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Numerical Integration of the Ostrovsky Equation Based on Its Geometric Structures

Yuto MIYATAKE*, Takaharu YAGUCHI†and Takayasu MATSUO‡

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Abstract

We consider structure preserving numerical schemes for the Ostrovsky equation, which describes gravity waves under the influence of Coriolis force. This equation has two associated invariants: an energy function and the L^2 norm. It is widely accepted that structure preserving methods such as invariants-preserving and multi-symplectic integrators generally yield qualitatively better numerical results. In this paper we propose five geometric integrators for this equation: energy-preserving and norm-preserving finite difference and Galerkin schemes, and a multi-symplectic integrator based on a newly found multi-symplectic formulation. A numerical comparison of these schemes is provided, which indicates that the energy-preserving finite difference schemes are more advantageous than the other schemes.

keyword
Ostrovsky equation, Conservation, Multi-symplecticity, Discrete variational derivative method, Discrete partial derivative method

1 Introduction

In this paper[1] we consider geometric numerical integration for the Ostrovsky equation [22] under the periodic boundary condition of length L:

$$u_t + \alpha u u_x - \beta u_{xxx} = \gamma {\partial_x}^{-1} u, \qquad u(t,x) = u(t, x+L), \tag{1}$$

where α, β, γ are real parameters and the subscript t (or x, respectively) denotes the differentiation with respect to time variable t (or x). This equation models gravity waves under the influence of Coriolis force. The parameter β measures the dispersion effects and γ measures the effect of the rotation. For example, for $\beta < 0$, the equation models surface and internal waves in the ocean and surface waves in a shallow channel with uneven bottom [2], and for $\beta > 0$, it models capillary waves on the surface of a liquid and magneto-acoustic

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[1] This work is a full version of our recent quick report [20], with detailed discussions including extra schemes and numerical results.

waves in a plasma [11, 12]. In the absence of rotation ($\gamma = 0$), (1) leads to the well-known Korteweg–de Vries (KdV) equation. Despite its physical importance for the case $\gamma \neq 0$, only a few numerical methods have been proposed. Hunter [14] used a finite difference scheme to investigate solutions under both positive and negative dispersion effects. In [6, 13] Fourier-pseudospectral and Fourier-Galerkin schemes are used to examine the evolutions of soliton-like solutions. Liu–Pelinovsky–Sakovich studied the wave breaking phenomena for the case $\beta = 0$ theoretically and numerically [15]. More recently Yaguchi–Matsuo–Sugihara proposed four numerical schemes that inherit the following conservation properties of the Ostrovsky equation [24].

The Ostrovsky equation has three first integrals [7]:

$$\int_0^L u \, \mathrm{d}x = \text{const.} = 0, \tag{2}$$

$$\int_0^L \left(\frac{\alpha}{6} u^3 + \frac{\beta}{2} u_x^2 + \frac{\gamma}{2} ({\partial_x}^{-1} u)^2 \right) \mathrm{d}x = \text{const.}, \tag{3}$$

$$\int_0^L \frac{u^2}{2} \, \mathrm{d}x = \text{const.} \tag{4}$$

In what follows, we call them the mass, energy, and L^2 norm conservation laws, respectively. Note that in the mass conservation (2), we claimed that u is a "zero-mean" function. This is necessary so that ${\partial_x}^{-1} u$ in the equation is also a periodic function. Generally, for mathematical convenience, the inverse operator ${\partial_x}^{-1}$ is defined for any zero-mean and L-periodic function as

$${\partial_x}^{-1} u = \int_0^x u(t,s) \, \mathrm{d}s - \frac{1}{L} \int_0^L \int_0^x u(t,s) \, \mathrm{d}s \, \mathrm{d}x \tag{5}$$

(see, for example, [14]). Notice this maps the function back to a zero-mean and L-periodic function:

$$\int_0^L {\partial_x}^{-1} u \, \mathrm{d}x = 0, \tag{6}$$

as far as integration is allowed.

For the PDEs with such invariants, it is widely accepted that structure preserving methods, for example numerical methods that preserve the invariants, generally yield qualitatively better numerical results, and in the last two decades much effort has been devoted in this topic to finally find out several unified frameworks. For example, the Discrete Variational Derivative Method (DVDM) was proposed by Furihata [9] (see also Furihata–Matsuo [10]), the Average Vector Field Method (AVFM) was proposed by Celledoni et al. [5] in finite difference context, and the Discrete Partial Derivative Method (DPDM) was proposed by Matsuo [17] in finite element context.

From these standpoints, Yaguchi et al. have proposed four conservative numerical schemes: a finite difference scheme and a pseudospectral scheme that conserve the energy (3), and the same types of schemes that conserve the norm (4). The energy-preserving schemes are based on the following Hamiltonian structure:

$$u_t = -\partial_x \frac{\delta G}{\delta u}, \qquad G(u) = \frac{\alpha}{6} u^3 + \frac{\beta}{2} u_x^2 + \frac{\gamma}{2} ({\partial_x}^{-1} u)^2, \tag{7}$$

and the norm-preserving schemes are based on the following variational structure:

$$u_t = \left(-\frac{\alpha}{3} (u \partial_x + \partial_x u) + \beta {\partial_x}^3 + \gamma {\partial_x}^{-1} \right) \frac{\delta H}{\delta u}, \qquad H(u) = \frac{u^2}{2}. \tag{8}$$

They derived the conservative schemes based on the DVDM, but there was a difficulty that the original DVDM did not support the non-local operator ${\partial_x}^{-1}$. In order to work around this, they extended the DVDM so that it can handle a discrete inverse operator $\tilde{\delta_x}^{\langle -1\rangle}$ that approximates ${\partial_x}^{-1}$ and correctly replicates the zero-mean property (6). In this sense their trial was quite successful. There remained, however, a small drawback that their operator $\tilde{\delta_x}^{\langle -1\rangle}$ failed to satisfy $\tilde{\delta_x}^{\langle -1\rangle}\delta_x^{\langle 1\rangle} = \text{id.}$, which is naturally expected corresponding to another basic property of ${\partial_x}^{-1}$: ${\partial_x}^{-1}\partial_x = \text{id.}$ (precise mathematical meaning of this identity will be described in Section 3.1.) This gave rise to unnatural average operators in the resulting schemes, which seemed to provoke undesirable oscillations in their numerical results (the detail will be reviewed later).

Taking these into consideration, we devote our effort to the following three points.

- Firstly, we propose new energy- or norm-preserving finite difference schemes where no such unnatural average operator appears. The key here is to introduce the potential function ϕ satisfying $\phi_x = u$, and rewrite the Hamiltonian and variational structures with it. The technique is natural in the context of dynamical systems. We then apply the DVDM to obtain conservative schemes. Our numerical experiment strongly suggests that the qualitative behavior of numerical solutions by the new schemes is better than the existing schemes.

- Secondly, we show that this equation has a multi-symplectic formulation, and provide a multi-symplectic scheme based on this formulation by applying the Preissman box scheme. This formulation is motivated by the multi-symplectic formulation of the KdV equation by Ascher–McLachlan [1]. Although this multi-symplectic scheme does not preserve the energy or the norm exactly, our numerical results show that the deviations are very small .

- Finally, we propose energy- or norm-preserving Galerkin schemes. The existing framework (the DPDM) does not directly apply to the Ostrovsky equation due to the non-local term ${\partial_x}^{-1}u$. We solve this difficulty by introducing an L^2-projection technique.

Remark 1. Note the difference between "Hamiltonian structure" (7) and "variational structure" (8). As is well known, the KdV equation ($\gamma = 0$) is a bi-Hamiltonian PDE; i.e., both (7) and (8) give Hamiltonian structures. On the contrary, for $\gamma \neq 0$, (8) should not give a Hamiltonian structure. In fact, if the Ostrovsky equation were a bi-Hamiltonian PDE, it would have infinitely many first integrals. But it has been reported that this equation is not completely integrable [7], and the above three are believed to be the only invariants. □

This paper is organized as follows. In Section 2 notation is introduced and the schemes by Yaguchi–Matsuo–Sugihara are summarized. In Section 3 two finite difference schemes are presented. In Section 4 a multi-symplectic formulation and a multi-symplectic scheme based on it are proposed. In Section 5 two Galerkin schemes are presented. In Section 6 some numerical results are provided. Concluding remarks and comments are given in Section 7.

2 Preliminaries

In this section we prepare notation and review the approach by Yaguchi–Matsuo–Sugihara [24].

2.1 Notation

We first prepare notation used in Section 2, 3 and 4. The interval $[0, L]$ is discretized by uniform grids with the space mesh size $\Delta x = L/N$, where N is the number of nodes. Numerical solutions are denoted by $U_k^{(n)} \simeq u(n\Delta t, k\Delta x)$ or $\Phi_k^{(n)} \simeq \phi(n\Delta t, k\Delta x)$, where Δt is the time mesh size. We often write the solutions as a vector $\boldsymbol{U}^{(n)} = (U_1^{(n)}, \dots, U_N^{(n)})^\top$. For simplicity, we also denote $U_k^{(n+\frac{1}{2})} = (U_k^{(n+1)} + U_k^{(n)})/2$. In order to treat the periodic boundary condition, we consider $\{U_k^{(n)}\}_{k=-\infty}^{\infty}$, an infinitely long vector, and then its N-dimensional restriction by the discrete periodic boundary condition $U_k^{(n)} = U_{k \bmod N}^{(n)}$ (for all $k \in \mathbb{Z}$). We denote the latter space by $X_\mathrm{d} := \{\boldsymbol{U} = \{U_k\}_{k\in\mathbb{Z}} \mid U_k \in \mathbb{R}, U_k = U_{k \bmod N}, \text{for all } k \in \mathbb{Z}\}$, and introduce its zero-mean subspace $\overline{X}_\mathrm{d} := \{\boldsymbol{U} \mid \sum_{k=0}^{N-1} U_k = 0,\ \boldsymbol{U} \in X_\mathrm{d}\}$. These are natural discretizations of continuous spaces X and $\overline{X}$, which will be introduced in Section 3.1.

The standard central difference operators that approximate $\partial_x, {\partial_x}^2, {\partial_x}^3, {\partial_x}^4$ are denoted by $\delta_x^{\langle 1\rangle}, \delta_x^{\langle 2\rangle}, \delta_x^{\langle 3\rangle}, \delta_x^{\langle 4\rangle}$ respectively:

$$\delta_x^{\langle 1\rangle} U_k^{(n)} = \frac{U_{k+1}^{(n)} - U_{k-1}^{(n)}}{2\Delta x}, \qquad \delta_x^{\langle 2\rangle} U_k^{(n)} = \frac{U_{k+1}^{(n)} - 2U_k^{(n)} + U_{k-1}^{(n)}}{(\Delta x)^2},$$
$$\delta_x^{\langle 3\rangle} = \delta_x^{\langle 2\rangle}\delta_x^{\langle 1\rangle}, \qquad \delta_x^{\langle 4\rangle} = (\delta_x^{\langle 2\rangle})^2,$$

and the forward and backward difference operators, which are also the approximations of ∂_x, are denoted by

$$\delta_x^{+} U_k^{(n)} = \frac{U_{k+1}^{(n)} - U_k^{(n)}}{\Delta x}, \qquad \delta_x^{-} U_k^{(n)} = \frac{U_k^{(n)} - U_{k-1}^{(n)}}{\Delta x}.$$

We denote the approximations of ${\partial_x}^{-1}$ by $\tilde{\delta_x}^{\langle -1\rangle}$ and $\delta_x^{\langle -1\rangle}$, whose explicit definitions will be provided later. We use the forward difference operator $\delta_t^+ = (U_k^{(n+1)} - U_k^{(n)})/\Delta t$ which is the discretization of ∂_t. In Section 5 we utilize the inner product defined by $(f, g) = \int_0^L fg \, \mathrm{d}x$.

As for the difference operators, the following summation-by-parts formulas are useful.

Lemma 2.1 ([9])**.** *The difference operators $\delta_x^{\langle 1\rangle}$ and $\delta_x^{\langle 3\rangle}$ are skew-symmetric in the sense that for any two sequences $\boldsymbol{U}$, $\boldsymbol{V} \in X_\mathrm{d}$,*

$$\sum_{k=0}^{N-1} U_k \delta_x^{\langle 1\rangle} V_k \Delta x + \sum_{k=0}^{N-1} (\delta_x^{\langle 1\rangle} U_k) V_k \Delta x = \sum_{k=0}^{N-1} U_k \delta_x^{\langle 3\rangle} V_k \Delta x + \sum_{k=0}^{N-1} (\delta_x^{\langle 3\rangle} U_k) V_k \Delta x = 0. \tag{9}$$

Eq. (9) corresponds to the integration-by-parts formula:

$$\int_0^L u\partial_x v \,\mathrm{d}x + \int_0^L (\partial_x u) v \,\mathrm{d}x = \int_0^L u{\partial_x}^3 v \,\mathrm{d}x + \int_0^L ({\partial_x}^3 u) v \,\mathrm{d}x = 0,$$

for any two L-periodic functions $u(t, \cdot)$ and $v(t, \cdot)$.

