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Naonori KAKIMURA and Kazuhisa MAKINO

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DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

WWW page: http://www.keisu.t.u-tokyo.ac.jp/research/techrep/index.html

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Naonori KAKIMURA[†‡] Kazuhisa MAKINO[‡§]

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Abstract

An independence system $\mathcal{F}$ is one of the most fundamental combinatorial concepts, which includes a variety of objects in graphs and hypergraphs such as matchings, stable sets, and matroids. We discuss the robustness for independence systems, which is a natural generalization of the greedy property of matroids. For a real number $\alpha > 0$, a set $X \in \mathcal{F}$ is said to be *α-robust* if for any k, it includes an α-approximation of the maximum k-independent set, where a set Y in $\mathcal{F}$ is called k-independent if the size $|Y|$ is at most k. In this paper, we show that every independence system has a $1/\sqrt{\mu(\mathcal{F})}$-robust independent set, where $\mu(\mathcal{F})$ denotes the *exchangeability* of $\mathcal{F}$. Our result contains a classical result for matroids and the ones of Hassin and Rubinstein [15] for matchings and Fujita, Kobayashi, and Makino [9] for matroid 2-intersections, and provides better bounds for the robustness for many independence systems such as b-matchings, hypergraph matchings, matroid p-intersections, and unions of vertex disjoint paths. Furthermore, we provide bounds of the robustness for nonlinear weight functions such as submodular and convex quadratic functions. We also extend our results to independence systems in the integral lattice with separable concave weight functions.

Key words: independence systems, matroids, exchangeability, robustness

1 Introduction

Let E be a finite set. A family $\mathcal{F}$ of subsets in E is an *independence system* if $\emptyset \in \mathcal{F}$, and $I \subseteq J \in \mathcal{F}$ implies $I \in \mathcal{F}$. A set F in $\mathcal{F}$ is called *independent*, and *k-independent* if $|F| \leq k$ holds in addition. For an independence system $\mathcal{F}$ with a nonnegative weight $w \in \mathbb{R}_+^E$ and a positive integer k, we consider the problem of finding a maximum weighted k-independent set, called *the maximum k-independent set problem.*

$$\text{Problem } \mathrm{P}_k(\mathcal{F}): \quad \begin{array}{ll} \text{maximize} & w(X) \\ \text{subject to} & |X| \leq k, \\ & X \in \mathcal{F}, \end{array}$$

where for $X \subseteq E$, we define $w(X) = \sum_{i \in X} w(i)$. If k is sufficiently large, e.g., $k \geq |E|$, then it is called *the maximum independent set problem*, and denoted by $\mathrm{P}(\mathcal{F})$. The maximum (k-)independent set problem is one of the most fundamental and important combinatorial

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†Department of Mathematical Informatics, Graduate School of Information Science and Technology, University of Tokyo, Tokyo 113-8656, Japan. kakimura@mist.i.u-tokyo.ac.jp

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§Department of Mathematical Informatics, Graduate School of Information Science and Technology, University of Tokyo, Tokyo 113-8656, Japan. makino@mist.i.u-tokyo.ac.jp

optimization problems, which includes a variety of graph and hypergraph problems such as matching, stable set, and matroid problems. See e.g., [21, 24, 31] for the details. The problem is NP-hard in general, while it is polynomially solvable if $\mathcal{F}$ belongs to some special classes. For example, it is well-known that, for a matroid $\mathcal{F}$, a greedy algorithm computes a maximum independent set [4, 30]. Note that this greedy solution X has a good property, called *robustness*, in the sense that it contains a maximum k-independent set for each k. More precisely, for each k, the heaviest k elements in X is an optimal solution for the maximum k-independent set problem. Thus the greedy solution X is adaptable to all the sizes k. This paper investigates this kind of structural properties for the independence systems.

For an independence system, a robust independent set does not always exist. Let X_k denote an optimal solution for Problem $\mathrm{P}_k(\mathcal{F})$, and for $X = \{x_1, \ldots, x_p\} \subseteq E$ with $w(x_i) \geq w(x_j)$ if $i < j$, we define

$$w_{\leq k}(X) = \sum_{i \leq k} w(x_i), \quad k = 1, 2, \ldots, |E|.$$

For a real number $\alpha > 0$, an independent set X is called *α-robust* (with respect to w) if $w_{\leq k}(X) \geq \alpha \cdot w(X_k)$ for all k's. By definition, a robust independent set is exactly 1-robust. We also say that an independence system $\mathcal{F}$ is *α-robust* if it has an α-robust solution for any nonnegative weight w.

Previous and Our Main Results for the Robustness

The α-robustness for independence systems was first introduced by Hassin and Rubinstein [15]. They proved that the greedy solution is ν-robust, where ν is the *rank quotient* defined in Section 2.2. Moreover, they showed that the maximum matching problem admits a $1/\sqrt{2}$-robust solution, and that $1/\sqrt{2}$-robustness is the best possible for the maximum matching problem. Fujita, Kobayashi, and Makino [9] extend their matching result to the matroid intersection problem, and show that computing an α-robust matching is NP-hard for any $\alpha\, (> 1/\sqrt{2})$. The robustness was also studied for several combinatorial optimization problems such as trees and paths [10, 16]. Another concept similar to the robustness, called the *incremental problems*, has been investigated for covering-type problems in connection with the online algorithms [26, 29].

In this paper, we analyze the robustness for an independence system $\mathcal{F}$ by using parameter $\mu(\mathcal{F})$, called the *exchangeability* of $\mathcal{F}$. For a nonnegative integer μ, we say that $\mathcal{F}$ is *μ-exchangeable* if

$$\forall X, Y \in \mathcal{F}, \forall i \in Y \setminus X, \exists Z \subseteq X \setminus Y \text{ s.t. } |Z| \leq \mu,\ X \cup \{i\} \setminus Z \in \mathcal{F}.$$

We denote by $\mu(\mathcal{F})$ the minimum μ satisfying the condition above. The exchangeability was introduced by Mestre [28][1] to measure the performance of the greedy algorithm. It is known [28] that $\mathcal{F}$ is a matroid if and only if $\mu(\mathcal{F}) \leq 1$, and that a number of independence systems arising from natural combinatorial optimization such as (hypergraph) matchings, stable sets, acyclic subgraphs, union of vertex disjoint paths, and matroid intersections, have bounded μ (see Section 2.1).

In this paper we obtain the following theorem.

Theorem 1. *Let $\mathcal{F}$ be an independence system on a finite set E. Then it is* $\min\{1, 1/\sqrt{\mu(\mathcal{F})}\}$*-robust. In particular, for any weight $w \in \mathbb{R}_+^E$, a w^2-optimal independent set is* $\min\{1, 1/\sqrt{\mu(\mathcal{F})}\}$*-robust with respect to w.*

Here, for a weight $u \in \mathbb{R}_+^E$, an independent set X of maximum weight $u(X)$ is called *u-optimal.* The vector u^b for $u \in \mathbb{R}_+^E$ is defined to be $(u^b)(i) = u(i)^b$ for $i \in E$.

[1] In fact, he introduced "μ-extendibility," which is equivalent to μ-exchangeability.

To obtain our robustness, we show a maximum weight independent set with respect to the b-th power weight w^b is $\min\{1, \mu(\mathcal{F})^{-1/b}, 1/\mu(\mathcal{F})^{-1+(1/b)}\}$-robust, where our main result is obtained by fixing $b = 2$. The statement is similar to Hassin and Rubinstein [15] and Fujita et al. [9], but different from their proofs, ours, described in Section 3, exploits a polyhedral description of the b-th power weight w^b such that a given set X is maximum with respect to w^b. Note that such constraints can be represented as (an exponential number of) inequalities which are linear in terms of w^b. We show that X is $\min\{1, \mu(\mathcal{F})^{-1/b}, 1/\mu(\mathcal{F})^{-1+(1/b)}\}$-robust with respect to any weight w satisfying the constraints, by considering the minimization of the total weight of the heaviest k elements in X subject to these inequalities.

We also show that the ratio $\min\{1, 1/\sqrt{\mu(\mathcal{F})}\}$ in Theorem 1 is tight.

Theorem 2. *For any positive integer μ, there exists an independence system $\mathcal{F}$ with $\mu(\mathcal{F}) = \mu$ and a weight $w \in \mathbb{R}_+^E$ such that for any $\alpha > 1/\sqrt{\mu}$, $\mathcal{F}$ has no α-robust independent set with respect to w.*

Since $\mathcal{F}$ is a matroid if and only if $\mu(\mathcal{F}) \leq 1$, Theorem 1 includes a classical result for the greediness of matroids [4, 30] (see Corollary 1 (i)), and it can be regarded as a generalization of Hassin and Rubinstein [15] and Fujita et al. [9] (see Corollary 1 (ii) with $b \equiv 1$ and (v) with $p = 2$, respectively), since independence systems from matchings and matroid 2-intersections are both 2-exchangeable. Moreover, our result implies the existence of highly robust independent sets for a variety of combinatorial optimization problems.

Corollary 1. **(i)** *Let $\mathcal{F}$ be a matroid. Then it is 1-robust.*

(ii) *For a graph $G = (V, E)$ with $b \in \mathbb{Z}_+^V$, let $\mathcal{F} \subseteq 2^E$ be the family of b-matchings in G. Then it is $1/\sqrt{2}$-robust.*

(iii) *For a complete directed graph $G = (V, E)$, we define $\mathcal{F} \subseteq 2^E$ to be the family of unions of vertex disjoint paths, i.e., $\mathcal{F} = \{P\,(= \bigcup_i P_i) \subseteq E \mid P_i$ are pairwise vertex disjoint paths$\}$. Then it is $1/\sqrt{3}$-robust.*

(iv) *For a hypergraph $\mathcal{E} \subseteq 2^V$ on a finite set V, let $\mathcal{F}$ be the family of matchings in $\mathcal{E}$, i.e., the family of disjoint hyperedges in $\mathcal{E}$. Let r denote the maximum number of disjoint neighbors of hyperedges. Then it is $1/\sqrt{r}$-robust. In particular, for any k-hypergraph $\mathcal{E}$ (i.e., $|E| \leq k$ for $E \in \mathcal{E}$), it is $1/\sqrt{k}$-robust.*

(v) *Let $\mathcal{F}$ be the intersection of $p\,(\geq 2)$ matroids. Then it is $1/\sqrt{p}$-robust.*

(vi) *For a d-dimensional knapsack problem, i.e., maximizing $w^{\mathrm{T}}x$ subject to $Ax \leq b$ and $x \in \{0,1\}^n$, where $A \in \mathbb{R}_+^{d\times n}$, $b \in \mathbb{R}_+^d$ and $w \in \mathbb{R}_+^n$, let $\mathcal{F}$ denote the independence system corresponding to the set of the feasible vectors. Define*

$$\mu(A) = \sum_{i=1}^{d} \left\lceil \frac{\max\{a_{ij} \mid 1 \leq j \leq n\}}{\min\{a_{ij} \mid a_{ij} \neq 0, 1 \leq j \leq n\}} \right\rceil.$$

Then it is $1/\sqrt{\mu(A)}$-robust.

