which implies $b \cap wb \neq \emptyset$. Since b and wb are both chambers, they coincide: $wb = b$. Thus $w \in W_b$ and we obtain $W_{z(x)} \subseteq W_b$. We also have the reverse inclusion because the average $z(x)$ is W_b-invariant by Part 1. $\square$

When W is the symmetric group $\mathfrak{S}_m = \mathfrak{S}_{\{1,\ldots,m\}}$, we have the following obvious fact (Proposition 2.8). Define

$$H_0 := \{x = (x_1, \ldots, x_m)^T \in \mathbb{R}^m \mid x_1 + \cdots + x_m = 0\}.$$

The group $W = \mathfrak{S}_m$ acts on $V = \mathbb{R}^m$ or $V = H_0$ by permuting coordinates. When $W = \mathfrak{S}_m$ and $V = \mathbb{R}^m$ or $V = H_0$, we agree that this action is considered.

Proposition 2.8. *Let $W = \mathfrak{S}_m$ and $V = \mathbb{R}^m$ or $V = H_0$. Then we have*

$$W_b = \mathfrak{S}_{k_1} \times \mathfrak{S}_{k_2} \times \cdots \times \mathfrak{S}_{k_\ell}, \quad b \in \mathbf{Ch}(\mathcal{B}),$$

where $k_1, \ldots, k_\ell$ ($k_1 + \cdots + k_\ell = m,\ 1 \leq \ell \leq m$) are defined by

$$z_{\sigma(1)} = \cdots = z_{\sigma(k_1)} > z_{\sigma(k_1+1)} = \cdots = z_{\sigma(k_1+k_2)} > \cdots > z_{\sigma(k_1+\cdots+k_{\ell-1}+1)} = \cdots = z_{\sigma(m)}$$

for $z = (z_1, \ldots, z_m)^T = z(x),\ x \in b$, and a permutation $\sigma \in \mathfrak{S}_{\{1,\ldots,m\}}$.

Remark 2.9. *Consider the map $b \mapsto W_b$ from $\mathbf{Ch}(\mathcal{B})$ to the set of subgroups of W. Since $W_{wb} = wW_bw^{-1},\ w \in W$, this induces a map τ from the set of W-orbits of $\mathbf{Ch}(\mathcal{B})$ to the set of conjugacy classes of subgroups of W:*

$$\tau(\mathcal{O}(b)) = [W_b], \quad b \in \mathbf{Ch}(\mathcal{B}), \tag{1}$$

where $[W_b] := \{wW_bw^{-1} \mid w \in W\}$. This map τ is not injective in general. See Remark 3.3 in Subsection 3.2.

3 Examples

In this section, we examine four examples. Those in the first three subsections are taken from problems in psychometrics—the arrangements in Subsections 3.1 and 3.2 (the braid arrangement in conjunction with the all-subset arrangement) appear naturally in the study of ranking patterns of unfolding models of codimension one (Kamiya, Takemura and Terao [9]), while the mid-hyperplane arrangement in Subsection 3.3 is needed in examining ranking patterns of unidimensional unfolding models (Kamiya, Orlik, Takemura and Terao [5]). In these three examples, the Coxeter group W is of type A_{m-1}. In Subsection 3.4, we provide an illustration with the Coxeter group of type B_m.

In Subsection 3.1, we also solve the open problem of [9].

3.1 Coxeter group of type A_{m-1} and restricted all-subset arrangement

Let $W = \mathfrak{S}_m$ and $V = H_0$. Then $\mathcal{A} = \mathcal{A}(W)$ is the braid arrangement in H_0, consisting of the hyperplanes defined by $x_i = x_j,\ 1 \le i < j \le m$. All the $|\mathcal{A}| = m(m-1)/2$ hyperplanes form one orbit under the action of W on $\mathcal{A}$.

Let $\mathcal{B}$ be the restricted all-subset arrangement (Kamiya, Takemura and Terao [9]):

$$\mathcal{B} = \{H_I^0 \mid \emptyset \neq I \subsetneq \{1, \dots, m\}\},$$

where

$$H_I^0 := \{x = (x_1, \dots, x_m)^T \in H_0 \mid \sum_{i \in I} x_i = 0\}.$$

Since $H_I^0 = H^0_{\{1,\dots,m\}\setminus I}$ for $I \neq \emptyset, \{1, \dots, m\}$, we have $|\mathcal{B}| = (2^m - 2)/2 = 2^{m-1} - 1$. The number of W-orbits of $\mathcal{B}$ is $(m-1)/2$ if m is odd and $m/2$ if m is even.

Theorem 2.6 applied to this case gives the affirmative answer to the open problem left in Section 6 of [9]. Using the terminology in Corollary 6.2 in [9], we state:

Theorem 3.1. *The number of inequivalent ranking patterns of unfolding models of codimension one is*

$$\frac{|\mathbf{Ch}(\mathcal{A} \cup \mathcal{B})|}{m!} - 1$$

for the braid arrangement $\mathcal{A}$ in H_0 and the restricted all-subset arrangement $\mathcal{B}$.

Proof. Part 1 of Theorem 2.6 implies that the number of W-orbits of $\mathbf{Ch}(\mathcal{B})$ for the restricted all-subset arrangement $\mathcal{B}$ is equal to $|\mathbf{Ch}(\mathcal{A} \cup \mathcal{B})|/(m!)$. This completes the proof because of the last sentence of Corollary 6.2 in [9]. □

Now, let us investigate the case $m = 3$.

The arrangement $\mathcal{A}$ consists of three lines in $V = H_0,\ \dim H_0 = 2$, each of which is defined by one of the following equations:

$$x_1 = x_2, \quad x_1 = x_3, \quad x_2 = x_3, \tag{2}$$

and $\mathcal{B}$ comprises three lines:

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0.$$

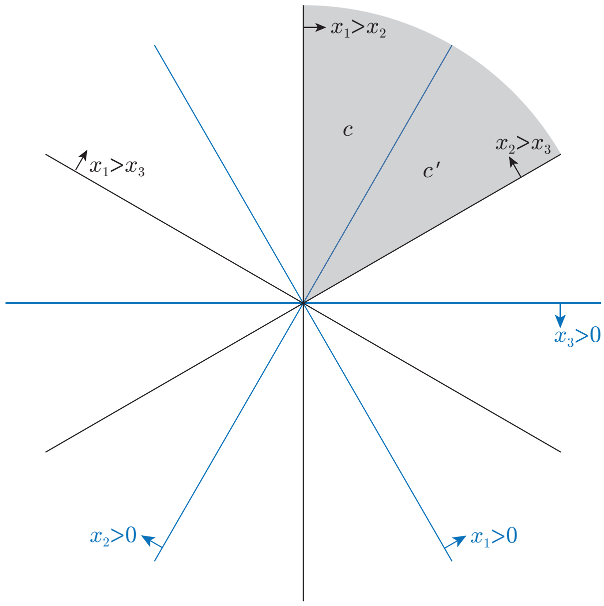

Figure 1: Braid plus restricted all-subset arrangement for $m = 3$.

Figure 1 displays $\mathcal{A}$ and $\mathcal{B}$ in $V = H_0$. Note that the action of W on $\mathcal{B}$ is transitive.

