MATHEMATICAL ENGINEERING
TECHNICAL REPORTS

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METR 2011–12 April 2011

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An Algorithm for Finding a Maximum t-Matching Excluding Complete Partite Subgraphs

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Abstract

For an integer t and a fixed graph H, we consider the problem of finding a maximum t-matching not containing H as a subgraph, which we call the H-free t-matching problem. This problem is a generalization of the problem of finding a maximum 2-matching with no short cycles, which has been well-studied as a natural relaxation of the Hamiltonian circuit problem. When H is a complete graph K_{t+1} or a complete bipartite graph $K_{t,t}$, in 2010, Bérczi and Végh gave a polynomial-time algorithm for the H-free t-matching problem in simple graphs with maximum degree at most $t+1$. A main contribution of this paper is to extend this result to the case when H is a t-regular complete partite graph. We also show that the problem is NP-complete when H is a connected t-regular graph that is not complete partite.

Keywords: b-matching, Complete partite graph, Polynomial-time algorithm, Shrinking

1 Introduction

For an undirected graph $G = (V, E)$ and an upper bound $b : V \to \mathbb{Z}_+$, an edge set $M \subseteq E$ is called a *simple b-matching* if the number of edges in M incident to v is at most $b(v)$ for each vertex $v \in V$. (We call such an edge subset just as a *b-matching* for remaining of the paper.) For a positive integer t, a b-matching with $b(v) = t$ for every vertex $v \in V$ is called a *t-matching*. In this paper, we deal with the problem of finding a maximum t-matching with some conditions in a given undirected graph.

When $t = 2$, the problem of finding a maximum 2-matching without short cycles has been studied as a natural relaxation of the Hamiltonian circuit problem. We say that a 2-matching is *$C_{\le k}$-free* if it contains no cycle of length k or less. The *$C_{\le k}$-free 2-matching problem* is to find a maximum $C_{\le k}$-free 2-matching. The case $k \le 2$ is exactly the classical simple 2-matching problem, which can be solved in polynomial time. Papadimitriou showed that the problem is NP-hard when $k \ge 5$ (see [3]). Hartvigsen provided an augmenting path algorithm for the $C_{\le 3}$-free 2-matching problem in his Ph.D. thesis [7]. The computational complexity of the C_4-free 2-matching problem is still open, and several results are known in some graph classes. For the $C_{\le 4}$-free 2-matching problem

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in bipartite graphs, a min-max formula [9] and polynomial-time algorithms [8, 15] are proposed. If each vertex of the input graph has degree at most three, a polynomial-time algorithm for finding a maximum 2-matching without cycles of length four and one for the $C_{\leq 4}$-free 2-matching problem are given in [1] and [2], respectively. Note that 2-matchings containing no cycle of length four are closely related to the vertex-connectivity augmentation problem (see [1, 18]).

The weighted versions of those problems are to find a $C_{\leq k}$-free 2-matching that maximizes the total weight of its edges for a given weighted graph. This problem is NP-complete when $k \geq 4$ (see [1, 4]), and it is left open when $k = 3$. The weighted $C_{\leq 4}$-free 2-matching problem in bipartite graphs is polynomially solvable if the weight function satisfies a certain condition called "vertex-induced on every square" [13, 16], and the weighted $C_{\leq 3}$-free 2-matching problem in graphs with maximum degree at most three can also be solved in polynomial time [10, 12].

The concept of $C_{\leq k}$-free 2-matchings can be extended to t-matchings in the following ways. The problem of finding a maximum t-matching not containing a $K_{t,t}$ as a subgraph, called the *$K_{t,t}$-free t-matching problem*, was first considered by Frank [4]. When $t = 2$ and an input graph is simple and bipartite, this problem is exactly the $C_{\leq 4}$-free 2-matching problem. Similarly, the notion of *K_{t+1}-free t-matchings*, which are t-matchings not containing a K_{t+1} as a subgraph, is a generalization of that of $C_{\leq 3}$-free 2-matchings. For the $K_{t,t}$-free t-matching problem in bipartite graphs, a min-max formula is given by Frank [4] and a combinatorial algorithm by Pap [14, 15]. Bérczi and Végh [2] considered a common generalization of the $K_{t,t}$-free t-matching problem and the K_{t+1}-free t-matching problem, where the forbidden subgraph list can include both $K_{t,t}$'s and K_{t+1}'s. They gave a min-max formula and a combinatorial algorithm for this problem in graphs with maximum degree at most $t + 1$.

As a further generalization of these problems, for a fixed graph H, we consider the problem of finding a maximum t-matching not containing H as a subgraph, which we call the *H-free t-matching problem.* Motivated by the result of Bérczi and Végh [2], in this paper, we focus on the case when H is a connected t-regular graph and the input graph has maximum degree at most $t+1$. Note that a graph is said to be *t-regular* if the degree of every vertex is t. Our main results are the following theorems, which draw a line between polynomially solvable cases and NP-hard cases.

Theorem 1. *Let t be a positive integer and H be a connected t-regular graph. If H is a complete partite graph, then the H-free t-matching problem in simple graphs with maximum degree at most $t + 1$ can be solved in polynomial time.*

Theorem 2. *Let t be a positive integer and H be a connected t-regular graph. If H is not a complete partite graph, then the H-free t-matching problem is NP-hard even if the input graph is simple and has maximum degree at most $t + 1$.*

Here, a graph $H = (V, E)$ is said to be a *complete partite graph* if there exists a partition $\{V_1, \dots, V_p\}$ of V such that $E = \{uv \mid u \in V_i, v \in V_j, i \neq j\}$ for some positive integer p. In other words, a complete partite graph is the complement of the disjoint union of the complete graphs. Since a K_{t+1} and a $K_{t,t}$ are t-regular complete partite graphs, Theorem 1 implies the polynomial-time solvability of the cases when $H = K_{t+1}$ and $H = K_{t,t}$.

We note that these theorems are also motivated by a relationship to jump systems. Recently, it was shown in [11] that, for a connected t-regular graph H, the degree sequences of all H-free t-matchings in a graph form a jump system if and only if H is a complete partite graph. Theorems 1 and 2 show that the polynomial-time solvability of the H-free t-matching problem is consistent with this condition.

This paper is organized as follows. Before we move onto the proof of our main results, the preliminary for this paper is presented in Section 2. In Section 3, we show a slight generalization of Theorem 1, and give a polynomial-time algorithm for it. To show the correctness of our proof, we need a key theorem (Theorem 5), whose proof is given in Section 4. A proof for Theorem 2 is given in Section 5.