(vii) *For a graph $G = (V, E)$, let $\mathcal{F} \subseteq 2^V$ denote the family of stable sets in G. Then it is $1/\sqrt{d_{\max}}$-robust, where $d_{\max}$ denotes the maximum degree of G.*

(viii) *For a directed graph $G = (V, E)$, let $\mathcal{F} \subseteq 2^E$ denote the family of acyclic subgraphs in G. Then it is $1/\sqrt{\lambda}$-robust, where λ denotes the maximum edge connectivity between two vertices.*

We remark that Theorem 1 improves upon the existing bounds for the robustness. For example, we have a stronger bound for the robustness of $\mathcal{F}$ if it is obtained from hypergraph matchings or unions of vertex disjoint paths in a complete directed graph (Corollary 1 (ii)(iii)(iv)). See Lemma 4 in Section 2. We also note that the bounds in the corollary are all tight as Theorem 2.

From a viewpoint of complexity, a robust solution stated above can be computed in polynomial time if the maximum independent set problem $\mathrm{P}(\mathcal{F})$ is polynomially solvable. For example, a $1/\sqrt{2}$-robust b-matching can be found in polynomial time, since the maximum b-matching problem can be computed in polynomial time [11]. If a given graph is perfect, we can find a $1/\sqrt{d_{\max}}$-robust stable set [14], and if a directed graph is planar, we can find a $1/\sqrt{\lambda}$-robust acyclic subgraph in polynomial time [7, 27]. However, it is often NP-hard to solve $\mathrm{P}(\mathcal{F})$. You might expect that a γ-approximation solution with respect to w^2 provides a $\sqrt{\frac{\gamma}{\mu}}$-robust set with respect to w. However, we show in Section 3.3 that a γ-approximation solution with respect to w^2 does not help to find a highly robust solution.

We further generalize Theorem 1 in the following two ways. First, we consider a non-linear weight $w : 2^E \to \mathbb{R}_+$. Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a one-dimensional function, and $q \in \mathbb{R}_+^E$ be a vector. We say that a weight set function $w : 2^E \to \mathbb{R}_+$ is *ρ-approximated by h and q* if

$$h(\sum_{i\in X} q(i)) \leq w(X) \leq \rho \cdot h(\sum_{i\in X} q(i)) \quad \text{for all } X \in 2^E.$$

If h is *monotone* (i.e., $h(x) \leq h(y)$ for any $x, y \in \mathbb{R}_+$ with $x \leq y$) and *submultiplicative* (i.e., $h(xy) \leq h(x)h(y)$ for any $x, y \in \mathbb{R}_+$), then we have the following result.

Theorem 3. *If a weight $w : 2^E \to \mathbb{R}_+$ is ρ-approximated by a monotone submultiplicative function h and $q \in \mathbb{R}_+^E$, then any independence system $\mathcal{F} \subseteq 2^E$ has a $\frac{1}{\rho \cdot h(\sqrt{\mu(\mathcal{F})})}$-robust independent set with respect to w.*

As corollaries, we have the robustness results for submodular and convex quadratic weight functions.

The second generalization is to independence systems in integral lattice $\mathbb{Z}_+^E$. We say that a function $w : \mathbb{Z}^E \to \mathbb{R}_+$ is *separable concave* if w can be written as $w(x) = \sum_{i\in E} w_i(x_i)$ for some one-dimensional concave functions $w_i : \mathbb{R} \to \mathbb{R}_+$ for $i \in E$. In this paper, we assume that $w_i(0) = 0$ and monotonicity.

Theorem 4. *Let $\mathcal{F}$ be a bounded independence system in $\mathbb{Z}_+^E$ and w be a separable concave function. Then $\mathcal{F}$ has a $\min\{1, 1/\sqrt{\mu(\mathcal{F})}\}$-robust independent vector with respect to w.*

This result leads to robustness for polymatroids, polymatroid intersections, and packing systems.

The rest of the paper is organized as follows. In Section 2, we define the exchangeability for independence systems and discuss its basic properties. In Section 3, we consider the robustness for independence systems in the Boolean lattice. In particular, we show Theorems 1 and 2 and Corollary 1. Section 4 is devoted to a nonlinear generalization of Theorem 1. Finally, Section 5 discusses extending Theorem 1 to separable concave maximization in integral lattice.

2 Exchangeable Independence Systems

For a nonnegative integer μ, we say that an independence system $\mathcal{F}$ is *μ-exchangeable* if

$$\forall X, Y \in \mathcal{F}, \forall i \in Y \setminus X, \exists Z \subseteq X \setminus Y \text{ with } |Z| \leq \mu \text{ and } X \cup \{i\} \setminus Z \in \mathcal{F}. \quad (1)$$

We denote by $\mu(\mathcal{F})$ the minimum μ satisfying (1). Note that $\mathcal{F}$ is 0-exchangeable if and only if $\mathcal{F} = 2^J$ for some $J \subseteq E$. Thus a 0-exchangeable independence system consists of the unique maximal independent set.

After providing a variety of examples of μ-exchangeable independence systems, in this section, we show further properties on μ-exchangeable independence systems. In particular, we prove that the exchangeability is at least the inverse of the rank quotient, and it is NP-hard to approximate $\mu(\mathcal{F})$ for a given $\mathcal{F}$.

2.1 Examples of Exchangeable Independence Systems

This subsection describes some basic independence systems with small μ.

Matroids: An independence system $\mathcal{M}$ is called a *matroid* if $M_1, M_2 \in \mathcal{M}$ and $|M_1| < |M_2|$ implies $M_1 \cup \{e\} \in \mathcal{M}$ for some $e \in M_2 \setminus M_1$. It is known in [28] that $\mathcal{F}$ is 1-exchangeable if and only if it is a matroid.

Matchings and b-Matchings of a Graph: For a graph $G = (V, E)$, let $\mathcal{F} \subseteq 2^E$ denote the family of matchings F in G, i.e., $e \cap e' = \emptyset$ holds for any distinct $e, e' \in F$. Note that for any edge $e \in E$ and any matching F, at most two edges in F intersect e. Thus $\mathcal{F}$ turns out to be 2-exchangeable. More generally, for a vector $b \in \mathbb{Z}_+^V$, we say that a subset $F \subseteq E$ is a *b-matching* if for all $v \in V$ the number of edges in F incident to v is at most b_v. Then the b-matchings also form a 2-exchangeable independence system.

Unions of Vertex Disjoint Paths and Asymmetric Traveling Salesman Systems: For a complete directed graph $G = (V, E)$, let $\mathcal{F}$ be the family of unions of vertex disjoint paths $P \subseteq E$, i.e., $\mathcal{F} = \{P\,(= \bigcup_i P_i) \subseteq E \mid P_i \text{ are pairwise vertex disjoint paths}\}$, and let $\mathcal{H}$ be the family of sets $H \subseteq E$ such that H is either unions of vertex disjoint paths or a Hamilton cycle. The family $\mathcal{H}$ is well studied to solve the maximum asymmetric traveling salesman problem, and it is known [20] that $\mathcal{H}$ is 3-exchangeable. Similarly to $\mathcal{H}$, we have $\mu(\mathcal{F}) \leq 3$.

Matchings in Hypergraphs: For a hypergraph $\mathcal{E} \subseteq 2^V$ on a finite set V, let $\mathcal{F}$ be the family of matchings in $\mathcal{E}$, i.e., the family of disjoint hyperedges in $\mathcal{E}$. Let r denote the maximum number of disjoint neighbors of hyperedges, i.e., $r = \max_{J \in \mathcal{E}} \{|\mathcal{M}| \mid \mathcal{M} \in \mathcal{F}, I \cap J \neq \emptyset \text{ for all } I \in \mathcal{M}\}$. Then it is not difficult to see that $\mathcal{F}$ is r-exchangeable. In particular, for any k-hypergraph $\mathcal{E}$ (i.e., $|E| \leq k$ for $E \in \mathcal{E}$), $\mathcal{F}$ is k-exchangeable. This problem is also known as *k-set packing* [3].

Mestre [28] provides a maximum profit scheduling problem as an example of the maximum independent set problem for $\mathcal{F}$ with $r = 2$. We also remark that the family of the triangles in a graph, i.e., complete subgraphs with size three, is an example with $r = 3$.

Intersections of p Matroids: For matroids $\mathcal{M}_i \subseteq 2^E\,(i = 1, \ldots, p)$, define $\mathcal{F} = \bigcap_{i=1}^p \mathcal{M}_i$. Then we can see that $\mathcal{F}$ is p-exchangeable [28].

Multidimensional Knapsack Systems: For a positive integer d, we consider the following *d-dimensional knapsack problem* with n items:

$$\begin{array}{ll} \text{maximize} & w^{\mathrm{T}} x \\ \text{subject to} & Ax \leq b, \\ & x \in \{0,1\}^n, \end{array}$$

where $A \in \mathbb{R}_+^{d \times n}$, $b \in \mathbb{R}_+^d$ and $w \in \mathbb{R}_+^n$. Let $\mathcal{F}$ denote the independence system corresponding to the set of the feasible vectors. Define

$$\mu(A) = \sum_{i=1}^{d} \left\lceil \frac{\max\{a_{ij} \mid 1 \leq j \leq n\}}{\min\{a_{ij} \mid a_{ij} \neq 0, 1 \leq j \leq n\}} \right\rceil.$$

Then we have

$$\mu(\mathcal{F}) \leq \mu(A) \leq d \left\lceil \frac{a_{\max}}{a_{\min}} \right\rceil,$$

where $a_{\min}$ and $a_{\max}$ denote the minimum and maximum values of nonzero entries in A, respectively.

Stable Sets of a Graph: For a graph $G = (V, E)$, let $\mathcal{F} \subseteq 2^V$ denote the family of stable sets (also called independent sets) in G, i.e., the set of vertices not directly connected by edges. Then it is $d_{\max}$-exchangeable, where $d_{\max}$ denotes the maximum degree of G. Note that if a graph G is claw-free, that is, a graph such that no vertex has a stable set of size three in its neighborhood, then it is not difficult to see that $\mathcal{F}$ is 2-exchangeable. The weighted stable set problem for claw-free graphs can be solved in polynomial time [5].

