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Algorithms for Finding a Maximum Non- k -linked Graph*

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Abstract

A graph with at least $2k$ vertices is said to be k -linked if for any ordered k -tuples (s_1, \dots, s_k) and (t_1, \dots, t_k) of $2k$ distinct vertices, there exist pairwise vertex-disjoint paths P_1, \dots, P_k such that P_i connects s_i and t_i for $i = 1, \dots, k$. For a given graph G , we consider the problem of finding a maximum induced subgraph of G that is not k -linked. This problem is a common generalization of computing the vertex-connectivity and testing the k -linkedness of G , and it is closely related to the concept of H -linkedness. In this paper, we give the first polynomial-time algorithm for the case of $k = 2$, whereas a similar problem that finds a maximum induced subgraph without 2-vertex-disjoint paths connecting fixed terminal pairs is NP-hard. For the case of general k , we give an $(8k - 2)$ -additive approximation algorithm. We also investigate the computational complexities of the edge-disjoint case and the directed case.

Key Words: k -linkedness, H -linkedness, disjoint paths, connectivity

1 Introduction

A graph is said to be k -linked if it has at least $2k$ vertices and for any ordered k -tuples (s_1, \dots, s_k) and (t_1, \dots, t_k) of $2k$ distinct vertices, there exist pairwise vertex-disjoint paths P_1, \dots, P_k such that P_i connects s_i and t_i for $i = 1, \dots, k$. The k -linkedness has been well-studied by many graph theorists, and there are many results on relationships between the k -linkedness and the vertex-connectivity of graphs [1, 8, 10, 14, 21]. From the algorithmic point of view, the k -linkedness has attracted attention because of similarities with the vertex-disjoint paths problem, which is one of the most important problems in computer science and algorithmic graph theory. In the *vertex-disjoint paths problem*, we are given a graph G and $2k$ distinct vertices $s_1, \dots, s_k, t_1, \dots, t_k$ called *terminals*, and the objective is to find pairwise vertex-disjoint paths P_1, \dots, P_k such that P_i connects s_i and t_i for $i = 1, \dots, k$. With the terminology of the vertex-disjoint paths problem, a graph is k -linked if and only if the vertex-disjoint paths problem has a solution for any choice of $2k$ terminals. In this paper, we consider the problem of finding a minimum number of vertices whose removal makes the graph non- k -linked, which can be stated as follows.

Max Non- k -Linked Induced Subgraph

Input. A graph $G = (V, E)$.

Problem. Find a vertex set $V_0 \subseteq V$ with maximum cardinality such that $G[V_0]$ (the subgraph induced by V_0) is not k -linked.

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We mainly discuss the case of $k = 2$, which is interesting because of its relation to the problem of finding a maximum planar induced subgraph. By a classical result on the 2 vertex-disjoint paths problem [18], it is well-known that the graph is not 2-linked if and only if it cannot be embedded in a plane up to “3-separations” (see Theorem 3.2 for the precise statement). That is, the non-2-linkedness is a similar concept to the planarity. The problem of finding a maximum planar induced subgraph is an important problem in theoretical computer science, because it amounts to computing a measure for non-planarity of graphs (see e.g. [2, 15]). **Max Non-2-Linked Induced Subgraph** can also be regarded as a problem of computing a measure for non-planarity of graphs, which is one of our motivations for studying **Max Non-k-Linked Induced Subgraph**. As we will describe later, we show that **Max Non-2-Linked Induced Subgraph** can be solved in polynomial time (Theorem 3.1). This result is surprising because most of all natural problems of computing measures for non-planarity, such as finding a maximum planar (induced) subgraph or computing the minimum number of crossings in an embedding in a plane, are known to be NP-hard (see [15]).

Max Non-k-Linked Induced Subgraph is motivated also by the concept of H -linkedness that has been studied [4, 6, 11, 12, 13] as a common generalization of the graph connectivity and the k -linkedness. For a multigraph H , an H -subdivision in a graph G is a pair of mappings $f : V(H) \rightarrow V(G)$ and $g : E(H) \rightarrow \mathcal{P}$, where \mathcal{P} is the set of paths in G , such that:

1. $f(u) \neq f(v)$ for all distinct $u, v \in V(H)$,
2. $g(uv)$ is a path connecting $f(u)$ and $f(v)$ in G for $uv \in E(H)$, and the paths are internally disjoint.

For a multigraph H , a graph G is H -linked if every injective mapping $f : V(H) \rightarrow V(G)$ can be extended to an H -subdivision in G . This is a generalization of the notions of k -linkedness and k -connectivity, because the H -linkedness is equivalent to the k -linkedness when H is a matching with k edges, and it is equivalent to the k -connectivity when H consists of $k + 1$ vertices and one edge.

For a multigraph H (or an integer k , respectively), determining whether a given graph G is H -linked (resp. k -linked) or not is a natural algorithmic problem. When a multigraph H (resp. an integer k) is fixed, Robertson and Seymour [16] gave a polynomial-time algorithm for this problem based on their seminal work on graph minor project, which spans 23 papers and gives several deep and profound results in discrete mathematics. On the other hand, when H or k is a part of the input, no polynomial-time algorithm is known for the problem. Thus, determining the H -linkedness for non-fixed multigraphs H is an interesting open problem.

Our second motivation for **Max Non-k-Linked Induced Subgraph** comes from the fact that it is a special case of the problem of determining the H -linkedness. More precisely, **Max Non-k-Linked Induced Subgraph** is equivalent to the case when H is a union of a matching of size k and l distinct vertices, i.e., H has $2k + l$ vertices and k edges. Let us emphasize that k and/or l are a part of the input throughout this paper, and so this problem setting is completely different from the case when H is fixed. Note that when $k = 1$, determining the H -linkedness is equivalent to testing the vertex-connectivity of an input graph. On the other hand, the polynomial-time solvability of the case of $k = 2$, which corresponds to **Max Non-2-Linked Induced Subgraph**, is non-trivial.