2 Preliminary

Let $G = (V, E)$ be an undirected graph (or simply a graph) with vertex set V and edge set E, and n and m denote the number of vertices and the number of edges, respectively. For a vertex $v \in V$, the set of vertices adjacent to v is denoted by $N(v)$. The *degree* of a vertex $v \in V$ in G, denoted by $d_G(v)$, is the number of edges incident with v. For a vertex $v \in V$ and an edge set $F \subseteq E$, $d_F(v)$ is the number of edges in F incident with v. Note that if a self-loop e is incident with v, e is counted twice. Recall that, for a vector $b : V \to \mathbb{Z}_+$, an edge set $M \subseteq E$ is said to be a *b-matching* if $d_M(v) \leq b(v)$ for every $v \in V$. In particular, for a positive integer t, a b-matching with $b(v) = t$ for every vertex $v \in V$ is called a *t-matching*. Note that these are often called a *simple* b-matching and a *simple* t-matching in the literature. For a subgraph H of G, the vertex set and edge set of H are denoted by $V(H)$ and $E(H)$, respectively.

For a positive integer p, we say that a graph $H = (V, E)$ is a *complete p-partite graph* if there exists a partition $\{V_1, \dots, V_p\}$ of V, where $V_i \neq \emptyset$ for each i, such that $E = \{uv \mid u \in V_i, v \in V_j, i \neq j\}$. Each V_i in this partition is called a *color class*. A graph is *complete partite* if it is a complete p-partite graph for some p. A graph is *t-regular* if the degree of every vertex is t. One can easily observe that, for positive integers t and p, a t-regular complete p-partite graph exists if and only if $q := t/(p-1)$ is an integer, and such a graph contains q vertices in each color class. For example, a t-regular complete 2-partite graph is a complete bipartite graph $K_{t,t}$, and a t-regular complete $(t+1)$-partite graph is a complete graph K_{t+1}.

For a set $\mathcal{K}$ of subgraphs of $G = (V, E)$, an edge set $F \subseteq E$ is said to be *$\mathcal{K}$-free* if the graph $G_F = (V, F)$ contains no member of $\mathcal{K}$ as a subgraph. If $\mathcal{K}$ consists of all subgraphs of G that are isomorphic to H for some graph H, then a $\mathcal{K}$-free edge set is said to be *H-free*. The *$\mathcal{K}$-free* (resp. *H-free*) *b-matching problem* is to find a maximum $\mathcal{K}$-free (resp. H-free) b-matching in a given graph.

3 Polynomial-Time Algorithm

As we mentioned in Section 1, Bérczi and Végh [2] gave a polynomial-time algorithm for the K_{t+1}-free t-matching problem and the $K_{t,t}$-free t-matching problem in graphs with maximum degree at most $t + 1$. Inspired by this result, we investigate the cases when the H-free t-matching problem can be solved in polynomial-time. The key observation is that a K_{t+1} and a $K_{t,t}$ are both t-regular subgraphs with a high degree of symmetry. Such a high symmetry appears in not only a K_{t+1} and a $K_{t,t}$, but also in other complete partite graphs. In this section, using symmetries of complete partite graphs, we prove the following theorem, which is a slight generalization of Theorem 1.

Theorem 3. *Let t and p be positive integers. Suppose that a graph $G = (V, E)$, a mapping $b : V \to \mathbb{Z}_+$, and a set $\mathcal{K}$ of t-regular complete p-partite subgraphs of G satisfy the following condition:*

$$\text{for any } K \in \mathcal{K} \text{ and } v \in V(K),\ V(K) \text{ spans no parallel edges, } b(v) = t, \text{ and } d_G(v) \leq t + 1. \quad (1)$$

Then, a maximum $\mathcal{K}$-free b-matching can be found in $\mathrm{O}(n \cdot t^3/p^2 + nm \log m)$ time.

By setting $b(v) = t$ for every $v \in V$ and by defining $\mathcal{K}$ as the set of all t-regular complete p-partite subgraphs, we obtain Theorem 1 as a special case of Theorem 3.

In our proof of Theorem 3, we define a "shrinking" operation and a new upper bound b° of degrees, and consider the correspondence of the maximum b°-matching problem in the "shrunk" graph with the maximum $\mathcal{K}$-free b-matching problem in the original graph. By utilizing such a correspondence, we reduce the $\mathcal{K}$-free b-matching problem to the classical b-matching problem.

As mentioned in Section 2, if a t-regular complete p-partite graph exists, then there exists an integer q such that $t = (p-1)q$. That is, the t-regular complete p-partite subgraph has q vertices in each color class. When $q = 1$, $\mathcal{K}$ consists of K_{t+1}-subgraphs, and this case is solved in [2]. For simplicity, we first consider the case of $q \geq 3$ in Sections 3.1 and 3.2. After that, in Section 3.3, we give remarks for the case of $q = 2$.

3.1 Shrinking operation for the case of $q \geq 3$

Let t and p be positive integers such that $q := t/(p-1)$ is an integer with $q \geq 3$. Suppose that a graph $G = (V, E)$, a mapping $b : V \to \mathbb{Z}_+$, and a set $\mathcal{K}$ of t-regular complete p-partite subgraphs of G satisfy the condition (1). We define the "shrinking" operation as follows:

Definition 4. (Shrinking a complete p-partite graph K) Let $K \in \mathcal{K}$ be a complete p-partite subgraph of G with color classes $U_1, \ldots, U_p$. *Shrinking* of K in G consists of the following operations:

Steps for shrinking a complete p-partite graph K

Step 1. For $i = 1, 2, \ldots, p$, identify the vertices in each color class U_i and denote the corresponding vertex by u_i, respectively (each edge in G whose end vertex is in U_i is replaced by an edge in G° incident to u_i);
Step 2. Remove one edge between each vertex pair $\{u_i, u_j\}$ for each $i, j = 1, 2, \ldots, p$ with $i \neq j$, and add the graph S shown in Figure 1 between $u_1, \ldots, u_p$ (see below).

Here, $S = (V_S, E_S)$ is the graph such that $V_S = \{u_1, \ldots, u_p, w_1, \ldots, w_p, c\}$ and E_S consists of one edge between c and w_i and $p-1$ parallel edges between w_i and u_i for $i = 1, \ldots, p$. For each edge $e \in E - E(K)$, it is left as it is and again denoted by e after shrinking.