Acyclic Subgraphs: For a directed graph $G = (V, E)$, let $\mathcal{F} \subseteq 2^E$ denote the family of acyclic subgraphs in G. Let λ denote the maximum edge connectivity between two vertices. Then $\mathcal{F}$ is λ-exchangeable.

2.2 Exchangeability and Rank Quotient

Let $\mathcal{F}$ be an independence system. For $X, Y \in \mathcal{F}$, a pair (Z_X, Z_Y), where $Z_X \subseteq X \setminus Y$ and $Z_Y \subseteq Y \setminus X$, is said to be (X, Y)-*admissible* if $X \setminus Z_X \cup Z_Y \in \mathcal{F}$. For $X, Y \in \mathcal{F}$ with $Y \setminus X \neq \emptyset$, we denote

$$\mu_{\mathcal{F}}(X, Y) = \max_{i \in Y \setminus X} \min\{|Z| \mid (Z, \{i\}) \text{ is } (X, Y)\text{-admissible}\}.$$

If $Y \setminus X = \emptyset$, we define $\mu_{\mathcal{F}}(X, Y) = 0$. Then we have

$$\mu(\mathcal{F}) = \max_{X, Y \in \mathcal{F}} \mu_{\mathcal{F}}(X, Y).$$

If there is no ambiguity, we simply use $\mu(X, Y)$ and μ instead of $\mu_{\mathcal{F}}(X, Y)$ and $\mu(\mathcal{F})$, respectively.

We first observe the following lemma. For an independence system $\mathcal{F}$, let $\mathcal{B}_{\mathcal{F}}$ be the family of maximal independent sets in $\mathcal{F}$.

Lemma 1. *It holds that* $\mu(\mathcal{F}) = \max_{X, Y \in \mathcal{B}_{\mathcal{F}}} \mu_{\mathcal{F}}(X, Y)$.

Proof. Let $\mu' = \max_{X, Y \in \mathcal{B}_{\mathcal{F}}} \mu_{\mathcal{F}}(X, Y)$. It is obvious that $\mu(\mathcal{F}) \geq \mu'$ by $\mathcal{B}_{\mathcal{F}} \subseteq \mathcal{F}$. Let $X, Y \in \mathcal{F}$ and $i \in Y \setminus X$. We arbitrarily take $X', Y' \in \mathcal{B}_{\mathcal{F}}$ with $X \subseteq X'$ and $Y \subseteq Y'$. If $i \in X'$, then we have $X \cup \{i\} \in \mathcal{F}$. On the other hand, if $i \notin X'$, then we have an (X', Y')-admissible pair $(Z, \{i\})$ with $|Z| \leq \mu'$. By $X' \setminus Z \cup \{i\} \in \mathcal{F}$, the pair $(X \cap Z, \{i\})$ is an (X, Y)-admissible. Since $|X \cap Z| \leq |Z| \leq \mu'$, we obtain $\mu_{\mathcal{F}}(X, Y) \leq \mu'$, which implies $\mu(\mathcal{F}) \leq \mu'$. □

We next consider the exchangeability of independence systems obtained by contraction and deletion. For $Z \subseteq E$, we define the *contraction* $\mathcal{F}/Z$ and *deletion* $\mathcal{F} \setminus Z$ of $\mathcal{F}$ by $\mathcal{F}/Z = \{X \subseteq E \setminus Z \mid X \cup Z \in \mathcal{F}\}$ and $\mathcal{F} \setminus Z = \{X \in \mathcal{F} \mid X \subseteq E \setminus Z\}$.

Lemma 2. *Let $\mathcal{F} \subseteq 2^E$ be an independence system on E. Then for any $Z \subseteq E$, it holds that $\mu(\mathcal{F} \setminus Z), \mu(\mathcal{F}/Z) \leq \mu(\mathcal{F})$.*

Proof. It is not difficult to see $\mu(\mathcal{F} \setminus Z) \leq \mu(\mathcal{F})$. We denote $\mathcal{F}' = \mathcal{F}/Z$. Let X', Y' be two independent sets in $\mathcal{F}'$, and $i \in Y' \setminus X'$. Then $\mathcal{F}$ has an $(X' \cup Z, Y' \cup Z)$-admissible pair $(Z', \{i\})$ with $|Z'| \leq \mu(\mathcal{F})$. Hence $(Z', \{i\})$ is (X', Y')-admissible with respect to $\mathcal{F}'$. Thus $\mu(\mathcal{F}') \leq \mu(\mathcal{F})$ holds. □

For an independence system $\mathcal{F} \subseteq 2^E$, we define two parameters $\nu(\mathcal{F})$ and $\kappa(\mathcal{F})$.

It is known [23] that $\mathcal{F}$ can be represented as $\mathcal{F} = \bigcap_{i=1}^k \mathcal{M}_i$ for some k matroids $\mathcal{M}_i$. We denote by $\kappa(\mathcal{F})$ the minimum number of matroids to describe $\mathcal{F}$ as the matroid intersection. For $J \subseteq E$, let $\rho(J)$ and $\gamma(J)$ be the minimum and maximum sizes of maximal independent sets in $\mathcal{F}$ contained in J, respectively. Define the *rank quotient* $\nu(\mathcal{F})$ to be

$$\nu(\mathcal{F}) = \min_{J \subseteq E} \frac{\rho(J)}{\gamma(J)}.$$

Jenkyns [19] and Korte and Hausmann [23] showed the greedy algorithm finds a $\nu(\mathcal{F})$-approximation solution for $\mathrm{P}(\mathcal{F})$. Hassin and Rubinstein [15] proved that the greedy solution is in fact $\nu(\mathcal{F})$-robust.

The three parameters $\mu(\mathcal{F})$, $\nu(\mathcal{F})$, and $\kappa(\mathcal{F})$ have the following relations, which was also mentioned in [1].

Lemma 3. *For an independence system $\mathcal{F} \subseteq 2^E$ with $\mu(\mathcal{F}) \geq 1$, we have*

$$\frac{1}{\nu(\mathcal{F})} \leq \mu(\mathcal{F}) \leq \kappa(\mathcal{F}).$$

Proof. Let $J \subseteq E$, and X, Y with $|X| \geq |Y|$ be two maximal independent sets in J. Then $\mu(\mathcal{F}) \cdot |Y \setminus X| \geq |X \setminus Y|$ holds, since otherwise there exists an independent set Z with $Y \subsetneq Z \subseteq X \cup Y$, which contradicts the maximality of Y. Hence we have

$$\frac{|Y|}{|X|} = \frac{|Y \setminus X| + |X \cap Y|}{|X \setminus Y| + |X \cap Y|} \geq \min\left\{1, \frac{|Y \setminus X|}{|X \setminus Y|}\right\} \geq \frac{1}{\mu(\mathcal{F})},$$

which implies the first inequality in this statement.

To prove the second inequality of Lemma 3, let X, Y be two maximal independent sets in $\mathcal{F}$ such that $\mu(X, Y) = \mu(\mathcal{F})$. That is, there exist $Z \subseteq X \setminus Y$ and $i \in Y \setminus X$ with $X \setminus Z \cup \{i\} \in \mathcal{F}$ and $|Z| = \mu(\mathcal{F})$. Let $W = X \setminus Z \cup \{i\}$. We denote by $\mathcal{M}_j$ $(j = 1, \ldots, \kappa)$ matroids whose intersection forms $\mathcal{F}$, where $\kappa = \kappa(\mathcal{F})$. Let W_j be an independent set of $\mathcal{M}_j$ such that (i) $W \subseteq W_j \subseteq X \cup \{i\}$ and (ii) it is maximal in $X \cup \{i\}$. Since $\mathcal{M}_j$ is a matroid and $X \in \mathcal{M}_j$, we have $|W_j| \geq |X|$, and hence $|W_j \setminus W| \geq |Z| - 1$ holds. This implies

$$\sum_{j=1}^{\kappa} |W_j \setminus W| \geq \sum_{j=1}^{\kappa} (|Z| - 1) = \kappa(\mu(\mathcal{F}) - 1).$$

Since each element in Z can be an element of at most $\kappa - 1$ W_j's, we obtain

$$\sum_{j=1}^{\kappa} |W_j \setminus W| \leq (\kappa - 1)|Z| \leq (\kappa - 1)\mu(\mathcal{F}).$$

Combining these two inequalities, we have $\mu(\mathcal{F}) \leq \kappa(\mathcal{F})$. □

We remark that Lemma 3, together with Korte and Hausmann [23], implies that the greedy algorithm provides a $1/\mu(\mathcal{F})$-approximation solution for $\mathrm{P}(\mathcal{F})$, which was also shown in [28].

We note that in many cases the first inequality in Lemma 3 attains the equality.

Lemma 4. *If an independence system $\mathcal{F}$ with $\mu(\mathcal{F}) \geq 1$ satisfies one of the following conditions, then we have $\frac{1}{\nu(\mathcal{F})} = \mu(\mathcal{F})$.*

(i) *it is a matroid,*

(ii) *for a graph $G = (V, E)$, it is the family of matchings in G,*

(iii) *for a complete directed graph $G = (V, E)$ with $|V| \geq 4$, it is the family of unions of vertex disjoint paths $P \subseteq E$,*

(iv) *for a hypergraph $\mathcal{E} \subseteq 2^V$, it is the family of matchings in $\mathcal{E}$.*

Proof. (i): This follows from [23, 28].

(ii) and (iv): By Lemma 3, it suffices to show that $\frac{1}{\nu(\mathcal{F})} \geq \mu(\mathcal{F})$. Let X, Y be two matchings such that $\mu_{\mathcal{F}}(X, Y) = \mu(\mathcal{F})$, i.e., for some $i \in Y \setminus X$, a minimum $Z \subseteq X \setminus Y$ with $X \setminus Z \cup \{i\} \in \mathcal{F}$ satisfies $|Z| = \mu$. Note that i intersects all the elements in Z. By letting $J = Z \cup \{i\}$, we have $\gamma(J) = |Z|$ and $\rho(J) = 1$. This proves $\frac{1}{\mu(\mathcal{F})} \geq \nu(\mathcal{F})$.

(iii): Let $G = (V, E)$ be a complete directed graph with $V = \{1, \ldots, n\}$, where $n \geq 4$. Define $J = \{(1, 2), (2, 3), (3, 2), (3, 4)\}$. Then we have $\gamma(J) = 3$ and $\rho(J) = 1$, implying $\nu(\mathcal{F}) \leq 1/3$. Since $\frac{1}{\nu(\mathcal{F})} \leq \mu(\mathcal{F}) \leq 3$, we have $\frac{1}{\nu(\mathcal{F})} = \mu(\mathcal{F}) = 3$. □

By this lemma, we can see that the bound for the robustness by Theorem 1 is *stronger* than the one obtained by using rank quotient $\nu(\mathcal{F})$ in [15].