We can also consider the edge-disjoint version. We say that a graph G is *weakly k -linked* if for any ordered k -tuples (s_1, \dots, s_k) and (t_1, \dots, t_k) of $2k$ vertices (not necessarily distinct), there exist pairwise edge-disjoint paths P_1, \dots, P_k such that P_i connects s_i and t_i for $i = 1, \dots, k$.

Max Weakly Non-k-Linked Subgraph

Input. A graph $G = (V, E)$.

Problem. Find an edge set $E_0 \subseteq E$ with maximum cardinality such that the subgraph $G_0 = (V, E_0)$ is not weakly k -linked.

Note that we find an edge set E_0 in this problem, whereas we find a vertex set V_0 in **Max Non- k -Linked Induced Subgraph**. This problem setting is natural because the weakly k -linkedness is closely related to the edge-connectivity rather than the vertex-connectivity. When $k = 1$, **Max Weakly Non- k -Linked Subgraph** is equivalent to computing the edge-connectivity of an input graph, that is, the edge-connectivity is c if and only if the optimal value is $|E| - c$. Thus, this problem is a generalization of computing the edge-connectivity and testing the weakly k -linkedness. We also note that in the same way as the relationship between **Max Non- k -Linked Induced Subgraph** and the H -linkedness, **Max Weakly Non- k -Linked Subgraph** is related to the concept of H -immersion studied in [3, 17].

Related work: Many graph theorists are interested in how much connectivity is necessary to ensure that a graph is k -linked [1, 8, 10, 14, 21]. It is shown (implicitly) in [21] that, every $10k$ -connected graph is k -linked, which is the currently best bound. Similar results are known for the edge-disjoint case, that is, it is shown in [7] that every $(k + 2)$ -edge-connected graph is weakly k -linked. In the same way as the k -linkedness, the main interest on the H -linkedness goes to sufficient conditions for graphs to be H -linked [4, 6, 11, 12, 13].

The k -linkedness is closely related to the vertex-disjoint paths problem with k terminal pairs (k -vertex-disjoint paths problem). When k is a part of the input of the problem, this is one of Karp's NP-complete problems [9]. In 1980, it was shown that the 2-vertex-disjoint paths problem is solvable in polynomial time [18, 20, 22]. In particular, the following characterization is shown for the existence of the 2-vertex-disjoint paths.

Theorem 1.1 ([18]). *Let $G = (V, E)$ be a graph and let s_1, t_1, s_2, t_2 be distinct terminals. The 2-vertex-disjoint paths problem has no solution if and only if there exists a partition U, A_1, \dots, A_l ($l \geq 0$) of V with $s_1, t_1, s_2, t_2 \in U$ such that*

- (1) for $1 \leq i, j \leq l$ with $i \neq j$, $N(A_i) \cap A_j = \emptyset$,
- (2) for $1 \leq i \leq l$, $|N(A_i)| \leq 3$ and $G[A_i]$ is connected, and
- (3) if G' is the graph obtained from G by deleting A_i and adding new edges joining every pair of distinct vertices in $N(A_i)$ for every i , then G' can be embedded in a plane so that s_1, s_2, t_1, t_2 are on the outer boundary of G' in this order.

The characterization will be used in our argument. On the other hand, the 2-vertex-disjoint paths problem (or the 2-edge-disjoint paths problem) in digraphs, in which we find directed paths P_1, P_2 such that P_i is from s_i to t_i for $i = 1, 2$, was shown to be NP-hard [5].

For fixed k , Robertson and Seymour [16] gave a polynomial-time algorithm for the k -vertex-disjoint (edge-disjoint) paths problem based on their graph minor theory. When k is fixed, by solving the k -vertex-disjoint (or edge-disjoint) paths problem for every choice of the terminals, the k -linkedness (resp. the weakly k -linkedness) of an input graph can be tested in polynomial time. Similarly, for any fixed multigraph H , we can determine whether an input graph is H -linked or not in polynomial time. We emphasize here that this algorithm runs in polynomial time only when H is fixed.

Our contributions: In this paper, we consider algorithms for **Max Non- k -Linked Induced Subgraph** and the corresponding edge-disjoint version (**Max Weakly Non- k -Linked Subgraph**). We summarize our results in Table 1.

Table 1: Our results on the problems

| Problems | Vertex-disjoint case | Edge-disjoint case |
|-----------------------------------|----------------------------------|-----------------------------|
| Max Non-2-Linked Induced Subgraph | P (Thm. 3.1) | P (Cor. 2.4) |
| Max 2-VDP-free Induced Subgraph | NP-hard (Thm. 4.1) | P (Thm. 2.3) |
| Max Non-k-Linked Induced Subgraph | $\text{OPT} - 8k + 2$ (Cor. 5.4) | $\text{OPT} - 2$ (Cor. 5.2) |

First, we show that **Max Non-2-Linked Induced Subgraph** can be solved in polynomial time (Theorem 3.1), which is one of the main results in this paper. This problem corresponds to the H -linkedness, where H consists of two edges and some isolated vertices. This is the first non-trivial case in which the H -linkedness can be determined in polynomial time when H is not fixed. We also give a polynomial-time algorithm for **Max Weakly Non-2-Linked Subgraph** (Corollary 2.4).

We now give some remarks on proof techniques for Theorem 3.1 and Corollary 2.4. A natural approach to solve **Max Non-2-Linked Induced Subgraph** is to consider the problem of finding a maximum vertex set whose inducing subgraph contains no two vertex-disjoint paths connecting fixed terminal pairs. We call it **Max 2-VDP-free Induced Subgraph**, whose formal description is as follows.

Max 2-VDP-free Induced Subgraph

Input. A graph $G = (V, E)$ and distinct terminals $s_1, t_1, s_2, t_2 \in V$.

Problem. Find a vertex set $V_0 \subseteq V$ with maximum cardinality such that $\{s_1, t_1, s_2, t_2\} \subseteq V_0$ and the vertex-disjoint paths problem with terminal pairs (s_1, t_1) and (s_2, t_2) has no solution in $G[V_0]$.