2.2 Approach by Yaguchi–Matsuo–Sugihara

In this subsection the previous approach by Yaguchi et al. [24] is summarized. They assumed that the initial condition is given such that it satisfies

$$\sum_{k=0}^{N-1} U_k^{(0)} \Delta x = 0,$$

which corresponds to (2), and defined the operator $\tilde{\delta_x}^{\langle -1\rangle}$ by a summation operator

$$\tilde{\delta_x}^{\langle -1\rangle} U_k^{(n)} = \Delta x \left(\frac{U_0^{(n)}}{2} + \sum_{j=1}^{k-1} U_j^{(n)} + \frac{U_k^{(n)}}{2} \right) - \frac{(\Delta x)^2}{L} \sum_{j=0}^{N-1} \left(\frac{U_0^{(n)}}{2} + \sum_{l=1}^{j-1} U_l^{(n)} + \frac{U_j^{(n)}}{2} \right), \tag{10}$$

which corresponds to (5). The second term of the right hand side of (10) enforces

$$\sum_{k=0}^{N-1} \tilde{\delta_x}^{\langle -1 \rangle} U_k^{(n)} \Delta x = 0,$$

which corresponds to the zero-mean property (6).

We review the energy-preserving scheme first. A discrete version of the energy $G = \alpha u^3/6 + \beta u_x^2/2 + \gamma(\partial_x{}^{-1} u)^2/2$, and accordingly a "discrete variational derivative" that approximates $\delta G/\delta u = \alpha u^2/2 - \beta u_{xx} - \gamma \partial_x{}^{-2} u$ are defined by

$$\begin{aligned} G_k^{(n)} &= \frac{\alpha}{6}(U_k^{(n)})^3 + \frac{\beta}{4}\left((\delta_x^+ U_k^{(n)})^2 + (\delta_x^- U_k^{(n)})^2\right) + \frac{\gamma}{2}(\tilde{\delta_x}^{\langle -1 \rangle} U_k^{(n)})^2, \\ \frac{\delta G}{\delta(\boldsymbol{U}^{(n+1)}, \boldsymbol{U}^{(n)})_k} &= \frac{\alpha}{6}\left((U_k^{(n+1)})^2 + U_k^{(n+1)} U_k^{(n)} + (U_k^{(n)})^2\right) \\ &\quad - \beta \delta_x^{\langle 2 \rangle} U_k^{(n+\frac{1}{2})} - \gamma(\tilde{\delta_x}^{\langle -1 \rangle})^2 U_k^{(n+\frac{1}{2})}. \end{aligned}$$

Then the scheme is defined as follows.

Scheme 1 (Yaguchi et al.'s energy-preserving finite difference scheme [24])**.** *Given an initial approximate solution $\boldsymbol{U}^{(0)} \in \overline{X}_{\mathrm{d}}$, we compute $\boldsymbol{U}^{(n)}$ $(n = 1, 2, \dots)$ by*

$$\frac{U_k^{(n+1)} - U_k^{(n)}}{\Delta t} = -\delta_x^{\langle 1 \rangle} \frac{\delta G}{\delta(\boldsymbol{U}^{(n+1)}, \boldsymbol{U}^{(n)})_k} \qquad (k = 0, \dots, N-1). \tag{11}$$

Eq. (11) corresponds to the Hamiltonian structure (7). Numerical solutions by Scheme 1 conserve both the total mass and the energy.

Theorem 2.2 (Conservation of the total mass and the energy [24])**.** *Under the periodic boundary condition, the numerical solutions by Scheme 1 conserve the total mass and the energy:*

$$\sum_{k=0}^{N-1} U_k^{(n)} \Delta x = \sum_{k=0}^{N-1} U_k^{(0)} \Delta x, \qquad \sum_{k=0}^{N-1} G_k^{(n)} \Delta x = \sum_{k=0}^{N-1} G_k^{(0)} \Delta x \qquad (n = 1, 2, \dots).$$

Next, the norm-preserving finite difference scheme is summarized. A discrete version of the norm $H = u^2/2$, and accordingly a "discrete variational derivative" that approximates $\delta H/\delta u = u$ are defined by

$$H_k^{(n)} = \frac{(U_k^{(n)})^2}{2}, \qquad \frac{\delta H}{\delta(\boldsymbol{U}^{(n+1)}, \boldsymbol{U}^{(n)})_k} = U_k^{(n+\frac{1}{2})}.$$

Then the scheme is defined as follows.

Scheme 2 (Yaguchi et al.'s norm-preserving finite difference scheme [24])**.** *Given an initial approximate solution $\boldsymbol{U}^{(0)} \in \overline{X}_{\mathrm{d}}$, we compute $\boldsymbol{U}^{(n)}$ $(n = 1, 2, \dots)$ by*

$$\frac{U_k^{(n+1)} - U_k^{(n)}}{\Delta t} = \left(-\frac{\alpha}{3}(U_k^{(n+\frac{1}{2})} \delta_x^{\langle 1 \rangle} + \delta_x^{\langle 1 \rangle} U_k^{(n+\frac{1}{2})}) + \beta \delta_x^{\langle 3 \rangle} + \gamma \tilde{\delta_x}^{\langle -1 \rangle}\right) \frac{\delta H}{\delta(\boldsymbol{U}^{(n+1)}, \boldsymbol{U}^{(n)})_k}$$

$$(k = 0, \dots, N-1). \tag{12}$$

Eq. (12) corresponds to the variational structure (8). Although Yaguchi et al. actually described the scheme in a different form

$$\frac{U_k^{(n+1)} - U_k^{(n)}}{\Delta t} + \frac{\alpha}{3}\left(U_k^{(n+\frac{1}{2})}\delta_x^{\langle 1\rangle}U_k^{(n+\frac{1}{2})} + \delta_x^{\langle 1\rangle}\left(U_k^{(n+\frac{1}{2})}\right)^2\right) - \beta\delta_x^{\langle 3\rangle}U_k^{(n+\frac{1}{2})} = \gamma\tilde{\delta_x}^{\langle -1\rangle}U_k^{(n+\frac{1}{2})}, \tag{13}$$

it is easy to show it is equivalent to (12). Numerical solutions by Scheme 2 conserve both the total mass and the norm.

Theorem 2.3 (Conservation of the total mass and the norm [24])**.** *Under the periodic boundary condition, the numerical solutions by Scheme 2 conserve the total mass and the norm:*

$$\sum_{k=0}^{N-1} U_k^{(n)}\Delta x = \sum_{k=0}^{N-1} U_k^{(0)}\Delta x, \qquad \sum_{k=0}^{N-1} H_k^{(n)}\Delta x = \sum_{k=0}^{N-1} H_k^{(0)}\Delta x, \qquad n = 0, 1, \ldots.$$

Remark 2. Actually, the scheme (13) corresponds to the discretization of

$$u_t + \frac{\alpha}{3}(u\partial_x u + \partial_x u^2) - \beta{\partial_x}^3 u = \gamma{\partial_x}^{-1}u,$$

by the midpoint rule. □

In this way, Yaguchi et al. succeeded in designing conservative schemes. There remained, however, a small disadvantage that although in an analogy with the continuous case the operator $\tilde{\delta_x}^{\langle -1\rangle}$ is expected to satisfy $\tilde{\delta_x}^{\langle -1\rangle}\delta_x^{\langle 1\rangle} = \mathrm{id.}$, it turned out not, and an unnatural operator appears as follows.

Lemma 2.4. *For any $\boldsymbol{U} \in \overline{X}_{\mathrm{d}}$, it holds*

$$\tilde{\delta_x}^{\langle -1\rangle}\delta_x^{\langle 1\rangle}U_k = \delta_x^{\langle 1\rangle}\tilde{\delta_x}^{\langle -1\rangle}U_k = \frac{U_{k-1} + 2U_k + U_{k+1}}{4}.$$

The unnecessary average operator broadens the stencil in the resulting schemes, and the behavior of the numerical solution can get slightly worse. Actually, Yaguchi et al. [24] found in their paper that the schemes can get slightly unstable when the space mesh size Δx is increased.

3 New conservative finite difference schemes

In view of the above background, we here propose two new conservative finite difference schemes free of such an unexpected average operator. This is done by reformulating the Hamiltonian and variational structures so that no inverse operator ${\partial_x}^{-1}$ is required explicitly. In this section, we first consider the reformulation and then derive schemes based on the reformulated structures.

3.1 Reformulation of the Hamiltonian and variational structures with the potential ϕ

We denote by X the set of L-periodic and sufficiently smooth functions, and by $\overline{X}$ its subset of all zero-mean functions. In order to eliminate the inverse operator ${\partial_x}^{-1}$, we here introduce a "potential" function $\phi = \partial_x{}^{-1} u$, where $\partial_x{}^{-1}$ is defined by (5). From this definition, it immediately follows that $\phi_x = u$, and that the equality (6) implies

$$\int_0^L \phi \, \mathrm{d}x = 0 \tag{14}$$

in terms of ϕ. This means that ϕ also belongs to $\overline{X}$. It is easy to check that $\partial_x{}^{-1}\partial_x = \partial_x \partial_x{}^{-1} = \text{id.}$ in $\overline{X}$; i.e., they map $u \in \overline{X}$ back to u itself.

The equation (1) can then be rewritten with ϕ to

$$\phi_{xt} + \alpha\phi_x\phi_{xx} - \beta\phi_{xxxx} = \gamma\phi. \tag{15}$$

Note that this does *not* include the inverse operator $\partial_x{}^{-1}$. By introducing an energy function, this can be cast into

$$\phi_{xt} = \frac{\delta G}{\delta \phi}, \qquad G(\phi) = \frac{\alpha}{6}\phi_x^3 + \frac{\beta}{2}\phi_{xx}^2 + \frac{\gamma}{2}\phi^2. \tag{16}$$

If we further multiply both sides by $\partial_x{}^{-1}$ (note that $\delta G/\delta\phi = -\alpha\phi_x\phi_{xx} + \beta\phi_{xxxx} + \gamma\phi \in \overline{X}$), we finally reach a Hamiltonian form:

$$\phi_t = \partial_x{}^{-1}\frac{\delta G}{\delta \phi}.$$

The skew-symmetry of $\partial_x{}^{-1}$ will be shown in Lemma 3.1 below. Although the Hamiltonian form includes $\partial_x{}^{-1}$, actually the previous expression (16) suffices to show the energy preservation (see Theorem 3.2 below). In the subsequent subsection, we will fully utilize this to construct an energy-preserving scheme avoiding inverse operators.

Similarly, Eq. (15) can be rewritten in another variational way:

$$\phi_{xt} = \left(\frac{\alpha}{3}(u\partial_x + \partial_x u) - \beta\partial_x{}^3 - \gamma\partial_x{}^{-1}\right)\left(\partial_x{}^{-1}\frac{\delta H}{\delta\phi}\right), \qquad H(\phi) = \frac{\phi_x^2}{2}. \tag{17}$$

By multiplying $\partial_x{}^{-1}$, we find

$$\phi_t = \partial_x{}^{-1}\left(\frac{\alpha}{3}(u\partial_x + \partial_x u) - \beta\partial_x{}^3 - \gamma\partial_x{}^{-1}\right)\partial_x{}^{-1}\frac{\delta H}{\delta\phi}.$$

As above, the expression (17) can imply the desired norm preservation property (see Theorem 3.2). But in this case the expression (17) still explicitly includes the inverse operator $\partial_x{}^{-1}$. Fortunately it can be avoided if we substitute $\delta H/\delta\phi = -\phi_{xx}$ into (17), and expand the right hand side to obtain

$$\phi_{xt} = -\frac{\alpha}{3}(u\partial_x + \partial_x u)\phi_x + \beta\phi_{xxxx} + \gamma\phi. \tag{18}$$

We will do exactly the same thing later in discrete setting to eliminate discrete inverse operators.

The skew-symmetry of the operators in the Hamiltonian and variational forms are summarized in the following lemma.