We finally remark that the gaps in the inequalities in Lemma 3 might be large in general.

The first example demonstrates that the gap between $1/\nu(\mathcal{F})$ and $\mu(\mathcal{F})$ becomes large with respect to $|E|$. Let $G = (V, E)$ be the directed graph with $V = \{1, 2, \ldots, n, a, b\}$ and $E = \{(a, j) \mid j = 1, 2, \ldots, n\} \cup \{(j, b) \mid j = 1, 2, \ldots, n\} \cup \{(b, a)\}$. We note that $|E| = 2n + 1$. Let $\mathcal{F} \subseteq 2^E$ be the independence system defined by $\mathcal{F} = \{F \subseteq E \mid \text{the subgraph } (V, F) \text{ is acyclic}\}$. Then we have $\mu(\mathcal{F}) = n$, because $\mu(E \setminus \{(b, a)\}, \{(b, a)\}) = n$ and $\mu(X, Y) \leq n$ for any X and Y in $\mathcal{F}$. However, we have $\nu(\mathcal{F}) = \frac{1}{2} + \frac{1}{2n}$. Indeed, it holds that $\frac{\rho(E)}{\gamma(E)} = \frac{n+1}{2n}$ and $\frac{\rho(J)}{\gamma(J)} \geq \frac{n+1}{2n}$ for any $J \subseteq E$. This shows $\mu(\mathcal{F})\nu(\mathcal{F}) = \Omega(n)$.

The next example shows that the gap between $\mu(\mathcal{F})$ and $\kappa(\mathcal{F})$ becomes $\Omega(|E|)$. Again, let $G = (V, E)$ and $\mathcal{F} \subseteq 2^E$ be the directed graph and independence system defined in the above paragraph, respectively. We denote $E_1 = \{(a, j) \mid j = 1, 2, \ldots, n\}$ and $E_2 = \{(j, b) \mid j = 1, 2, \ldots, n\}$. Define $\mathcal{H} \subseteq 2^E$ to be the independence system $\mathcal{H} = \{H \in \mathcal{F} \mid |H \cap E_i| \leq 2, \forall i \in \{1, 2\}\}$. Then it is not difficult to see $\mu(\mathcal{H}) = 2$. On the other hand, we claim that $\kappa(\mathcal{H}) = n + 1$. Let $C_j = \{(b, a), (a, j), (j, b)\}$ for $j = 1, \ldots, n$. Note that they are all circuits (i.e., minimal dependent sets) of $\mathcal{H}$. Since $\mathcal{F}$ is the intersection of n matroids each of which has exactly one circuit C_j, we know $\kappa(\mathcal{F}) \leq n$. Moreover, since the cardinality constraints for E_1 and E_2 form one matroid, we have $\kappa(\mathcal{H}) \leq n + 1$. Conversely, let $\mathcal{M}_p$ $(p = 1, \ldots, \kappa(\mathcal{H}))$ be matroids such that $\mathcal{H} = \bigcap_{p=1}^{\kappa(\mathcal{H})} \mathcal{M}_p$. Since C_j is a circuit of $\mathcal{H}$ for $j = 1, \ldots, n$, it must be a circuit of some matroid, say $\mathcal{M}_p$. If $\mathcal{M}_p$ has two distinct C_i and C_j as circuits, then the circuit axiom of matroids implies that $C = \{(a, i), (i, b), (a, j), (j, b)\}$ contains a circuit of $\mathcal{M}_p$. However, C is independent in $\mathcal{H}$, which is a contradiction. Hence each matroid $\mathcal{M}_p$ has at most one C_j as a circuit. Similarly, C_j and a circuit $\{(a, i), (a, j), (a, k)\}$ of $\mathcal{H}$ for distinct i, j, and k cannot be represented as one matroid. Thus at least $n + 1$ matroids are necessary to represent $\mathcal{H}$.

Before concluding this section, we show that computing $\mu(\mathcal{F})$ which is useful for the robustness by Theorem 1 is intractable. More precisely, we prove that it is NP-hard to approximate $\mu(\mathcal{F})$ by reducing the maximum stable set problem [17].

Theorem 5. *For an independence system $\mathcal{F}$ on E with $n = |E|$, $\mu(\mathcal{F})$ is not approximable within $n^{1/2-\varepsilon}$ for any $\varepsilon > 0$, unless $P = NP$.*

Proof. We reduce the maximum stable set problem to computing $\mu(\mathcal{F})$. Let $G = (V, E)$ be an undirected graph. We add a new vertex v and an edge connecting v and u for all $u \in V$. Let $\mathcal{F}$ denote the family of the stable sets in the resulting graph. Then $\mu(X, \{v\}) = |X|$ holds for any $X \in \mathcal{F}$ with $X \neq \{v\}$. Moreover, for any $X, Y \in \mathcal{F}$, we have $\mu(X, Y) \leq |X|$. Hence $\mu(\mathcal{F}) = \max_{X \in \mathcal{F}} |X|$, and thus the problem of computing $\mu(\mathcal{F})$ is equivalent to the maximum stable set problem for G. □

3 Robust Independence Systems in Boolean Lattice

In this section, we investigate the robustness for independence systems in Boolean lattice. Especially, we show Theorems 1, 2 and Corollary 1. Let us first consider Theorem 1.

Theorem 6. *Let $\mathcal{F}$ be an independence system on a finite set E and $w \in \mathbb{R}_+^E$ be a weight vector on E. Then, for $b \geq 1$, a w^b-optimal independent set is $\min\{1, 1/\mu(\mathcal{F})^{1/b}, 1/\mu(\mathcal{F})^{1-1/b}\}$-robust with respect to w.*

When b is sufficiently large, a w^b-optimal independent set can be obtained by a greedy algorithm for the original weight w. Thus the theorem implies that a greedy solution is $1/\mu$-robust. Theorem 1 is obtained by maximizing the formula in the theorem, i.e., when $b = 2$.

It should be noted that the ratio $1/\sqrt{\mu(\mathcal{F})}$ cannot be improved to $\sqrt{\nu(\mathcal{F})}$ in Theorem 1. Consider the same example $\mathcal{F}$ as in Section 2.2, that is, $\mathcal{F} = \{F \subseteq E \mid \text{the subgraph } (V, F) \text{ is acyclic}\}$, where $G = (V, E)$ is the directed graph with $V = \{1, 2, \ldots, n, a, b\}$ and $E = \{(a, j) \mid j = 1, 2, \ldots, n\} \cup \{(j, b) \mid j = 1, 2, \ldots, n\} \cup \{(b, a)\}$. Define a weight $w \in \mathbb{R}_+^E$ as $w(b, a) = 2$ and $w(e) = 1$ for $e \neq (b, a)$. Then, for any $X \in \mathcal{F}$, if $(b, a) \notin X$ the set X has robustness $\leq 1/2$. Otherwise (i.e., if $(b, a) \in X$), X has robustness $\leq \frac{n+1}{2n}$, since $E \setminus \{(b, a)\}$ is a unique w-optimal independent set. By $\mu(\mathcal{F}) = n$ and $\nu(\mathcal{F}) = \frac{n+1}{2n}$, $\mathcal{F}$ has a $1/\sqrt{\mu(\mathcal{F})}$-robust independent set, but no one with at least $\sqrt{\nu(\mathcal{F})}$-robustness.

We remark that it is natural to ask whether or not a given μ-exchangeable independence system has an α-robust independent set for a given $\alpha > 1/\sqrt{\mu}$. It is, however, NP-hard even when an independence system is the family of matchings in a bipartite graph [9].

3.1 The Proof of Theorem 6

In order to prove Theorem 6, we show Lemma 5 below. For two subsets $X, Y \in \mathcal{F}$, we denote $\mathcal{F}_{X,Y} = \{Z \in \mathcal{F} \mid X \cap Y \subseteq Z \subseteq X \cup Y\}$. We say that a weight vector $w \in \mathbb{R}_+^E$ is (X, Y)-*optimal* if w satisfies

$$w(X) \geq w(Z) \quad \text{for any } Z \in \mathcal{F}_{X,Y}. \tag{2}$$

Lemma 5. *Let $\mathcal{F}$ be an independence system on E, and X, Y be two sets in $\mathcal{F}$ with $X \cap Y = \emptyset$ and $|Y| = k$. If a weight vector $w \in \mathbb{R}_+^E$ is (X, Y)-optimal, then for any β with $0 \leq \beta \leq 1$,*

$$(w^\beta)_{\leq k}(X) \geq \min\left\{1, \frac{1}{\mu^\beta}, \frac{1}{\mu^{1-\beta}}\right\} w^\beta(Y).$$

Here we show that Lemma 5 immediately implies Theorem 6. We henceforth denote $\alpha(\mathcal{F}) = \min\{1, \frac{1}{\mu^\beta}, \frac{1}{\mu^{1-\beta}}\}$.

Proof of [Lemma 5 $\Longrightarrow$ Theorem 6]. Let k be a positive integer with $k \leq |E|$. Let X^* be a w^b-optimal independent set and X_k be a w-optimal k-independent set. We contract $X^* \cap X_k$, denoted by Z, as follows. Define $\mathcal{H} = \mathcal{F}/Z$ and $u(Y) = w(Y \cup Z)$ for $Y \in \mathcal{H}$. We denote $Y^* = X^* \setminus Z$ and $Y_k = X_k \setminus Z$. Then u^b is a (Y^*, Y_k)-optimal weight vector with respect to $\mathcal{H}$. This follows from that for any $J \in \mathcal{H}$ with $J \subseteq Y^* \cup Y_k$, we have $u(J) = w(J \cup Z) \leq w(X^*) = u(Y^*)$ by (2). Since $Y^* \cap Y_k = \emptyset$, it follows from Lemma 5 with $\beta = 1/b$ that

$$u_{\leq \ell}(Y^*) \geq \alpha(\mathcal{H}) u(Y_k),$$

where $\ell = |Y_k|$. Hence, we obtain, together with Lemma 2,

$$\begin{aligned} w_{\leq k}(X^*) &\geq w_{\leq \ell}(Y^*) + w(Z) \\ &= u_{\leq \ell}(Y^*) \\ &\geq \alpha(\mathcal{H}) u(Y_k) = \alpha(\mathcal{H}) w(X_k) \\ &\geq \alpha(\mathcal{F}) w(X_k). \end{aligned}$$

This means that X^* is $\alpha(\mathcal{F})$-robust. □

The Proof of Lemma 5

The rest of this subsection is devoted to the proof of Lemma 5. For that purpose, we first observe the following lemma about the sizes of two independent sets X and Y. For a weight vector w, we denote $\Gamma(w) = \{i \in E \mid w(i) \neq 0\}$.

