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Abstract

In combinatorial studies of the Kronecker form of matrix pencils, two linear-algebraic characteristics have been featured: degrees of subdeterminants and ranks of expanded matrices. This paper shows the Legendre duality between the two and their combinatorial counterparts for matroid pencils, which serve as upper bounds on the corresponding linear-algebraic quantities. Tightness of one of the combinatorial bounds is shown to be equivalent to that of the other. A sufficient condition for the tightness is given, and its application to electric networks is indicated. Furthermore, the proposed approach is extended to mixed matrix pencils.

Key words: combinatorial matrix theory, matrix pencil, Kronecker canonical form, matroid, valuated matroid, convexity, Legendre transformation
AMS subject classifications: 15A22, 05C50, 52B40

1 Introduction

A matrix pencil, i.e., a polynomial matrix of the form $sA+B$ with an indeterminate variable s, is a fundamental linear-algebraic concept used in many application areas including electric network theory, control theory, and numerical analysis. A matrix pencil can be brought to a block-diagonal canonical form, known as the Kronecker form [6].

Besides numerical methods [1, 2, 3, 24, 25] to compute the Kronecker form, a number of combinatorial methods have been proposed [8, 10, 11, 12, 13, 16, 18, 19] to determine the structural indices, in particular, the indices of nilpotency, in the

Kronecker form. The combinatorial methods share a common feature that, under certain genericity assumptions on the matrices A and B, the structural indices are expressed in terms of combinatorial objects, like matchings and linkings.

Technically, however, two different approaches can be distinguished in combinatorial methods. The first approach, initiated by Murota [16] and pursued by [8, 10, 13, 18, 19], uses maximum degrees of subdeterminants of specified orders, denoted δ_k in this paper. Combinatorial properties inherent in the degrees of subdeterminants are discussed in terms of valuated bimatroids in [17] and [19, Chapter 5]; see Section 3.2 for valuated bimatroids. The second approach, initiated by Iwata–Shimizu [11] and pursued by [12], uses ranks of the expanded matrices, denoted θ_k in this paper. As a combinatorial abstraction along this approach, Iwata [9] studies a pair of linking systems [23] (or bimatroids [14]) under the name of matroid pencils. In particular, a combinatorial counterpart of θ_k is introduced for matroid pencils; see Section 4.1. The two characteristics, δ_k and θ_k, have so far been considered rather independently, and no explicit statement about their relationship has been made in the literature.

This paper is to shed a new light on combinatorial studies of matrix pencils by featuring the discrete Legendre transformation as a methodological pivot. While technical issues are detailed in Remark 2.1 in Section 2.2, the fundamental construction of the discrete Legendre transformation is as follows. For an integer sequence (α_k) in general, the discrete Legendre (concave) transform of (α_k) is another integer sequence (β_k) defined by $\beta_k = \min_{\ell} (\alpha_\ell - k\ell)$, which is *concave* in the sense that $\beta_{k-1} + \beta_{k+1} \leq 2\beta_k$ for all k. The discrete Legendre (convex) transform of (β_k) is a sequence (γ_k) given by $\gamma_k = \max_{\ell} (\beta_\ell + k\ell)$, which is *convex* in the sense that $\gamma_{k-1} + \gamma_{k+1} \geq 2\gamma_k$ for all k. If (α_k) is convex, then (γ_k) coincides with (α_k). Therefore, the discrete Legendre transformation establishes a one-to-one correspondence between convex and concave integer sequences.

In this paper we start with an easy observation that $(\delta_0, \delta_1, \delta_2, \cdots)$ and $(\theta_0, \theta_1, \theta_2, \cdots)$ for a matrix pencil are mutually dual sequences with respect to the discrete Legendre transformation (see Proposition 2.2 for a precise statement). By introducing a combinatorial counterpart of δ_k for matroid pencils in addition to the combinatorial counterpart of θ_k considered in [9], we extend this Legendre duality to a matroid pencil. For a given matrix pencil $sA + B$, we may associate a matroid pencil in a natural way, and δ_k and θ_k for $sA + B$ are upper-bounded by their combinatorial counterparts, say, $\hat{\delta}_k$ and $\hat{\theta}_k$ for the associated matroid pencil. It is shown that $\hat{\delta}_k = \delta_k$ for all k if and only if $\hat{\theta}_k = \theta_k$ for all k (cf. Theorem 5.4), and that this is the case if A or B is a generic matrix (cf. Theorem 5.5). For a generic matrix pencil, of which both A and B are generic, $\hat{\delta}_k$ is obviously tight for all k (see, e.g., [19, Theorem 6.2.2]), and hence we can derive the tightness of $\hat{\theta}_k$, established in [11], as an immediate corollary of our result. Our result also has a significant implication in combinatorial methods [7, 22] for electric networks. The Legendre duality is extended to the combinatorial bounds for mixed matrix pencils, providing a novel insight into the methods of Murota [18] and Iwata–Takamatsu [12].