We can easily see that by solving **Max 2-VDP-free Induced Subgraph** for every choice of the terminals s_1, t_1, s_2, t_2 , we obtain a solution of **Max Non-2-Linked Induced Subgraph**. However, we show that **Max 2-VDP-free Induced Subgraph** is NP-hard (Theorem 4.1), which suggests that this reduction does not work for solving **Max Non-2-Linked Induced Subgraph**. Therefore, we need another approach to **Max Non-2-Linked Induced Subgraph**.

In the same way as **Max 2-VDP-free Induced Subgraph**, we consider the edge-disjoint version of the problem, which we call **Max 2-EDP-free Subgraph**.

Max 2-EDP-free Subgraph

Input. A graph $G = (V, E)$ and terminals $s_1, t_1, s_2, t_2 \in V$.

Problem. Find an edge set $E_0 \subseteq E$ with maximum cardinality such that the edge-disjoint paths problem with terminal pairs (s_1, t_1) and (s_2, t_2) has no solution in the subgraph $G_0 = (V, E_0)$.

We give a polynomial-time algorithm for this problem (Theorem 2.3), and consequently, we show the polynomial solvability of **Max Weakly Non-2-Linked Subgraph** (Corollary 2.4). Since most problems on vertex-disjoint paths and their corresponding edge-disjoint versions are equivalent with respect to their polynomial solvability, it is interesting to note that **Max 2-EDP-free Subgraph** is solvable in polynomial time, whereas **Max 2-VDP-free Induced Subgraph** is NP-hard.

Second, for general k , we show that there exists an $(8k - 2)$ -additive approximation algorithm for **Max Non-k-Linked Induced Subgraph** (Corollary 5.4) by using a known sufficient condition for a graph to be k -linked, that is, our algorithm finds a feasible solution V_0 whose cardinality

is at least the optimum value minus $8k - 2$. Similarly, for general k , we give a 2-additive approximation algorithm for Max Weakly Non- k -Linked Subgraph (Corollary 5.2).

Finally, we also consider the directed versions of these problems. It is well-known that the directed versions of Max 2-VDP-free Induced Subgraph and Max 2-EDP-free Subgraph are NP-hard [5]. By observing that the weakly k -linkedness is equivalent to the k -connectivity for digraphs, we see that Directed Max Weakly Non- k -Linked Subgraph can be solved in polynomial time for general k . On the other hand, based on the arguments in [23], we show that Directed Max Non- k -Linked Induced Subgraph is NP-hard even when $k = 2$ (Theorem 6.2).

Notation: In this paper, we use the following notations. Let $G = (V, E)$ be a graph with a vertex set V and an edge set E . For a vertex set $X \subseteq V$, let $\delta_G(X)$ be the set of edges between X and $V \setminus X$, and such an edge set is called a *cut*. Let $N_G(X)$ denote the set of vertices in $V \setminus X$ that are adjacent to X . We simply denote $\delta(X)$ and $N(X)$ if no confusion may arise. For $X \subseteq V$, the subgraph induced by X is denoted by $G[X]$, and the graph $G[V \setminus X]$ is denoted by $G - X$. For an edge set $F \subseteq E$, let $G - F = (V, E \setminus F)$ and let G/F denote the graph obtained from G by contracting all edges in F . For $s, t \in V$, a cut $\delta(X)$ is called an *s - t cut* if exactly one of s and t is contained in X , and a vertex set X is called an *s - t vertex cut* if s and t are contained in different connected components of $G - X$.

Organization: The rest of this paper is organized as follows. In Section 2, we deal with the edge-disjoint case and give polynomial-time algorithms for Max 2-EDP-free Subgraph and Max Weakly Non-2-Linked Subgraph. In Section 3, we give a polynomial-time algorithm for Max Non-2-Linked Induced Subgraph, which is the main part of this paper. In Section 4, we show the NP-hardness of Max 2-VDP-free Induced Subgraph. In Section 5, we give approximation algorithms for Max Non- k -Linked Induced Subgraph and Max Weakly Non- k -Linked Subgraph for general k . In Section 6, we discuss the directed variants.

2 Weakly 2-linkedness of Graphs

In this section, we show that Max 2-EDP-free Subgraph and Max Weakly Non-2-Linked Subgraph can be solved in polynomial time.

First, we consider Max 2-EDP-free Subgraph. Let $G = (V, E)$ be a graph, and fix terminal pairs (s_1, t_1) and (s_2, t_2) . In this section, 2 edge-disjoint paths mean 2 edge-disjoint paths with respect to those terminals, and a graph is said to be *2-EDP-free* if it does not have 2 edge-disjoint paths.

We now give an upper bound on the number of edges we must remove to make G 2-EDP-free. Let C_1, C_2 be minimum cuts separating s_1 from t_1 and s_2 from t_2 , respectively. It is easy to see that G becomes 2-EDP-free after removing C_1 or C_2 . Let C_{12}, C'_{12} be minimum cuts separating $\{s_1, s_2\}$ from $\{t_1, t_2\}$ and $\{s_1, t_2\}$ from $\{t_1, s_2\}$, respectively. Let F_{12} (resp., F'_{12}) be an edge set obtained from C_{12} (resp., C'_{12}) by discarding an edge. It is also easy to see that G becomes 2-EDP-free after removing F_{12} or F'_{12} . This motivates us to define $c = \min\{|C_1|, |C_2|, |F_{12}|, |F'_{12}|\}$, and let F be the edge set that attains the minimum. When $c = 0$, we say that G violates the *cut condition*. Clearly, c is the upper bound on the number of edges we must remove to make G 2-EDP-free. Now, we show that the converse also holds when $c \geq 4$.

Theorem 2.1. *If $c \geq 4$, $G \setminus F$ is an optimal solution for Max 2-EDP-free Subgraph.*

To prove Theorem 2.1, We use the following theorem by Seymour [18] on the feasibility of the 2-edge-disjoint paths problem.