Lemma 6. *Let X, Y be two independent sets in $\mathcal{F}$ with $X \cap Y = \emptyset$. If a weight vector $w \in \mathbb{R}_+^E$ is (X, Y)-optimal, then we have $\mu(\mathcal{F})\,|X| \geq |\Gamma(w) \cap Y|$.*

Proof. By the definition of $\mu(\mathcal{F})$, there exists a (Y, X)-admissible pair (Z, X) such that $Y \setminus Z \cup X \in \mathcal{F}$ and $|Z| \leq \mu(\mathcal{F})\,|X|$. Since $|X| \leq |Y \setminus Z \cup X|$ and $w(X) \geq w(Y \setminus Z \cup X)$ by (2), we have $w(Y \setminus Z) = 0$. This means $(Y \setminus Z) \cap \Gamma(w) = \emptyset$, and hence we have $\mu(\mathcal{F})\,|X| \geq |\Gamma(w) \cap Y|$. □

We will show Lemma 5 by induction on $|X|$. The following lemma asserts that it is true when $|X| = 1$.

Lemma 7. *Let $X \in \mathcal{F}$ with $|X| = 1$, and $Y \in \mathcal{F}$ with $X \cap Y = \emptyset$. Then $w^\beta(X) \geq \alpha(\mathcal{F}) w^\beta(Y)$ for any (X, Y)-optimal w.*

Proof. Let i be the index with $X = \{i\}$. The (X, Y)-optimality implies that $w(i) \geq w(Y)$. If $w(Y) = 0$, i.e., $q = |\Gamma(w) \cap Y| = 0$, then the lemma is clearly true. Otherwise, by maximizing $w^\beta(Y)$ subject to $w(Y) \leq w(i)$ and $q = |\Gamma(w) \cap Y|$, we have $w^\beta(Y) \leq w(i)^\beta q^{1-\beta}$. Since $q \leq \mu(\mathcal{F})$ by Lemma 6, we obtain

$$\frac{w^\beta(i)}{w^\beta(Y)} \geq \frac{w(i)^\beta}{w(i)^\beta q^{1-\beta}} \geq \frac{1}{\mu^{1-\beta}(\mathcal{F})} \geq \alpha(\mathcal{F}).$$

□

We assume that Lemma 5 is true when $|X| \leq p - 1$ and consider the case in which $|X| = p\,(\geq 2)$. By induction hypothesis, the following two lemmas hold for any (X, Y)-optimal w.

Lemma 8. *If $w(i) = 0$ for some $i \in X$, then we have $(w^\beta)_{\leq k}(X) \geq \alpha(\mathcal{F}) w^\beta(Y)$.*

Proof. For $i \in X$ with $w(i) = 0$, let $X' = X \setminus \{i\}$. Then we claim that w is (X', Y)-optimal. Indeed, for $Z \in \mathcal{F}_{X',Y}$, the set Z is contained in $\mathcal{F}_{X,Y}$. Hence $w(X) \geq w(Z)$ by (2), which implies that $w(X') \geq w(Z)$ holds by $w(i) = 0$. Therefore, w is (X', Y)-optimal, and hence by the induction hypothesis, we have

$$(w^\beta)_{\leq k}(X) = (w^\beta)_{\leq k}(X') \geq \alpha(\mathcal{F}) w^\beta(Y).$$

This proves Lemma 8. □

Lemma 9. *If there exists a set $Z \in \mathcal{F}_{X,Y}$ with $w(X) = w(Z)$, $X \cap Z \neq \emptyset$, and $X \cap Z \subsetneq X$. Then we have $w^\beta_{\leq k}(X) \geq \alpha(\mathcal{F}) w^\beta(Y)$.*

Proof. We denote $X_1 = X \cap Z$ and $Y_1 = Y \setminus (Y \cap Z)$. Let $X_2 = X \setminus X_1$ and $Y_2 = Y \setminus Y_1$. Define the two independence systems $\mathcal{F}_1 = \mathcal{F}/Y_2$ and $\mathcal{F}_2 = \mathcal{F}/X_1$.

We claim that w is (X_1, Y_1)-optimal with respect to $\mathcal{F}_1$. Let $Z_1 \in (\mathcal{F}_1)_{X_1,Y_1}$. By $Z_1 \cup Y_2 \in \mathcal{F}_{X,Y}$, we have $w(X) \geq w(Z_1 \cup Y_2)$. Since $w(X) = w(Z)$, it holds that $w(Z) \geq w(Z_1 \cup Y_2)$, and hence $w(X_1) \geq w(Z_1)$. Thus this claim holds. Moreover, since $X_2 \neq \emptyset$, it holds that $|X_1| < |X|$.

We next show that w is (X_2, Y_2)-optimal with respect to $\mathcal{F}_2$. Let $Z_2 \in (\mathcal{F}_2)_{X_2,Y_2}$. By $Z_2 \cup X_1 \in \mathcal{F}_{X,Y}$, we have $w(X) \geq w(Z_2 \cup X_1)$, and hence $w(X_2) \geq w(Z_2)$. Thus this claim holds. Moreover, since $X \setminus X_2 \neq \emptyset$, it holds that $|X_2| < |X|$.

Therefore, by applying the induction hypothesis to $\mathcal{F}_1$ and $\mathcal{F}_2$, together with Lemma 2, we obtain

$$\begin{aligned} (w^\beta)_{\leq k-\ell}(X_1) &\geq \alpha(\mathcal{F}_1) w^\beta(Y_1) \geq \alpha(\mathcal{F}) w^\beta(Y_1), \\ (w^\beta)_{\leq \ell}(X_2) &\geq \alpha(\mathcal{F}_2) w^\beta(Y_2) \geq \alpha(\mathcal{F}) w^\beta(Y_2), \end{aligned}$$

where $\ell = |Y_2|$. Hence we have

$$\begin{aligned} (w^\beta)_{\leq k}(X) &\geq (w^\beta)_{\leq k-\ell}(X_1) + (w^\beta)_{\leq \ell}(X_2) \\ &\geq \alpha(\mathcal{F}) w^\beta(Y_1) + \alpha(\mathcal{F}) w^\beta(Y_2) \\ &\geq \alpha(\mathcal{F}) w^\beta(Y). \end{aligned}$$

Thus the statement holds. □

For a vector $u \in \mathbb{R}^E_+$, we define a function $f_X : \mathbb{R}^E_+ \to \mathbb{R}_+$ to be $f_X(u) = (u^\beta)_{\leq k}(X)$. Given an (X, Y)-optimal vector $w \in \mathbb{R}^E_+$, let w^* be an (X, Y)-optimal vector such that $f_X(w^*)$ is minimum over $w^*(i) = w(i)$ for $i \in E \setminus X$. Note that such a w^* exists, because the feasible region represented by linear inequalities is nonempty, and $f_X(u)$ is continuous and nonnegative. We call such w^* a *minimizer* of f_X.

By Lemmas 8 and 9, we may assume that w^* satisfies

$$\begin{aligned} & w^*(X) > w^*(Z) \quad \text{for any } Z \in \mathcal{F}_{X,Y} \text{ with } X \cap Z \neq \emptyset \text{ and } X \cap Z \subsetneq X, \\ & w^*(X) \geq w^*(Z) \quad \text{for any } Z \in \mathcal{F}_{X,Y} \text{ with } X \subseteq Z \text{ or } X \cap Z = \emptyset, \\ & w^*(i) > 0, \qquad\quad \text{for any } i \in X. \end{aligned} \tag{3}$$

Note that the second inequality is equivalent to $w^*(X) \geq w^*(Y)$ and $w^*(X) \geq w^*(Z)$ for any $Z \in \mathcal{F}_{X,Y}$ with $X \subseteq Z$.

For a minimizer w^* of f_X satisfying (3), we first observe the following lemma.

Lemma 10. *Assume that a minimizer w^* of f_X satisfies* (3). *Then it holds that $w^*(X) = w^*(Y)$.*

Proof. Assume to the contrary that $w^*(X) > w^*(Y)$. We replace w^* with the vector u defined to be $u(i) = w^*(i) - \varepsilon$ for some small ε if $i \in X$ with $w^*(i) > 0$, and $u(i) = w^*(i)$ otherwise. Then u is still (X, Y)-optimal and $f_X(u) < f_X(w^*)$, a contradiction. Thus $w^*(X) = w^*(Y)$. □

We denote $W = w^*(X) = w^*(Y)$. Let $k' = \min\{p, k\}$, where we recall that $p = |X|$ and $k = |Y|$. For $i \in E$, the vector $\chi_i \in \mathbb{Z}^E$ is a unit vector such that $\chi_i(i) = 1$ and $\chi_i(j) = 0$ for $j \neq i$. Then the following lemma holds by the concavity of the β-th power of numbers in f_X.

Lemma 11. *Assume that a minimizer w^* of f_X satisfies* (3). *Then there exist $i_1 \in X$ and $s, t \in \mathbb{R}_+$ $(s \geq t)$ such that $w^*(i_1) = s$ and $w^*(j) = t$ for any $j \in X \setminus \{i_1\}$.*

Proof. We let $X = \{i_1, \dots, i_p\}$ in order that $w^*(i_1) \geq \cdots \geq w^*(i_p)$. Define $t = w(i_{k'})$.

First assume that there exists an index ℓ with $2 \leq \ell \leq k'$ such that $w^*(i_\ell) > t$. Among such ℓ, take the maximum ℓ, and denote $\delta = w^*(i_\ell) - t$. Let $\lambda = w^*(X) - \max\{w^*(Z) \mid i_1 \in Z, i_\ell \notin Z, Z \in \mathcal{F}_{X,Y}\}$. Then $\lambda > 0$ by the first inequality of (3). Define $\varepsilon = \min\{\lambda, \delta/2\} > 0$, and a weight vector $u = w^* + \varepsilon(\chi_{i_1} - \chi_{i_\ell})$. Then u remains (X, Y)-optimal. This is because $u(X) = w^*(X)$ and, for any $Z \in \mathcal{F}_{X,Y}$, if $u(Z) > w^*(Z)$ then $i_1 \in Z$ and $i_\ell \notin Z$, and hence we have $u(Z) \leq w^*(Z) + \lambda \leq w^*(X) = u(X)$. Moreover, since $u(i_1) \geq \cdots \geq u(i_p)$ by $2\varepsilon \leq \delta$, we obtain $f_X(w^*) - f_X(u) = (w^*(i_1))^\beta + (w^*(i_\ell))^\beta - (w^*(i_1) + \varepsilon)^\beta - (w^*(i_\ell) - \varepsilon)^\beta > 0$, because $a^\beta + b^\beta > (a + c)^\beta + (b - c)^\beta$ if $a \geq b$ and $c > 0$. This contradicts that w^* is a minimizer of f_X. Thus w^* satisfies $w^*(i_\ell) = t$ for any $\ell \in \{2, \dots, k'\}$.