This paper is organized as follows. Section 2 describes the Kronecker form of matrix pencils, and Section 3 explains matroid-theoretic concepts such as bimatroids and valuated bimatroids as preliminaries. Sections 4 and 5 present the main ideas of this paper, the former dealing with the combinatorial analogues of δ_k and θ_k for matroid pencils and the latter discussing the relationship between the linear-algebraic quantities and their combinatorial counterparts. Section 6 indicates a possible application to electric networks. Section 7 describes the existing methods for mixed matrix pencils, and Section 8 shows the extension of the Legendre duality approach to mixed matrix pencils.

2 Matrix Pencils

Let $D(s) = sA + B$ be a *matrix pencil*, which means that A and B are constant matrices of the same size, and s is an indeterminate variable. We assume that $D(s)$ is an $m \times n$ matrix of rank r. A matrix pencil $D(s) = sA + B$ is said to be *regular* if it is square and $\det D(s)$ is a nonvanishing polynomial in s; it is *strictly regular* if both A and B are nonsingular.

2.1 Kronecker canonical form

A matrix pencil can be brought to a canonical block-diagonal form, known as the *Kronecker form*, through a strict equivalence transformation, i.e., a transformation $UD(s)V$ with constant nonsingular matrices U and V.

For positive integers ρ and μ, we define a $\rho \times \rho$ matrix $K_\rho(s)$ and a $\mu \times \mu$ matrix $N_\mu(s)$ as

$$K_\rho(s) = \begin{bmatrix} s & 1 & 0 & \cdots & 0 \\ 0 & s & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & s & 1 \\ 0 & \cdots & \cdots & 0 & s \end{bmatrix}, \quad N_\mu(s) = \begin{bmatrix} 1 & s & 0 & \cdots & 0 \\ 0 & 1 & s & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & s \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

For a positive integer ε, we define an $\varepsilon \times (\varepsilon + 1)$ matrix $L_\varepsilon(s)$ as

$$L_\varepsilon(s) = \begin{bmatrix} s & 1 & 0 & \cdots & 0 \\ 0 & s & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & s & 1 \end{bmatrix},$$

and, for a positive integer η, we define an $(\eta+1)\times\eta$ matrix $U_\eta(s)$ to be the transpose of $L_\eta(s)$.

The following is the theorem for the Kronecker form. See [6, Chapter XII] and [19, Chapter 5] for proofs.

Theorem 2.1 (Kronecker, Weierstrass)**.** *For a matrix pencil $D(s)$ there exist nonsingular constant matrices U and V such that*

$$\begin{aligned} UD(s)V &= \text{block-diag}\,(H_\nu; K_{\rho_1}(s), \cdots, K_{\rho_c}(s); N_{\mu_1}(s), \cdots, N_{\mu_d}(s); \\ &\qquad L_{\varepsilon_1}(s), \cdots, L_{\varepsilon_p}(s); U_{\eta_1}(s), \cdots, U_{\eta_q}(s)), \end{aligned} \tag{1}$$

where $\rho_1 \geq \cdots \geq \rho_c \geq 1$, $\mu_1 \geq \cdots \geq \mu_d \geq 1$, $\varepsilon_1 \geq \cdots \geq \varepsilon_p \geq 1$, $\eta_1 \geq \cdots \geq \eta_q \geq 1$, and H_ν is a strictly regular matrix pencil of size ν. The numbers c, d, p, q, ν, $\rho_1, \ldots, \rho_c$, $\mu_1, \ldots, \mu_d$, $\varepsilon_1, \ldots, \varepsilon_p$, $\eta_1, \ldots, \eta_q$, are uniquely determined.

In this paper we are particularly interested in $\mu_1, \ldots, \mu_d$, which we call the *indices of nilpotency*. Note that ρ_i for $sA + B$ is equal to μ_i for $sB + A$ and hence all the arguments of this paper, developed for μ_i, translate to those for ρ_i.