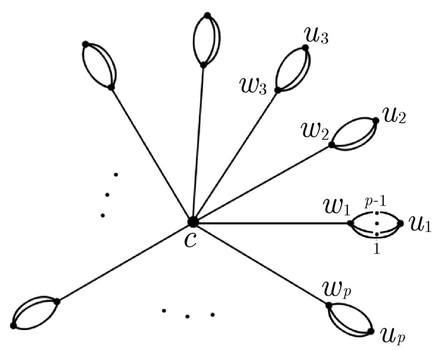

Figure 1: Added graph S

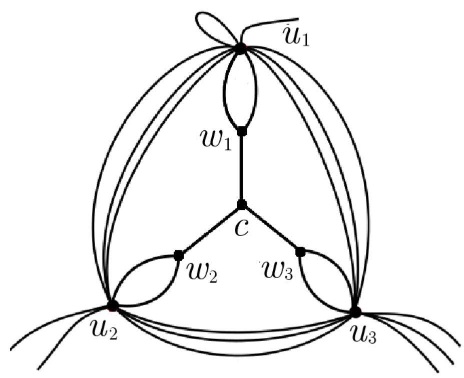

Figure 2: Example of a shrunk graph ($p = 3$)

Let G° be the shrunk graph obtained by these operations (see Figure 2), and denote the vertex set of G° by V°. For a given vector $b : V \to \mathbb{Z}_+$, we define $b^\circ : V^\circ \to \mathbb{Z}_+$ in the following way:

$$b^\circ(v) = \begin{cases} 2 & \text{if } v = c, \\ p-1 & \text{if } v \in \{w_1, \dots, w_p\}, \\ qt & \text{if } v \in \{u_1, \dots, u_p\}, \\ b(v) & \text{otherwise.} \end{cases}$$

We further define $\mathcal{K}^\circ$ as the list of subgraphs in $\mathcal{K}$ that are disjoint from K. Note that the obtained triple G°, b°, and $\mathcal{K}^\circ$ satisfies the condition (1).

With the above construction, we prove the following theorem.

Theorem 5. *Let t and p be positive integers such that $q := t/(p-1)$ is an integer with $q \geq 3$. Suppose that a graph $G = (V, E)$, $b : V \to \mathbb{Z}_+$, and a list $\mathcal{K}$ of forbidden subgraphs of G consisting of t-regular complete p-partite subgraphs satisfy the condition (1). For a subgraph $K \in \mathcal{K}$, let G° be the graph obtained from G by shrinking K as in Definition 4, and let $b^\circ : V^\circ \to \mathbb{Z}_+$ and $\mathcal{K}^\circ$ be defined as above. Then,*

$$\text{(Maximum size of a } \mathcal{K}\text{-free } b\text{-matching } M \text{ in } G\text{)} =$$
$$\text{(Maximum size of a } \mathcal{K}^\circ\text{-free } b^\circ\text{-matching } M^\circ \text{ in } G^\circ\text{)} - \binom{p}{2} - 1.$$

Furthermore, given a maximum $\mathcal{K}^\circ$-free b°-matching M° in G°, we can construct a maximum $\mathcal{K}$-free b-matching M in G in $\mathrm{O}(m)$ time.

The proof of this theorem is given in Section 4.

3.2 Algorithm

In this subsection, we give an algorithm for the $\mathcal{K}$-free b-matching problem for the case of $q \geq 3$ with the aid of Theorem 5. Suppose that we are given a graph $G = (V, E)$, $b : V \to \mathbb{Z}_+$, and a list $\mathcal{K}$ of t-regular complete p-partite subgraphs of G satisfying the condition (1). Then, we can find a maximum $\mathcal{K}$-free b-matching by the following algorithm.

Algorithm for finding a maximum $\mathcal{K}$-free b-matching

Step 1. Greedily choose an inclusionwise maximal subset $\mathcal{H} = \{H_1, \dots, H_k\} \subseteq \mathcal{K}$ of disjoint forbidden subgraphs.
Step 2. Shrink the members of $\mathcal{H}$ simultaneously, resulting in graph $G^\circ = (V^\circ, E^\circ)$ with a bound $b^\circ : V^\circ \to \mathbb{Z}_+$ with no forbidden graph;
Step 3. Find a maximum b°-matching M° of G° (with existing algorithm such as in [6]);
Step 4. Modify M° into a maximum $\mathcal{K}$-free b-matching M using the construction described in Theorem 5 k times.

The correctness of this algorithm is ensured by Theorem 5. Now we estimate the running time of the algorithm. If we are given the list $\mathcal{K}$ explicitly, we can execute Step 1 in $\mathrm{O}(tq|\mathcal{K}|)$ time, because one has to check whether each vertex of K is already used for each $K \in \mathcal{K}$, and $|V(K)| = tq$. However, in the H-free t-matching problem, the list $\mathcal{K}$ is not necessarily given explicitly. Even in such a case, the following claim ensures that the algorithm can still check for all the possible forbidden t-regular complete p-partite subgraphs in G in polynomial-time.

Claim 6. *Let $G = (V, E)$ be a graph and $\mathcal{K}$ be a set of t-regular complete p-partite subgraphs in G satisfying the condition (1). Then, $|\{V(K) \mid K \in \mathcal{K}\}| \leq n \cdot (t+1)/p$.*

Proof. Suppose that a vertex v is contained in $V(K)$ for some $K \in \mathcal{K}$ and $U \subseteq V(K)$ be the vertex set of K in different color classes from v. Since $|N(v)| \leq t+1$, $|U| = t$, and $U \subseteq N(v)$, we have at most $\binom{t+1}{t} = t+1$ choices for U. For each choice, we take a vertex $u \in U$ arbitrarily. Since $|N(u) \setminus (U \cup \{v\})| \leq q$, $|V(K) \setminus (U \cup \{v\})| = q-1$, and $V(K) \setminus U \subseteq N(u)$, we have at most $\binom{q}{q-1} = q$ choices for $V(K) \setminus U$. Thus, $|\{V(K) \mid K \in \mathcal{K},\ v \in V(K)\}|$ is bounded by $(t+1) \cdot q$. Since each $V(K)$ contains pq vertices, $|\{V(K) \mid K \in \mathcal{K}\}|$ is bounded by $((t+1) \cdot q \cdot n)/(pq) = n \cdot (t+1)/p$, which completes the proof. □

By this claim, even if $\mathcal{K}$ is not given explicitly, Step 1 can be executed in $\mathrm{O}(n \cdot t^2 q/p)$ time, which can be also represented as $\mathrm{O}(n \cdot t^3/p^2)$. On the other hand, each of the shrinking can be done in $\mathrm{O}(m)$ time, the running time of Step 2 is bounded by $\mathrm{O}(nm)$, while Step 3 can be done in $O(nm \log m)$ time using the algorithm in [6]. Step 4, just as shrinking the graph, can be done in $\mathrm{O}(m)$ time for each $H_1, \ldots, H_k$, and therefore have a time bound of $\mathrm{O}(nm)$. Therefore, the total running time of the algorithm is $\mathrm{O}(n \cdot t^3/p^2 + nm \log m)$, which is strongly polynomial.