Lemma 3.1. *If $u(t,\cdot) \in X$, then the operator $\left(\frac{\alpha}{3}(u\partial_x + \partial_x u) - \beta\partial_x{}^3 - \gamma\partial_x{}^{-1}\right)$ is skew-symmetric in the sense that for any $v, w \in \overline{X}$, it holds*

$$\int_0^L v_x\left(\left(\frac{\alpha}{3}(u\partial_x + \partial_x u) - \beta\partial_x{}^3 - \gamma\partial_x{}^{-1}\right)w_x\right)\,\mathrm{d}x$$
$$= -\int_0^L \left(\left(\frac{\alpha}{3}(u\partial_x + \partial_x u) - \beta\partial_x{}^3 - \gamma\partial_x{}^{-1}\right)v_x\right)w_x\,\mathrm{d}x.$$

Proof. The skew-symmetry of $(u\partial_x + \partial_x u)$ is well-known (see, for example, [18]). Since ${\partial_x}^3$ is obviously skew-symmetric, it remains to show the skew-symmetry of ${\partial_x}^{-1}$. From the skew-symmetry of ∂_x,

$$\int_0^L v_x {\partial_x}^{-1} w_x \ \mathrm{d}x = \int_0^L v_x w \ \mathrm{d}x = -\int_0^L v w_x \ \mathrm{d}x = -\int_0^L ({\partial_x}^{-1} v_x) w_x \ \mathrm{d}x.$$

□

We can prove the energy and norm conservation laws based on the expression (16) and (17) as follows.

Theorem 3.2. *Let $\phi \in \overline{X}$ be a solution to* (15)*. Then it preserves the following quantities:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L G \ \mathrm{d}x = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_0^L H \ \mathrm{d}x = 0.$$

Proof. The conservation of the energy G is immediate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L G \ \mathrm{d}x = \int_0^L \frac{\delta G}{\delta \phi}\phi_t \ \mathrm{d}x = \int_0^L \phi_{xt}\phi_t \ \mathrm{d}x = 0.$$

The second equality follows from the expression (16). We next prove the norm H conservation property:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L H \ \mathrm{d}x &= \int_0^L \frac{\delta H}{\delta \phi}\phi_t \ \mathrm{d}x \\
&= -\int_0^L \left({\partial_x}^{-1}\frac{\delta H}{\delta \phi}\right)\phi_{xt} \ \mathrm{d}x \\
&= -\int_0^L \left({\partial_x}^{-1}\frac{\delta H}{\delta \phi}\right)\left(\frac{\alpha}{3}(u\partial_x + \partial_x u) - \beta{\partial_x}^3 - \gamma{\partial_x}^{-1}\right)\left({\partial_x}^{-1}\frac{\delta H}{\delta \phi}\right) \mathrm{d}x \\
&= 0.
\end{aligned}$$

Note that ${\partial_x}^{-1}\delta H/\delta\phi = -\phi_x$. The first equality is just the chain rule. The second is from the skew-symmetry of ∂_x, and the third is from the variational structure (17). The last follows from the skew-symmetry of $\left(\frac{\alpha}{3}(u\partial_x + \partial_x u) - \beta{\partial_x}^3 - \gamma{\partial_x}^{-1}\right)$. □

Based on these Hamiltonian and variational structures, we derive energy-preserving and norm-preserving finite difference schemes in the following two subsections.

3.2 The new energy-preserving scheme

In this subsection a finite difference scheme that conserves both the total mass and the energy is presented. We define a discrete version of the energy $G = \alpha\phi_x^3/6 + \beta\phi_{xx}^2/2 + \gamma\phi^2/2$, and accordingly a "discrete variational derivative" that approximates $\delta G/\delta\phi = -\alpha\phi_x\phi_{xx} + \beta\phi_{xxxx} + \gamma\phi$ by

$$\begin{aligned}
G_k^{(n)} &= \frac{\alpha}{6}(\delta_x^{\langle 1\rangle}\Phi_k^{(n)})^3 + \frac{\beta}{2}(\delta_x^{\langle 2\rangle}\Phi_k^{(n)})^2 + \frac{\gamma}{2}(\Phi_k^{(n)})^2, \\
\frac{\delta G}{\delta(\mathbf{\Phi}^{(n+1)}, \mathbf{\Phi}^{(n)})_k} &= -\frac{\alpha}{6}\delta_x^{\langle 1\rangle}\left((\delta_x^{\langle 1\rangle}\Phi_k^{(n+1)})^2 + (\delta_x^{\langle 1\rangle}\Phi_k^{(n+1)})(\delta_x^{\langle 1\rangle}\Phi_k^{(n)}) + (\delta_x^{\langle 1\rangle}\Phi_k^{(n)})^2\right) \\
&\qquad + \beta\delta_x^{\langle 4\rangle}\Phi_k^{(n+\frac{1}{2})} + \gamma\Phi_k^{(n+\frac{1}{2})}.
\end{aligned}$$

We can easily check the following key equality:

$$\frac{1}{\Delta t}\sum_{k=0}^{N-1}(G_k^{(n+1)} - G_k^{(n)})\Delta x = \sum_{k=0}^{N-1}\frac{\delta G}{\delta(\mathbf{\Phi}^{(n+1)},\mathbf{\Phi}^{(n)})_k}\frac{\Phi_k^{(n+1)}-\Phi_k^{(n)}}{\Delta t}\Delta x. \tag{19}$$

Using the above discrete variational derivative, we define the energy-preserving scheme as follows.

Scheme 3 (Energy-preserving finite difference scheme)**.** *Given an initial approximate solution* $\mathbf{\Phi}^{(0)} \in \overline{X}_{\mathrm{d}}$, *we compute* $\mathbf{\Phi}^{(n)}$ $(n = 1, 2, \ldots)$ *by*

$$\delta_x^{\langle 1\rangle}\frac{\Phi_k^{(n+1)}-\Phi_k^{(n)}}{\Delta t} = \frac{\delta G}{\delta(\mathbf{\Phi}^{(n+1)},\mathbf{\Phi}^{(n)})_k} \qquad (k = 0, \ldots, N-1). \tag{20}$$

Eq. (20) corresponds to the Hamiltonian structure (16). Notice that in this scheme we claimed that the initial solution $\mathbf{\Phi}^{(0)}$ should satisfy $\mathbf{\Phi}^{(0)} \in \overline{X}_{\mathrm{d}}$, which is natural in view of the continuous case (14). The initial value $\mathbf{\Phi}^{(0)}$ can be generated either by integrating $u(0, x)$ analytically or by the summing of $\boldsymbol{U}^{(0)}$ via (10). Scheme 3 defines a map $\overline{X}_{\mathrm{d}} \to \overline{X}_{\mathrm{d}}$ in the following sense.

Lemma 3.3. *Consider one step of Scheme 3 starting from* $\mathbf{\Phi}^{(n)} \in \overline{X}_{\mathrm{d}}$*. If* $\mathbf{\Phi}^{(n+1)}$ *solves* (20)*, then* $\mathbf{\Phi}^{(n+1)} \in \overline{X}_{\mathrm{d}}$*.*

Proof. By applying $\sum_{k=0}^{N-1}(\cdot)\,\Delta x$ to both sides of (20), we find $\sum_{k=0}^{N-1}\Phi_k^{(n+\frac{1}{2})}\Delta x = 0$. Then the assumption $\mathbf{\Phi}^{(n)} \in \overline{X}_{\mathrm{d}}$ implies $\mathbf{\Phi}^{(n+1)} \in \overline{X}_{\mathrm{d}}$. □

Numerical solutions by Scheme 3 preserve both the total mass and the energy.

Theorem 3.4 (Scheme 3 : Conservation of the total mass and the energy)**.** *Under the periodic boundary condition, the numerical solutions by Scheme 3 conserve the total mass and the energy:*

$$\sum_{k=0}^{N-1}\delta_x^{\langle 1\rangle}\Phi_k^{(n)}\Delta x = \sum_{k=0}^{N-1}\delta_x^{\langle 1\rangle}\Phi_k^{(0)}\Delta x = 0, \qquad \sum_{k=0}^{N-1}G_k^{(n)}\Delta x = \sum_{k=0}^{N-1}G_k^{(0)}\Delta x, \qquad n = 0, 1, \ldots.$$

Proof. The conservation of the total mass follows from the property of the difference operator $\delta_x^{\langle 1\rangle}$ and the discrete periodic boundary condition. We prove the conservation of the energy:

$$\begin{aligned}
&\frac{1}{\Delta t}\sum_{k=0}^{N-1}(G_k^{(n+1)} - G_k^{(n)})\Delta x \\
=&\sum_{k=0}^{N-1}\frac{\delta G}{\delta(\mathbf{\Phi}^{(n+1)},\mathbf{\Phi}^{(n)})_k}\frac{\Phi_k^{(n+1)}-\Phi_k^{(n)}}{\Delta t}\Delta x \\
=&\sum_{k=0}^{N-1}\left(\delta_x^{\langle 1\rangle}\left(\frac{\Phi_k^{(n+1)}-\Phi_k^{(n)}}{\Delta t}\right)\right)\left(\frac{\Phi_k^{(n+1)}-\Phi_k^{(n)}}{\Delta t}\right)\Delta x \\
=&0.
\end{aligned}$$

The first equality is from (19), the second is from (20), and the third follows from the skew-symmetry of $\delta_x^{\langle 1\rangle}$. □

3.3 The new norm-preserving scheme

In this subsection a finite difference scheme that conserves both the total mass and the norm is presented. The story becomes slightly more complicated than the previous case.

We define a discrete version of norm $H = \phi_x^2/2$, and accordingly a "discrete variational derivative" that approximates $\delta H/\delta\phi = -\phi_{xx}$ by

$$H_k^{(n)} = \frac{(\delta_x^{\langle 1\rangle}\Phi_k^{(u)})^2}{2}, \qquad \frac{\delta H}{\delta(\mathbf{\Phi}^{(n+1)}, \mathbf{\Phi}^{(n)})_k} = -(\delta_x^{\langle 1\rangle})^2\Phi_k^{(n+\frac{1}{2})}. \tag{21}$$

Note that these satisfy the key equality:

$$\frac{1}{\Delta t}\sum_{k=0}^{N-1}(H_k^{(n+1)} - H_k^{(n)})\Delta x = \sum_{k=0}^{N-1}\frac{\delta H}{\delta(\mathbf{\Phi}^{(n+1)}, \mathbf{\Phi}^{(n)})_k}\frac{\Phi_k^{(n+1)} - \Phi_k^{(n)}}{\Delta t}\Delta x. \tag{22}$$

With this discrete variational derivative, we can formally define a prototype of the norm-preserving scheme, analogously to the continuous variational structure (17).

Scheme 4 (Norm-preserving finite difference scheme: a prototype)**.** *Given an initial approximate solution $\mathbf{\Phi}^{(0)} \in \overline{X}_{\mathrm{d}}$, we compute $\mathbf{\Phi}^{(n)}$ $(n = 1, 2, \ldots)$ by*

$$\delta_x^{\langle 1\rangle}\frac{\Phi_k^{(n+1)} - \Phi_k^{(n)}}{\Delta t} = \left(\frac{\alpha}{3}(U_k^{(n+\frac{1}{2})}\delta_x^{\langle 1\rangle} + \delta_x^{\langle 1\rangle}U_k^{(n+\frac{1}{2})}) - \beta\delta_x^{\langle 3\rangle} - \gamma\delta_x^{\langle -1\rangle}\right)\delta_x^{\langle -1\rangle}\frac{\delta H}{\delta(\mathbf{\Phi}^{(n+1)}, \mathbf{\Phi}^{(n)})_k}$$
$$(k = 0, \ldots, N-1), \tag{23}$$

where $U_k^{(n+\frac{1}{2})} = \delta_x^{\langle 1\rangle}\Phi_k^{(n+\frac{1}{2})}$, and $\delta_x^{\langle -1\rangle} : \overline{X}_{\mathrm{d}} \to \overline{X}_{\mathrm{d}}$ is a discrete inverse operator approximating ${\partial_x}^{-1}$ such that $\delta_x^{\langle -1\rangle}\delta_x^{\langle 1\rangle} = \delta_x^{\langle 1\rangle}\delta_x^{\langle -1\rangle} = \mathrm{id.}$ in $\overline{X}_{\mathrm{d}}$.

Note that here we used a new discrete inverse operator $\delta_x^{\langle -1\rangle}$, which is different from $\tilde{\delta_x}^{\langle -1\rangle}$ used in Yaguchi et al.'s scheme. Such an operator $\delta_x^{\langle -1\rangle}$ in fact exists.