2.2 Characterizations of the indices

In combinatorial studies of the Kronecker canonical form, two different characterizations of the indices of nilpotency μ_i have been employed. They are described here with an observation about their relationship.

For $k = 0, 1, 2, \cdots$ we denote by δ_k the highest degree in s of a minor (subdeterminant) of order k of $D(s) = sA + B$:

$$\delta_k = \delta_k(sA + B) = \max\{\deg_s \det D[X, Y] \mid |X| = |Y| = k\}, \tag{2}$$

where $D[X, Y]$ denotes the submatrix of D with row set X and column set Y; we put $\delta_0 = 0$, and $\delta_k = -\infty$ for $k > r$ or $k < 0$. It holds that

$$\delta_k = k - \sum_{i=r-k+1}^{d} \mu_i \qquad (0 \leq k \leq r). \tag{3}$$

In particular, $\delta_k = k$ if $0 \leq k \leq r - d$. The identity (3), characterizing μ_i in terms of δ_k, forms the basis of combinatorial studies of μ_i via δ_k initiated by Murota [16] and pursued by [8, 10, 13, 18, 19]. The sequence (δ_k) is concave:

$$\delta_{k-1} + \delta_{k+1} \leq 2\delta_k \qquad (1 \leq k \leq r - 1). \tag{4}$$

The second characterization of μ_i refers to larger matrices, called *expanded matrices*. For a positive integer k, let Θ_k be a $km \times kn$ matrix defined by

$$\Theta_k = \Theta_k(sA + B) = \begin{bmatrix} A & O & \cdots & \cdots & O \\ B & A & \ddots & & \vdots \\ O & B & \ddots & \ddots & O \\ \vdots & \ddots & \ddots & A & O \\ O & \cdots & O & B & A \end{bmatrix}. \tag{5}$$

We denote the rank of $\Theta_k(sA+B)$ by $\theta_k = \theta_k(sA+B)$, where $\theta_0 = 0$ and $\theta_k = -\infty$ for $k < 0$. It holds that

$$\theta_k = kr - \sum_{i=1}^{d} \min(k, \mu_i) \qquad (k \geq 0). \tag{6}$$

The sequence of μ_i determines the sequence of θ_k, and vice versa. The identity (6), characterizing μ_i in terms of θ_k, forms the basis of combinatorial studies of μ_i via θ_k initiated by Iwata–Shimizu [11] and pursued by [12]. The sequence (θ_k) is convex:

$$\theta_{k-1} + \theta_{k+1} \geq 2\theta_k \qquad (k \geq 1). \tag{7}$$

The relationship between the two expressions (3) and (6) can be identified as the discrete Legendre transformation (see Remark 2.1 below). This is an easy observation, but no explicit statement seems to have been made in the literature of combinatorial analysis of the Kronecker form.

Proposition 2.2. *For $\delta_k = \delta_k(sA+B)$ and $\theta_k = \theta_k(sA+B)$ we have*

$$\begin{aligned} \delta_k &= \min_{\ell \geq 0} (\theta_{\ell+1} - k\ell) \qquad (0 \leq k \leq r), &(8)\\ \theta_{\ell+1} &= \max_{0 \leq k \leq r} (\delta_k + k\ell) \qquad (\ell \geq 0). &(9) \end{aligned}$$

Proof. Since δ_k is concave and θ_ℓ is convex, the two expressions (8) and (9) are equivalent; see Remark 2.1 below. The latter expression (9) can be verified as follows. For $\ell \geq 0$ let $\hat{k} = \hat{k}(\ell)$ be the minimum index k such that $\mu_{r-k} \geq \ell + 1 \geq \mu_{r-k+1}$, where $\mu_0 = +\infty$. Using (3) we obtain

$$\max_k (\delta_k + k\ell) = \max_k \left(k(\ell+1) - \sum_{i=r-k+1}^{d} \mu_i \right) = \hat{k}(\ell+1) - \sum_{i=r-\hat{k}+1}^{d} \mu_i,$$

in which

$$\sum_{i=r-\hat{k}+1}^{d} \mu_i = \sum_{i=r-\hat{k}+1}^{d} \min(\ell+1, \mu_i) = \sum_{i=1}^{d} \min(\ell+1, \mu_i) - (r-\hat{k})(\ell+1).$$

Therefore,

$$\max_k (\delta_k + k\ell) = r(\ell+1) - \sum_{i=1}^{d} \min(\ell+1, \mu_i) = \theta_{\ell+1},$$

where the last equality is due to (6). $\square$

Combinatorial (or matroid-theoretic) analogues of the linear-algebraic characteristics above will be considered in Section 4 as the main topic of this paper, and the relationship between linear-algebraic quantities and combinatorial quantities is discussed in Section 5 in terms of the Legendre duality. This approach is followed by an extension to mixed matrix pencils in Section 8.