3.3 Modification for the Case of $q = 2$

In this subsection, we give an algorithm for the case of $q = 2$. This case is different from the case of $q \geq 3$ in the following sense:

> if both K and K' are t-regular complete p-partite subgraphs with the same vertex set, then $K = K'$ when $q \geq 3$ (see the proof of Claim 14), but it is not true when $q = 2$.

To see the latter half of this fact, suppose that $t = 2(p-1)$ and $G = (V, E)$ is a complete graph with $t + 2 = 2p$ vertices. Even if we remove any matching of size at most p from G, the obtained graph contains a t-regular complete p-partite subgraph. This means that there are many t-regular complete p-partite subgraphs with the same vertex set V.

Because of this difference, we have to modify the shrinking operation given in Definition 4. Suppose that a graph $G = (V, E)$, a mapping $b : V \to \mathbb{Z}_+$, and a set $\mathcal{K}$ of t-regular complete p-partite subgraphs of G satisfy the condition (1). Suppose also that $q = t/(p-1) = 2$. Let $K \in \mathcal{K}$ be a forbidden t-regular complete p-partite subgraph of G.

If there exists a t-regular complete p-partite subgraph K' of G such that $V(K') = V(K)$ and $K' \notin \mathcal{K}$, then removing K from $\mathcal{K}$ does not affect the optimal value of the $\mathcal{K}$-free b-matching problem. This is because, for any b-matching M containing K, a b-matching $M' = (M \setminus K) \cup K'$ not containing K satisfies $|M'| = |M|$. Therefore, we only consider the problem with the following condition:

$$\text{for any } K \in \mathcal{K}, \text{ every } t\text{-regular complete } p\text{-partite graph with the vertex set } V(K) \text{ is in } \mathcal{K}. \quad (2)$$

On the other hand, when G contains a complete graph H with $t + 2$ vertices as a subgraph, it forms a connected component, and hence we can deal with the $\mathcal{K}$-free b-matching problem in H and in $G - H$, independently. Therefore, we only consider the problem with the following condition:

$$G \text{ contains no complete graph with } t+2 \text{ vertices as a subgraph.} \quad (3)$$

We define the shrinking operation under these conditions as follows.

Definition 7. (Shrinking operation for the case of $q = 2$) Suppose that the conditions (2) and (3) hold. Let $K \in \mathcal{K}$ be a forbidden t-regular complete p-partite subgraph of G with color classes $U_1, \ldots, U_p$, where $|U_i| = q = 2$ for $i = 1, \ldots, p$. Suppose that the two vertices in U_i are not connected by an edge for $i = 1, \ldots, r$ and they are connected by an edge for $i = r+1, \ldots, q$, and let $U_0 = U_{r+1} \cup \cdots \cup U_q$. Note that $r \geq 1$ by the condition (3). *Shrinking* of K in G consists of the following operations:

Shrinking operation for the case of $q = 2$

Step 1. For $i = 0, 1, 2, \ldots, r$, identify the vertices in each U_i and denote the corresponding vertex by u_i, respectively (each edge in G whose end vertex is in U_i is replaced by an edge in G° incident to u_i);
Step 2. Remove one edge between each vertex pair $\{u_i, u_j\}$ for each $i, j = 0, 1, 2, \ldots, r$ with $i \neq j$, remove $q - r$ self-loops incident to u_0, and add the graph S' between $u_0, u_1, \ldots, u_r$ (defined below).

Here, $S' = (V'_S, E'_S)$ is the graph such that $V'_S = \{u_0, u_1, \ldots, u_r, w_0, w_1, \ldots, w_r, c\}$ and E'_S consists of one edge between c and w_i and r parallel edges between w_i and u_i for $i = 0, 1, \ldots, r$. For each edge $e \in E - E(K)$, it is left as it is and again denoted by e after shrinking.

Let G° be the shrunk graph obtained by these operations, and denote the vertex set of G° by V°. For a given vector $b : V \to \mathbb{Z}_+$, we define $b^\circ : V^\circ \to \mathbb{Z}_+$ in the following way:

$$b^\circ(v) = \begin{cases} 2 & \text{if } v = c, \\ r & \text{if } v \in \{w_0, w_1, \ldots, w_r\}, \\ t|U_i| & \text{if } v = u_i \text{ for } i = 0, 1, \ldots, r, \\ b(v) & \text{otherwise.} \end{cases}$$

We define $\mathcal{K}^\circ$ as the list of subgraphs in $\mathcal{K}$ that are disjoint from K.

With this modification, we can prove almost the same result as Theorem 5 for the case of $q = t/(p-1) = 2$. Note that we apply the shrinking operation as in Definition 7 instead of Definition 4, and the constant "$-\binom{p}{2} - 1$" in Theorem 5 becomes "$-\binom{r+1}{2} - 1$" in this case. Since the proof is almost the same as Theorem 5, we omit it. We only mention here that we use the following observations in our proof.

- There exists a maximum $\mathcal{K}$-free b-matching M in G such that at least one edge between U_i and U_j is not contained in M for some $i, j \in \{0, 1, \ldots, r\}$ with $i \neq j$ (the condition (2) is used here).
- For a b-matching M in G, if at least one edge between U_i and U_j is not contained in M for some $i, j \in \{0, 1, \ldots, r\}$ with $i \neq j$, then M does not contain a t-regular complete p-partite subgraph $K' \in \mathcal{K}$ with $V(K') = V(K)$.

Based on this variant of Theorem 5, almost the same algorithm as in Section 3.2 works for the case of $q = 2$. We only need to execute the following preprocessing in order to obtain an instance with the conditions (2) and (3).

Preprocessing for the case of $q = 2$

Step 1. While there exist t-regular complete p-partite subgraphs K, K' of G such that $V(K) = V(K')$, $K \in \mathcal{K}$, and $K' \notin \mathcal{K}$, remove K from $\mathcal{K}$.
Step 2. If G contains a complete graph with $t + 2$ vertices as a subgraph, we deal with this component separately.