Next assume that there exists an index $r \in \{k' + 1, \dots, p\}$ such that $w^*(i_r) < t$. Take the minimum r among such r, and denote $\delta = t - w^*(i_r) > 0$. Let $\lambda = w^*(X) - \max\{w^*(Z) \mid i_r \in Z, L \setminus Z \neq \emptyset, Z \in \mathcal{F}_{X,Y}\}$, where $L = \{i_1, \dots, i_{r-1}\}$. Then $\lambda > 0$ holds by (3). Let $\varepsilon = \frac{\min\{\lambda,\delta\}}{r} > 0$, and define a weight vector $u = w^* + \varepsilon \sum_{d=1}^{r-1} (\chi_{i_r} - \chi_{i_d})$. Then u is (X, Y)-optimal, because $u(X) = w^*(X)$ and for any $Z \in \mathcal{F}_{X,Y}$ with $i_r \in Z$ and $L \setminus Z \neq \emptyset$, $u(Z) \leq w^*(Z) + (r - 1)\varepsilon \leq w^*(X) = u(X)$. Since $u(i_1) \geq \cdots \geq u(i_p)$ by $\varepsilon \leq \delta/r$, we have $f_X(u) = f_X(w^*) - k'\varepsilon < f_X(w^*)$, which contradicts that w^* is a minimizer of f_X.

Therefore, w^* satisfies that $w^*(i_1) \geq t$ and $w^*(j) = t$ for any $j \in X \setminus \{i_1\}$. □

We further show the following lemma.

Lemma 12. *Assume that a minimizer w^* of f_X satisfies* (3). *Then $w^*(j) = W/p$ holds for $j \in X$.*

Proof. By Lemma 11, it holds that $w^*(i_1) = s$ for some $i_1 \in X$ and $w^*(j) = t\,(s \geq t)$ for any $j \in X \setminus \{i_1\}$. We will further show that $t = W/p$ or $t = 0$. The function f_X is equal to

$$f_X(w) = (W - (p - 1)t)^\beta + (k' - 1)t^\beta,$$

denoted by a one-dimensional function $g(t)$ for $0 \leq t \leq W/p$. By differentiating $g(t)$, we know that g is concave and

$$\min\left\{g(t) \mid 0 \leq t \leq \frac{W}{p}\right\} \geq \min\left\{k'(\frac{W}{p})^\beta, W^\beta\right\},$$

where the minimum value is $k'(W/p)^\beta$ if and only if $t = W/p$, and W^β if and only if $t = 0$.

Assume that $W/p > t > 0$. We denote $L = X \setminus \{i_1\}$. For $\varepsilon \in \mathbb{R}$, define the weight vector u_ε by $u_\varepsilon(i_1) = W - (p - 1)(t + \varepsilon)$ and $u_\varepsilon(j) = t + \varepsilon$ for any $j \in L$. Then, since w^* satisfies (3), both u_ε and $u_{-\varepsilon}$ are (X, Y)-optimal for some sufficiently small $\varepsilon > 0$. Indeed, if u_ε is not (X, Y)-optimal for any $\varepsilon > 0$, then there exists a set $Z \in \mathcal{F}_{X,Y}$ such that $w^*(X) = w^*(Z)$, $i_1 \notin Z$, and $Z \cap L \neq \emptyset$. Similarly, if $u_{-\varepsilon}$ is not, then there exists a set $Z \in \mathcal{F}_{X,Y}$ such that $w^*(X) = w^*(Z)$, $i_1 \in Z$, and $L \setminus Z \neq \emptyset$. In both cases, we have a contradiction to (3). Since one of $g(t-\varepsilon)$ or $g(t+\varepsilon)$ takes a smaller value than $g(t)$ for small $\varepsilon > 0$, this contradicts that w^* is a minimizer. Thus we know that $t = 0$ or $t = W/p$. By (3), we have $t > 0$, and hence $t = W/p$. □

We then prove the following lemma, which completes the proof of Lemma 5.

Lemma 13. *If $w^*(j) = W/p$ for $j \in X$, then it holds that $(w^\beta)_{\leq k}(X) \geq \alpha(\mathcal{F})w^\beta(Y)$.*

Proof. First consider the case of $p \geq k$. Each $i \in Y$ has an (X, Y)-admissible pair $(Z, \{i\})$ for some Z with $|Z| \leq \mu(\mathcal{F})$. Hence the equation (2) implies that $w^*(i) \leq w(Z) \leq \mu\frac{W}{p}$ for any $i \in Y$. Hence

$$\frac{(w^{*\beta})_{\leq k}(X)}{w^{*\beta}(Y)} \geq \frac{k(W/p)^\beta}{k(\mu W/p)^\beta} \geq \frac{1}{\mu^\beta} \geq \alpha(\mathcal{F}).$$

Next consider the case of $p < k$. Then, by letting $q = |\Gamma(w) \cap Y|$, we obtain $w^{*\beta}(Y) \leq q \cdot (W/q)^\beta$ by maximizing $w^{*\beta}(Y)$ subject to $w^*(Y) \leq W$. Hence

$$\frac{(w^{*\beta})_{\leq k}(X)}{w^{*\beta}(Y)} \geq \frac{p(W/p)^\beta}{q(W/q)^\beta} \geq (\frac{p}{q})^\beta \geq \frac{1}{\mu^\beta} \geq \alpha(\mathcal{F}),$$

where the second to last inequality follows from Lemma 6. □

3.2 Tight Examples for Theorem 1

This subsection discusses the tightness of the ratio in Theorem 1 and Corollary 1. Especially we prove Theorem 2.

Let p denote an integer with $p \geq 2$. For $i = 1, \ldots, p$, let $V_i = \{v_1^i, \ldots, v_p^i\}$, and $V = \bigcup_{i=1}^p V_i$. By definition, $|V| = p^2$. Let $\mathcal{E} \subseteq 2^V$ be the hypergraph defined as $\mathcal{E} = \{e_0, e_1, \ldots, e_p\}$, where $e_0 = (v_1^1, v_2^2, \ldots, v_p^p)$ and $e_j = (v_j^1, \ldots, v_j^p)$ for $j = 1, \ldots, p$. Let $\mathcal{F}$ be the family of matchings in $\mathcal{E}$. It follows that $\mu(\mathcal{F}) = p$. We define a weight $w \in \mathbb{R}_+^E$ as $w(e_0) = \sqrt{p}$ and $w(e_j) = 1$ for $j = 1, \ldots, p$.

We can see that $\mathcal{F}$ has exactly two maximal independent sets $I = \{e_1, \ldots, e_p\}$ and $J = \{e_0\}$. For a positive integer k, J is a w-optimal k-independent set if $k \leq \lfloor\sqrt{p}\rfloor$, and so is $\{e_j \mid j = 1, \ldots, k\}$ if $k > \sqrt{p}$. Hence, for any $\alpha > 1/\sqrt{p}$, J is not α-robust, since $w_{\leq p}(J)/w(I) = 1/\sqrt{p}$. Similarly, I is not α-robust, since $w_{\leq 1}(I)/w(J) = 1/\sqrt{p}$. Thus no independent set is α-robust. This proves Theorem 2.

Note that the above example also shows the tightness for Corollary 1 (ii), (iv) and (v), since $\mathcal{F}$ can be regarded as the family of matchings (if $p = 2$), hypergraph matchings, and p-matroid intersection. Since Corollary 1 (i) is clearly tight, we next show the tightness for (iii), (vi), (vii) and (viii).

(iii): Let $|V| \geq 4$, and let a, b, c, d be distinct four vertices in V. Define $w(e) = 1$ if $e = (a,b), (b,c), (c,d)$, $\sqrt{3}$ if $e = (c,b)$, and 0 otherwise. Then $\mathcal{F}$ is $1/\sqrt{3}$-robust, but not α-robust for any $\alpha > 1/\sqrt{3}$.

(vi): For $d = 1$, we consider the problem of maximizing $\sqrt{p}x(0) + \sum_{i=1}^p x(i)$ subject to $px(0) + \sum_{i=1}^p x(i) \leq p$ and $x(i) \in \{0,1\}$ for $i = 0, \ldots, p$. Then $\mathcal{F}$ is $1/\sqrt{p}$-robust, but not α-robust for any $\alpha > 1/\sqrt{p}$.

(vii): For a positive integer d, consider the star $G = (V, E)$ with $V = \{v, u_1, \ldots, u_d\}$ and $E = \{(v, u_i) \mid i = 1, \ldots, d\}$. Define a weight $w \in \mathbb{R}_+^V$ as $w(v) = \sqrt{d}$ and $w(u_i) = 1$ for $i = 1, \ldots, d$. This example shows the tightness for (vii).

(viii): Consider the directed graph $G = (V, E)$ with $V = \{v, u\}$ and $E = \{e_i \mid i = 0, \ldots, \lambda\}$ such that $e_0 = (v, u)$ and $e_i = (u, v)$ for $i = 1, \ldots, \lambda$. Define a weight $w \in \mathbb{R}_+^E$ to be $w(e_0) = \sqrt{\lambda}$ and $w(e_i) = 1$ for $i = 1, \ldots, \lambda$. This example shows the tightness for (viii).

3.3 Using Approximation Solution to Find a Robust Solution

This section discusses computational complexity to find a robust solution. We mentioned in Section 1 that if the maximum independent set problem $P(\mathcal{F})$ is polynomially solvable, then a robust solution stated in Theorem 1 can be computed in polynomial time. However, it is often NP-hard to solve $P(\mathcal{F})$, for example, if $\mathcal{F}$ is the family of matchings in a hypergraph [2, 3], the intersection of p matroids [23], or the family of stable sets in a graph. You might expect that a γ-approximation solution with respect to w^2 provides a $\sqrt{\frac{\gamma}{\mu}}$-robust set with respect to w. Here a set J in $\mathcal{F}$ is *γ-approximation with respect to u*, where $\gamma < 1$, if $u(J) \geq \gamma \cdot u(J^*)$ holds for a u-optimal set J^*. However, this is not the case.