Remark 2.1. The discrete Legendre transformation is explained here in a way suitable for this paper; see [20] for details. A (discrete) function $f : \mathbb{Z} \to \mathbb{Z} \cup \{+\infty\}$ is called *convex* if

$$f(x-1) + f(x+1) \geq 2f(x) \qquad \text{for all } \ x \in \mathbb{Z}.$$

A function $g : \mathbb{Z} \to \mathbb{Z} \cup \{-\infty\}$ is called *concave* if

$$g(y-1) + g(y+1) \leq 2g(y) \qquad \text{for all } \ y \in \mathbb{Z}.$$

For a function $f : \mathbb{Z} \to \mathbb{Z} \cup \{+\infty\}$ in general, convex or not, the discrete Legendre (concave) transform of f is a function $f^{\circ} : \mathbb{Z} \to \mathbb{Z} \cup \{-\infty\}$ defined by

$$f^{\circ}(y) = \inf\{f(x) - xy \mid x \in \mathbb{Z}\} \qquad (y \in \mathbb{Z}), \tag{10}$$

where it is assumed that $f(x) \in \mathbb{Z}$ (i.e., finite-valued) for some $x \in \mathbb{Z}$. The function f° is concave. For a function $g : \mathbb{Z} \to \mathbb{Z} \cup \{-\infty\}$ in general, the discrete Legendre (convex) transform of g is a function $g^{\bullet} : \mathbb{Z} \to \mathbb{Z} \cup \{+\infty\}$ defined by

$$g^{\bullet}(x) = \sup\{g(y) + xy \mid y \in \mathbb{Z}\} \qquad (x \in \mathbb{Z}), \tag{11}$$

where it is assumed that $g(y) \in \mathbb{Z}$ (i.e., finite-valued) for some $y \in \mathbb{Z}$. The function $g^{\bullet}$ is convex. For a convex function f and a concave function g we have

$$(f^{\circ})^{\bullet} = f, \qquad (g^{\bullet})^{\circ} = g. \tag{12}$$

Hence the discrete Legendre transformation establishes a one-to-one correspondence between convex and concave integer-valued discrete functions.

An integer sequence (α_k) indexed by $k \in K$ can be identified with a discrete function $K \to \mathbb{Z}$. Furthermore, it can be identified with $\check{\alpha} : \mathbb{Z} \to \mathbb{Z} \cup \{+\infty\}$ by defining $\check{\alpha} = +\infty$ outside K, or alternatively, with $\hat{\alpha} : \mathbb{Z} \to \mathbb{Z} \cup \{-\infty\}$ by defining $\hat{\alpha} = -\infty$ outside K. In this paper we identify $(\delta_0, \delta_1, \cdots, \delta_r)$ with a function $g : \mathbb{Z} \to \mathbb{Z} \cup \{-\infty\}$ by $g(k) = \delta_k$ for $k = 0, 1, \ldots, r$, and $(\theta_0, \theta_1, \theta_2, \cdots)$ with $f : \mathbb{Z} \to \mathbb{Z} \cup \{+\infty\}$ by $f(k) = \theta_{k+1}$ for $k \geq -1$. Then, (8) is the form of $g = f^{\circ}$ and (9) is $f = g^{\bullet}$, where $\theta_0 = 0$ is a tacit understanding. ∎

3 Bimatroids and Valuated Bimatroids

This section introduces basic concepts that we need for the matroid-theoretic constructions in this paper.

3.1 Bimatroids

The concept of bimatroids was introduced first by Schrijver [23] under the name of linking system, and by Kung [14] under the name of bimatroid.