Lemma 3.5. *There exists an operator $\delta_x^{\langle -1\rangle} : \overline{X}_{\mathrm{d}} \to \overline{X}_{\mathrm{d}}$ such that $\delta_x^{\langle -1\rangle}\delta_x^{\langle 1\rangle} = \delta_x^{\langle 1\rangle}\delta_x^{\langle -1\rangle} = \mathrm{id.}$ in $\overline{X}_{\mathrm{d}}$.*

Proof. The Moore–Penrose pseudo-inverse of $\delta_x^{\langle 1\rangle}$ is an example of $\delta_x^{\langle -1\rangle}$. In fact, the matrix expression of $\delta_x^{\langle 1\rangle}$, say $D^{\langle 1\rangle}$, is a circulant matrix which can be diagonalized via the discrete Fourier transform. The eigenvalues are $\mathrm{i}\sin(2\pi j/N)/\Delta x$ $(j = 0, \ldots, N-1)$, which becomes 0 for the eigenvector $(1, \ldots, 1)^\top$. This corresponds to $\partial_x 1 = 0$ (in other words, the vector is orthogonal to $\overline{X}_{\mathrm{d}}$). The Moore–Penrose pseudo-inverse of $D^{\langle 1\rangle}$, which we denote by $(D^{\langle 1\rangle})^+$, is also a circulant matrix with the same eigenvectors, and the corresponding eigenvalues are the reciprocals and the zero (see, for example, [23]). This means that both $(D^{\langle 1\rangle})^+D^{\langle 1\rangle}$ and $D^{\langle 1\rangle}(D^{\langle 1\rangle})^+$ have the zero eigenvalue for $(1, \ldots, 1)^\top$, while all other eigenvalues are equal to 1. This implies the claim. □

Given such an operator $\delta_x^{\langle -1\rangle}$, Scheme 4 is in fact conservative.

Lemma 3.6 (Scheme 4 : Conservation of the total mass and the norm)**.** *Under the periodic boundary condition, the numerical solutions by Scheme 4 conserve the total mass and the norm:*

$$\sum_{k=0}^{N-1}\delta_x^{\langle 1\rangle}\Phi_k^{(n)}\Delta x = \sum_{k=0}^{N-1}\delta_x^{\langle 1\rangle}\Phi_k^{(0)}\Delta x = 0, \qquad \sum_{k=0}^{N-1}H_k^{(n)}\Delta x = \sum_{k=0}^{N-1}H_k^{(0)}\Delta x, \qquad n = 0, 1, \ldots. \tag{24}$$

For the proof, we need the following lemma (cf. Lemma 3.1).

Lemma 3.7. *If $\boldsymbol{U} \in X_{\mathrm{d}}$, then an operator $\left(\frac{\alpha}{3}(U_k^{(n+\frac{1}{2})}\delta_x^{\langle 1\rangle} + \delta_x^{\langle 1\rangle}U_k^{(n+\frac{1}{2})}) - \beta\delta_x^{\langle 3\rangle} - \gamma\delta_x^{\langle -1\rangle}\right)$ is skew-symmetric in the sense that for any sequences $\boldsymbol{V}, \boldsymbol{W} \in \overline{X}_{\mathrm{d}}$,*

$$\begin{aligned}
&\sum_{k=0}^{N-1}(\delta_x^{\langle 1\rangle}V_k)\left(\left(\frac{\alpha}{3}(U_k^{(n+\frac{1}{2})}\delta_x^{\langle 1\rangle} + \delta_x^{\langle 1\rangle}U_k^{(n+\frac{1}{2})}) - \beta\delta_x^{\langle 3\rangle} - \gamma\delta_x^{\langle -1\rangle}\right)(\delta_x^{\langle 1\rangle}W_k)\right)\Delta x \\
&= -\sum_{k=0}^{N-1}\left(\left(\frac{\alpha}{3}(U_k^{(n+\frac{1}{2})}\delta_x^{\langle 1\rangle} + \delta_x^{\langle 1\rangle}U_k^{(n+\frac{1}{2})}) - \beta\delta_x^{\langle 3\rangle} - \gamma\delta_x^{\langle -1\rangle}\right)(\delta_x^{\langle 1\rangle}V_k)\right)(\delta_x^{\langle 1\rangle}W_k)\Delta x.
\end{aligned}$$

Proof. The skew-symmetry of $(U_k^{(n+\frac{1}{2})}\delta_x^{\langle 1\rangle} + \delta_x^{\langle 1\rangle}U_k^{(n+\frac{1}{2})})$ is straightforward (for example, see [18]). Since $\delta_x^{\langle 3\rangle}$ is also skew-symmetric, it remains to show the skew-symmetry of $\delta_x^{\langle -1\rangle}$. From the skew-symmetry of $\delta_x^{\langle 1\rangle}$,

$$\begin{aligned}
\sum_{k=0}^{N-1}(\delta_x^{\langle 1\rangle}V_k)\delta_x^{\langle -1\rangle}(\delta_x^{\langle 1\rangle}W_k)\Delta x &= \sum_{k=0}^{N-1}(\delta_x^{\langle 1\rangle}V_k)W_k\Delta x \\
= -\sum_{k=0}^{N-1}V_k\delta_x^{\langle 1\rangle}W_k\Delta x &= -\sum_{k=0}^{N-1}(\delta_x^{\langle -1\rangle}\delta_x^{\langle 1\rangle}V_k)\delta_x^{\langle 1\rangle}W_k\Delta x.
\end{aligned}$$

□

Proof. (Proof of Lemma 3.6). The conservation of the total mass is the same as before. The conservation of the norm can be shown as follows.

$$\begin{aligned}
&\frac{1}{\Delta t}\sum_{k=0}^{N-1}(H_k^{(n+1)} - H_k^{(n)})\Delta x \\
&= \sum_{k=0}^{N-1}\frac{\delta H}{\delta(\boldsymbol{\Phi}^{(n+1)}, \boldsymbol{\Phi}^{(n)})_k}\frac{\Phi_k^{(n+1)} - \Phi_k^{(n)}}{\Delta t}\Delta x \\
&= -\sum_{k=0}^{N-1}\left(\delta_x^{\langle -1\rangle}\frac{\delta H}{\delta(\boldsymbol{\Phi}^{(n+1)}, \boldsymbol{\Phi}^{(n)})_k}\right)\left(\delta_x^{\langle 1\rangle}\frac{\Phi_k^{(n+1)} - \Phi_k^{(n)}}{\Delta t}\right)\Delta x \\
&= -\sum_{k=0}^{N-1}\left(\delta_x^{\langle -1\rangle}\frac{\delta H}{\delta(\boldsymbol{\Phi}^{(n+1)}, \boldsymbol{\Phi}^{(n)})_k}\right) \\
&\qquad \times\left(\left(\frac{\alpha}{3}(U_k^{(n+\frac{1}{2})}\delta_x^{\langle 1\rangle} + \delta_x^{\langle 1\rangle}U_k^{(n+\frac{1}{2})}) - \beta\delta_x^{\langle 3\rangle} - \gamma\delta_x^{\langle -1\rangle}\right)\delta_x^{\langle -1\rangle}\frac{\delta H}{\delta(\boldsymbol{\Phi}^{(n+1)}, \boldsymbol{\Phi}^{(n)})_k}\right)\Delta x \\
&=0.
\end{aligned}$$

The first equality is from (22). In the second equality, we have used the assumption $\delta_x^{\langle 1\rangle}\delta_x^{\langle -1\rangle} = \mathrm{id.}$ and the summation-by-parts formula (9) (also note that $\delta H/\delta(\boldsymbol{\Phi}^{(n+1)}, \boldsymbol{\Phi}^{(n)}) \in \overline{X}_{\mathrm{d}}$). Then the scheme (23) and Lemma 3.7 were used. □

Scheme 4 is, however, not satisfactory as is in the following senses: (i) it still contains the inverse operator $\delta_x^{\langle -1\rangle}$, and (ii) it seems not straightforward to show for Scheme 4 a counterpart of Lemma 3.3. In order to work around these difficulties, we consider slightly modifying the scheme: by combining (23) with (21), we see that Scheme 4 is equivalent to

$$\delta_x^{\langle 1\rangle}\frac{\Phi_k^{(n+1)} - \Phi_k^{(n)}}{\Delta t} \tag{25}$$

$$= -\frac{\alpha}{3}(U_k^{(n+\frac{1}{2})}\delta_x^{\langle 1\rangle} + \delta_x^{\langle 3\rangle}U_k^{(n+\frac{1}{2})})\delta_x^{\langle 1\rangle}\Phi_k^{(n+\frac{1}{2})} + \beta\delta_x^{\langle 4\rangle}\Phi_k^{(n+\frac{1}{2})} + \gamma\delta_x^{\langle -1\rangle}\delta_x^{\langle 1\rangle}\Phi_k^{(n+\frac{1}{2})}. \tag{26}$$

Note that the assumption $\delta_x^{\langle -1\rangle}\delta_x^{\langle 1\rangle} = \mathrm{id.}$ applies only to those in $\overline{X}_\mathrm{d}$, and until $\Phi_k^{(n+\frac{1}{2})} \in \overline{X}_\mathrm{d}$ is proved, the operator $\delta_x^{\langle -1\rangle}\delta_x^{\langle 1\rangle}$ in the right hand side cannot be erased. But let us put this issue aside for the moment, and consider the following modified scheme, which corresponds to (18).

Scheme 5 (Norm-preserving finite difference scheme)**.** *Given an initial approximate solution* $\mathbf{\Phi}^{(0)} \in \overline{X}_\mathrm{d}$*, we compute* $\mathbf{\Phi}^{(n)}$ $(n = 1, 2, \dots)$ *by*

$$\delta_x^{\langle 1\rangle}\frac{\Phi_k^{(n+1)} - \Phi_k^{(n)}}{\Delta t} = -\frac{\alpha}{3}(U_k^{(n+\frac{1}{2})}\delta_x^{\langle 1\rangle} + \delta_x^{\langle 3\rangle}U_k^{(n+\frac{1}{2})})U_k^{(n+\frac{1}{2})} + \beta\delta_x^{\langle 4\rangle}\Phi_k^{(n+\frac{1}{2})} + \gamma\Phi_k^{(n+\frac{1}{2})}$$
$$(k = 0, \dots, N-1). \tag{27}$$

This scheme is completely free of the inverse operator $\delta_x^{\langle -1\rangle}$. Furthermore, the solution of Scheme 5 turns out to happily remain in $\overline{X}_\mathrm{d}$ as follows.

Lemma 3.8. *Assume* $\mathbf{\Phi}^{(n)} \in \overline{X}_\mathrm{d}$*, and consider one step of Scheme 5. Then* $\mathbf{\Phi}^{(n+1)} \in \overline{X}_\mathrm{d}$*.*

Proof. From (27), we obtain

$$\sum_{k=0}^{N-1}\left(\delta_x^{\langle 1\rangle}\frac{\Phi_k^{(n+1)} - \Phi_k^{(n)}}{\Delta t}\right)\Delta x$$
$$= \sum_{k=0}^{N-1}\left(-\frac{\alpha}{3}(U_k^{(n+\frac{1}{2})}\delta_x^{\langle 1\rangle} + \delta_x^{\langle 1\rangle}U_k^{(n+\frac{1}{2})})U_k^{(n+\frac{1}{2})} + \beta\delta_x^{\langle 3\rangle}\Phi_k^{(n+\frac{1}{2})} + \gamma\Phi_k^{(n+\frac{1}{2})}\right)\Delta x.$$

Since the left hand side, and the first and second terms of the right hand side vanish from the skew-symmetry of $\delta_x^{\langle 1\rangle}$, $(U_k^{(n+\frac{1}{2})}\delta_x^{\langle 1\rangle} + \delta_x^{\langle 1\rangle}U_k^{(n+\frac{1}{2})})$ and $\delta_x^{\langle 3\rangle}$, we get $\sum_{k=0}^{N-1}\Phi_k^{(n+\frac{1}{2})}\Delta x = 0$. This implies the claim. □

Consequently, the solution of Scheme 5 also satisfy (26), and thus it is also a solution of Scheme 4. This implies the conservation properties of Scheme 5; we summarize this in the next theorem.

Theorem 3.9. *Under the periodic boundary condition, the numerical solution by Scheme 5 enjoys the conservation properties* (24)*.*

4 Multi-symplectic integrator

In this section a new multi-symplectic formulation and the associated local conservation laws are shown. The multi-symplectic discretization is also proposed by means of the Preissman box scheme.