As a chamber of $\mathcal{A}$, let us take $a \in \mathcal{A}$ defined by $x_1 > x_2 > x_3$ (a is shaded in Figure 1). This chamber a of $\mathcal{A}$ contains exactly two chambers c, c' of $\mathcal{C}$: $\varphi_{\mathcal{A}}^{-1}(a) = \{c, c'\}$. These two chambers c, c' have the following walls:

$$\text{Walls of } c : x_1 = x_2,\ x_2 = 0\ \ (x_1 > x_2 > 0),$$
$$\text{Walls of } c' : x_3 = x_2,\ x_2 = 0\ \ (x_3 < x_2 < 0).$$

Note that c and c' are obtained from each other by changing (x_1, x_2, x_3) to $(-x_3, -x_2, -x_1)$. Since $a \in \mathbf{Ch}(\mathcal{A})$ consists of $|\varphi_{\mathcal{A}}^{-1}(a)| = 2$ chambers c, c' of $\mathcal{C}$, we have $|\mathbf{Ch}(\mathcal{C})| = |\varphi_{\mathcal{A}}^{-1}(a)| \times |W| = 2 \times 3! = 12$. The set $\varphi_{\mathcal{A}}^{-1}(a) = \{c, c'\} \subset \mathbf{Ch}(\mathcal{C})$ is a complete set of representatives of the W-orbits of $\mathbf{Ch}(\mathcal{C})$, i.e., $\mathbf{Ch}(\mathcal{C})$ has exactly two orbits $\mathcal{O}(c), \mathcal{O}(c')$ under the action of W.

Now, the chambers $b := \varphi_{\mathcal{B}}(c),\ b' := \varphi_{\mathcal{B}}(c')$ of $\mathcal{B}$ containing c, c' have the following walls:

$$\text{Walls of } b : x_1 = 0,\ x_2 = 0\ \ (x_1 > 0,\ x_2 > 0),$$
$$\text{Walls of } b' : x_3 = 0,\ x_2 = 0\ \ (x_3 < 0,\ x_2 < 0).$$

See Figure 2. The chamber $b \in \mathbf{Ch}(\mathcal{B})$ (resp. $b' \in \mathbf{Ch}(\mathcal{B})$) is divided by the line of $\mathcal{A}$ defined by $x_1 = x_2$ (resp. $x_3 = x_2$) into two chambers of $\mathcal{C}$, $|\varphi_{\mathcal{B}}^{-1}(b)| = |\varphi_{\mathcal{B}}^{-1}(b')| = 2$. The W_b-invariant points z of b are $z = d(1, 1, -2)^T,\ d > 0$, while the $W_{b'}$-invariant points z of b' are $z = d(2, -1, -1)^T,\ d > 0$, so we have $W_b = \mathfrak{S}_{\{1,2\}},\ W_{b'} = \mathfrak{S}_{\{3,2\}}$. Thus we confirm $|\varphi_{\mathcal{B}}^{-1}(b)| = |W_b|,\ |\varphi_{\mathcal{B}}^{-1}(b')| = |W_{b'}|$ (Part 2 of Theorem 2.6).

We have $\{W\text{-orbits of } \mathbf{Ch}(\mathcal{B})\} = \{\mathcal{O}(b), \mathcal{O}(b')\}$, so

$$|\{W\text{-orbits of } \mathbf{Ch}(\mathcal{B})\}| = 2.$$

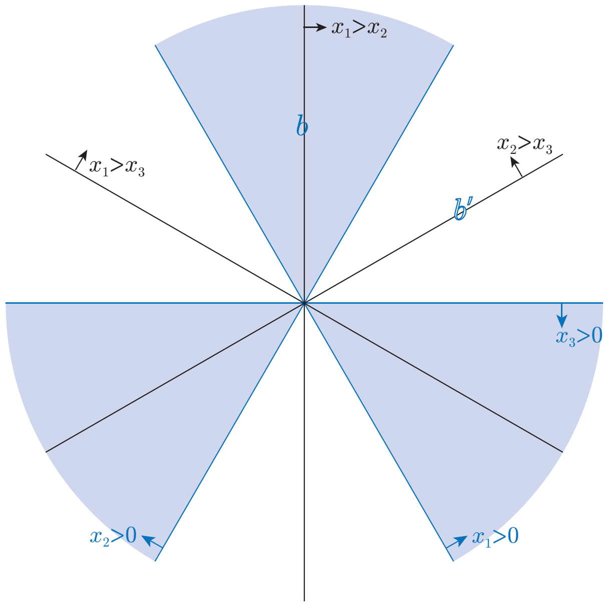

Figure 2: Braid plus restricted all-subset arrangement for $m = 3$. $\mathcal{O}(b)$ and $\mathcal{O}(b')$.

The chambers in $\mathcal{O}(b)$ are shaded in Figure 2, while those in $\mathcal{O}(b')$ are not. The chambers of $\mathcal{B}$ on the same W-orbit as b (resp. b') have walls of the form $x_i > 0,\ x_j > 0,\ i \neq j$ (resp. $x_i < 0,\ x_j < 0,\ i \neq j$). Thus the orbit sizes are $|\mathcal{O}(b)| = |\mathcal{O}(b')| = \binom{3}{2} = 3$ $(= |W|/|W_b| = |W|/|W_{b'}|)$. Therefore, $|\mathbf{Ch}(\mathcal{C})|$ can be computed also as

$$
\begin{aligned}
|\mathbf{Ch}(\mathcal{C})| &= |\varphi_{\mathcal{B}}^{-1}(b)| \times |\mathcal{O}(b)| + |\varphi_{\mathcal{B}}^{-1}(b')| \times |\mathcal{O}(b')| \\
&= \big(|\varphi_{\mathcal{B}}^{-1}(b)| \times |\mathcal{O}(b)|\big) \times 2 = (2 \times 3) \times 2 = 12 \\
&= \left(|W_b| \times \frac{|W|}{|W_b|}\right) \times 2 = |W| \times |\{W\text{-orbits of } \mathbf{Ch}(\mathcal{B})\}|.
\end{aligned}
$$

In addition, $|\mathbf{Ch}(\mathcal{B})| = |\mathcal{O}(b)| + |\mathcal{O}(b')| = 2|\mathcal{O}(b)| = 2 \times 3 = 6$.

We have found

$$
\begin{aligned}
&\varphi_{\mathcal{A}}^{-1}(a) = \{c, c'\}, \qquad \{W\text{-orbits of } \mathbf{Ch}(\mathcal{C})\} = \{\mathcal{O}(c), \mathcal{O}(c')\}, \\
&\{W\text{-orbits of } \mathbf{Ch}(\mathcal{B})\} = \{\mathcal{O}(b), \mathcal{O}(b')\}, \qquad |\mathbf{Ch}(\mathcal{C})| = 12,
\end{aligned}
$$

so we see that Part 1 of Theorem 2.6 holds true:

$$
|\varphi_{\mathcal{A}}^{-1}(a)| = |\{W\text{-orbits of } \mathbf{Ch}(\mathcal{C})\}| = |\{W\text{-orbits of } \mathbf{Ch}(\mathcal{B})\}| = \frac{|\mathbf{Ch}(\mathcal{C})|}{|W|} = 2.
$$

Alternatively, we can compute $|\mathbf{Ch}(\mathcal{C})|$ by using Zaslavsky's result ([13]) on the chamber-counting problem as follows. The characteristic polynomial $\chi(\mathcal{C}, t)$ of $\mathcal{C}$ (Orlik and Terao [11, Definition 2.52]) is

$$
\chi(\mathcal{C}, t) = (t-1)(t-5)
$$

(Kamiya,Takemura and Terao [9, Sec. 6.2.1]). Together with this polynomial, Zaslavsky's result yields

$$
|\mathbf{Ch}(\mathcal{C})| = (-1)^2 \chi(\mathcal{C}, -1) = 12
$$

([13, Theorem A], [11, Theorem 2.68]), which is consistent with our observation above.

Let us move on to the case $m = 4$.