A *bimatroid* (or *linking system*) is a triple $\mathbf{L} = (S, T, \Lambda)$, where S and T are disjoint finite sets, and Λ is a nonempty subset of $2^S \times 2^T$ such that (L-1)–(L-3) below are satisfied:

(L-1) If $(X, Y) \in \Lambda$ and $x \in X$, then there exists $y \in Y$ such that $(X - x, Y - y) \in \Lambda$;

(L-2) If $(X, Y) \in \Lambda$ and $y \in Y$, then there exists $x \in X$ such that $(X - x, Y - y) \in \Lambda$;

(L-3) If $(X_i, Y_i) \in \Lambda$ $(i = 1, 2)$, then there exist $X \subseteq S$ and $Y \subseteq T$ such that $(X, Y) \in \Lambda,\ X_1 \subseteq X \subseteq X_1 \cup X_2,\ Y_2 \subseteq Y \subseteq Y_1 \cup Y_2$,

where $X - x$ is a short-hand notation for $X \setminus \{x\}$. We call S the *row set* and T the *column set* of $\mathbf{L}$. A member (X, Y) of Λ is called a *linked pair*. It follows from (L-1) and (L-2) that $(\emptyset, \emptyset) \in \Lambda$ and that $|X| = |Y|$ for any linked pair $(X, Y) \in \Lambda$. The maximum size of a linked pair in $\mathbf{L}$ is referred to as the *rank* of $\mathbf{L}$. We sometimes write $(X, Y) \in \mathbf{L}$ to mean $(X, Y) \in \Lambda$.

For two bimatroids $\mathbf{L}_i = (S_i, T_i, \Lambda_i)$ $(i = 1, 2)$, the *union* of $\mathbf{L}_1$ and $\mathbf{L}_2$ is a bimatroid $\mathbf{L}_1 \vee \mathbf{L}_2 = (S_1 \cup S_2, T_1 \cup T_2, \Lambda_1 \vee \Lambda_2)$ with

$$\begin{aligned} \Lambda_1 \vee \Lambda_2 \quad = \quad & \{(X_1 \cup X_2, Y_1 \cup Y_2) \mid \\ & \qquad (X_1, Y_1) \in \Lambda_1,\ (X_2, Y_2) \in \Lambda_2,\ X_1 \cap X_2 = \emptyset,\ Y_1 \cap Y_2 = \emptyset\}. \end{aligned}$$

It is mentioned that $S_1 \cap S_2 \neq \emptyset$ and $T_1 \cap T_2 \neq \emptyset$ in general.

Remark 3.1. A canonical example of a bimatroid arises from a matrix. Let A be a matrix over a field, with row set S and column set T. Define Λ to be the family of all pairs $(X, Y) \in 2^S \times 2^T$ such that $|X| = |Y|$ and the corresponding submatrix $A[X, Y]$ is nonsingular. Then (S, T, Λ) is a bimatroid, which we denote by $\mathbf{L}(A)$. ∎

Remark 3.2. Another bimatroid arises from a matrix. Let A be a matrix over a field, with row set S and column set T. Define Λ to be the family of all pairs $(X, Y) \in 2^S \times 2^T$ such that $|X| = |Y|$ and the corresponding submatrix $A[X, Y]$ is term-nonsingular, which means that there exists a one-to-one mapping $\sigma : X \to Y$ such that the $(i, \sigma(i))$-entry of A is nonzero for all $i \in X$. In other words, $(X, Y) \in \Lambda$ if and only if a perfect matching exists between X and Y in the associated bipartite graph $(S, T; E)$, where $E = \{(i, j) \mid (i, j)\text{-entry of } A \text{ is nonzero}\}$. Then (S, T, Λ) is a bimatroid, which we denote by $\mathbf{G}(A)$. Every linked pair in $\mathbf{L}(A)$ is a linked pair in $\mathbf{G}(A)$, which we denote as $\mathbf{L}(A) \subseteq \mathbf{G}(A)$ by abuse of notation. We have $\mathbf{L}(A) = \mathbf{G}(A)$ if A is a generic matrix, i.e., if the nonzero entries of A are independent parameters. ∎

Remark 3.3. A bimatroid union corresponds roughly to a matrix sum. Consider $C = A + B$, and an expansion of its subdeterminant:

$$\det C[X, Y] = \sum_{I \subseteq X, J \subseteq Y} \pm \det A[I, J] \cdot \det B[X \setminus I, Y \setminus J]. \tag{13}$$