Theorem 2.2 ([18]). *Let $G = (V, E)$ be a graph and let s_1, t_1, s_2, t_2 be terminals. Then, the 2-edge-disjoint paths problem has no solution if and only if it violates the cut condition or there exists an edge set $E' \subseteq E$ such that*

- (1) *each vertex of G/E' has degree at most 3,*
- (2) *terminals of G/E' are distinct, and have degree at most 2,*
- (3) *G/E' can be embedded in a plane so that s_1, s_2, t_1, t_2 are on the outer boundary of G/E' in this order.*

Proof of Theorem 2.1. Since we have seen that $G - F$ is a solution for Max 2-EDP-free Subgraph, it suffices to show that it is optimal. Assume that there exists an edge set $F' \subseteq E$ with $|F'| \leq c-1$ such that $G - F'$ is 2-EDP-free for terminal pairs (s_1, t_1) and (s_2, t_2) .

Since $G - F'$ satisfies the cut condition, there exists an edge set $E' \subseteq E \setminus F'$ satisfying the three conditions of Theorem 2.2. Let $G' = (G - F')/E'$, and let S_1, T_1, S_2, T_2 be connected subgraphs of G' that correspond to s_1, t_1, s_2, t_2 in G' , respectively. Note that, since s_1, t_1, s_2, t_2 are distinct in G' , the subgraphs S_1, T_1, S_2, T_2 are disjoint. By the condition (2), we have $\delta_{G-F'}(S_1) \leq 2$, $\delta_{G-F'}(T_1) \leq 2$, $\delta_{G-F'}(S_2) \leq 2$, and $\delta_{G-F'}(T_2) \leq 2$. On the other hand, since the size of a minimum s_i-t_i edge cut is at least c for $i = 1, 2$, we have $\delta_G(S_1) \geq c$, $\delta_G(T_1) \geq c$, $\delta_G(S_2) \geq c$, and $\delta_G(T_2) \geq c$. Observing that removing $|F'|$ edges decreases the value $\delta_G(S_1) + \delta_G(T_1) + \delta_G(S_2) + \delta_G(T_2)$ by at most $2|F'|$, we have $4c - 8 \leq 2|F'|$. This contradicts that $|F'| \leq c - 1$ and $c \geq 4$. \square

Theorem 2.3. *Max 2-EDP-free Subgraph is solvable in polynomial time.*

Proof. First, we compute c and F using any polynomial-time algorithm for the minimum cut problem. If $c \geq 4$, by Theorem 2.1, $G - F$ is an optimal solution. If $c \leq 3$, for every set F' of at most c edges, we test whether $G - F'$ is 2-EDP-free, which can be done in polynomial time. \square

Corollary 2.4. *Max Weakly Non-2-Linked Subgraph is solvable in polynomial time.*

Proof. We solve Max 2-EDP-free Subgraph for each terminal pairs (s_1, t_1) and (s_2, t_2) , and we take the maximum of them. \square

3 2-linkedness of Graphs

This section is devoted to proving the following theorem.

Theorem 3.1. *Max Non-2-Linked Induced Subgraph is solvable in polynomial time.*

Since we consider vertex-disjoint paths in this section, we assume that all graphs are simple. By Theorem 1.1, we immediately see the following theorem.

Theorem 3.2. *A graph $G = (V, E)$ is not 2-linked if and only if there exists a partition U, A_1, \dots, A_l ($l \geq 0$) of V such that*

- (1) *for $1 \leq i, j \leq l$ with $i \neq j$, $N(A_i) \cap A_j = \emptyset$,*
- (2) *for $1 \leq i \leq l$, $|N(A_i)| \leq 3$ and $G[A_i]$ is connected, and*
- (3) *if G' is the graph obtained from G by deleting A_i and adding new edges joining every pair of distinct vertices in $N(A_i)$ for every $i = 1, \dots, l$, then G' can be embedded in a plane so that the boundary of some face contains at least four vertices.*

We note that, from the conditions (1) and (2), A_1, \dots, A_l must induce connected components in $G - U$.

First, we observe that if a graph G' is not 3-connected, there exist two vertices x and y separated by a vertex cut $\{v_1, v_2\}$ of size two. Then, we can easily see that G' is not 2-linked by setting $U = \{x, y, v_1, v_2\}$ in Theorem 3.2. Suppose that $|V| \geq 4$ and the vertex-connectivity of the input graph G is $c \geq 2$. By the above observation, we can make G non-2-linked by removing at most $c - 2$ vertices. Thus, it suffices to give algorithms for finding a vertex set $X \subseteq V$ with $|X| \leq c - 3$ such that $G - X$ is not 2-linked. We consider the cases $|X| \leq c - 4$ and $|X| = c - 3$ in Sections 3.1 and 3.2, separately (see Propositions 3.3 and 3.9).

We note that when c is bounded by a fixed constant, the problem can be solved in polynomial time by enumerating all possible vertex sets X . Thus, in what follows, we suppose that c is sufficiently large (e.g. $c \geq 50$).

3.1 Finding a vertex set X with $|X| \leq c - 4$

Suppose that $G - X$ is not 2-linked for some vertex set $X \subseteq V$ with $|X| \leq c - 4$. In this case, $G - X$ is 4-connected, and hence by Theorem 3.2, $G - X$ is not 2-linked if and only if $G - X$ can be embedded in a plane so that the boundary of some face contains at least four vertices.

We observe that $G - X$ contains a vertex v of degree at most five when $G - X$ is planar, which implies that $d_G(v) \leq c + 1$. Recall that we have assumed that G is simple. On the other hand, since G is c -connected, the degree of v in G is at least c , and $|N_G(v) \cap X| \geq c - 5$. With this observation, we can find all possible vertex sets with at most $c - 4$ vertices by executing the following procedure:

For every $v \in V$ with degree at most $c + 1$, and for every vertex set X with $|X| \leq c - 4$ and $|N_G(v) \cap X| \geq c - 5$, test the 2-linkedness of $G - X$.

Since the number of the choices of X is at most $n \cdot \binom{c+1}{c-5} \cdot n = O(n^8)$, this procedure can be done in polynomial time, and we have the following proposition.