4.1 Multi-symplectic partial differential equations and their integrators

We start by briefly reviewing the concept of multi-symplecticity in a general context [3, 4, 16, 21]. A partial differential equation $F(u, u_t, u_x, u_{tx}, \dots) = 0$ is said to be multi-symplectic if it can be written as a system of first order equations:

$$Mz_t + Kz_x = \nabla_z S(z), \tag{28}$$

with $z \in \mathbb{R}^d$ a vector of state variables, typically consisting of the original variable u as one of its components. The constant $d \times d$ -matrices M and K are skew-symmetric, and S is a smooth function depending on z. A key observation for the multi-symplectic formulation (28) is that it has a multi-symplectic conservation law:

$$\partial_t \omega + \partial_x \kappa = 0, \tag{29}$$

where ω and κ are differential two forms:

$$\omega = \mathrm{d}z \wedge M\mathrm{d}z, \qquad \kappa = \mathrm{d}z \wedge K\mathrm{d}z.$$

Another key property is the following conservation laws. The system (28) has local conservation laws:

$$\partial_t E(z) + \partial_x F(z) = 0, \qquad \partial_t I(z) + \partial_x G(z) = 0, \tag{30}$$

where $E(z)$, $F(z)$, $I(z)$, and $G(z)$ are the density functions defined by

$$\begin{aligned} E(z) &= S(z) - \frac{1}{2} z_x^\top K^\top z, \qquad & F(z) &= \frac{1}{2} z_t^\top K^\top z, \\ G(z) &= S(z) - \frac{1}{2} z_t^\top M^\top z, \qquad & I(z) &= \frac{1}{2} z_x^\top M^\top z. \end{aligned}$$

Thus integrating the densities $E(z)$ and $I(z)$ over the spatial domain (under appropriate assumptions on $F(z)$ and $G(z)$) leads to the global invariants:

$$\mathcal{E}(z) = \int E(z)\ \mathrm{d}x, \qquad \mathcal{I}(z) = \int I(z)\ \mathrm{d}x.$$

A scheme is called to be multi-symplectic if it satisfies some discrete version of the multi-symplectic conservation law (29). As multi-symplectic schemes, the Preissman box scheme and the Euler box scheme are widely known. We adopt the Preissman box scheme in this paper. The Preissman box scheme (also known as the centered box scheme) was first introduced by Preissman in 1960, and widely used in hydraulics. It reads[2]

$$M\delta_t^+ Z_{k+\frac{1}{2}}^{(n)} + K\delta_x^+ Z_k^{(n+\frac{1}{2})} = \nabla_z S(Z_{k+\frac{1}{2}}^{(n+\frac{1}{2})}), \tag{31}$$

where the sub-/super-indices $k + \frac{1}{2}$ and $(n + \frac{1}{2})$ mean the abbreviations:

$$\begin{aligned} Z_k^{(n+\frac{1}{2})} &= \frac{Z_k^{(n+1)} + Z_k^{(n)}}{2}, \quad Z_{k+\frac{1}{2}}^{(n)} = \frac{Z_{k+1}^{(n)} + Z_k^{(n)}}{2}, \\ Z_{k+\frac{1}{2}}^{(n+\frac{1}{2})} &= \frac{1}{4}\left(Z_k^{(n)} + Z_{k+1}^{(n)} + Z_k^{(n+1)} + Z_{k+1}^{(n+1)}\right). \end{aligned}$$

In what follows we use the same abbreviation also for other quantities (i.e., all the quantities with $+1/2$ index mean the averages; not the quantities on staggered grid).

The scheme was proved to be multi-symplectic by Bridges–Reich [4]; the result is summarized in the next theorem. For the detail, including the definition of discrete symbols, readers may refer [4].

[2] In the context of multi-symplectic integration, the discretization of z is usually denoted by $Z^{k,n}$ and the numerical solution by $U^{k,n}$. But in this paper we use the same notation $Z_k^{(n)}$ or $U_k^{(n)}$ as in the finite difference schemes.

Theorem 4.1 ([4]). *Let us define discrete differential two forms by*

$$\omega_{k,n}(U_k^{(n)}, V_k^{(n)}) = \langle MU_k^{(n)}, V_k^{(n)}\rangle, \qquad \kappa_{k,n}(U_k^{(n)}, V_k^{(n)}) = \langle KU_k^{(n)}, V_k^{(n)}\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean scalar product on $\mathbb{R}^d$, and suppose that $U_k^{(n)}$ and $V_k^{(n)}$ are any two solutions of the discrete variational equation

$$M\delta_t^+ \mathrm{d}Z_{k+\frac{1}{2}}^{(n)} + K\delta_x^+ \mathrm{d}Z_k^{(n+\frac{1}{2})} = D_{zz}S(Z_{k+\frac{1}{2}}^{(n+\frac{1}{2})})\mathrm{d}Z_{k+\frac{1}{2}}^{(n+\frac{1}{2})}$$

associated with (31)*. The Preissman box scheme* (31) *is multi-symplectic in the sense that the discretization satisfies*

$$\frac{\omega_{k+\frac{1}{2},n+1} - \omega_{k+\frac{1}{2},n}}{\Delta t} + \frac{\omega_{k+1,n+\frac{1}{2}} - \omega_{k,n+\frac{1}{2}}}{\Delta x} = 0.$$

Remark 3. It is sometimes possible to derive schemes preserving discrete versions of the local conservation laws (30), when $S(z)$ is quadratic [8]. Unfortunately, however, it is not the case for the Ostrovsky equation, as shown below. □

4.2 A multi-symplectic formulation and an integrator for the Ostrovsky equation

In this subsection, a multi-symplectic formulation for the Ostrovsky equation is presented. With $z = (\phi, u, v, w)^\top$, it is given by

$$\begin{pmatrix} 0 & -1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} z_t + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} z_x = \begin{pmatrix} -\gamma\phi \\ w + V'(u) \\ \dfrac{1}{\beta}v \\ u \end{pmatrix}, \tag{32}$$

where $S(z) = uw + V(u) + v^2/2\beta - \gamma\phi^2/2$ and $V(u) = -\alpha u^3/6$. This formulation is motivated by the multi-symplectic formulation for the KdV equation [1]. In fact, for $\gamma = 0$, (32) reduces to the one in [1].

From (32), the density functions E and I defined above are explicitly given by

$$E(z) = S(z) - \frac{1}{2}z_x^\top K^\top z = -\frac{\alpha}{6} + \frac{\beta}{2}u_x^2 - \frac{\gamma}{2}\phi^2 + uw - \frac{1}{2}\left[\phi_x w + u_x v - v_x u - w_x \phi\right],$$

$$I(z) = \frac{1}{2}z_x^\top M^\top z = \frac{1}{4}(\phi_x u - u_x \phi).$$

Under periodic (or vanishing) boundary condition, we have the following two global conserved quantities:

$$\mathcal{E}(z) = -\int \left(\frac{\alpha}{6}u^3 + \frac{\beta}{2}u_x^2 + \frac{\gamma}{2}\phi^2\right)\,\mathrm{d}x, \qquad \mathcal{I}(z) = \frac{1}{2}\int u^2\,\mathrm{d}x.$$

Based on this expression, we derive the following multi-symplectic scheme.

Scheme 6 (The multi-symplectic scheme)**.** *Given an initial approximate solution $\mathbf{\Phi}^{(0)} \in \overline{X}_\mathrm{d}$, we compute $\mathbf{\Phi}^{(n)}$ $(n = 1, 2, \dots)$ by*

$$\frac{U_{k+\frac{1}{2}}^{(n+1)} - U_{k+\frac{1}{2}}^{(n)}}{\Delta t} + \alpha U_{k+\frac{1}{2}}^{(n+\frac{1}{2})}\frac{U_{k+1}^{(n+\frac{1}{2})} - U_k^{(n+\frac{1}{2})}}{\Delta x}$$
$$- \beta\frac{U_{k+2}^{(n+\frac{1}{2})} - 3U_{k+1}^{(n+\frac{1}{2})} + 3U_k^{(n+\frac{1}{2})} - U_{k-1}^{(n+\frac{1}{2})}}{(\Delta x)^3} = \gamma\Phi_{k+\frac{1}{2}}^{(n+\frac{1}{2})} \quad (k = 0, \dots, N-1),$$

where $U_{k+\frac{1}{2}}^{(n+\frac{1}{2})} = (\Phi_{k+1}^{(n+\frac{1}{2})} - \Phi_k^{(n+\frac{1}{2})})/\Delta x$.

5 Conservative Galerkin schemes

In this section, we propose conservative Galerkin (finite element) schemes for the Ostrovsky equation. As a framework to derive conservative Galerkin schemes, the DPDM [17] is known, which also fully utilizes Hamiltonian or variational structures. But unfortunately one can not apply the DPDM directly to the Ostrovsky equation due to the non-local operator ${\partial_x}^{-1}$. Even worse, the Hamiltonian/variational structures with the potential $\phi = {\partial_x}^{-1}u$ are not so useful in this case, since that would require expensive smooth elements.

We circumvent this situation by incorporating the idea of L^2-projection. We here only show the resulting schemes, and leave the discussion on the general framework to our future work [19]. Below, we first consider two weak formulations for the energy and the norm preservations, respectively. Then conservative Galerkin schemes are shown based on these formulations.

5.1 Weak formulations

5.1.1 A weak formulation for the energy preservation

Recall that the energy function G is defined by $G(\phi, \phi_x, \phi_{xx}) = \alpha\phi_x^3/6 + \beta\phi_{xx}^2/2 + \gamma\phi^2/2$, and the partial derivatives are

$$\frac{\partial G}{\partial \phi} = \gamma\phi, \qquad \frac{\partial G}{\partial \phi_x} = \frac{\alpha}{2}\phi_x^2, \qquad \frac{\partial G}{\partial \phi_{xx}} = \beta\phi_{xx}.$$

Then we consider the following weak formulation: Find $\phi(t,\cdot), p(t,\cdot) \in H^1(\mathbb{S})$ such that for any $v_1, v_2 \in H^1(\mathbb{S})$,

$$(\phi_{xt}, v_1) = \left(\frac{\partial G}{\partial \phi}, v_1\right) + \left(\frac{\partial G}{\partial \phi_x}, (v_1)_x\right) - (p, (v_1)_x), \tag{33}$$

$$(p, v_2) = -\left(\frac{\partial G}{\partial \phi_{xx}}, (v_2)_x\right). \tag{34}$$

The symbol $\mathbb{S}$ means the torus of length L, and $H^1(\mathbb{S})$ is the standard first order Sobolev space on the torus (we often drop $(\mathbb{S})$ where no confusion occurs). From the weak forms (33), (34), the desired conservation property can be deduced as shown in the next theorem.

Theorem 5.1 (Energy conservation property of the weak forms (33), (34))**.** *Suppose $\phi(t,\cdot)$, $p(t,\cdot) \in H^1(\mathbb{S})$ are the solutions of the weak forms* (33), (34)*. Also assume that $\phi_t(t,\cdot)$, $\phi_{xt}(t,\cdot) \in H^1(\mathbb{S})$, then it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L G \;\mathrm{d}x = 0, \qquad \text{with} \quad G(\phi, \phi_x, \phi_{xx}) = \frac{\alpha}{6}\phi_x^3 + \frac{\beta}{2}\phi_{xx}^2 + \frac{\gamma}{2}\phi^2.$$

Proof.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L G(\phi, \phi_x, \phi_{xx}) \;\mathrm{d}x &= \left(\frac{\partial G}{\partial \phi}, \phi_t\right) + \left(\frac{\partial G}{\partial \phi_x}, \phi_{xt}\right) + \left(\frac{\partial G}{\partial \phi_{xx}}, \phi_{xxt}\right) \\ &= \left(\frac{\partial G}{\partial \phi}, \phi_t\right) + \left(\frac{\partial G}{\partial \phi_x}, \phi_{xt}\right) - (p, \phi_{xt}) \\ &= (\phi_{xt}, \phi_x) \\ &= 0.\end{aligned}$$

The first equality is just the chain rule. The second and third equalities follow from (34) with $v_2 = \phi_{xt}$ and (33) with $v_1 = \phi_t$, respectively. The last is from the skew-symmetry of ∂_x. □

5.1.2 A weak formulation for the norm preservation

We next consider the following weak formulation: Find $\phi(t,\cdot), p(t,\cdot) \in H^1(\mathbb{S})$ such that for any $v_1, v_2 \in H^1(\mathbb{S})$,

$$(\phi_{xt}, v_1) = \left(\frac{\alpha}{2}\phi_x^2, (v_1)_x\right) - (p, (v_1)_x) + (\gamma\phi, v_1)\,, \tag{35}$$

$$(p, v_2) = -\left(\beta\phi_{xx}, (v_2)_x\right). \tag{36}$$

From the weak forms (35), (36), the desired conservation property can be deduced as shown in the next theorem.