By the arguments in Sections 3.2 and 3.3, we complete the proof for Theorem 3. □

4 Proof for Theorem 5

In this section, we give a proof of Theorem 5. To help establish the relationship between M and M°, we consider another shrinking of a complete p-partite graph, say G_N, that is similar to G°. Suppose that G° is obtained from G by shrinking $K \in \mathcal{K}$ with color classes $U_1, \ldots, U_p$. Let G_N be the graph obtained from G by identifying the vertices in each color class U_i and denote the corresponding vertex by u_i, respectively. In other words, compared to G°, G_N does not have S attached to it, but has one additional parallel edge between each of the vertex pair instead. Let K_N be the subgraph of G_N consisting of the p vertices $\{u_1, \ldots, u_p\}$ and all the edges between them. The vertex set of G_N is denoted as V_N, and a mapping $b_N : V_N \to \mathbb{Z}_+$ is defined by

$$b_N(v) = b^\circ(v) = \begin{cases} qt & \text{if } v \in \{u_1, \ldots, u_p\}, \\ b(v) & \text{otherwise.} \end{cases}$$

We define $\mathcal{K}_N$ by joining $\{K_N\}$ and the list of forbidden subgraphs in $\mathcal{K}$ disjoint from K.

4.1 Correspondence of M to M°

In this subsection, we prove the following proposition, which shows the correspondence of M to M°.

Proposition 8. *Suppose that $G = (V, E)$, $b : V \to \mathbb{Z}_+$, $\mathcal{K}$, G°, $b^\circ : V^\circ \to \mathbb{Z}_+$, and $\mathcal{K}^\circ$ are as in Theorem 5. For a maximum $\mathcal{K}$-free b-matching M in G, there exists a $\mathcal{K}^\circ$-free b°-matching M° in G° with*

$$|M^\circ| = |M| + \binom{p}{2} + 1.$$

To prove this proposition, we first show the correspondence of M to a maximum $\mathcal{K}_N$-free b_N-matching M_N in G.

Claim 9. (Correspondence of M to M_N) *For a maximum $\mathcal{K}$-free b-matching M in G, there exists a $\mathcal{K}_N$-free b_N-matching M_N in G_N with $|M| = |M_N|$ such that*

$$\text{for every } i, j \in \{1, \ldots, p\} \text{ with } i \neq j,\ M_N \text{ contains at least one edge connecting } u_i \text{ and } u_j. \quad (4)$$

Proof. Let M_N be the edge subset in G_N corresponding to M. That is, each edge in M whose end vertex is in U_i $(i = 1, \ldots, p)$ is replaced by an edge in G_N incident to u_i. Then, $|M| = |M_N|$ holds by the construction. It is obvious that $d_{M_N}(u_i) = \sum_{v \in U_i} d_M(v) \leq qt = b_N(v)$ for $i = 1, \ldots, p$, and $d_{M_N}(v) = d_M(v) \leq b(v) = b_N(v)$ for any $v \in V_N \setminus \{u_1, \ldots, u_p\}$. Therefore, M_N is a $\mathcal{K}_N$-free b_N-matching in G_N.

If M contains no edge between U_i and U_j for some i, j, then at most $t + 1 - q$ edges in M are incident to each vertex in $U_i \cup U_j$. In this case, we can obtain a larger $\mathcal{K}$-free b-matching by adding an edge between U_i and U_j to M, which contradicts the maximality of M. Therefore, M contains at least one edge between U_i and U_j for each i, j, and hence M_N defined as above satisfies the condition (4). This completes the proof. □

Next, we show the correspondence of M_N to M°.

Claim 10. (Correspondence of M_N to M°) *For a $\mathcal{K}_N$-free b_N-matching M_N in G_N satisfying the condition (4), there exists a $\mathcal{K}^\circ$-free b°-matching M° in G° with $|M^\circ| = |M_N| + \binom{p}{2} + 1$.*

Proof. Let $F \subseteq M$ be an edge set that contains exactly one edge connecting u_i and u_j for every pair i, j with $i \neq j$, that is, F spans a complete graph containing $\binom{p}{2}$ edges. The existence of $F \subseteq M$ is guaranteed by the condition (4). Let $e \in F$ be an edge connecting u_1 and u_2. Define an edge subset M° of G° by

$$M^\circ = (M_N \setminus F) \cup (E(S) \setminus \{u_1w_1, u_2w_2, cw_3, cw_4, \ldots, cw_p\}) \cup \{e\}.$$

Then, $|M^\circ| = |M_N| - \binom{p}{2} + (p^2 - p) + 1 = |M_N| + \binom{p}{2} + 1$ holds. Since $d_{M^\circ}(v) = d_{M_N}(v) \leq b_N(v) = b^\circ(v)$ holds for any $v \in V_N$ and $d_{M^\circ}(v) \leq b^\circ(v)$ obviously holds for any $v \in V^\circ \setminus V_N$, M° is a $\mathcal{K}^\circ$-free b°-matching in G°. □

Proof for Proposition 8. By Claims 9 and 10, for a maximum $\mathcal{K}$-free b-matching M in G, there exists a $\mathcal{K}^\circ$-free b°-matching M° in G° with

$$|M^\circ| = |M_N| = |M| + \binom{p}{2} + 1.$$

□

4.2 Correspondence of M° to M

In this subsection, we prove the following proposition, which shows the correspondence of M° to M.

Proposition 11. *Suppose that $G = (V, E)$, $b : V \to \mathbb{Z}_+$, $\mathcal{K}$, G°, $b^\circ : V^\circ \to \mathbb{Z}_+$, and $\mathcal{K}^\circ$ are as in Theorem 5. For a maximum $\mathcal{K}^\circ$-free b°-matching M° in G°, there exists a $\mathcal{K}$-free b-matching M in G with*

$$|M| = |M^\circ| - \binom{p}{2} - 1.$$

Furthermore, we can construct M from M° in $\mathrm{O}(m)$ time.

We start the proof for this proposition with the following claim.

Claim 12. *There exists a maximum $\mathcal{K}^\circ$-free b°-matching M° in G° such that $d_{M^\circ}(c) = 2$ and $d_{M^\circ}(w_i) = p - 1$ for $i = 1, \ldots, p$.*

Proof. Let $\hat{M}$ be a maximum b°-matching in G° that maximizes the value $f(\hat{M}) = 2d_{\hat{M}}(c) + \sum_{i=1}^{p} d_{\hat{M}}(w_i)$ among all maximum b°-matchings.