The elements of $\mathcal{A}$ are the six planes in $V = H_0$, $\dim H_0 = 3$, defined by the following equations:

$$x_1 = x_2, \quad x_1 = x_3, \quad x_1 = x_4, \quad x_2 = x_3, \quad x_2 = x_4, \quad x_3 = x_4, \tag{3}$$

whereas those of $\mathcal{B}$ are the seven planes below:

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 0; \tag{4}$$

$$x_1 + x_2 = 0, \quad x_1 + x_3 = 0, \quad x_1 + x_4 = 0. \tag{5}$$

Figure 3 shows the intersection with the unit sphere $\mathbb{S}^2 = \{(x_1, \ldots, x_4)^T \in H_0 \mid x_1^2 + \cdots + x_4^2 = 1\}$ in H_0. Note that $\mathcal{B}$ has two orbits under the action of W—the four planes in (4) constitute one orbit, and the three in (5) form the other one. (In Figure 3, the planes in (4) are drawn in blue and those in (5) are sketched in red.)

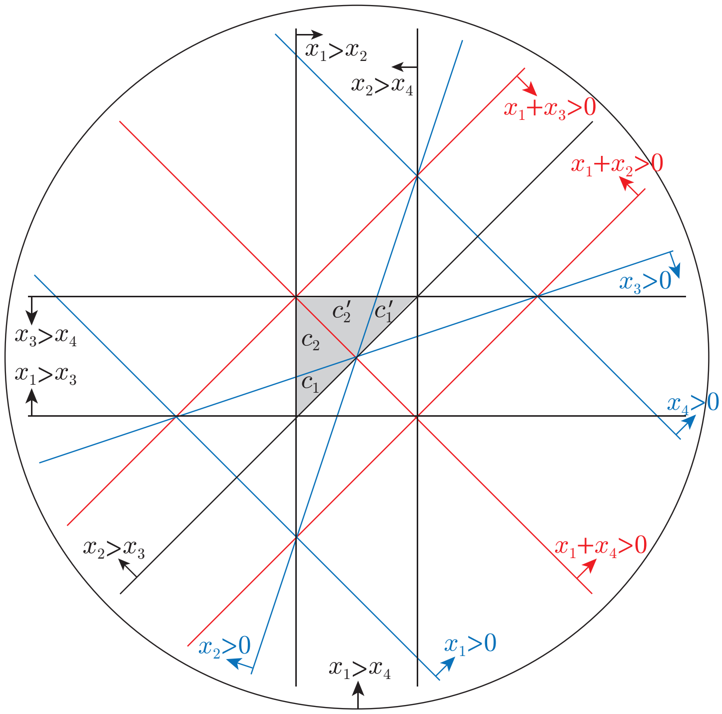

Figure 3: Braid plus restricted all-subset arrangement for $m = 4$.

As a chamber of $\mathcal{A}$, let us take $a \in \mathbf{Ch}(\mathcal{A})$ defined by $x_1 > x_2 > x_3 > x_4$ (a is shaded in Figure 3). This chamber a of $\mathcal{A}$ contains exactly four chambers c_1, c_2, c_1', c_2' of $\mathcal{C}$: $\varphi_{\mathcal{A}}^{-1}(a) = \{c_1, c_2, c_1', c_2'\}$. These chambers have the following walls:

$$\begin{aligned}
&\text{Walls of } c_1 : x_1 = x_2,\ x_2 = x_3,\ x_3 = 0 \quad (x_1 > x_2 > x_3 > 0),\\
&\text{Walls of } c_2 : x_1 = x_2,\ x_1 + x_4 = 0,\ x_3 = 0 \quad (-x_4 > x_1 > x_2,\ x_3 < 0),\\
&\text{Walls of } c_1' : x_4 = x_3,\ x_3 = x_2,\ x_2 = 0 \quad (x_4 < x_3 < x_2 < 0),\\
&\text{Walls of } c_2' : x_4 = x_3,\ x_4 + x_1 = 0,\ x_2 = 0 \quad (-x_1 < x_4 < x_3,\ x_2 > 0).
\end{aligned}$$

Note that c_1' (resp. c_2') is obtained from c_1 (resp. c_2) by changing (x_1, x_2, x_3, x_4) to $(-x_4, -x_3, -x_2, -x_1)$. Since $|\varphi_{\mathcal{A}}^{-1}(a)| = 4$, we have $|\mathbf{Ch}(\mathcal{C})| = |\varphi_{\mathcal{A}}^{-1}(a)| \times |W| = 4 \times 4! = 96$. The set $\varphi_{\mathcal{A}}^{-1}(a) = \{c_1, c_2, c_1', c_2'\} \subset \mathbf{Ch}(\mathcal{C})$ is a complete set of representatives of the W-orbits of $\mathbf{Ch}(\mathcal{C})$, namely, $\mathbf{Ch}(\mathcal{C})$ has exactly four orbits $\mathcal{O}(c_1), \mathcal{O}(c_2), \mathcal{O}(c_1'), \mathcal{O}(c_2')$.

The chambers $b_i := \varphi_{\mathcal{B}}(c_i),\ b_i' := \varphi_{\mathcal{B}}(c_i'),\ i = 1, 2$, of $\mathcal{B}$ containing c_1, c_2, c_1', c_2' have the following walls:

$$\begin{aligned}
&\text{Walls of } b_1 : x_1 = 0,\ x_2 = 0,\ x_3 = 0\ \ (x_1 > 0,\ x_2 > 0,\ x_3 > 0),\\
&\text{Walls of } b_2 : x_1 + x_3 = 0,\ x_1 + x_4 = 0,\ x_3 = 0\ \ (-x_4 > x_1 > -x_3 > 0),\\
&\text{Walls of } b_1' : x_4 = 0,\ x_3 = 0,\ x_2 = 0\ \ (x_4 < 0,\ x_3 < 0,\ x_2 < 0),\\
&\text{Walls of } b_2' : x_4 + x_2 = 0,\ x_4 + x_1 = 0,\ x_2 = 0\ \ (-x_1 < x_4 < -x_2 < 0)
\end{aligned}$$

(Figure 4). The chamber $b_1 \in \mathbf{Ch}(\mathcal{B})$ is divided by the three planes of $\mathcal{A}$ defined by $x_1 = x_2,\ x_1 = x_3,\ x_2 = x_3$ into six chambers of $\mathcal{C}$, whereas b_2 is divided by the plane $x_1 = x_2$ into two chambers of $\mathcal{C}$. For b_1, we have $W_{b_1} = \mathfrak{S}_{\{1,2,3\}}$ (the W_{b_1}-invariant points z of b_1 are $z = d(1, 1, 1, -3)^T,\ d > 0$), and for b_2, we find $W_{b_2} = \mathfrak{S}_{\{1,2\}}$ (the W_{b_2}-invariant points z of b_2 are $z = d_1(1, 1, -1, -1)^T + d_2(1, 1, 0, -2)^T,\ d_1, d_2 > 0$). So we see $|W_{b_1}| = |\varphi_{\mathcal{B}}^{-1}(b_1)|$ $(= 6)$ and $|W_{b_2}| = |\varphi_{\mathcal{B}}^{-1}(b_2)|$ $(= 2)$ hold true.

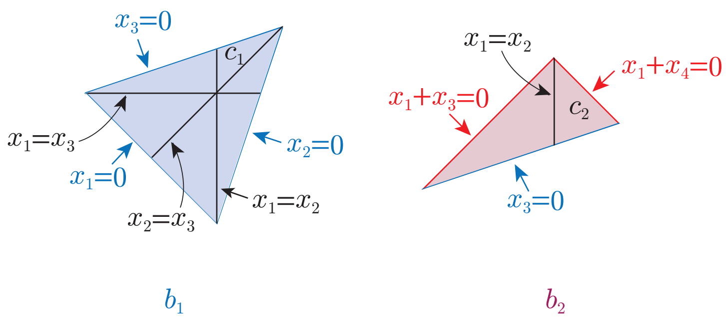

Figure 4: Braid plus restricted all-subset arrangement for $m = 4$. b_1 and b_2.