Proposition 3.3. *We can enumerate all vertex sets X such that $G - X$ is not 2-linked and $|X| \leq c - 4$ in polynomial time.*

3.2 Finding a vertex set X with $|X| = c - 3$

In this subsection, we give an algorithm for finding a vertex set X such that $|X| = c - 3$ and $G - X$ is not 2-linked. If there exists a vertex set X with $|X| \leq c - 3$ such that $G - X$ can be embedded in a plane so that the boundary of some face contains at least four vertices, then such a set X can be found in polynomial time in the same way as Section 3.1. Thus, it suffices to consider the case when there exist a *positive* integer l and a partition U, A_1, \dots, A_l of $V \setminus X$ satisfying the conditions of Theorem 3.2. Note that, since $|X| = c - 3$, we must have $N_{G-X}(A_i) = 3$ for every i . The main idea behind our algorithm is to guess a vertex $r \in A_1$ and a set of three vertices $W = N_{G-X}(A_1)$ and then check whether a required partition exists under this condition.

Case 1: When $|U| \leq 6$

First, we find a vertex set X and a partition U, A_1, \dots, A_l of $V \setminus X$ with $|U| \leq 6$. We note that an algorithm in this part can be applied even if $|U|$ is more than six but bounded by a fixed constant. In order to find such vertex sets, we consider the following subproblem.

Problem A

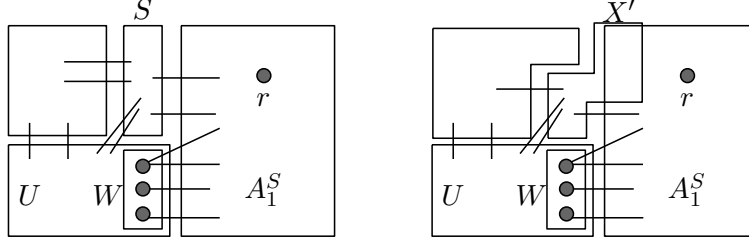


Figure 1: An example of Case 1.

Input. A c -connected graph $G = (V, E)$, a vertex set U , a vertex $r \in V \setminus U$, and a set of three vertices $W \subseteq U$.

Problem. Find a vertex set $X \subseteq V \setminus (U \cup \{r\})$ with $|X| = c - 3$ satisfying the following conditions: $V \setminus X$ can be partitioned into U, A_1, \dots, A_l ($l \geq 1$) such that they satisfy the conditions of Theorem 3.2, $r \in A_1$, and $N_{G-X}(A_1) = W$.

Lemma 3.4. *Problem A is solvable in polynomial time.*

Proof. If $|U| \leq 3$, then the graph G' defined as in the condition (3) in Theorem 3.2 has at most three vertices, which violates the condition (3). Hence, a desired set X obviously does not exist. Suppose that $|U| \geq 4$. We compute a minimum vertex cut separating $U \setminus W$ and r in $G - W$. Among them, let $S \subseteq V$ be the minimum vertex cut such that the connected component of $G - W - S$ containing r is maximum. Let A_1^S be the vertex set of the connected component containing r . If $|S| \geq c - 2$, then we can conclude that the required X does not exist. Thus, since G is c -connected, we may assume that $|S| = c - 3$ and $S \cup W$ is a minimum vertex cut of G .

In this case, let $X = S$ and $A_1 = A_1^S$, and define A_2, \dots, A_l as the vertex sets of the connected components of $G - S - U - A_1^S$. Let \mathcal{P} be the partition $U, A_1^S, A_2, \dots, A_l$ of $V \setminus S$. If \mathcal{P} satisfies the conditions of Theorem 3.2, then $X = S$ is a desired set. Now we show the following claim, which says that we do not have to consider other sets.

Claim 3.5. *If $X = S$ is not a solution of Problem A, then there exists no solution.*

Proof of the claim. Assume that S is not a solution of Problem A, but $X' \subseteq V \setminus (U \cup \{r\})$ is a solution. Since G is c -connected and $|X'| = c - 3$, X' is a minimum vertex cut separating $U \setminus W$ and r . Let A'_1 be the vertex set of the connected component of $G - W - X'$ containing r , and define $A'_2, \dots, A'_{l'}$ as the vertex sets of the connected components of $G - X' - U - A'_1$. Let \mathcal{P}' be the partition $U, A'_1, A'_2, \dots, A'_{l'}$ of $V \setminus X'$. Since X' is a solution of Problem A, \mathcal{P}' satisfies the conditions of Theorem 3.2. Note that, by the maximality of A_1 , $G - X' - U - A'_1 = G[A'_2 \cup \dots \cup A'_{l'}]$ contains $G - S - U - A_1^S = G[A_2 \cup \dots \cup A_l]$ as a subgraph (see Fig. 1).

Since S is not a solution, the partition \mathcal{P} violates the conditions of Theorem 3.2. Assume that $|N_{G-S}(A_i)| \geq 4$ for some $i = 2, 3, \dots, l$. Since A_i is contained in A'_j for some $j \in \{2, 3, \dots, l'\}$, we have $|N_{G-X'}(A'_j)| \geq |N_{G-S}(A_i)| \geq 4$, which contradicts that the partition \mathcal{P}' satisfies the conditions of Theorem 3.2.

Therefore, $|N_{G-S}(A_i)| = 3$ for every $i = 2, 3, \dots, l$, and \mathcal{P} violates the condition (3) if we apply Theorem 3.2 to $G - S$. That is, the graph G^S obtained by the operations in the condition (3) is either a non-planar graph or a planar graph whose every face has three vertices. On the other hand, for any $i = 2, 3, \dots, l$, $N_{G-S}(A_i) \subseteq N_{G-X'}(A'_j)$ holds for some $j \in \{2, 3, \dots, l'\}$. Thus, if we apply Theorem 3.2 to $G - X'$, then either the partition \mathcal{P}' violates the condition

(2), or the graph G' defined as in the condition (3) contains G^S as a subgraph and $V(G') = U = V(G^S)$. This means that the partition \mathcal{P}' violates the conditions of Theorem 3.2, which contradicts the assumption. \square

By this claim, in order to solve Problem A, it suffices to test whether $X = S$ is a desired set or not, which can be done in polynomial time. \square

We can find a solution X and a partition U, A_1, \dots, A_l of $V \setminus X$ with $|U| \leq 6$ by solving Problem A for every choice of U, r and W , which can be done in polynomial time by Lemma 3.4.