Theorem 7. *Let E be a finite set with $|E| \geq 3$. There exist an independence system $\mathcal{F} \subseteq 2^E$ and a weight $w \in \mathbb{R}_+^E$ such that a γ-approximation solution with respect to w^2 is not $1/\sqrt{(\gamma^{-1}-1)(|E|-1)}$-robust.*

Proof. Let $G = (U, V; E)$ be a bipartite graph with $U = \{u_1, \ldots, u_n\}$, $V = \{v_1, \ldots, v_n\}$, and $E = \{e_i = (u_i, v_i) \mid i = 1, \ldots, n\} \cup \{f = (u_1, v_2)\}$. Let $\mathcal{F}$ be the family of matchings in G. Define a weight w on E as $w(e_i) = 1$ for $i = 1, \ldots, n$ and $w(f) = \sqrt{(\gamma^{-1}-1)n+2}$, where $\gamma < 1$. Then $\mathcal{F}$ contains exactly two maximal independent sets $M_1 = \{e_i \mid i = 1, \ldots, n\}$ and $M_2 = \{e_i \mid i = 3, \ldots, n\} \cup \{f\}$. Note that $w^2(M_1) = n$ and $w^2(M_2) = (n-2) + w(f)^2 = \gamma^{-1}n$. Since $\gamma < 1$, this implies M_2 is w^2-optimal and M_1 is a γ-approximation solution with respect to w^2. The ratio between $w_{\leq 1}(M_1)$ and the w-optimal 1-independent set $\{f\}$ is

$$\frac{w_{\leq 1}(M_1)}{w(f)} = \frac{1}{\sqrt{(\gamma^{-1}-1)n+2}} < \frac{1}{\sqrt{(\gamma^{-1}-1)(|E|-1)}}.$$

Thus M_1 is not $1/\sqrt{(\gamma^{-1}-1)(|E|-1)}$-robust. □

4 Maximizing Set Functions over Independence Systems

In this section, we generalize Theorem 1 to a non-linear weight $w : 2^E \to \mathbb{R}_+$, and prove Theorem 3. For a general weight function $w : 2^E \to \mathbb{R}_+$, we define functions $w_{\leq k}(J) : 2^E \to \mathbb{R}_+$ as

$$w_{\leq k}(J) = \max\{w(I) \mid I \subseteq J,\, |I| \leq k\}, \quad k = 1, \ldots, |E|.$$

Then α-robustness can be defined in a similar way to the linear weight case, that is, J is called *α-robust* if $w_{\leq k}(J) \geq \alpha \cdot w(X_k)$, where X_k is a w-optimal k-independent set.

Proof of Theorem 3. It follows from Theorem 1 that we have a $1/\sqrt{\mu(\mathcal{F})}$-robust set X^* with respect to the weight q. That is, X^* satisfies $q_{\leq k}(X^*) \geq q(X)/\sqrt{\mu(\mathcal{F})}$ for any $k \in \mathbb{Z}_+$ and $X \in \mathcal{F}$ with $|X| \leq k$.

For $k = 1, \ldots, |E|$, let $X^{(k)}$ be the subset of X^* with $q_{\leq k}(X^*) = q(X^{(k)})$. Then by the definition of h and q, we have $w(X^{(k)}) \geq h(q(X^{(k)}))$. This, together with $w_{\leq k}(X^*) \geq w(X^{(k)})$ and the robustness of X^* with respect to q, implies

$$w_{\leq k}(X^*) \;\geq\; h(q(X^{(k)})) \;\geq\; h\Big(q(X)/\sqrt{\mu(\mathcal{F})}\Big)$$

for any $X \in \mathcal{F}$ with $|X| \leq k$. Since $w(X) \leq \rho \cdot h(q(X))$ for any $X \subseteq E$, the submultiplicativity of h implies

$$w_{\leq k}(X^*) \geq \frac{w(X)}{\rho \cdot h\big(\sqrt{\mu(\mathcal{F})}\big)}.$$

Thus X^* has the desired robustness. □

As corollaries, we have the robustness results for submodular and convex quadratic weight functions as below.

We say that a set function $w : 2^E \to \mathbb{R}_+$ is *monotone* if $w(X) \leq w(Y)$ for any $X, Y \subseteq E$ with $X \subseteq Y$, and *submodular* if, for any $X, Y \subseteq E$,

$$w(X) + w(Y) \geq w(X \cap Y) + w(X \cup Y).$$

Submodular function optimization under combinatorial structure has been studied recently [12, 18]. We here discuss maximizing monotone submodular functions over independence systems. It is known that a greedy algorithm finds a $1/(\mu(\mathcal{F})+1)$-approximation solution [1, 6], and hence a greedy algorithm returns a $1/(\mu(\mathcal{F})+1)$-robust solution. A local search algorithm computes $1/(\kappa(\mathcal{F})+\varepsilon)$-approximation one for any $\varepsilon > 0$ [25].

For a monotone submodular function w, Goemans, Harvey, Iwata, and Mirrokni [13] presented a polynomial time algorithm for constructing an approximation of w with polynomially many queries. More precisely, their algorithm constructs a set function $\hat{w}(X) = \sqrt{\sum_{i \in X} q(i)}$ for some $q \in \mathbb{R}_+^E$ such that $\hat{w}(X) \leq w(X) \leq \rho \cdot \hat{w}(X)$ for any $X \subseteq E$, where $\rho = O(\sqrt{n} \log n)$ with $n = |E|$. If w is a matroid rank function, then their algorithm returns such a $\hat{w}$ with $\rho = \sqrt{n+1}$. Since $h(x) = \sqrt{x}$ is monotone and submultiplicative, the results in [13] imply Corollary 2.

Corollary 2. *Let $\mathcal{F}$ be an independence system on E with $n = |E|$. If a weight function $w : 2^E \to \mathbb{R}_+$ is monotone submodular, then $\mathcal{F}$ has a $\frac{1}{\mu(\mathcal{F})^{1/4} O(\sqrt{n} \log n)}$-robust independent set with respect to w. In particular, if w is a matroid rank function, it has a $\frac{1}{\mu(\mathcal{F})^{1/4}\sqrt{n+1}}$-robust independent set. In either case, such robust sets can be computed in polynomial time and number of queries if* $\mathrm{P}(\mathcal{F})$ *is polynomially solvable.*

In addition, we obtain a corollary when a weight w is convex and quadratic.

Corollary 3. *Let $\mathcal{F}$ be an independence system on E with $n = |E|$, and let $w : 2^E \to \mathbb{R}_+$ be a convex quadratic function, i.e., a function defined by $w(X) = \sum_{i,j \in X} a_{ij}$ for $X \subseteq E$ with a positive definite matrix $A = (a_{ij}) \in \mathbb{R}^{E \times E}$. Then $\mathcal{F}$ has a $\frac{\lambda_{\min}}{\sqrt{\mu(\mathcal{F})\lambda_{\max}}}$-robust independent set with respect to w, where $\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and minimum eigenvalues of A, respectively.*

Proof. It is not difficult to see that

$$\lambda_{\min}|X| \;\leq\; w(X) \;\leq\; \lambda_{\max} \cdot |X|,$$

where $\lambda_{\max}$ and $\lambda_{\min}$ denote the maximum and minimum eigenvalues of A, respectively. Therefore, by taking $h(x) = \lambda_{\min} \cdot x$ and $q = 1$, we have Corollary 3. □

5 Robust Independence Systems with Separable Concave Weight

This section deals with an *independence system in* $\mathbb{Z}_+^E$, that is, the family $\mathcal{F}$ of vectors in $\mathbb{Z}_+^E$ such that $0 \in \mathcal{F}$, and, for $x, y \in \mathbb{Z}_+^E$ with $x \leq y$, $y \in \mathcal{F}$ implies $x \in \mathcal{F}$. Here, $x \leq y$ for two vectors x, y denotes $x(i) \leq y(i)$ for all $i \in E$. A vector $x \in \mathcal{F}$ is called *independent*, and *k-independent* if $x(E) \leq k$ in addition, where $x(J) = \sum_{i \in J} x(i)$ for $J \subseteq E$.

For two vectors x, y in $\mathbb{Z}_+^E$, we define $x \wedge y$ and $x \vee y$ to be vectors such that $(x \wedge y)(i) = \min(x(i), y(i))$ and $(x \vee y)(i) = \max(x(i), y(i))$ for $i \in E$, respectively. Then, in a similar way to one in the Boolean lattice, we introduce the *μ-exchangeability* of an independence system $\mathcal{F}$ in $\mathbb{Z}_+^E$. We say that $\mathcal{F}$ is *μ-exchangeable* if it satisfies

$$\forall x, y \in \mathcal{F}, \forall i \in \Gamma(y - x \wedge y), \exists z \in \mathbb{Z}_+^E \text{ such that } z \leq x - x \wedge y, z(E) \leq \mu \text{ and } x + \chi_i - z \in \mathcal{F},$$

where χ_i for $i \in E$ is a unit vector such that $\chi_i(i) = 1$ and $\chi_j(j) = 0$ for $j \neq i$. The minimum μ that satisfies the condition above is denoted by $\mu(\mathcal{F})$.

In this section, we focus on maximizing a separable concave function over an independence system $\mathcal{F}$ in $\mathbb{Z}^E$. We say that a function $w : \mathbb{Z}^E \to \mathbb{R}_+$ is *separable concave* if w can be written as $w(x) = \sum_{i \in E} w_i(x(i))$ for some one-dimensional concave functions $w_i : \mathbb{R} \to \mathbb{R}_+$ for $i \in E$. In this paper, we assume that $w_i(0) = 0$ and monotonicity.

For a separable concave function w with a positive integer k, we define a function $w_{\leq k} : \mathbb{Z}^E \to \mathbb{R}_+$ as

$$w_{\leq k}(x) = \max\{w(y) \mid y \leq x,\, y(E) \leq k\}.$$

For a real number $\alpha > 0$, we say that an independent vector x is *α-robust* if for each k, $w_{\leq k}(x)$ is at least α times the maximum weight of k-independent vectors in $\mathcal{F}$.

We then extend Theorem 6 as follows. For a separable concave function w and $b \in \mathbb{R}_+$, we define $w^b : \mathbb{Z}_+^E \to \mathbb{R}_+$ as follows:

$$w^b(x) = \sum_{i \in E} \sum_{t=1}^{x(i)} (w_i(t) - w_i(t-1))^b.$$

Note that if w is linear, i.e., $w(x) = \sum_{i \in E} w(i)x(i)$ for some vector $w \in \mathbb{R}_+^E$, then w^b is equal to $w^b(x) = \sum_{i \in E} w(i)^b x(i)$. The following theorem is a generalization of Theorem 6.