We have $\{W\text{-orbits of } \mathbf{Ch}(\mathcal{B})\} = \{\mathcal{O}(b_1), \mathcal{O}(b_2), \mathcal{O}(b_1'), \mathcal{O}(b_2')\}$ and hence

$$|\{W\text{-orbits of } \mathbf{Ch}(\mathcal{B})\}| = 4.$$

The chambers $b \in \mathbf{Ch}(\mathcal{B})$ on the same W-orbit as b_1, $\mathcal{O}(b) = \mathcal{O}(b_1)$, have walls of the form $x_i > 0,\ x_j > 0,\ x_k > 0$ (i, j, k are all distinct), while $b \in \mathbf{Ch}(\mathcal{B})$ such that $\mathcal{O}(b) = \mathcal{O}(b_2)$ have walls of the form $x_i < 0,\ x_i + x_j > 0,\ x_i + x_k > 0$ (i, j, k are all distinct). Thus the orbit sizes are $|\mathcal{O}(b_1)| = \binom{4}{3} = 4$ $(= |W|/|W_{b_1}|)$ and $|\mathcal{O}(b_2)| = \binom{4}{3} \times 3 = 12$ $(= |W|/|W_{b_2}|)$. Accordingly, $|\mathbf{Ch}(\mathcal{C})|$ is again

$$\begin{aligned}
|\mathbf{Ch}(\mathcal{C})| &= \left(|\varphi_{\mathcal{B}}^{-1}(b_1)| \cdot |\mathcal{O}(b_1)| + |\varphi_{\mathcal{B}}^{-1}(b_2)| \cdot |\mathcal{O}(b_2)|\right) \times 2 = (6 \times 4 + 2 \times 12) \times 2 = 96\\
&= \left(|W_{b_1}| \cdot \frac{|W|}{|W_{b_1}|} + |W_{b_2}| \cdot \frac{|W|}{|W_{b_2}|}\right) \times 2 = |W| \times |\{W\text{-orbits of } \mathbf{Ch}(\mathcal{B})\}|.
\end{aligned}$$

Besides, $|\mathbf{Ch}(\mathcal{B})| = 2(|\mathcal{O}(b_1)| + |\mathcal{O}(b_2)|) = 2(4 + 12) = 32$.

The characteristic polynomial $\chi(\mathcal{C}, t)$ of $\mathcal{C}$ is

$$\chi(\mathcal{C}, t) = (t-1)(t-5)(t-7)$$

(Kamiya,Takemura and Terao [9, Sec. 6.2.2]). This polynomial yields

$$|\mathbf{Ch}(\mathcal{C})| = (-1)^3 \chi(\mathcal{C}, -1) = 96$$

in agreement with our observation above.

For $5 \le m \le 9$, we used the finite-field method (Athanasiadis [1, 2], Stanley [12, Lecture 5], Crapo and Rota [4], Kamiya, Takemura and Terao [6, 7, 8]) to calculate the characteristic polynomials $\chi(\mathcal{C}, t)$ of $\mathcal{C}$, and obtained the numbers of W-orbits of $\mathbf{Ch}(\mathcal{B})$ by using Zaslavsky's result [13, Theorem A] and Part 1 of Theorem 2.6: $|\{W\text{-orbits of } \mathbf{Ch}(\mathcal{B})\}| = |\varphi_{\mathcal{A}}^{-1}(a)| = |\mathbf{Ch}(\mathcal{C})|/|W|$ as follows.

$$\begin{aligned}
m = 5: \quad & \chi(\mathcal{C}, t) = (t-1)(t-7)(t-8)(t-9), \quad |\mathbf{Ch}(\mathcal{C})| = 1440, \quad |\varphi_{\mathcal{A}}^{-1}(a)| = 12, \\
m = 6: \quad & \chi(\mathcal{C}, t) = (t-1)(t-7)(t-11)(t-13)(t-14), \\
& \qquad |\mathbf{Ch}(\mathcal{C})| = 40320, \quad |\varphi_{\mathcal{A}}^{-1}(a)| = 56, \\
m = 7: \quad & \chi(\mathcal{C}, t) = (t-1)(t-11)(t-13)(t-17)(t-19)(t-23), \\
& \qquad |\mathbf{Ch}(\mathcal{C})| = 2903040, \quad |\varphi_{\mathcal{A}}^{-1}(a)| = 576, \\
m = 8: \quad & \chi(\mathcal{C}, t) = (t-1)(t-19)(t-23)(t-25)(t-27)(t-29)(t-31), \\
& \qquad |\mathbf{Ch}(\mathcal{C})| = 670924800, \quad |\varphi_{\mathcal{A}}^{-1}(a)| = 16640, \\
m = 9: \quad & \chi(\mathcal{C}, t) = (t-1)(t^7 - 290t^6 + 36456t^5 - 2573760t^4 + 110142669t^3 \\
& \qquad\qquad - 2855339970t^2 + 41492561354t - 260558129500), \\
& \qquad |\mathbf{Ch}(\mathcal{C})| = 610037568000, \quad |\varphi_{\mathcal{A}}^{-1}(a)| = 1681100.
\end{aligned}$$

Note that the characteristic polynomial $\chi(\mathcal{C}, t)$ factors into polynomials of degree one over $\mathbb{Z}$ for $m \le 8$.

Remark 3.2. *For $m = 5$ and $m = 6$, Kamiya, Takemura and Terao [9] identified all the elements c of $\varphi_{\mathcal{A}}^{-1}(a)$ for $a : x_1 > \cdots > x_m$ and gave an example of the W_b-invariant points z of $b = \varphi_{\mathcal{B}}(c)$ for each c. From those z, we immediately obtain W_b by Proposition 2.8.*

3.2 Coxeter group of type A_{m-1} and unrestricted all-subset arrangement

Let $W = \mathfrak{S}_m$ and $V = \mathbb{R}^m$. Then $\mathcal{A} = \mathcal{A}(W)$ is the braid arrangement in $\mathbb{R}^m$. Let $\mathcal{B}$ be the (unrestricted) all-subset arrangement ([9]):

$$\mathcal{B} = \{H_I \mid \emptyset \ne I \subseteq \{1, \ldots, m\}\},$$

where

$$H_I := \{x = (x_1, \ldots, x_m)^T \in \mathbb{R}^m \mid \sum_{i \in I} x_i = 0\}.$$

Note $|\mathcal{B}| = 2^m - 1$. The number of orbits of $\mathcal{B}$ under the action of W is m.

We will examine the case $m = 3$.

The arrangement $\mathcal{A}$ has exactly the three planes in $V = \mathbb{R}^3$ defined by the same equations as those in (2). On the other hand, $\mathcal{B}$ consists of the seven planes defined by

$$\begin{gathered} x_1 = 0, \quad x_2 = 0, \quad x_3 = 0; \\ x_1 + x_2 = 0, \quad x_1 + x_3 = 0, \quad x_2 + x_3 = 0; \\ x_1 + x_2 + x_3 = 0 \end{gathered}$$

with each line corresponding to one orbit under the action of W on $\mathcal{B}$. Figure 5 exhibits the intersection with the unit sphere in $V = \mathbb{R}^3$.