If $d_{\hat{M}}(c) < 2$, then by adding an edge cw_i to $\hat{M}$ and removing w_iu_i (such an edge exists by the maximality of $\hat{M}$) from it, we obtain another maximum b°-matching M° such that $f(M^\circ) > f(\hat{M})$, which contradicts the definition of $\hat{M}$. If $d_{\hat{M}}(w_i) < p - 1$ for some i, then by adding an edge w_iu_i to $\hat{M}$ and removing an edge incident to u_i (such an edge exists by the maximality of $\hat{M}$) from it, we obtain another maximum b°-matching M° such that $f(M^\circ) > f(\hat{M})$, which contradicts the definition of $\hat{M}$. Thus, G° contains a desired maximum $\mathcal{K}^\circ$-free b°-matching. We note that if $\hat{M}$ is $\mathcal{K}^\circ$-free, then M° defined as above is also $\mathcal{K}^\circ$-free, because no member of $\mathcal{K}^\circ$ contains edges incident to w_i. □

We will examine the correspondence between a maximum $\mathcal{K}$-free b-matching M in G and a maximum $\mathcal{K}^\circ$-free b°-matching M° in G° with the help of a maximum $\mathcal{K}_N$-free b_N-matching M_N in G_N.

Claim 13. (Correspondence of M° to M_N) *For each $\mathcal{K}^\circ$-free b°-matching M° in G° satisfying $d_{M^\circ}(c) = 2$ and $d_{M^\circ}(w_i) = p-1$ for $i = 1, \dots, p$, there exists a corresponding $\mathcal{K}_N$-free b_N-matching M_N in G_N such that*

$$|M_N| = |M^\circ| - \binom{p}{2} - 1.$$

Proof. Define $M' = M^\circ \setminus E(S)$, and denote the corresponding edge subset in G_N also by M'. By definition, G_N has one edge between u_i and u_j is not contained in G° for each i, j with $i \neq j$, which implies that it is not contained in M'.

Suppose that M° satisfies that $d_{M^\circ}(c) = 2$ and $d_{M^\circ}(w_i) = p - 1$ for $i = 1, \dots, p$. By symmetry, we may assume that M° contains cw_1 and cw_2. Then, we can see that $p - 2$ edges in $M^\circ \cap S$ are incident to u_i for $i = 1, 2$, and $p - 1$ edges in $M^\circ \cap S$ are incident to u_i for $i = 3, \dots, p$. In this case, let M_N be an edge set of G_N obtained from M' by adding one edge u_iu_j for each pair $\{i, j\}$ except for u_1u_2. Since M_N satisfies $d_{M_N}(v) = d_{M^\circ}(v)$ for all $v \in V_N$, it is a $\mathcal{K}_\mathcal{N}$-free b_N-matching in G_N, and obviously, $|M_N| = |M^\circ| - \binom{p}{2} - 1$. □

It remains for us to consider the correspondence of M_N to M in order to prove the correspondence of M° to M.

Claim 14. (Correspondence of M_N to M) *For each $\mathcal{K}_N$-free b_N-matching M_N in G_N, there exists a corresponding $\mathcal{K}$-free b-matching M in G, such that $|M| = |M_N|$.*

Proof. It suffices to consider the case when M_N is a maximum $\mathcal{K}_N$-free b_N-matching in G_N. In this case, $d_{M_N}(u_i) \geq qt - 1$ for $i = 1, \dots, p$ by the maximality of M_N. Let $M' = M_N - E(K_N)$, and let r_{ij} be the number of edges in M_N between u_i and u_j, where $i \neq j$. For each edge in M' incident to u_i $(i = 1, 2, \dots, p)$, replace it with a corresponding edge in E. Then, M' can be regarded as an edge set of G. We construct M from M' by adding r_{ij} edges between U_i and U_j for every pair $\{i, j\}$ with $i \neq j$. Since $\sum_{j \neq i} r_{ij} + d_{M'}(u_i) \leq b_N(u_i) = \sum_{v \in U_i} b(v)$ holds for any i, and $U_i \cup U_j$ spans a complete bipartite graph for any i, j, by choosing r_{ij} edges between U_i and U_j appropriately, we obtain a b-matching M. Furthermore, it is obvious that $|M| = |M_N|$, and so we only have to verify that M is also $\mathcal{K}$-free.

Recall that we consider the case of $q \geq 3$. Assume there exists a forbidden complete p-partite subgraph $K' \in \mathcal{K}$ in M. Since K' must contain a vertex in $V(K)$, without loss of generality, suppose that $u \in U_1$ is contained in K'. Observing that K' forms a connected component of the subgraph induced by M, we find vertices that are contained in K'.

Since $d_M(u) = t$, the degree of u is at most $t + 1$, and u is adjacent to all the vertices in $U_2 \cup \cdots \cup U_p$, it holds that at least $t-1$ vertices in $U_2 \cup \cdots \cup U_p$ are connected with u by an edge in M. Thus, these $t - 1$ vertices are contained in K', which implies that $|V(K') \cap V(K)| \geq t$. On the other hand, we have $d_M(v) \geq t - 1$ for each $v \in V(K)$, since $d_{M_N}(u_i) \geq qt - 1$ for each i. Then, by a similar argument as above, at least $t-2$ vertices in $V(K)$ are connected with v by an edge in M. Since $|V(K') \cap V(K)| + t - 2 \geq 2t - 2 > |V(K)|$, there exists a vertex in $V(K') \cap V(K)$ connected with v by an edge in M, and hence $v \in V(K')$. Therefore, we have $V(K') = V(K)$.

Now we show that $V(K') = V(K)$ implies $K' = K$ when $q \geq 3$. Let $G[K]$ be the subgraph of G induced by $V(K)$. Since $q \geq 3$ and the degree of each vertex is at most $t+1$ in $G[K]$, each color class U_i of K forms a connected component in the complement graph of $G[K]$. This means that U_i is contained in the same color class in K', which shows $K' = K$.

By the above arguments, M is $\mathcal{K}$-free, because M does not contain K. □

Proof for Proposition 11. By Claims 12, 13, and 14, for a maximum $\mathcal{K}^\circ$-free b°-matching M° in G°, there exists a $\mathcal{K}$-free b-matching M in G with

$$|M| = |M_N| = |M^\circ| - \binom{p}{2} - 1.$$

Since all the proofs for Claims 12, 13 and 14 are constructive, by following the proofs, we can construct M from M° in $\mathrm{O}(m)$ time. □

By combining Propositions 8 and 11, we now give a proof for Theorem 5.

Proof for Theorem 5. Let M be a maximum $\mathcal{K}$-free b-matching in G and M° be a maximum $\mathcal{K}^\circ$-free b°-matching in G°. We have

$$|M^\circ| \geq |M| + \binom{p}{2} + 1$$

by Proposition 8, and

$$|M^\circ| \leq |M| + \binom{p}{2} + 1$$

by Proposition 11. By combining these two inequalities, Theorem 5 stands. □

5 NP-Hardness of Other t-Regular Forbidden Subgraphs

In this section, we show the NP-hardness of the H-free t-matching problem, where H is a forbidden connected t-regular subgraph that is not complete partite. When $t = 2$, the following theorem is an important special case of this result.