Theorem 5.2 (Norm conservation property of the weak forms (35), (36))**.** *Suppose $\phi(t,\cdot)$, $p(t,\cdot) \in H^1(\mathbb{S})$ are the solutions of the weak forms (35), (36). Also assume that $\phi_x(t,\cdot)$, $\phi_{xx}(t,\cdot) \in H^1(\mathbb{S})$, then it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L H\ \mathrm{d}x = 0, \qquad \text{with} \qquad H = \frac{\phi_x^2}{2}.$$

Proof.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L H\ \mathrm{d}x = (\phi_{xt}, \phi_x)\\
=&\left(\frac{\alpha}{2}\phi_x^2, \phi_{xx}\right) - (p, \phi_{xx}) + (\gamma\phi, \phi_x)\\
=&\left(\frac{\alpha}{2}\phi_x^2, \phi_{xx}\right) + (\beta\phi_{xx}, \phi_{xxx}) + (\gamma\phi, \phi_x)\\
=&0.
\end{aligned}$$

The first equality is just the chain rule. The second and third equalities follow from (35) with $v_1 = \phi_x$ and (36) with $v_2 = \phi_{xx}$, respectively. The last is from the periodic boundary condition and the skew-symmetry of ∂_x. □

5.2 An energy-preserving Galerkin scheme

In this subsection, an energy-preserving Galerkin scheme based on the weak forms (33) and (34) is constructed and its properties are shown. Before presenting schemes, we define the L^2-projection operator $\mathcal{P}_W$ onto the finite-dimensional approximation space $W \subseteq H^1 \subset L^2$. We also denote $\mathcal{P}_W u_x$ by $\mathcal{D}_W u$ for convenience. Roughly speaking, $(\mathcal{D}_W)^p$ $(p \geq 1)$ is the operator that approximates ${\partial_x}^p$. As for the above operators, the following formulas are straightforward.

Lemma 5.3. *For any $u \in L^2$ and $v \in W$, it holds*

$$(u, v) = (\mathcal{P}_W u, v)\,, \tag{37}$$

and for any $u \in H^1$ and $v \in W$, it holds

$$((\mathcal{D}_W)^p u, v) = \left(((\mathcal{D}_W)^{p-1}u)_x, v\right) \qquad (p \geq 1). \tag{38}$$

Utilizing the above operator, we define the following Galerkin scheme. Let $S \subset H^1(\mathbb{S})$ be an appropriately chosen trial space, and $W \subset H^1(\mathbb{S})$ a test space. For example, we can simply choose the standard periodic piecewise linear function space. In what follows, we basically assume this.

Recall that the energy function $G(\phi, \phi_x, \phi_{xx})$ is defined as $G = \alpha\phi_x^3/6 + \beta\phi_{xx}^2/2 + \gamma\phi^2/2$, and the corresponding partial derivatives are $\partial G/\partial\phi = \gamma\phi, \partial G/\partial\phi_x = \frac{\alpha}{2}\phi_x^2, \partial G/\partial\phi_{xx} = \beta\phi_{xx}$. In view of this, we define discrete partial derivatives by

$$
\begin{aligned}
\frac{\partial G_{\mathrm{d}}}{\partial(\phi^{(n+1)}, \phi^{(n)})} &= \gamma\frac{\phi^{(n+1)} + \phi^{(n)}}{2}, \\
\frac{\partial G_{\mathrm{d}}}{\partial(\mathcal{D}_W\phi_x^{(n+1)}, \mathcal{D}_W\phi_x^{(n)})} &= \frac{\alpha}{2}\frac{(\mathcal{D}_W\phi^{(n+1)})^2 + (\mathcal{D}_W\phi^{(n+1)})(\mathcal{D}_W\phi^{(n)}) + (\mathcal{D}_W\phi^{(n)})^2}{3}, \\
\frac{\partial G_{\mathrm{d}}}{\partial((\mathcal{D}_W)^2\phi_{xx}^{(n+1)}, (\mathcal{D}_W)^2\phi_{xx}^{(n)})} &= \beta\frac{(\mathcal{D}_W)^2\phi^{(n+1)} + (\mathcal{D}_W)^2\phi^{(n)}}{2}.
\end{aligned}
$$

It is easy to check that they satisfy the following discrete chain rule

$$
\begin{aligned}
&\frac{1}{\Delta t}\int_0^L \left(G(\phi^{(n+1)}, \mathcal{D}_W\phi^{(n+1)}, (\mathcal{D}_W)^2\phi^{(n+1)}) - G(\phi^{(n)}, \mathcal{D}_W\phi^{(n)}, (\mathcal{D}_W)^2\phi^{(n)})\right) \mathrm{d}x \\
&= \left(\frac{\partial G_{\mathrm{d}}}{\partial(\phi^{(n+1)}, \phi^{(n)})}, \frac{\phi^{(n+1)} - \phi^{(n)}}{\Delta t}\right) + \left(\frac{\partial G_{\mathrm{d}}}{\partial(\mathcal{D}_W\phi^{(n+1)}, \mathcal{D}_W\phi^{(n)})}, \frac{\mathcal{D}_W\phi^{(n+1)} - \mathcal{D}_W\phi^{(n)}}{\Delta t}\right) \\
&\quad + \left(\frac{\partial G_{\mathrm{d}}}{\partial((\mathcal{D}_W)^2\phi^{(n+1)}, (\mathcal{D}_W)^2\phi^{(n)})}, \frac{(\mathcal{D}_W)^2\phi^{(n+1)} - (\mathcal{D}_W)^2\phi^{(n)}}{\Delta t}\right). \qquad (39)
\end{aligned}
$$

Scheme 7 (Energy-preserving Galerkin scheme)**.** *Suppose $\phi^{(n)}(x) \in S$ is given. Find $\phi^{(n+1)} \in S$ and $p^{(n+\frac{1}{2})} \in S$ such that, for any $v_1 \in W$, $v_2 \in W$,*

$$
\begin{aligned}
\left(\frac{\phi_x^{(n+1)} - \phi_x^{(n)}}{\Delta t}, v_1\right) &= \left(\frac{\partial G_{\mathrm{d}}}{\partial(\phi^{(n+1)}, \phi^{(n)})}, v_1\right) + \left(\mathcal{P}_W\frac{\partial G_{\mathrm{d}}}{\partial(\mathcal{D}_W\phi^{(n+1)}, \mathcal{D}_W\phi^{(n)})}, (v_1)_x\right) \\
&\quad - \left(p^{(n+\frac{1}{2})}, (v_1)_x\right) \qquad (40)
\end{aligned}
$$

$$
\left(p^{(n+\frac{1}{2})}, v_2\right) = -\left(\frac{\partial G_{\mathrm{d}}}{\partial((\mathcal{D}_W)^2\phi^{(n+1)}, (\mathcal{D}_W)^2\phi^{(n)})}, (v_2)_x\right). \qquad (41)
$$

Obviously (40) and (41) correspond to (33) and (34), respectively. Numerical solutions by Scheme 7 conserve both the total mass and the energy.

Theorem 5.4 (Scheme 7 : Conservation of the total mass and the energy)**.** *Assume that the trial and test spaces S, W are given such that $S \subseteq W$. Then Scheme 7 is conservative in the sense that*

$$
\begin{aligned}
&\int_0^L \phi_x^{(n+1)} \, \mathrm{d}x = \int_0^L \phi_x^{(n)} \, \mathrm{d}x = 0, \\
&\int_0^L G(\phi^{(n+1)}, \mathcal{D}_W\phi^{(n+1)}, (\mathcal{D}_W)^2\phi^{(n+1)}) \, \mathrm{d}x = \int_0^L G(\phi^{(n)}, \mathcal{D}_W\phi^{(n)}, (\mathcal{D}_W)^2\phi^{(n)}) \, \mathrm{d}x.
\end{aligned}
$$

Proof. Since

$$
\int_0^L \phi_x^{(n)} \, \mathrm{d}x = \left[\phi^{(n)}\right]_0^L = 0,
$$

the conservation of the total mass is obvious. We next prove the conservation of the energy:

$$
\frac{1}{\Delta t}\int_0^L (G(\phi^{(n+1)}, \mathcal{D}_W\phi^{(n+1)}, (\mathcal{D}_W)^2\phi^{(n+1)}) - G(\phi^{(n)}, \mathcal{D}_W\phi^{(n)}, (\mathcal{D}_W)^2\phi^{(n)})) \, \mathrm{d}x
$$

$$
\begin{aligned}
&= \left(\frac{\partial G_{\mathrm{d}}}{\partial(\phi^{(n+1)}, \phi^{(n)})}, \frac{\phi^{(n+1)} - \phi^{(n)}}{\Delta t} \right) + \left(\frac{\partial G_{\mathrm{d}}}{\partial(\mathcal{D}_W \phi^{(n+1)}, \mathcal{D}_W \phi^{(n)})}, \frac{\mathcal{D}_W \phi^{(n+1)} - \mathcal{D}_W \phi^{(n)}}{\Delta t} \right) \\
&\quad + \left(\frac{\partial G_{\mathrm{d}}}{\partial((\mathcal{D}_W)^2 \phi^{(n+1)}, (\mathcal{D}_W)^2 \phi^{(n)})}, \frac{(\mathcal{D}_W)^2 \phi^{(n+1)} - (\mathcal{D}_W)^2 \phi^{(n)}}{\Delta t} \right) \\
&= \left(\frac{\partial G_{\mathrm{d}}}{\partial(\phi^{(n+1)}, \phi^{(n)})}, \frac{\phi^{(n+1)} - \phi^{(n)}}{\Delta t} \right) + \left(\mathcal{P}_W \frac{\partial G_{\mathrm{d}}}{\partial(\mathcal{D}_W \phi^{(n+1)}, \mathcal{D}_W \phi^{(n)})}, \frac{\mathcal{D}_W \phi^{(n+1)} - \mathcal{D}_W \phi^{(n)}}{\Delta t} \right) \\
&\quad + \left(\frac{\partial G_{\mathrm{d}}}{\partial((\mathcal{D}_W)^2 \phi^{(n+1)}, (\mathcal{D}_W)^2 \phi^{(n)})}, \frac{(\mathcal{D}_W \phi^{(n+1)} - \mathcal{D}_W \phi^{(n)})_x}{\Delta t} \right) \\
&= \left(\frac{\partial G_{\mathrm{d}}}{\partial(\phi^{(n+1)}, \phi^{(n)})}, \frac{\phi^{(n+1)} - \phi^{(n)}}{\Delta t} \right) + \left(\mathcal{P}_W \frac{\partial G_{\mathrm{d}}}{\partial(\mathcal{D}_W \phi^{(n+1)}, \mathcal{D}_W \phi^{(n)})}, \frac{(\phi^{(n+1)} - \phi^{(n)})_x}{\Delta t} \right) \\
&\quad - \left(p^{(n+\frac{1}{2})}, \frac{\mathcal{D}_W \phi^{(n+1)} - \mathcal{D}_W \phi^{(n)}}{\Delta t} \right) \\
&= \left(\frac{\partial G_{\mathrm{d}}}{\partial(\phi^{(n+1)}, \phi^{(n)})}, \frac{\phi^{(n+1)} - \phi^{(n)}}{\Delta t} \right) + \left(\mathcal{P}_W \frac{\partial G_{\mathrm{d}}}{\partial(\mathcal{D}_W \phi^{(n+1)}, \mathcal{D}_W \phi^{(n)})}, \frac{(\phi^{(n+1)} - \phi^{(n)})_x}{\Delta t} \right) \\
&\quad - \left(p^{(n+\frac{1}{2})}, \frac{(\phi^{(n+1)} - \phi^{(n)})_x}{\Delta t} \right) \\
&= \left(\frac{\phi_x^{(n+1)} - \phi_x^{(n)}}{\Delta t}, \frac{\phi^{(n+1)} - \phi^{(n)}}{\Delta t} \right) \\
&= 0.
\end{aligned}
$$

The first equality is just the discrete chain rule (39). In the second equality, we have used (37) and (38) with $p = 2$. In the third equality, (38) with $p = 1$, and (41) with $v_2 = (\mathcal{D}_W \phi^{(n+1)} - \mathcal{D}_W \phi^{(n)})/\Delta t \in W$ were used. The fourth equality follows from (38) with $p = 1$, and the fifth from (40) with $v_1 = (\phi^{(n+1)} - \phi^{(n)})/\Delta t \in S \subseteq W$. □

5.3 A norm-preserving Galerkin scheme

In this subsection a norm-preserving Galerkin scheme based on the weak forms (35) and (36) is constructed, and its properties are shown.