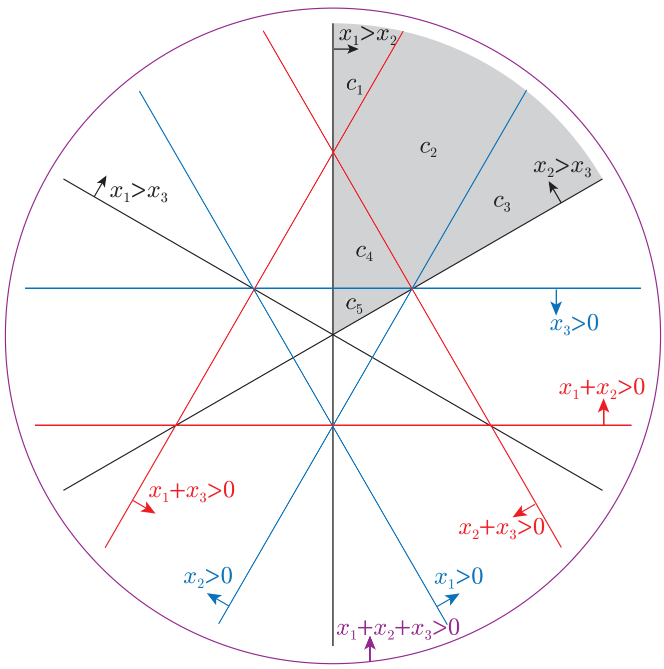

Figure 5: Braid plus unrestricted all-subset arrangement.

Let us take $a \in \mathbf{Ch}(\mathcal{A})$ defined by $x_1 > x_2 > x_3$ (a with $x_1 + x_2 + x_3 > 0$ is shaded in Figure 5). Then $\varphi_{\mathcal{A}}^{-1}(a) = \{c_1, c_2, c_3, c_4, c_5, c'_1, c'_2, c'_3, c'_4, c'_5\}$, where

$$\begin{aligned} &c_1 : x_1 > x_2, \ x_1 + x_3 < 0, \ x_1 + x_2 + x_3 > 0, \\ &c_2 : x_2 > 0, \ x_1 + x_3 > 0, \ x_2 + x_3 < 0, \\ &c_3 : x_2 > x_3, \ x_2 < 0, \ x_1 + x_2 + x_3 > 0, \\ &c_4 : x_1 > x_2, \ x_3 < 0, \ x_2 + x_3 > 0, \\ &c_5 : x_1 > x_2, \ x_2 > x_3, \ x_3 > 0, \end{aligned}$$

and c'_i, $i = 1, \ldots, 5$, are the chambers obtained from c_i, $i = 1, \ldots, 5$, by changing (x_1, x_2, x_3) to $(-x_3, -x_2, -x_1)$. Thus $|\varphi_{\mathcal{A}}^{-1}(a)| = |\{W\text{-orbits of } \mathbf{Ch}(\mathcal{C})\}| = 10$ and $|\mathbf{Ch}(\mathcal{C})| = |\varphi_{\mathcal{A}}^{-1}(a)| \times |W| = 10 \times 3! = 60$.

Table 1: W_b and z for $b = b_1, \ldots, b_5$.

b	W_b	W_b-invariant points z of b
b_1	$\mathfrak{S}_{\{1,2\}}$	$d_1(1,1,-1)^T + d_2(1,1,-2)^T,\ d_1, d_2 > 0$
b_2	$\{1\}$	$d_1(1,1,-1)^T + d_2(1,0,-1)^T + d_3(1,0,0)^T,\ d_1, d_2, d_3 > 0$
b_3	$\mathfrak{S}_{\{2,3\}}$	$d_1(1,0,0)^T + d_2(2,-1,-1)^T,\ d_1, d_2 > 0$
b_4	$\mathfrak{S}_{\{1,2\}}$	$d_1(1,1,-1)^T + d_2(1,1,0)^T,\ d_1, d_2 > 0$
b_5	$\mathfrak{S}_{\{1,2,3\}}$	$d(1,1,1)^T,\ d > 0$

The chambers $b_i := \varphi_{\mathcal{B}}(c_i) \in \mathbf{Ch}(\mathcal{B}),\ i = 1, \ldots, 5$, are

$$\begin{aligned} &b_1 : x_1 + x_3 < 0,\ x_2 + x_3 < 0,\ x_1 + x_2 + x_3 > 0, \\ &b_2 : x_2 > 0,\ x_1 + x_3 > 0,\ x_2 + x_3 < 0, \\ &b_3 : x_2 < 0,\ x_3 < 0,\ x_1 + x_2 + x_3 > 0, \\ &b_4 : x_3 < 0,\ x_1 + x_3 > 0,\ x_2 + x_3 > 0, \\ &b_5 : x_1 > 0,\ x_2 > 0,\ x_3 > 0, \end{aligned} \tag{6}$$

and $b'_i := \varphi_{\mathcal{B}}(c'_i) \in \mathbf{Ch}(\mathcal{B}),\ i = 1, \ldots, 5$, can be obtained from $b_i,\ i = 1, \ldots, 5$, by the above-mentioned rule. See Figure 6. The chamber b_1 is divided by the plane $x_1 = x_2$ into two chambers; b_2 is not divided by any plane in $\mathcal{A}$; b_3 is divided by $x_2 = x_3$ into two; b_4 is divided by $x_1 = x_2$ into two; and b_5 is divided by the three planes $x_1 = x_2,\ x_1 = x_3,\ x_2 = x_3$ into six. The isotropy subgroups W_b and the W_b-invariant points z of b for $b = b_1, \ldots, b_5$ are given in Table 1. We can confirm $|W_{b_1}| = 2! = |\varphi_{\mathcal{B}}^{-1}(b_1)|,\ |W_{b_2}| = 1 = |\varphi_{\mathcal{B}}^{-1}(b_2)|,\ |W_{b_3}| = 2! = |\varphi_{\mathcal{B}}^{-1}(b_3)|,\ |W_{b_4}| = 2! = |\varphi_{\mathcal{B}}^{-1}(b_4)|,\ |W_{b_5}| = 3! = |\varphi_{\mathcal{B}}^{-1}(b_5)|$.

We have $\{W\text{-orbits of } \mathbf{Ch}(\mathcal{B})\} = \{\mathcal{O}(b_1), \ldots, \mathcal{O}(b_5), \mathcal{O}(b'_1), \ldots, \mathcal{O}(b'_5)\}$ and thus

$$|\{W\text{-orbits of } \mathbf{Ch}(\mathcal{B})\}| = 10.$$

From (6), we see $|\mathcal{O}(b_1)| = 3,\ |\mathcal{O}(b_2)| = 3 \times 2 = 6,\ |\mathcal{O}(b_3)| = 3,\ |\mathcal{O}(b_4)| = 3,\ |\mathcal{O}(b_5)| = 1$, which coincide with $|W|/|W_{b_i}|,\ i = 1, \ldots, 5$. Hence, $|\mathbf{Ch}(\mathcal{C})| = 60$ can be obtained also from

$$|\mathbf{Ch}(\mathcal{C})| = 2 \sum_{i=1}^{5} |\varphi_{\mathcal{B}}^{-1}(b_i)| \cdot |\mathcal{O}(b_i)| = 2\,(2 \cdot 3 + 1 \cdot 6 + 2 \cdot 3 + 2 \cdot 3 + 6 \cdot 1) = 60.$$

We can also get $|\mathbf{Ch}(\mathcal{B})| = 2 \sum_{i=1}^{5} |\mathcal{O}(b_i)| = 2\,(3 + 6 + 3 + 3 + 1) = 32$.

Remark 3.3. *In Table 1, we find that $W_{b_1} = W_{b_4} = \mathfrak{S}_{\{1,2\}},\ W_{b_3} = \mathfrak{S}_{\{2,3\}}$ are all conjugate to one another, although b_1, b_3, b_4 are on different orbits. Using τ in (1), we have*

$$\tau^{-1}([\mathfrak{S}_{\{1,2\}}]) = \{\mathcal{O}(b_1), \mathcal{O}(b_3), \mathcal{O}(b_4)\},$$

and τ is not injective. The chambers b_1, b_3, b_4 are triangular cones (triangles in Figure 6) cut by a single plane (line) $x_i = x_j$ from the braid arrangement. However, these chambers are easily seen to be on different orbits, since their three walls (edges) are of different combinations of orbits of $\mathcal{B}$.