If $\det C[X, Y] \neq 0$, there exists a pair (I, J) such that $\det A[I, J] \neq 0$ and $\det B[X \setminus I, Y \setminus J] \neq 0$; the converse is also true provided no cancellation occurs among nonzero terms in the summation. Therefore, every linked pair in $\mathbf{L}(A + B)$ is a linked pair in $\mathbf{L}(A) \vee \mathbf{L}(B)$, which we denote as

$$\mathbf{L}(A + B) \subseteq \mathbf{L}(A) \vee \mathbf{L}(B). \tag{14}$$

In particular we have

$$\operatorname{rank} \mathbf{L}(A + B) \leq \operatorname{rank} (\mathbf{L}(A) \vee \mathbf{L}(B)). \tag{15}$$

Furthermore, we have $\mathbf{L}(A + B) = \mathbf{L}(A) \vee \mathbf{L}(B)$ in a situation that guarantees no-cancellation in (13). ∎

3.2 Valuated bimatroids

As a variant of valuated matroids of Dress–Wenzel [4, 5] the concept of valuated bimatroids was introduced by Murota [17]; see [19, Chapter 5] for an exposition.

Let (S, T, Λ) be a bimatroid of rank r. A function $f : \Lambda \to \mathbb{R}$ is called a *valuated bimatroid* if (VB-1) and (VB-2) below hold for any $(X, Y) \in \Lambda$ and $(X', Y') \in \Lambda$:

(VB-1) For any $x' \in X' \setminus X$, either (a1) or (b1) (or both) holds, where
(a1) there exists $y' \in Y' \setminus Y$ such that
$f(X, Y) + f(X', Y') \leq f(X + x', Y + y') + f(X' - x', Y' - y')$,
(b1) there exists $x \in X \setminus X'$ such that
$f(X, Y) + f(X', Y') \leq f(X - x + x', Y) + f(X' - x' + x, Y')$.

(VB-2) For any $y \in Y \setminus Y'$, either (a2) or (b2) (or both) holds, where
(a2) there exists $x \in X \setminus X'$ such that
$f(X, Y) + f(X', Y') \leq f(X - x, Y - y) + f(X' + x, Y' + y)$,
(b2) there exists $y' \in Y' \setminus Y$ such that
$f(X, Y) + f(X', Y') \leq f(X, Y - y + y') + f(X', Y' - y' + y)$.

Here $X - x$, $X + x'$ and $X - x + x'$ are short-hand notations for $X \setminus \{x\}$, $X \cup \{x'\}$ and $(X \setminus \{x\}) \cup \{x'\}$, respectively.

For any $p : S \to \mathbb{R}$ and $q : T \to \mathbb{R}$, define a function $f_{pq} : \Lambda \to \mathbb{R}$ by

$$f_{pq}(X, Y) = f(X, Y) - \sum_{x \in X} p(x) - \sum_{y \in Y} q(y). \tag{16}$$

If f is a valuated bimatroid, then f_{pq} is also a valuated bimatroid. Denote by argmax f_{pq} the set of maximizers of f_{pq}.

Proposition 3.1. *Let (S, T, Λ) be a bimatroid. A function $f : \Lambda \to \mathbb{R}$ is a valuated bimatroid if and only if* argmax f_{pq} *forms a bimatroid for every (p, q).*

Proof. Since (S, T, Λ) is a bimatroid, $\mathcal{B} = \{X \cup Y \mid (S \setminus X, Y) \in \Lambda\}$ is the basis family of a matroid. Then $\omega : \mathcal{B} \to \mathbb{R}$ defined by $\omega(X \cup Y) = f(S \setminus X, Y)$ is a valuated matroid if and only if f is a valuated bimatroid [19, Section 5.2.5]. The assertion for f is a straightforward translation of the corresponding statement for ω; see [19, Theorem 5.2.26]. □

To consider the maximum f-value over linked pairs (X, Y) of a specified size k we define

$$\delta_k = \max\{f(X, Y) \mid |X| = |Y| = k, (X, Y) \in \Lambda\} \qquad (0 \leq k \leq r). \tag{17}$$

Proposition 3.2 ([17]; also [19, Theorem 5.2.13])**.** *The sequence (δ_k) defined by* (17) *is concave, i.e., $\delta_{k-1} + \delta_{k+1} \leq 2\delta_k$ for each k with $1 \leq k \leq r - 1$.* ∎

A union operation can be defined for valuated bimatroids compatibly with the bimatroid union. The function $f_1 \vee f_2$ below is called the *union* of f_1 and f_2.