Scheme 8 (Norm-preserving Galerkin scheme)**.** *Suppose $\phi^{(n)}(x) \in S$ is given. Find $\phi^{(n+1)} \in S$ and $p^{(n+\frac{1}{2})} \in S$ such that, for any $v_1 \in W$, $v_2 \in W$,*

$$
\left(\frac{\phi_x^{(n+1)} - \phi_x^{(n)}}{\Delta t}, v_1 \right) = \left(\frac{\alpha}{2} \left(\mathcal{D}_W \phi^{(n+\frac{1}{2})} \right)^2, (v_1)_x \right) - \left(p^{(n+\frac{1}{2})}, (v_1)_x \right) + \left(\gamma \phi^{(n+\frac{1}{2})}, v_1 \right), \tag{42}
$$

$$
\left(p^{(n+\frac{1}{2})}, v_2 \right) = - \left(\beta (\mathcal{D}_W)^2 \phi^{(n+\frac{1}{2})}, (v_2)_x \right), \tag{43}
$$

where $\phi^{(n+\frac{1}{2})} = (\phi^{(n+1)} + \phi^{(n)})/2$.

Eq. (42) and (43) correspond to (35) and (36), respectively. Then the scheme enjoys the following the total mass and the norm conservation properties.

Theorem 5.5 (Scheme 8 : Conservation of the total mass and the norm)**.** *Assume that the trial and test spaces S, W are set such that $S \subseteq W$. Then Scheme 8 is conservative in the sense that*

$$\int_0^L \phi_x^{(n+1)}\ \mathrm{d}x = \int_0^L \phi_x^{(n)}\ \mathrm{d}x = 0, \qquad \int_0^L H^{(n+1)}\ \mathrm{d}x = \int_0^L H^{(n)}\ \mathrm{d}x = 0,$$

where $H^{(n)} = (\mathcal{D}_W\phi^{(n)})^2/2$.

Proof. Similar to Scheme 7, the conservation of the total mass is obvious. The conservation of the norm goes as follows.

$$\begin{aligned}
&\frac{1}{\Delta t}\int_0^L (H^{(n+1)} - H^{(n)})\ \mathrm{d}x\\
&= \left(\frac{\mathcal{D}_W\phi^{(n+1)} - \mathcal{D}_W\phi^{(n)}}{\Delta t}, \frac{\mathcal{D}_W\phi^{(n+1)} + \mathcal{D}_W\phi^{(n)}}{2}\right)\\
&= \left(\frac{\phi_x^{(n+1)} - \phi_x^{(n)}}{\Delta t}, \mathcal{D}_W\phi^{(n+\frac{1}{2})}\right)\\
&= \left(\frac{\alpha}{2}\left(\mathcal{D}_W\phi^{(n+\frac{1}{2})}\right)^2, \left(\mathcal{D}_W\phi^{(n+\frac{1}{2})}\right)_x\right) - \left(p^{(n+\frac{1}{2})}, \left(\mathcal{D}_W\phi^{(n+\frac{1}{2})}\right)_x\right) + \left(\gamma\phi^{(n+\frac{1}{2})}, \mathcal{D}_W\phi^{(n+\frac{1}{2})}\right)\\
&= \left(\frac{\alpha}{2}\left(\mathcal{D}_W\phi^{(n+\frac{1}{2})}\right)^2, \left(\mathcal{D}_W\phi^{(n+\frac{1}{2})}\right)_x\right) - \left(p^{(n+\frac{1}{2})}, (\mathcal{D}_W)^2\phi^{(n+\frac{1}{2})}\right) + \left(\gamma\phi^{(n+\frac{1}{2})}, \phi_x^{(n+\frac{1}{2})}\right)\\
&= \left(\frac{\alpha}{2}\left(\mathcal{D}_W\phi^{(n+\frac{1}{2})}\right)^2, \left(\mathcal{D}_W\phi^{(n+\frac{1}{2})}\right)_x\right) - \left(\beta(\mathcal{D}_W)^2\phi^{(n+\frac{1}{2})}, (\mathcal{D}_W)^2\phi_x^{(n+\frac{1}{2})}\right)\\
&\quad + \left(\gamma\phi^{(n+\frac{1}{2})}, \phi_x^{(n+\frac{1}{2})}\right)\\
&=0.
\end{aligned}$$

□

6 Numerical examples

Table 1: Numerical schemes used in this paper.

Scheme 1	The Energy-Preserving Finite Difference Scheme by Yaguchi et al.
Scheme 2	The Norm-Preserving Finite Difference Scheme by Yaguchi et al.
Scheme 3	The Energy-Preserving Finite Difference Scheme
Scheme 5	The Norm-Preserving Finite Difference Scheme
Scheme 6	The Multi-Symplectic Scheme
Scheme 7	The Energy-Preserving Galerkin Scheme
Scheme 8	The Norm-Preserving Galerkin Scheme

We compare the proposed schemes numerically. The parameters are set to $\alpha = 1, \beta = -0.01, \gamma = -1$. The length of the spatial period was set to $L = 2\pi$. The initial condition was set to $u(0, x) = \sin(x)$, and accordingly the potential to $\phi(0, x) = -\cos(x)$. Hunter reported that oscillations were observed in this setting [14], and Yaguchi et al. confirmed this with the above parameters [24]. We took mesh sizes $\Delta t = 0.1$ and $\Delta x = L/N$, with $N = 101$ for Scheme 1, 2, 3, 5, 6, and $N = 102$ for Scheme 7, 8 (the difference is to avoid singularities in the coefficient matrices). Although in principle we can use non-uniform grids in the Galerkin schemes, we did not do this since in the following numerical examples non-uniform grids does not necessarily seem advantageous. We employed the P1 elements, for which it is easy to check the assumptions in Theorem 5.4 and 5.5. In order to solve the proposed implicit schemes, we used "fsolve" in MATLAB with the tolerances $TolFun$ and $TolX$ set to 10^{-16}.

Figs. 1 and 2 show the evolutions of the energies and the norms. As shown in the theorems in the previous sections, Scheme 1, 3 and 7 conserve the energies, and Scheme 2, 5 and 8 the norms. Scheme 6 conserve neither of them, but the deviations are small. In fact, as for the norm, the deviation by Scheme 6 is smaller than those by Scheme 1, 3 and 7 which do not conserve the norm. A similar result is also observed for the energy. Thus Scheme 6 seems to be the best of all the proposed schemes. We also notice that if we compare Scheme 1, 3 and 7, Scheme 3 and 7 seem better than Scheme 1, because the deviations of the energy are smaller. As for the energy-preserving schemes, in a similar way, Scheme 5 and 8 seem better than Scheme 2.

Next, let us evaluate each scheme in view of qualitative behaviors. The numerical solutions are shown in Figs. 3–9 respectively. All figures show the oscillation that Hunter reported and Yaguchi et al. observed. But as Yaguchi et al. observed, there exist small differences in the qualities of the solutions. The results by the conservative finite difference schemes (see Figs. 5 and 6) are smoother, especially in $t > 2$, than other schemes. From this standpoint, conservative finite difference schemes, especially the energy-preserving finite difference scheme (Scheme 3), are the most advantageous in all schemes. These results seem to support the effectiveness of the schemes expressed in the potential ϕ.

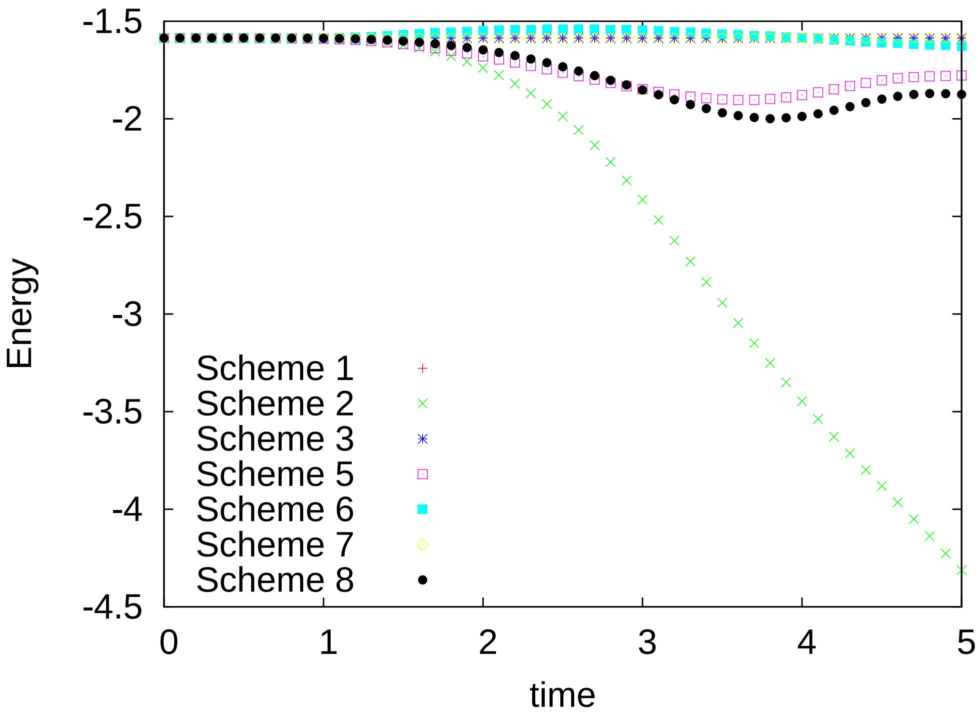

Figure 1: Evolution of the energy for each scheme.

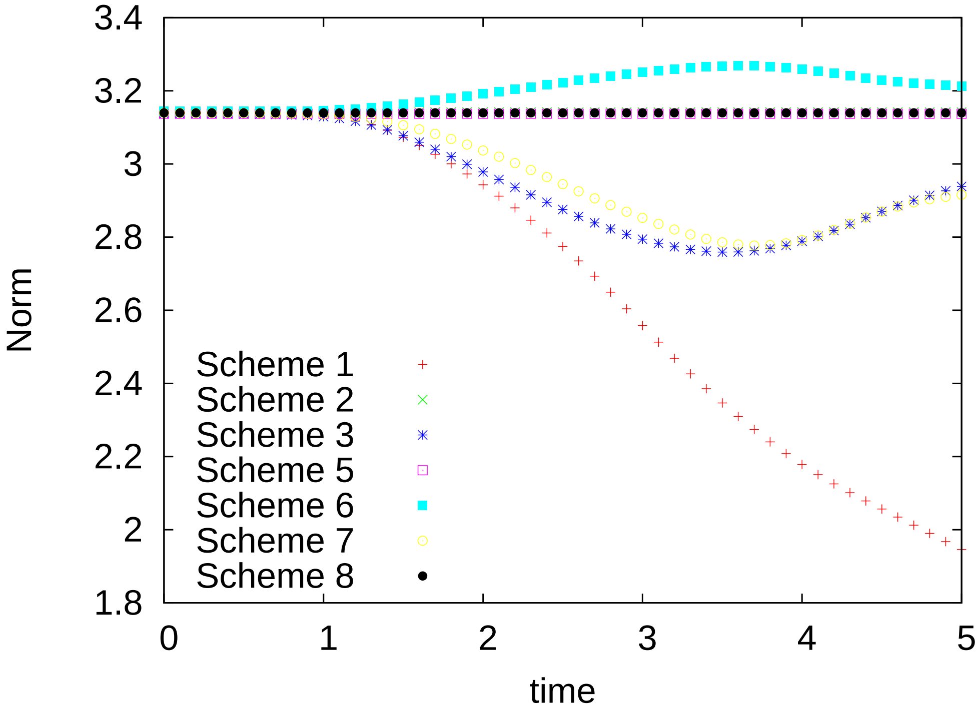

Figure 2: Evolution of the norm for each scheme.

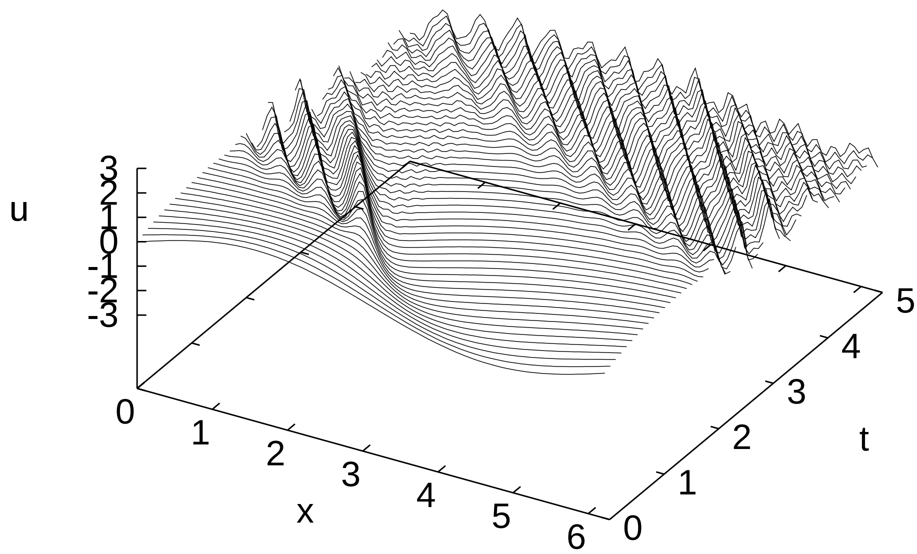

Figure 3: The numerical solution obtained by Scheme 1 (the energy-preserving finite difference scheme by Yaguchi et al.) with $N = 101$ and $\Delta t = 0.1$.