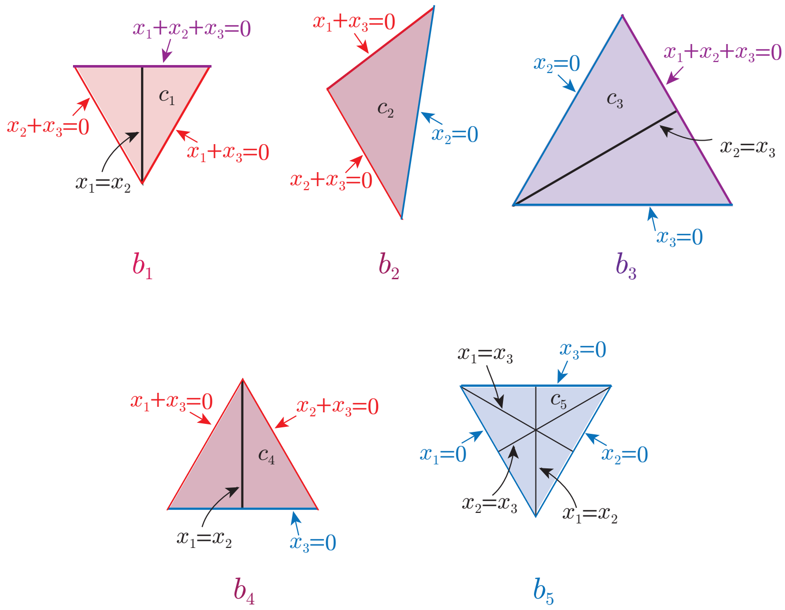

Figure 6: Braid plus unrestricted all-subset arrangement. $b_1, \ldots, b_5$.

For $m \leq 7$, we computed $\chi(\mathcal{C}, t)$ using the finite-field method, and obtained the numbers of W-orbits of $\mathbf{Ch}(\mathcal{B})$ as follows:

$$\begin{aligned}
m=3: &\quad \chi(\mathcal{C},t) = (t-1)(t-4)(t-5), \quad |\mathbf{Ch}(\mathcal{C})| = 60, \quad |\varphi_{\mathcal{A}}^{-1}(a)| = 10,\\
m=4: &\quad \chi(\mathcal{C},t) = (t-1)(t-5)(t-7)(t-8), \quad |\mathbf{Ch}(\mathcal{C})| = 864, \quad |\varphi_{\mathcal{A}}^{-1}(a)| = 36,\\
m=5: &\quad \chi(\mathcal{C},t) = (t-1)(t-7)(t-9)(t-11)(t-13),\\
&\qquad\qquad |\mathbf{Ch}(\mathcal{C})| = 26880, \quad |\varphi_{\mathcal{A}}^{-1}(a)| = 224,\\
m=6: &\quad \chi(\mathcal{C},t) = (t-1)(t-11)(t-13)(t-17)^2(t-19),\\
&\qquad\qquad |\mathbf{Ch}(\mathcal{C})| = 2177280, \quad |\varphi_{\mathcal{A}}^{-1}(a)| = 3024,\\
m=7: &\quad \chi(\mathcal{C},t) = (t-1)(t-19)(t-23)(t^4-105t^3+4190t^2-75180t+510834),\\
&\qquad\qquad |\mathbf{Ch}(\mathcal{C})| = 566697600, \quad |\varphi_{\mathcal{A}}^{-1}(a)| = 112440.
\end{aligned}$$

Note that the characteristic polynomial $\chi(\mathcal{C}, t)$ factors into polynomials of degree one over $\mathbb{Z}$ for $m \leq 6$.

3.3 Mid-hyperplane arrangement

Let $W = \mathfrak{S}_m$ and $V = H_0$, so $\mathcal{A} = \mathcal{A}(W)$ is the braid arrangement in H_0. We take

$$\mathcal{B} = \{H_{ijkl} \mid 1 \leq i < j \leq m,\ 1 \leq k < l \leq m,\ i < k,\ |\{i,j,k,l\}| = 4\},$$

where

$$H_{ijkl} := \{x = (x_1, \ldots, x_m)^T \in H_0 \mid x_i + x_j = x_k + x_l\},$$

so that $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ is an essentialization of the mid-hyperplane arrangement (Kamiya, Orlik, Takemura and Terao [5]). We have $|\mathcal{B}| = 3\binom{m}{4}$, and the action of W on $\mathcal{B}$ is transitive.

Let us consider the case $m = 4$.

The elements of $\mathcal{A}$ are the six planes in $V = H_0$, $\dim H_0 = 3$, defined by the equations in (3), whereas those of $\mathcal{B}$ are the three planes defined by the following equations:

$$x_1 + x_2 = x_3 + x_4, \quad x_1 + x_3 = x_2 + x_4, \quad x_1 + x_4 = x_2 + x_3.$$

Figure 7 shows the intersection with the unit sphere $\mathbb{S}^2$ in H_0.

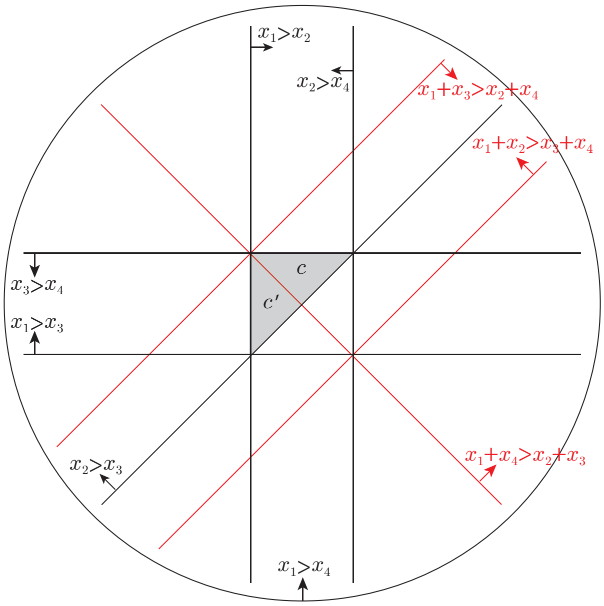

Figure 7: Essentialization of mid-hyperplane arrangement.

Let us take $a \in \mathbf{Ch}(\mathcal{A})$ defined by $x_1 > x_2 > x_3 > x_4$ (a is shaded in Figure 7). Then $\varphi_{\mathcal{A}}^{-1}(a) = \{c, c'\}$, where

$$\begin{aligned} c &: x_2 > x_3,\ x_3 > x_4,\ x_1 + x_4 > x_2 + x_3, \\ c' &: x_3 < x_2,\ x_2 < x_1,\ x_4 + x_1 < x_3 + x_2. \end{aligned}$$

Note that c' is obtained from c by changing (x_1, x_2, x_3, x_4) to $(-x_4, -x_3, -x_2, -x_1)$. We have $|\varphi_{\mathcal{A}}^{-1}(a)| = |\{W\text{-orbits of } \mathbf{Ch}(\mathcal{C})\}| = 2$ and $|\mathbf{Ch}(\mathcal{C})| = |\varphi_{\mathcal{A}}^{-1}(a)| \times |W| = 2 \times 4! = 48$.