Case 2: When $|U| \geq 7$

Second, we find a solution X and a partition U, A_1, \dots, A_l of $V \setminus X$ with $|U| \geq 7$. To find such a solution, we consider the following subproblem.

Problem B

Input. A c -connected graph $G = (V, E)$, a vertex r , a set of three vertices $W \subseteq V$, and a set of four vertices $\tilde{U} \subseteq V$ such that r, W, \tilde{U} are disjoint.

Problem. Find a vertex set $X \subseteq V \setminus (\{r\} \cup W \cup \tilde{U})$ with $|X| = c - 3$ satisfying the following conditions: $V \setminus X$ can be partitioned into U, A_1, \dots, A_l ($l \geq 1$) such that they satisfy the conditions of Theorem 3.2, $r \in A_1$, $W \subseteq U$, $\tilde{U} \subseteq U$ and $N_{G-X}(A_1) = W$.

Lemma 3.6. *Problem B is solvable in polynomial time.*

Proof. We compute minimum vertex cuts separating \tilde{U} and r in $G - W$. Among them, let $S \subseteq V$ be the minimum vertex cut such that the connected component of $G - W - S$ containing r is maximum. Let A_1^S be the vertex set of the connected component containing r . If $|S| \geq c - 2$, then we can conclude that such X does not exist. Thus, we may assume that $|S| = c - 3$. Let $G' = G - W - S - A_1^S$. We consider the following two cases separately.

Case 2-1: When G' is not planar

When G' is not planar, we show the following claim.

Claim 3.7. *Every solution X of Problem B contains at least $c - 10$ vertices of S .*

Proof of the claim. Assume that a solution X of Problem B satisfies that $|X \cap S| \leq c - 11$. Then, there exists a set of eight vertices $Y \subseteq S \setminus X$. Since $|W \cup (S \setminus Y)| = c - 8$, $G - W - (S \setminus Y)$ is 8-connected. Thus, for every pair of vertices $v_1, v_2 \in V(G')$, there exist eight internally vertex-disjoint paths P_1, \dots, P_8 in $G - W - (S \setminus Y)$.

On the other hand, since X is a minimum vertex cut separating \tilde{U} and r in $G - W$, every vertex of X is contained in $S \cup A_1^S$ by the maximality of A_1^S (see Fig. 2). Thus, if a path P_i intersects with X , then P_i contains at least two vertices of Y . Hence, at least four paths of P_1, \dots, P_8 do not intersect with X , which means that there is no cut of size three in $G - X$ separating v_1 and v_2 .

This means that U contains all vertices of G' when we consider the partition of $V \setminus X$, which contradicts the planarity of the graph $G[U]$. \square

By this claim, when G' is not planar we can find all solutions of Problem B. That is, for every vertex set $X \subseteq V$ with $|X| = c - 3$ and $|X \cap S| \geq c - 10$, we test whether X is a solution of Problem B or not by using an algorithm for the 2-vertex-disjoint paths problem. Since the number of choices of X is at most $n^7 \cdot \binom{c-3}{c-10} = O(n^{14})$, this procedure can be done in polynomial time.

Case 2-2: When G' is planar

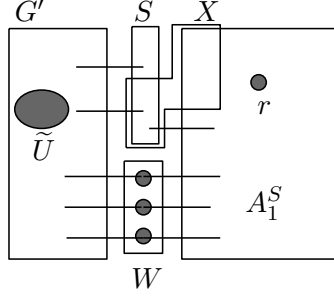


Figure 2: An example of Case 2-1.

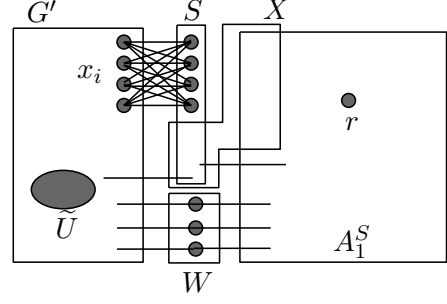


Figure 3: An example of Case 2-2.

When G' is planar, at least four vertices of G' have degree at most six since the average degree of all vertices in a planar graph is at most six. Note that every vertex has degree at most six when G' has at most seven vertices. Let x_1, x_2, x_3, x_4 be vertices of G' with degree at most six.

For $i = 1, \dots, 4$, since the degree of x_i is at least c and $N_G(x_i) \cap A_1^S = \emptyset$, it holds that $|N_G(x_i) \cap S| \geq c - 6 - |W| = |S| - 6$. Let $S_0 = \bigcap_i N_G(x_i) \cap S$. Then, we have $|S_0| \geq |S| - 6 \cdot 4 = c - 27$. Now we show the following claim.

Claim 3.8. *Every solution X of Problem B contains at least $|S_0| - 3$ vertices of S_0 .*

Proof of the claim. Assume that a solution X of Problem B satisfies that $|X \cap S_0| \leq |S_0| - 4$. Then, there exists a set of four vertices $Y = \{y_1, y_2, y_3, y_4\}$ in $S_0 \setminus X$. By definition, x_i and y_j are connected by an edge for any $i, j \in \{1, 2, 3, 4\}$, that is, they form a complete bipartite graph $K_{4,4}$ (see Fig. 3).

Since $|W \cup (S \setminus Y)| = c - 4$, $G - W - (S \setminus Y)$ is 4-connected, and hence, there exists no vertex cut of size three in $G - X$ separating the $K_{4,4}$ and \tilde{U} . Note that every vertex of X is contained in $S \cup A_1^S$ by the maximality of A_1^S since X is a minimum vertex cut separating \tilde{U} and r in $G - W$. Therefore, the $K_{4,4}$ is contained in U in the partition, which contradicts the planarity of $G[U]$. \square

By this claim, when G' is planar we can find all solutions of Problem B. That is, for every vertex set $X \subseteq V$ with $|X| = c - 3$ and $|X \cap S_0| \geq |S_0| - 3$, we test whether X is a solution of Problem B or not by using an algorithm for the 2-vertex-disjoint paths problem. Since the number of the choices of X is at most $n^{c-|S_0|} \cdot \binom{|S_0|}{|S_0|-3} = O(n^{30})$, this procedure can be done in polynomial time.