Theorem 15. *Let C_5 be the cycle of length five. Then the C_5-free 2-matching problem in graphs with maximum degree at most three is NP-hard.*

The NP-hardness of the C_5-free 2-matching problem in general graphs is shown in [3]. We prove Theorem 15 by a similar construction to [3].

Proof. We say that a 2-matching M in $G = (V, E)$ is *perfect* if $d_M(v) = 2$ for any $v \in V$. It suffices to show the NP-hardness of determining whether a given graph with maximum degree at most three has a perfect C_5-free 2-matching or not. We now give a polynomial reduction from the 3-SAT problem to this problem.

As in [3], let us consider q boolean variables $x_1, \ldots, x_q$ and a boolean expression $F = C^1 \wedge C^2 \wedge \cdots \wedge C^r$. Here each of the clauses C^i contains exactly three literals, i.e., $C^i = y^i_1 \vee y^i_2 \vee y^i_3$, where for each $k \in \{1, 2, 3\}$, $y^i_k = x_s$ or $\bar{x}_s$ for some $s = 1, 2, \ldots, n$, and $\bar{x}_s$ is the negation of x_s.

For each x_s, we make a corresponding graph γ_s as shown in Figure 3(a); and for each C^i we make a corresponding graph Γ^i as shown in Figure 3(b). Note that for each γ_s, there are two different perfect 2-matchings possible, and we associate the one containing e_{s1} with $x_s = 1$ (or $\bar{x}_s = 0$), and the one containing e_{s0} with $x_s = 0$ (or $\bar{x}_s = 1$). On the other hand, the key observation regarding Γ^i is stated in the following claim.

Claim 16. ([3, Proposition 5.2]) *Let $T \subseteq \{1, 2, 3\}$ and $\overline{T} = \{1, 2, 3\} \setminus T$. Then there exists a perfect C_5-free 2-matching in Γ^i containing all the edges j^i_k for $k \in T$, while containing none of the edges j^i_r for $r \in \overline{T}$ if and only if T is nonempty.*

(a) γ_s (b) Γ^i

Figure 3: Definitions of γ_s and Γ^i given in [3]

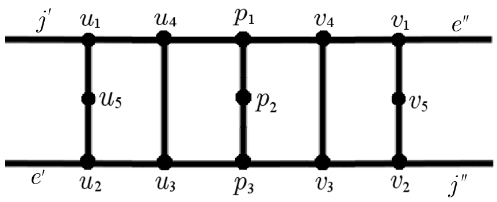

Figure 4: Definition of connector α

This claim associates the existence of a perfect C_5-free 2-matching in Γ^i with the boolean value of the clause $C^i = y_1^i \vee y_2^i \vee y_3^i$ being 1.

Now we use the connector α shown in Figure 4 to make y_k^i and x_s consistent. The connector α consists of a vertex set V_α and an edge set E_α defined as follows:

$$V_\alpha = \{u_1, u_2, u_3, u_4, u_5, v_1, v_2, v_3, v_4, v_5, p_1, p_2, p_3\},$$
$$E_\alpha = \{u_1u_5, u_5u_2, u_2u_3, u_3u_4, u_4u_1, v_1v_5, v_5v_2, v_2v_3, v_3v_4, v_4v_1, u_4p_1, u_3p_3, v_4p_1, v_3p_3, p_1p_2, p_2p_3\}.$$

Note that although another connector is given in [3], to construct a graph with maximum degree at most three, we use α in our proof.

For two edges $j = u'u''$ and $e = v'v''$, we say that α *connects* j and e if j and e are replaced by a new graph isomorphic to α and edges $j' = u'u_1$, $j'' = u''v_2$, $e' = v'v_1$, and $e'' = v''u_2$. The connector α connects the corresponding j and e by either connecting j_k^i and e_{s0} if $y_k^i = x_s$, or connecting j_k^i and e_{s1} if $y_k^i = \bar{x}_s$. Note that when a perfect C_5-free 2-matching in α includes either e' or e'', then it contains both e' and e''. The same thing can be said for j' and j''. Furthermore, there is no perfect C_5-free 2-matching containing all four e', e'', j', j'', i.e., the edges j and e are not in a perfect C_5-free 2-matching simultaneously.

Let G be the graph constructed by combining the above three types of graphs γ_s, Γ^i, and α. Notice that G has a maximum degree at most three. Then G contains a perfect C_5-free 2-matching if and only if $F = C^1 \wedge C^2 \wedge \cdots \wedge C^r = 1$ for some boolean assignment. Therefore, finding a perfect C_5-free 2-matching in a given graph with maximum degree at most three is NP-hard, which implies the theorem. □

Based on the techniques used in this proof, we give a proof for Theorem 2, which generalizes

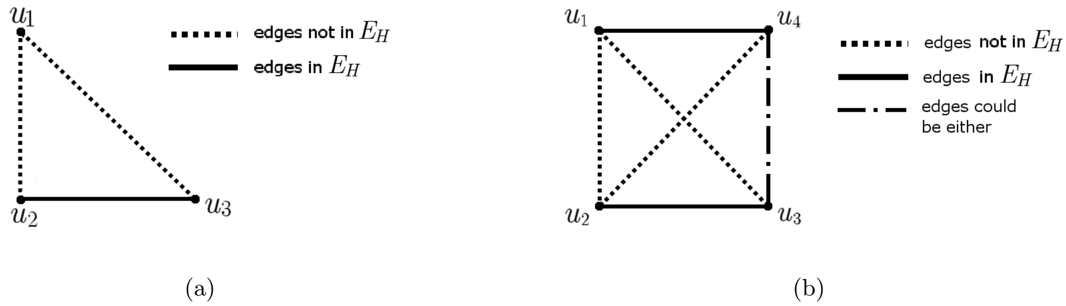

Figure 5: Characteristic subgraphs of non-complete partite t-regular graph

Theorem 15. Here, we restate Theorem 2.

Theorem. *Let H be a connected t-regular graph. If H is not a complete partite graph, then the H-free t-matching problem is NP-hard even if the input graph is simple and has maximum degree at most $t+1$.*

We first prove some properties of non-complete partite graphs that play an essential role in proving Theorem 2.

Claim 17. *A connected t-regular graph $H = (V_H, E_H)$ is not complete partite if and only if it contains three vertices u_1, u_2, u_3 such that $u_1u_2 \notin E_H$, $u_1u_3 \notin E_H$, and $u_2u_3 \in E_H$ (see Figure 5(a)).*

Proof. By the definition of complete partite graphs, the complement graph of a complete partite graph is a union of disjoint cliques (complete graphs). It is also trivial that when a graph has a complement graph that is a union of cliques, the graph itself is a complete partite graph.