Theorem 8. *Let $\mathcal{F}$ be a bounded independence system in $\mathbb{Z}_+^E$ and w be a separable concave function. Then, for $b \geq 1$, a w^b-optimal independent vector is $\min\{1, 1/\mu^\beta, 1/\mu^{1-\beta}\}$-robust with respect to w, where $\beta = 1/b$.*

Proof. We reduce to the case of an independence system in the Boolean lattice. Let $N_i = \max_{x \in \mathcal{F}} x(i)$ for $i \in E$. Define an independence system $\mathcal{H}$ as follows. The ground set S is defined to be $S = \bigcup_{i \in E} S_i$, where $S_i = \{e_i^1, \ldots, e_i^{N_i}\}$, and $\mathcal{H} \subseteq 2^S$ is to be $\mathcal{H} = \bigcup_{x \in \mathcal{F}} \{X \subseteq S \mid |X \cap S_i| = x(i), \forall i \in E\}$. It is not difficult to see $\mu(\mathcal{F}) = \mu(\mathcal{H})$.

For a given separable concave function w, define a weight vector $u \in \mathbb{R}_+^S$ to be $u(e_i^j) = w_i(j) - w_i(j-1)$ for $i \in E$ and $j \in \{1, \ldots, N_i\}$. Note that, since w is separable concave, $u(e_i^j) \geq u(e_i^{j+1})$ holds for $i \in E$ and $j \in \{1, \ldots, N_i - 1\}$. Then a w^b-optimal independent vector, say x^*, corresponds to the set X^* in $\mathcal{H}$ with $X^* \cap S_i = \{e_i^1, \ldots, e_i^{x^*(i)}\}$ for $i \in E$ and $u^b(X^*) = w^b(x^*)$. This set X^* is in fact u^b-optimal, because $\mathcal{H}$ has a u^b-optimal independent set J with $J \cap S_i = \{e_i^1, \ldots, e_i^{|J \cap S_i|}\}$. Similarly, for each size k, we can take a w-optimal k-independent vector $x^{(k)}$ and the corresponding u-optimal k-independent set $X^{(k)}$ with $u(X^{(k)}) = w(x^{(k)})$.

It follows from Theorem 6 that X^* is $\alpha(\mathcal{H})$-robust with respect to u, where $\alpha(\mathcal{H})$ is equal to $\min\{1, 1/\mu(\mathcal{H})^\beta, 1/\mu(\mathcal{H})^{1-\beta}\}$. That is, for each size k, it holds that $u_{\leq k}(X^*) \geq \alpha(\mathcal{H}) u(X^{(k)})$. By $u_{\leq k}(X^*) = w_{\leq k}(x^*)$, we have $w_{\leq k}(x^*) \geq \alpha(\mathcal{H}) w(x^{(k)}) = \alpha(\mathcal{F}) w(x^{(k)})$. Thus Theorem 4 holds. □

By setting $b = 2$ in the above theorem, we obtain Theorem 4.

Theorem 4 has applications as the following corollary. To prove (i), we exploits well-known properties of polymatroids. See e.g., [8] for the properties.

Corollary 4. (i) *Let $f : 2^E \to \mathbb{Z}_+$ be a monotone submodular function, and $\mathcal{F}$ be the set of integral vectors in $\mathcal{P}(f)$, where $\mathcal{P}(f) = \{x \in \mathbb{R}_+^E \mid \sum_{i \in J} x(i) \leq f(J), \forall J \subseteq E\}$ denotes the* polymatroid. *Then it is* 1*-robust.*

(ii) *Let $f_i : 2^E \to \mathbb{Z}_+ \, (i = 1, \ldots, p)$ be monotone submodular functions, and $\mathcal{F}$ be the set of integral vectors in $\bigcap_{i=1}^p \mathcal{P}(f_i)$. Then it is $1/\sqrt{p}$-robust.*

(iii) *Let $\mathcal{F}$ be the set of feasible vectors of $\{x \in \mathbb{Z}_+^n \mid Ax \leq b\}$, where $A \in \mathbb{R}_+^{d\times n}$ and $b \in \mathbb{R}_+^d$. Then $\mathcal{F}$ is $1/\sqrt{\mu(A)}$-robust.*

Proof. (i) We first claim that $\mathcal{F}$ is 1-exchangeable. Since Lemma 1 also holds for bounded independence systems in integral lattice, it suffices to discuss the exchangeability of two maximal vectors x, y in $\mathcal{F}$. Note that x and y are known to be contained in the *base polytope*, i.e., $\mathcal{B}(f) = \{x \in \mathcal{P}(f) \mid x(E) = f(E)\}$. This fact implies that, for $i \in \Gamma(y - x \wedge y)$, there exists $j \in \Gamma(x - x \wedge y)$ such that $x + \chi_i - \chi_j \in \mathcal{F}$. Thus $\mathcal{F}$ is 1-exchangeable, and hence, by Theorem 4, $\mathcal{F}$ has a 1-robust independent vector. Thus (i) holds. (ii) easily follows from (i). (iii) is the same as the Boolean case. □

References

[1] G. Calinescu, C. Chekuri, M. Pal, and J. Vondrak, Maximizing a submodular set function subject to a matroid constraint, *SIAM Journal on Computing*, to appear.

[2] Y. H. Chan and L. C. Lau, On linear and semidefinite programming relaxations for hypergraph matching, *Proceedings of the 21st Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2010)*, 2010, pp. 1500–1511.

[3] B. Chandra and M. Halldórsson, Greedy local improvement and weighted set packing approximation, *Journal of Algorithms*, 39 (2001), pp. 223–240.

[4] J. Edmonds, Matroids and the greedy algorithm, *Mathematical Programming*, 1 (1971), pp. 127–136.

[5] Y. Faenza, G. Oriolo, and G. Stauffer, An algorithmic decomposition of claw-free graphs leading to an $O(n^3)$-algorithm for the weighted stable set problem, *Proceedings of the 22nd Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2011)*, 2011, pp. 630–647.

[6] M. L. Fisher, G. L. Nemhauser, and L. A. Wolsey, An analysis of approximations for maximizing submodular set functions ii, *Mathematical Programming Study*, 8 (1978), pp. 73–87.

[7] A. Frank, How to make a digraph strongly connected, *Combinatorica*, 1 (1981), pp. 145–153.

[8] S. Fujishige, *Submodular Functions and Optimization*, 2nd ed., Elsevier, 2005.

[9] R. Fujita, Y. Kobayashi, and K. Makino, Robust matchings and matroid intersections, *Proceedings of the 18th Annual European Symposium on Algorithms (ESA 2010)*, Lecture Notes in Computer Science 6347, Springer-Verlag, pp. 123–134.

[10] T. Fukunaga, M. Halldórsson, and H. Nagamochi, Robust cost colorings, *Proceedings of the 19th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2008)*, 2008, pp. 1204–1212.

[11] H. N. Gabow, An efficient reduction technique for degree-constrained subgraph and bidirected network flow problems, *Proceedings of the 15th ACM Symposium on Theory of Computing (STOC 1983)*, 1983, pp. 448–456.

[12] G. Goel, C. Karande, R. Tripathi, and L. Wang, Approximability of combinatorial problems with multi-agent submodular cost functions, *Proceedings of the 50th Annual Symposium on Foundations of Computer Science (FOCS 2009)*, 2009.

[13] M. X. Goemans, N. J. A. Harvey, S. Iwata, and V. Mirrokni, Approximating submodular functions everywhere, *Proceedings of the 20th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2009)*, 2009, pp. 535–544.

[14] M. Grötschel, L. Lovász, and A. Schrijver, *Geometric Algorithms and Combinatorial Optimization*, Springer-Verlag, 2nd ed., 1993.

[15] R. Hassin and S. Rubinstein, Robust matchings, *SIAM Journal on Discrete Mathematics*, 15 (2002), pp. 530–537.

[16] R. Hassin and D. Segev, Robust subgraphs for trees and paths, *ACM Transaction on Algorithms*, 2 (2006), pp. 263–281.

[17] J. Hásted, Clique is hard to approximate within $n^{1-\varepsilon}$, *Acta Mathematica*, 182 (1999), pp. 105–142.

[18] S. Iwata and K. Nagano, Submodular function minimization under covering constraints, *Proceedings of the 50th Annual Symposium on Foundations of Computer Science (FOCS 2009)*, 2009.

[19] T. A. Jenkyns, The efficacy of the "greedy" algorithm, *Proceedings of the 7th Southeastern Conference on Combinatorics, Graph Theory and Computing*, pp. 341–350, 1976.

[20] T. A. Jenkyns, The greedy traveling salesman's problem, *Networks*, 9 (1979), pp. 363–373.

[21] D. Jungnickel, *Graphs, Networks, and Algorithms*, vol. 5 of Algorithms and Computation in Mathematics, Springer-Verlag, 2nd ed., 2002.

[22] N. Kakimura and K. Makino, Robust independence systems, *Proceedings of the 38th International Colloquium on Automata, Languages and Programming (ICALP 2011)*, Lecture Notes in Computer Science, to appear, 2011.

[23] B. Korte and D. Hausmann, An analysis of the greedy heuristic for independence systems, *Annals of Discrete Mathematics*, 2 (1978), pp. 65–74.

[24] B. Korte and J. Vygen, *Combinatorial Optimization: Theory and Algorithms*, Springer-Verlag, 2006.

[25] J. Lee, M. Sviridenko, and J. Vondrák, Submodular maximization over multiple matroids via generalized exchange properties, *Mathematics of Operations Research*, to appear (2010).

[26] G. Lin, C. Nagarajan, R. Rajarama, and D. Williamson, A general approach for incremental approximation and hierarchical clustering, *SIAM Journal on Computing*, 39 (2010), pp. 3633–3669.

[27] C. L. Lucchesi, *A Minimax Equality for Directed Graphs*, PhD thesis, University of Waterloo, 1976.

[28] J. Mestre, Greedy in approximation algorithms, *Lecture Notes in Computer Science* 4168 *(ESA 2006)*, 2006, pp. 528–539.

[29] R. R. Mettu and C. G. Plaxton, The online median problem, *SIAM Journal on Computing*, 32 (2003), pp. 816–832.

[30] R. Rado, Note on independence relations, *Proceedings of the London Mathematical Society*, 7 (1957), pp. 300–320.

[31] A. Schrijver, *Combinatorial Optimization — Polyhedra and Efficiency*, Springer-Verlag, 2003.