By combining Cases 2-1 and 2-2, we complete the proof of Lemma 3.6. \square

We can find a solution X and a partition U, A_1, \dots, A_l of $V \setminus X$ with $|U| \geq 7$ by solving Problem B for every choice of r, W and \tilde{U} , which can be done in polynomial time by Lemma 3.6.

Thus, by Cases 1 and 2, we have the following proposition.

Proposition 3.9. *We can find a vertex set X such that $G - X$ is not 2-linked and $|X| = c - 3$ in polynomial time (if one exists).*

4 NP-Hardness of Max 2-VDP-free Induced Subgraph

Theorem 4.1. *Max 2-VDP-free Induced Subgraph is NP-hard.*

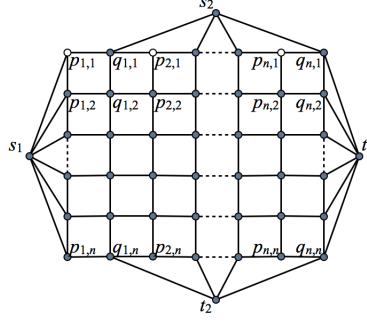


Figure 4: The graph G' obtained from G by the reduction. For each edge uv in G , there is an edge $p_{u,1}p_{v,1}$ in G' (edges may be added among empty vertices).

Proof. We show that Vertex Cover can be reduced to Max 2-VDP-free Induced Subgraph. Note that Vertex Cover is an NP-hard problem, in which we are given a simple graph $G = (V, E)$ and the objective is to find a vertex set $X \subseteq V$ with minimum cardinality such that every edge in E is incident to at least one vertex in X . Suppose that we are given an instance $G = (V, E)$ of Vertex Cover, where $V = \{1, \dots, n\}$. We construct a new graph $G' = (V', E')$ as follows (see Fig. 4):

$$\begin{aligned}
V' &= \{s_1, t_1, s_2, t_2\} \cup \{p_{i,j}, q_{i,j} \mid i, j \in \{1, \dots, n\}\}, \\
E' &= \{p_{i,j}q_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq n\} \cup \{p_{i+1,j}q_{i,j} \mid 1 \leq i \leq n-1, 1 \leq j \leq n\} \\
&\quad \cup \{p_{i,j}p_{i,j+1}, q_{i,j}q_{i,j+1} \mid 1 \leq i \leq n, 1 \leq j \leq n-1\} \\
&\quad \cup \{s_1p_{1,j}, t_1q_{n,j} \mid 1 \leq j \leq n\} \cup \{s_2q_{i,1}, q_{i,n}t_2, \mid 1 \leq i \leq n\} \\
&\quad \cup \{p_{u,1}p_{v,1} \mid u, v \in V, uv \in E\}.
\end{aligned}$$

We show that, for a positive integer $k \leq n-1$, G has a vertex cover of size k if and only if Max 2-VDP-free Induced Subgraph in G' has a feasible solution of size $|V'| - k$.

Suppose that $G = (V, E)$ has a vertex cover X with $|X| = k$. If we remove the vertex set $X' = \{p_{v,1} \mid v \in X\} \subseteq V'$ from G' , then the obtained graph is 2-VDP-free (i.e., it does not have 2 vertex-disjoint paths) by the planarity of $G' - X'$. This means that $V' \setminus X'$ is a feasible solution of Max 2-VDP-free Induced Subgraph whose size is $|V'| - k$.

Conversely, suppose that Max 2-VDP-free Induced Subgraph in G' has a feasible solution of size $|V'| - k$, that is, there exists a vertex set $X' \subseteq V'$ with $|X'| = k$ such that $G' - X'$ is 2-VDP-free. First, we note that $|X'| \leq n-1$ since the graph obtained from G' by removing $p_{1,1}, \dots, p_{n-1,1}$ clearly avoids 2 vertex-disjoint paths. Define $X \subseteq V$ by

$$X = \{v \in V \mid X' \cap \{p_{v,1}, \dots, p_{v,n}, q_{v,1}, \dots, q_{v,n}\} \neq \emptyset\}.$$

We now show that X is a vertex cover of G . In order to derive a contradiction, assume that there exist $u, v \in V$ such that $u < v$, $uv \in E$, $X' \cap \{p_{u,1}, \dots, p_{u,n}, q_{u,1}, \dots, q_{u,n}\} = \emptyset$ and $X' \cap \{p_{v,1}, \dots, p_{v,n}, q_{v,1}, \dots, q_{v,n}\} = \emptyset$. Since $|X'| \leq n-1$, $G' - X'$ contains a path P_s from s_1 to $p_{u,1}$ and a path P_t from $p_{v,1}$ to t_1 which are both not intersecting with $\{q_{u,1}, \dots, q_{u,n}\}$. This means that G' contains two vertex-disjoint paths $P_1 = P_s \cup \{p_{u,1}q_{v,1}\} \cup P_t$ and $P_2 = (s_2, q_{u,1}, \dots, q_{u,n}, t_2)$, which contradicts the definition of X' . By the above argument, G contains a vertex cover X of size at most k .

Therefore, Vertex Cover in G can be reduced to Max 2-VDP-free Induced Subgraph in G' , which shows the NP-hardness of Max 2-VDP-free Induced Subgraph. \square

5 Weakly k -linkedness and k -linkedness of Graphs

In this section, we show that there exist polynomial-time approximation algorithms for Max Weakly Non-2-Linked Subgraph and Max Non-2-Linked Induced Subgraph with constant additive error. The following lemma is useful to connect weakly linkedness and connectivity.

Lemma 5.1 ([7]). *A graph is weakly k -linked if it is $(k + 2)$ -edge-connected.*

Corollary 5.2. *There exists a polynomial-time algorithm that finds a feasible solution of Max Weakly Non- k -Linked Subgraph whose cardinality is at least the optimal value minus two.*

Proof. Let G be an input graph and c be the size of a minimum edge cut of G . From Lemma 5.1, we must remove at least $\max(c - k - 2, 0)$ edges to make G weakly non- k -linked. However, we can make G weakly non- k -linked by removing $\max(c - k, 0)$ edges from a minimum edge cut. Thus, the claim holds. \square

A lemma similar to Lemma 5.1 holds for linkedness.