On the other hand, being a union of cliques, the complement graph of a complete partite graph is transitive, meaning if each of two vertices is connected to a same vertex by an edge, then there must be an edge directly connecting them. Thus, we could see that the complement graph of H is not a union of cliques if and only if H has three vertices u_1, u_2, u_3 such that $u_1u_2 \notin E_H$, $u_1u_3 \notin E_H$, and $u_2u_3 \in E_H$. This completes the proof for Claim 17. □

Claim 18. *Let $H = (V_H, E_H)$ be a connected t-regular graph. Then, H is not complete partite if and only if it contains four vertices u_1, u_2, u_3, u_4 such that E_H contains u_1u_4 and u_2u_3, but does not contain u_1u_2, u_2u_4, u_1u_3. (Note that whether the edge u_3u_4 is contained is not concerned. See Figure 5(b).)*

Proof. The sufficiency ("if" part) is obvious by Claim 17. To show the necessity ("only if" part), suppose that $H = (V_H, E_H)$ is a connected t-regular graph that is not complete partite. By Claim 17, H has three vertices u_1, u_2, u_3 such that $u_1u_2 \notin E_H$, $u_1u_3 \notin E_H$, and $u_2u_3 \in E_H$. Since $|N(u_1)| = |N(u_2)| = t$ and $u_3 \in N(u_2) \setminus N(u_1)$, there exists a vertex u_4 which is adjacent to u_1 but not u_2, i.e., $u_4 \in N(u_1) \setminus N(u_2)$. □

The existence of such four vertices in connected t-regular non-complete partite graphs enables us to prove Theorem 2.

Proof for Theorem 2. We say that a t-matching M in $G = (V, E)$ is *perfect* if $d_M(v) = t$ for any $v \in V$. It suffices to show the NP-hardness of determining whether a given simple graph with maximum degree at most $t + 1$ has a perfect H-free t-matching or not. We now give a polynomial reduction from the 3-SAT problem to this problem.

Let a boolean expression $F = C^1 \wedge C^2 \wedge \cdots \wedge C^r$ be an instance of the 3-SAT problem. With the same argument as the proof for Theorem 15, for each boolean variable x_s and each clause C^i, associate them again with the graphs γ_s and Γ^i, respectively. Then, to each vertex v in them, we attach a complete graph with $t - 2$ vertices (a K_{t-2}) to v. More precisely, for each vertex v, we add $t - 2$ vertices $v_1, v_2, \ldots, v_{t-2}$ and $\binom{t-1}{2}$ edges each connecting a pair of distinct vertices in $\{v, v_1, v_2, \ldots, v_{t-2}\}$. We denote the obtained graphs by $\tilde{\gamma}_s$ and $\tilde{\Gamma}^i$, respectively. Such a construction maintains the property that the graph $\tilde{\gamma}_s$ has only two perfect H-free t-matchings either containing e_{s1} or e_{s0}, and we associate those with the value of x_s in the same way as before. The graph $\tilde{\Gamma}^i$ also has the property of Γ^i stated in Claim 18.

To ensure the consistency of y^i_k and x_s, we construct a new connector $\tilde{\beta}$ obtained from α by replacing cycles $(u_1, u_5, u_2, u_3, u_4)$ and $(v_1, v_5, v_2, v_3, v_4)$ with two graphs isomorphic to H and by attaching a K_{t-2} to each p_i for $i = 1, 2, 3$. More precisely, the new connector $\tilde{\beta}$ is defined in the following way. Let H_1 and H_2 be new graphs each isomorphic to H, and u_1, u_2, u_3, u_4 (resp. v_1, v_2, v_3, v_4) be vertices in H_1 (resp. H_2) satisfying the conditions in Claim 18. We define $\beta = (V_\beta, E_\beta)$ by

$$V_\beta = V(H_1) \cup V(H_2) \cup \{p_1, p_2, p_3\},$$
$$E_\beta = E(H_1) \cup E(H_2) \cup \{u_4p_1, u_3p_3, v_4p_1, v_3p_3, p_1p_2, p_2p_3\}.$$

The connector $\tilde{\beta}$ is obtained from β by attaching a K_{t-2} to p_i for $i = 1, 2, 3$. For two edges $j = u'u''$ and $e = v'v''$, we say that $\tilde{\beta}$ *connects* j and e if j and e are replaced by a new graph isomorphic to $\tilde{\beta}$ and edges $j' = u'u_1$, $j'' = u''v_2$, $e' = v'v_1$, and $e'' = v''u_2$, where $u_1, u_2 \in V(H_1)$ and $v_1, v_2 \in V(H_2)$. In the same way as the proof for Theorem 15, this connector $\tilde{\beta}$ connects the corresponding j and e by either connecting j^i_k and e_{s0} if $y^i_k = x_s$, or connecting j^i_k and e_{s1} if $y^i_k = \bar{x}_s$.

Now we show that the connector $\tilde{\beta}$ ensures the consistency of y^i_k and x_s. Since $d_{E \setminus E(H_1)}(u_i) = 1$ for $i = 1, 2, 3, 4$ and $d_{E \setminus E(H_1)}(u) = 0$ for each $u \in V(H_1) \setminus \{u_1, u_2, u_3, u_4\}$, for a perfect H-free t-matching M, the sequence $(d_{M \setminus E(H_1)}(u_1), d_{M \setminus E(H_1)}(u_2), d_{M \setminus E(H_1)}(u_3), d_{M \setminus E(H_1)}(u_4))$ is one of $(1, 0, 0, 1), (0, 1, 1, 0), (0, 0, 1, 1)$, and $(1, 1, 1, 1)$. By symmetry, the same thing holds for H_2. Then, since a perfect H-free t-matching M must contain p_1p_2 and p_2p_3, there are only two possible ways of M as shown in Figure 6.

(a) (b)

Figure 6: Possible matching in U

This shows that when a perfect H-free t-matching M includes either e' or e'', it contains both e' and e''. The same thing can be said for j' and j''. Furthermore, there is no perfect H-free t-matching M containing all four e', e'', j', j'', i.e., the edges j and e are not in M simultaneously.

Let G be the graph constructed by combining the above three types of graphs. Notice that G has a maximum degree at most $t+1$. Then G contains a perfect H-free t-matching if and only if $F = C^1 \wedge C^2 \wedge \cdots \wedge C^r = 1$ for some boolean assignment. Therefore, finding a perfect H-free t-matching in a given graph with maximum degree at most $t+1$ is NP-hard, which implies the theorem. □

Acknowledgement

The authors thank Kazuo Murota for his helpful comments.

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