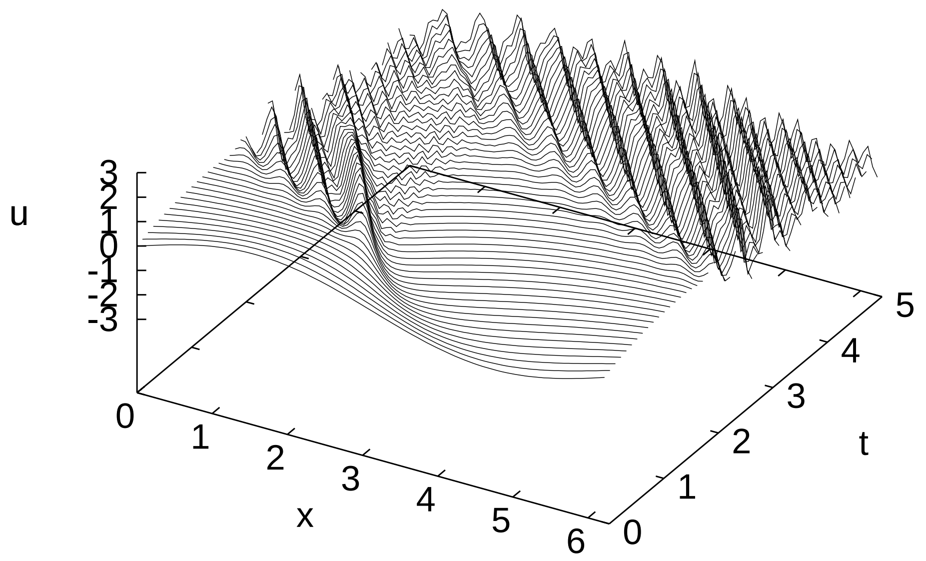

Figure 4: The numerical solution obtained by Scheme 2 (the norm-preserving finite difference scheme by Yaguchi et al.) with $N = 101$ and $\Delta t = 0.1$.

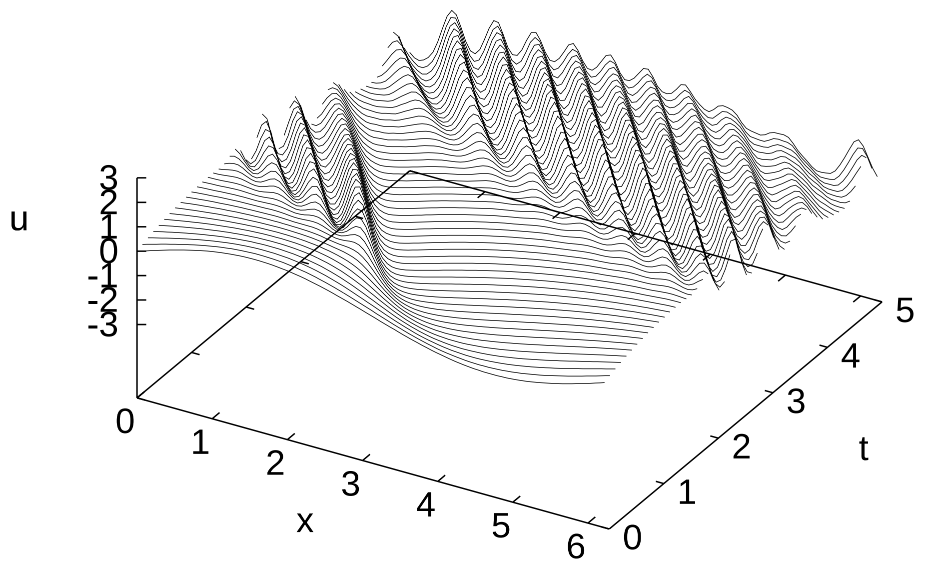

Figure 5: The numerical solution obtained by Scheme 3 (the energy-preserving finite difference scheme) with $N = 101$ and $\Delta t = 0.1$.

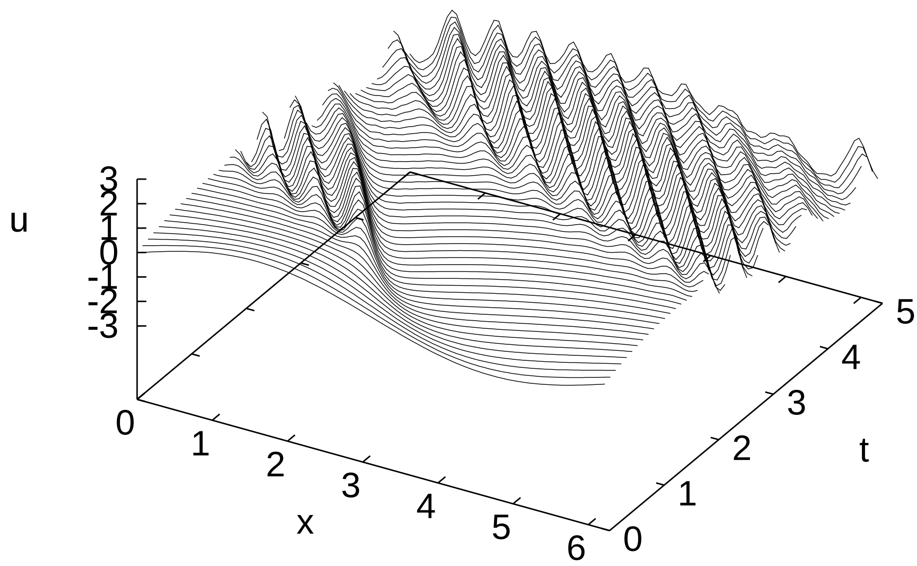

Figure 6: The numerical solution obtained by Scheme 5 (the norm-preserving finite difference scheme) with $N = 101$ and $\Delta t = 0.1$.

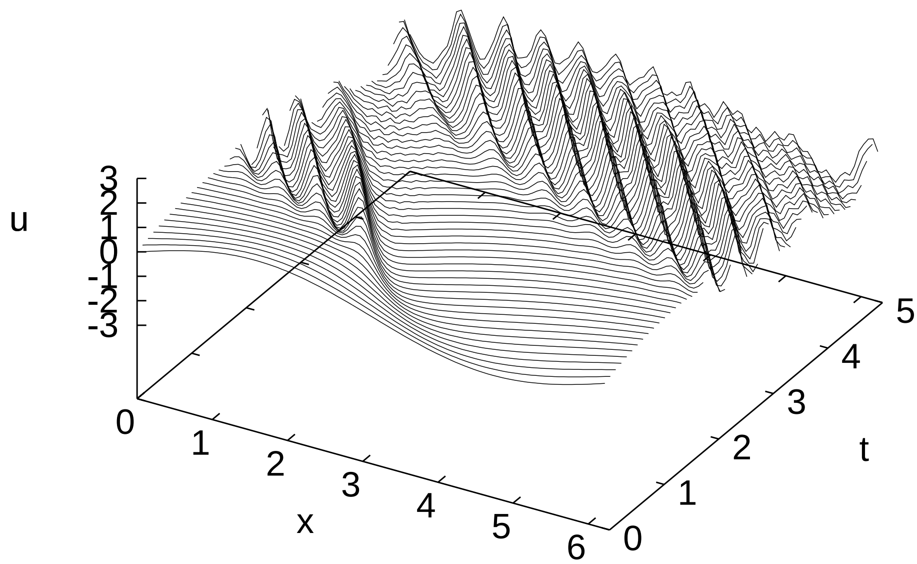

Figure 7: The numerical solution obtained by Scheme 6 (the multi-symplectic scheme) with $N = 101$ and $\Delta t = 0.1$.

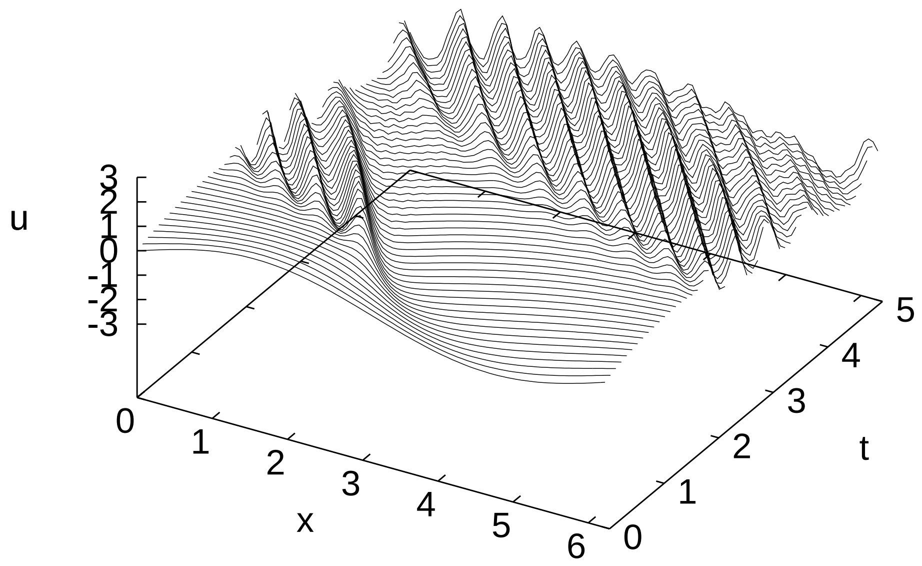

Figure 8: The numerical solution obtained by Scheme 7 (the energy-preserving Galerkin scheme) with $N = 102$ and $\Delta t = 0.1$.

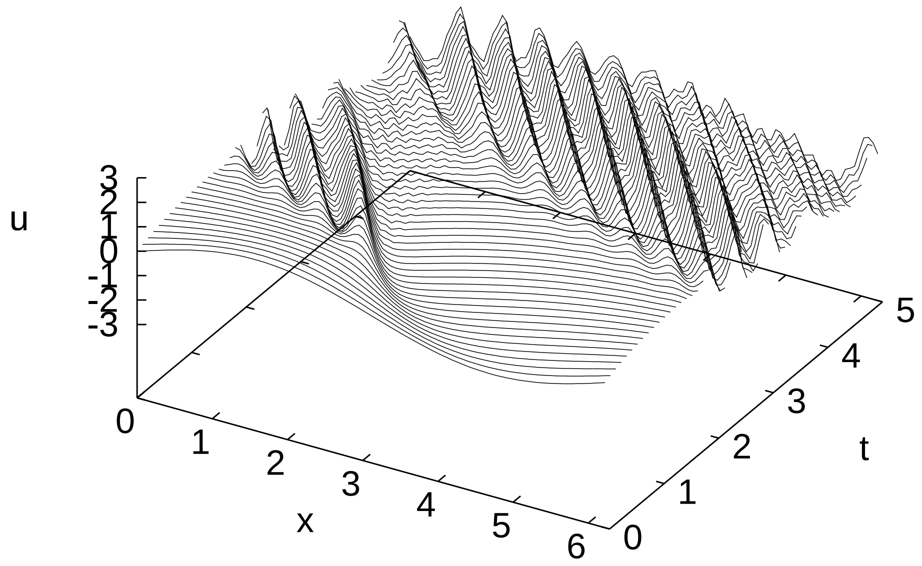

Figure 9: The numerical solution obtained by Scheme 8 (the norm-preserving Galerkin scheme) with $N = 102$ and $\Delta t = 0.1$.

7 Concluding remarks

We have proposed five new structure preserving numerical schemes for the Ostrovsky equation: the energy-preserving finite difference scheme, the norm-preserving finite difference scheme, the multi-symplectic integrator, the energy-preserving Galerkin scheme and the norm-preserving Galerkin scheme. We also like to emphasize that in this paper we have also found a multi-symplectic formulation of the Ostrovsky equation. Numerical examples confirmed the effectiveness of the proposed schemes, in particular, the finite difference schemes based on the potential expression. The reason why this approach improved the results, and whether or not it works also for other PDEs, such as the KdV, are interesting future research topics. For the Galerkin schemes, in the present paper we have introduced the idea of L^2-projection. This technique seems useful in more general context; the discussion and the extension of the DPDM are left to our future work[19], which will be reported soon elsewhere.

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