Lemma 5.3 ([21]). *A graph is k -linked if it is $10k$ -vertex-connected.*

Corollary 5.4. *There exists a polynomial-time algorithm that finds a feasible solution of Max Non- k -Linked Induced Subgraph whose cardinality is at least the optimal value minus $8k - 2$.*

Proof. If a graph contains a vertex cut X of size at most $2k - 2$, then we can choose $s_1, \dots, s_k, t_1, \dots, t_k$ such that $X = \{s_2, \dots, s_k, t_2, \dots, t_k\}$ and s_1 and t_1 are contained in different components of $G - X$, which means that the graph is not k -linked. Thus, every k -linked graph is $2k - 1$ connected.

Let G be an input graph and c be the size of a minimum vertex cut of G . From Lemma 5.3, we must remove at least $\max(c - 10k, 0)$ vertices to make G non- k -linked. However, by the observation above, we can make G non- k -linked by removing $\max(c - 2k + 2, 0)$ vertices from a minimum vertex cut. Thus, the claim holds. \square

6 k -linkedness and Weakly k -linkedness of Digraphs

In this section, we consider the directed versions of Max Non- k -Linked Induced Subgraph and Max Weakly Non- k -Linked Subgraph, which we call Directed Max Non- k -Linked Induced Subgraph and Directed Max Weakly Non- k -Linked Subgraph, respectively. We can easily solve Directed Max Weakly Non- k -Linked Subgraph with the aid of the following lemma.

Lemma 6.1 ([19]). *For any integer $k \geq 1$, a digraph is weakly k -linked if and only if it is k -arc-connected.*

Thus, given a digraph G and an integer k , by computing a minimum arc-cut C of G and discarding $\max(|C| - k + 1, 0)$ arcs among them, we have an optimal solution.

Thomassen [23] showed that the directed version of 2-vertex-disjoint paths problem in k -connected digraphs is NP-hard for any constant k as a generalization of the result of [5]. Appealing to this result, we can easily show that Directed Max Non- k -Linked Induced Subgraph is NP-hard. For completeness, we give the proof in the following.

Theorem 6.2. *Directed Max Non- k -Linked Induced Subgraph is NP-hard even if $k = 2$.*

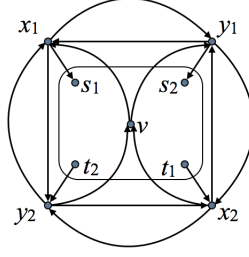


Figure 5: The graph G'

Proof. We show that the directed 2-vertex-disjoint paths problem in 2-connected digraphs can be reduced to Directed Max Non-2-Linked Induced Subgraph. Suppose that we are given an instance of the directed 2-vertex-disjoint paths problem, which consists of a 2-connected digraph $G = (V, E)$ and its four distinct vertices $s_1, t_1, s_2,$ and t_2 . We define a new digraph $G' = (V', E')$ (see Fig. 5) by

$$\begin{aligned} V' &= V \cup \{x_1, x_2, y_1, y_2\} \\ E' &= E \cup \{(x_i, y_j), (y_j, x_i) \mid i, j \in \{1, 2\}\} \\ &\quad \cup \{(v, x_1), (v, y_1) \mid v \in V\} \cup \{(x_2, v), (y_2, v) \mid v \in V\} \\ &\quad \cup \{(x_1, s_1), (t_1, x_2), (y_1, s_2), (t_2, y_2)\}. \end{aligned}$$

Now we show that G' is 2-linked if and only if G contains two disjoint paths P_1 and P_2 such that P_i is a path from s_i to t_i for $i = 1, 2$.

Let s'_1, t'_1, s'_2, t'_2 be four distinct vertices of G' and consider whether G' has two disjoint paths P'_1 and P'_2 such that P'_i is a path from s'_i to t'_i for $i = 1, 2$. If $\{(s'_1, t'_1), (s'_2, t'_2)\} \neq \{(x_1, x_2), (y_1, y_2)\}$, then we can see that G' always has desired disjoint paths P'_1 and P'_2 as follows.

Case 1. Suppose that $\{x_1, x_2, y_1, y_2\} \cap \{s'_1, t'_1, s'_2, t'_2\} = \emptyset$. In this case, G' contains desired disjoint paths $P'_1 = (s'_1, x_1, y_2, t'_1)$ and $P'_2 = (s'_2, y_1, x_2, t'_2)$.

Case 2. Suppose that $1 \leq |\{x_1, x_2, y_1, y_2\} \cap \{s'_1, t'_1, s'_2, t'_2\}| \leq 3$. In this case, by alternating (s'_1, t'_1) and (s'_2, t'_2) if necessary, we can take a dipath P'_1 from s'_1 to t'_1 such that $|V(P'_1) \cap V| \leq 1$ and $s'_2, t'_2 \notin V(P'_1)$. Since G is 2-connected, there exists a dipath P'_2 from s'_2 to t'_2 in $G' - V(P'_1)$.

Case 3. Suppose that $\{x_1, x_2, y_1, y_2\} = \{s'_1, t'_1, s'_2, t'_2\}$. When $\{(s'_1, t'_1), (s'_2, t'_2)\} \neq \{(x_1, x_2), (y_1, y_2)\}$, we can easily take desired disjoint paths P'_1 and P'_2 in G' .

Thus, in order to determine the 2-linkedness of G' , it suffices to consider the 2-vertex-disjoint paths problem in G' in which the terminals $s'_1, t'_1, s'_2,$ and t'_2 satisfy that $\{(s'_1, t'_1), (s'_2, t'_2)\} = \{(x_1, x_2), (y_1, y_2)\}$. In this case, G' has desired disjoint paths P'_1 and P'_2 if and only if G contains two disjoint paths P_1 and P_2 such that P_i is a path from s_i to t_i for $i = 1, 2$. This means that the directed 2-vertex-disjoint paths problem in 2-connected digraphs can be reduced to Directed Max Non-2-Linked Induced Subgraph which completes the proof. \square

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