The chamber $b := \varphi_{\mathcal{B}}(c) \in \mathbf{Ch}(\mathcal{B})$ is

$$b : x_1 + x_2 > x_3 + x_4,\ x_1 + x_3 > x_2 + x_4,\ x_1 + x_4 > x_2 + x_3, \tag{7}$$

which is divided by the three planes $x_2 = x_3$, $x_2 = x_4$, $x_3 = x_4$ into six chambers (Figure 8). We find the W_b-invariant points z of b to be $z = d(3, -1, -1, -1)^T$, $d > 0$, so $W_b = \mathfrak{S}_{\{2,3,4\}}$ and $|W_b| = 3! = |\varphi_{\mathcal{B}}^{-1}(b)|$.

We have $\{W\text{-orbits of } \mathbf{Ch}(\mathcal{B})\} = \{\mathcal{O}(b), \mathcal{O}(b')\}$ with $b' := \varphi_{\mathcal{B}}(c')$. Thus

$$|\{W\text{-orbits of } \mathbf{Ch}(\mathcal{B})\}| = 2.$$

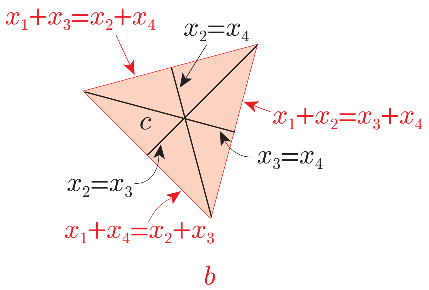

Figure 8: Essentialization of mid-hyperplane arrangement. b.

From (7), we see $|\mathcal{O}(b)| = 4$ $(= |W|/|W_b| = 4!/(3!))$, so we can calculate $|\mathbf{Ch}(\mathcal{C})|$ alternatively as

$$|\mathbf{Ch}(\mathcal{C})| = 2(|\varphi_{\mathcal{B}}^{-1}(b)| \cdot |\mathcal{O}(b)|) = 2 \times 6 \cdot 4 = 48.$$

Moreover, we get $|\mathbf{Ch}(\mathcal{B})| = 2|\mathcal{O}(b)| = 2 \times 4 = 8$.

For $m \leq 10$, the characteristic polynomials of $\mathcal{C}$ are known ([5], [10]), so we can find the numbers of W-orbits of $\mathbf{Ch}(\mathcal{B})$:

$$\begin{aligned}
m = 4: \quad & \chi(\mathcal{C}, t) = (t-1)(t-3)(t-5), \quad |\mathbf{Ch}(\mathcal{C})| = 48, \quad |\varphi_{\mathcal{A}}^{-1}(a)| = 2,\\
m = 5: \quad & \chi(\mathcal{C}, t) = (t-1)(t-7)(t-8)(t-9), \quad |\mathbf{Ch}(\mathcal{C})| = 1440, \quad |\varphi_{\mathcal{A}}^{-1}(a)| = 12,\\
m = 6: \quad & \chi(\mathcal{C}, t) = (t-1)(t-13)(t-14)(t-15)(t-17),\\
& \qquad |\mathbf{Ch}(\mathcal{C})| = 120960, \quad |\varphi_{\mathcal{A}}^{-1}(a)| = 168,\\
m = 7: \quad & \chi(\mathcal{C}, t) = (t-1)(t-23)(t-24)(t-25)(t-26)(t-27),\\
& \qquad |\mathbf{Ch}(\mathcal{C})| = 23587200, \quad |\varphi_{\mathcal{A}}^{-1}(a)| = 4680,\\
m = 8: \quad & \chi(\mathcal{C}, t) = (t-1)(t-35)(t-37)(t-39)(t-41)(t^2-85t+1926),\\
& \qquad |\mathbf{Ch}(\mathcal{C})| = 9248117760, \quad |\varphi_{\mathcal{A}}^{-1}(a)| = 229386,\\
m = 9: \quad & \chi(\mathcal{C}, t) = (t-1)(t^7 - 413t^6 + 73780t^5 - 7387310t^4 + 447514669t^3\\
& \qquad\qquad - 16393719797t^2 + 336081719070t - 2972902161600),\\
& \qquad |\mathbf{Ch}(\mathcal{C})| = 6651665153280, \quad |\varphi_{\mathcal{A}}^{-1}(a)| = 18330206,\\
m = 10: \quad & \chi(\mathcal{C}, t) = (t-1)(t^8 - 674t^7 + 201481t^6 - 34896134t^5 + 3830348179t^4\\
& \qquad\qquad - 272839984046t^3 + 12315189583899t^2\\
& \qquad\qquad\qquad - 321989533359786t + 3732690616086600),\\
& \qquad |\mathbf{Ch}(\mathcal{C})| = 4067272044460800, \quad |\varphi_{\mathcal{A}}^{-1}(a)| = 1120831141.
\end{aligned}$$

3.4 Signed all-subset arrangement

Let W be the Coxeter group of type B_m, i.e., the semidirect product of $\mathfrak{S}_m$ by $(\mathbb{Z}/2\mathbb{Z})^m$: $W = (\mathbb{Z}/2\mathbb{Z})^m \rtimes \mathfrak{S}_m$, $|W| = 2^m \cdot m!$. Then W acts on $V = \mathbb{R}^m$ by permuting coordinates by $\mathfrak{S}_m$ and changing signs of coordinates by $(\mathbb{Z}/2\mathbb{Z})^m$. The Coxeter arrangement $\mathcal{A} =$

$\mathcal{A}(W)$ consists of the hyperplanes defined by

$$x_i = 0, \quad 1 \le i \le m; \tag{8}$$
$$x_i + x_j = 0,\ x_i - x_j = 0, \quad 1 \le i < j \le m. \tag{9}$$

We have $|\mathcal{A}| = m + m(m-1) = m^2$. Moreover, the number of orbits of $\mathcal{A}$ under the action of W is two: one consisting of the m hyperplanes in (8) and the other made up of the $m(m-1)$ hyperplanes in (9).

Let

$$\mathcal{B} = \{H_{(\epsilon_1,\dots,\epsilon_m)} \mid \epsilon_1, \dots, \epsilon_m \in \{-1, 0, 1\},\ \sum_{i=1}^{m} |\epsilon_i| \ge 3\},$$

where

$$H_{(\epsilon_1,\dots,\epsilon_m)} := \{x = (x_1, \dots, x_m)^T \in \mathbb{R}^m \mid \sum_{i=1}^{m} \epsilon_i x_i = 0\}.$$

Note $H_{(\epsilon_1,\dots,\epsilon_m)} = H_{(-\epsilon_1,\dots,-\epsilon_m)}$ so that $|\mathcal{B}| = \sum_{i=3}^{m} 2^{i-1}\binom{m}{i}$. The number of W-orbits of $\mathcal{B}$ is $m-2$.

Let us study the case $m = 3$.

In this case, $\mathcal{A}$ comprises the nine planes in $V = \mathbb{R}^3$ defined by

$$x_1 = 0,\ x_2 = 0,\ x_3 = 0; \tag{10}$$
$$x_1 + x_2 = 0,\ x_1 - x_2 = 0,\ x_1 + x_3 = 0,\ x_1 - x_3 = 0,\ x_2 + x_3 = 0,\ x_2 - x_3 = 0 \tag{11}$$

with each line corresponding to one orbit, and $\mathcal{B}$ consists of one orbit containing the four planes defined by

$$-x_1 + x_2 + x_3 = 0,\ x_1 - x_2 + x_3 = 0,\ x_1 + x_2 - x_3 = 0,\ x_1 + x_2 + x_3 = 0. \tag{12}$$

Figure 9 shows the intersection with the unit sphere in $V = \mathbb{R}^3$. (In Figure 9, the planes in (10), (11) and (12) are drawn in blue, black and purple, respectively.)

Let us take $a \in \mathbf{Ch}(\mathcal{A})$ defined by $x_1 > x_2 > x_3 > 0$ (this chamber is shaded in Figure 9). Then $\varphi_{\mathcal{A}}^{-1}(a) = \{c_1, c_2\}$, where

$$c_1 : x_1 - x_2 > 0,\ x_2 - x_3 > 0,\ -x_1 + x_2 + x_3 > 0,$$
$$c_2 : x_3 > 0,\ x_2 - x_3 > 0,\ -x_1 + x_2 + x_3 < 0.$$

So $|\varphi_{\mathcal{A}}^{-1}(a)| = |\{W\text{-orbits of } \mathbf{Ch}(\mathcal{C})\}| = 2$ and $|\mathbf{Ch}(\mathcal{C})| = |\varphi_{\mathcal{A}}^{-1}(a)| \times |W| = 2 \times 2^3 \cdot 3! = 96$.

The chambers $b_i := \varphi_{\mathcal{B}}(c_i) \in \mathbf{Ch}(\mathcal{B}),\ i = 1, 2$, are

$$b_1 : -x_1 + x_2 + x_3 > 0,\ x_1 - x_2 + x_3 > 0,\ x_1 + x_2 - x_3 > 0,$$
$$b_2 : -x_1 + x_2 + x_3 < 0,\ x_1 - x_2 + x_3 > 0,\ x_1 + x_2 - x_3 > 0,\ x_1 + x_2 + x_3 > 0$$

(Figure 10). The chamber b_1 is divided by the three planes $x_1 - x_2 = 0,\ x_1 - x_3 = 0,\ x_2 - x_3 = 0$ into six chambers, and b_2 is divided by the four planes $x_2 = 0,\ x_3 = 0,\ x_2 + x_3 = 0,\ x_2 - x_3 = 0$ into eight. We see $W_{b_1} = \mathfrak{S}_{\{1,2,3\}}$ is the Coxeter group of type A_2 (the W_{b_1}-invariant points z of b_1 are $z = d(1, 1, 1)^T,\ d > 0$), and that

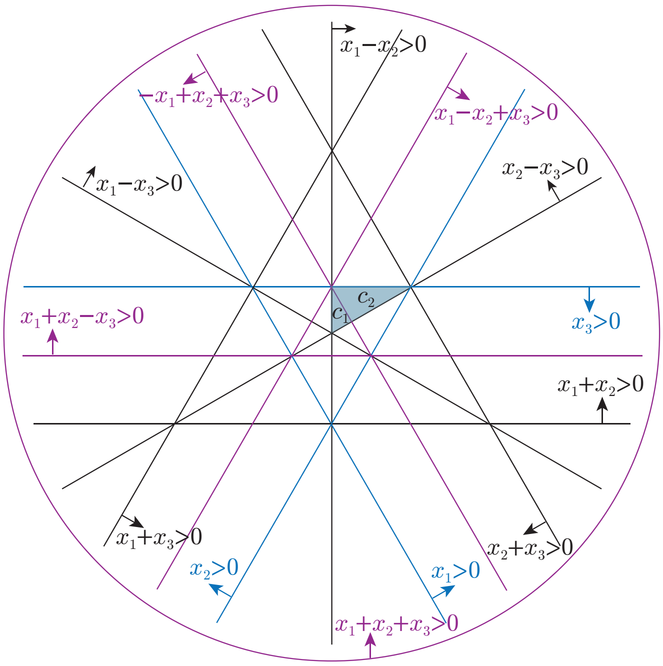

Figure 9: Signed all-subset arrangement.

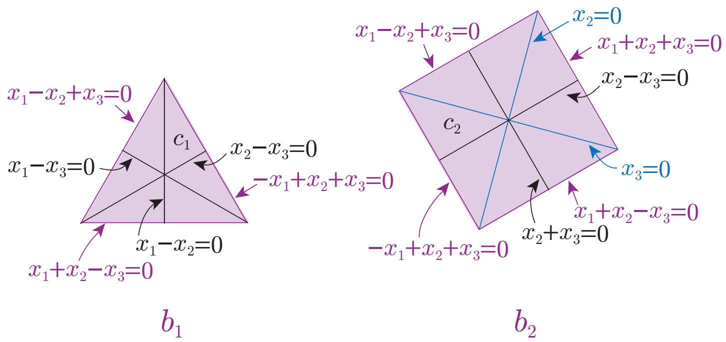

Figure 10: Signed all-subset arrangement. b_1 and b_2.

$W_{b_2} = (\mathbb{Z}/2\mathbb{Z})^2 \rtimes \mathfrak{S}_{\{2,3\}}$ is the Coxeter group of type B_2 (the W_{b_2}-invariant points z of b_2 are $z = d(1,0,0)^T,\ d > 0$). Hence $|W_{b_1}| = 3! = |\varphi_{\mathcal{B}}^{-1}(b_1)|,\ |W_{b_2}| = 2^2 \cdot 2! = |\varphi_{\mathcal{B}}^{-1}(b_2)|$.

We have $\{W\text{-orbits of } \mathbf{Ch}(\mathcal{B})\} = \{\mathcal{O}(b_1), \mathcal{O}(b_2)\}$, so

$$|\{W\text{-orbits of } \mathbf{Ch}(\mathcal{B})\}| = 2.$$

From Figures 9 and 10, we see $|\mathcal{O}(b_1)| = 4\times 2 = 8$ $(= |W|/|W_{b_1}| = (2^3\cdot 3!)/(3!))$, $|\mathcal{O}(b_2)| = 3 \times 2 = 6$ $(= |W|/|W_{b_2}| = (2^3 \cdot 3!)/(2^2 \cdot 2!))$, so $|\mathbf{Ch}(\mathcal{C})|$ can be computed also as

$$|\mathbf{Ch}(\mathcal{C})| = |\varphi_{\mathcal{B}}^{-1}(b_1)| \cdot |\mathcal{O}(b_1)| + |\varphi_{\mathcal{B}}^{-1}(b_2)| \cdot |\mathcal{O}(b_2)| = 6 \cdot 8 + 8 \cdot 6 = 96.$$

Furthermore, we can get $|\mathbf{Ch}(\mathcal{B})| = |\mathcal{O}(b_1)| + |\mathcal{O}(b_2)| = 8 + 6 = 14$.

For $m \leq 6$, we computed $\chi(\mathcal{C}, t)$ and obtained the numbers of W-orbits of $\mathbf{Ch}(\mathcal{B})$ as follows:

$$
\begin{aligned}
m = 3: \quad & \chi(\mathcal{C}, t) = (t-1)(t-5)(t-7), \quad |\mathbf{Ch}(\mathcal{C})| = 96, \quad |\varphi_{\mathcal{A}}^{-1}(a)| = 2, \\
m = 4: \quad & \chi(\mathcal{C}, t) = (t-1)(t-11)(t-13)(t-15), \quad |\mathbf{Ch}(\mathcal{C})| = 5376, \quad |\varphi_{\mathcal{A}}^{-1}(a)| = 14, \\
m = 5: \quad & \chi(\mathcal{C}, t) = (t-1)(t-29)(t-31)(t^2 - 60t + 971), \\
& \qquad\qquad |\mathbf{Ch}(\mathcal{C})| = 1981440, \quad |\varphi_{\mathcal{A}}^{-1}(a)| = 516, \\
m = 6: \quad & \chi(\mathcal{C}, t) = (t-1)(t^5 - 363t^4 + 54310t^3 - 4182690t^2 + 165591769t - 2691439347), \\
& \qquad\qquad |\mathbf{Ch}(\mathcal{C})| = 5722536960, \quad |\varphi_{\mathcal{A}}^{-1}(a)| = 124187.
\end{aligned}
$$

Note that the characteristic polynomial $\chi(\mathcal{C}, t)$ factors into polynomials of degree one over $\mathbb{Z}$ for $m \leq 4$.

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