Proposition 3.3 ([19, Theorem 5.2.24])**.** *For two valuated bimatroids $f_i : \Lambda_i \to \mathbb{R}$ $(i = 1, 2)$, the function $f_1 \vee f_2 : \Lambda_1 \vee \Lambda_2 \to \mathbb{R}$ defined by*

$$\begin{aligned}(f_1 \vee f_2)(X, Y) \quad = \quad & \max\{f_1(X_1, Y_1) + f_2(X_2, Y_2) \mid (X_1, Y_1) \in \Lambda_1,\ (X_2, Y_2) \in \Lambda_2, \\ & X_1 \cup X_2 = X,\ Y_1 \cup Y_2 = Y,\ X_1 \cap X_2 = \emptyset,\ Y_1 \cap Y_2 = \emptyset\}\end{aligned}$$

for $(X, Y) \in \Lambda_1 \vee \Lambda_2$ is a valuated bimatroid. ∎

Remark 3.4. A canonical example of a valuated bimatroid arises from a polynomial matrix. Let $D(s)$ be a polynomial matrix in variable s with coefficients from a field, and $\mathbf{L}(D) = (S, T, \Lambda)$ be the associated bimatroid (cf. Remark 3.1). Define $f : \Lambda \to \mathbb{Z}$ by $f(X, Y) = \deg_s \det D[X, Y]$. Then f is a valuated bimatroid. ∎

4 Valuated Bimatroids Associated with Matroid Pencils

4.1 Matroid pencils

A *matroid pencil* is a pair of bimatroids having the row/column sets in common, which was introduced by Iwata [9] as a combinatorial abstraction of matrix pencils. A matroid pencil $(\mathbf{A}, \mathbf{B})$ with $\mathbf{A} = (S, T, \Lambda)$ and $\mathbf{B} = (S, T, \Xi)$ is also denoted as

$(S, T; \Lambda, \Xi)$, and the rank of $(\mathbf{A}, \mathbf{B})$ is defined as the rank of $\mathbf{A} \vee \mathbf{B}$, to be denoted by r.

A combinatorial counterpart of the expanded matrix Θ_k in (5) can be defined as follows. For a positive integer j, let S_j and T_j be disjoint copies of S and T, respectively. Furthermore, let $\mathbf{A}_j = (S_j, T_j, \Lambda_j)$ and $\mathbf{B}_j = (S_{j+1}, T_j, \Xi_j)$ be the copies of $\mathbf{A}$ and $\mathbf{B}$, respectively. For each positive integer k, consider the union

$$\Theta_k = \Theta_k(\mathbf{A}, \mathbf{B}) = \mathbf{A}_1 \vee \mathbf{B}_1 \vee \mathbf{A}_2 \vee \cdots \vee \mathbf{B}_{k-1} \vee \mathbf{A}_k \tag{18}$$

and denote the rank of $\Theta_k(\mathbf{A}, \mathbf{B})$ by $\theta_k = \theta_k(\mathbf{A}, \mathbf{B})$, where $\theta_0 = 0$.

Proposition 4.1 ([9])**.** *The sequence $\theta_k = \theta_k(\mathbf{A}, \mathbf{B})$ has the following properties.*

(i) $\theta_{k-1} + \theta_{k+1} \geq 2\theta_k$ *for every $k \geq 1$ (convexity).*

(ii) $0 \leq \theta_{k+1} - \theta_k \leq r$ *for every $k \geq 0$.*

(iii) $\theta_{k+1} - \theta_k = r$ *for every $k \geq r$.* ∎

Theorem 4.2 ([9])**.** *For $\theta_k = \theta_k(\mathbf{A}, \mathbf{B})$ associated with a matroid pencil $(\mathbf{A}, \mathbf{B}) = (S, T; \Lambda, \Xi)$, we have*

$$\theta_k = \max\{k|X| + (k-1)|\tilde{X}| \mid (X, Y) \in \Lambda, (\tilde{X}, \tilde{Y}) \in \Xi, X \cap \tilde{X} = \emptyset, Y \cap \tilde{Y} = \emptyset\}.$$

∎

4.2 Associated valuated bimatroids

Given a matroid pencil $(\mathbf{A}, \mathbf{B}) = (S, T; \Lambda, \Xi)$, we define two functions $f = f_{(\mathbf{A},\mathbf{B})} : \Lambda \vee \Xi \to \mathbb{R}$ and $g = g_{(\mathbf{A},\mathbf{B})} : \Lambda \vee \Xi \to \mathbb{R}$ by

$$\begin{aligned} f(X, Y) &= \max\{|\hat{X}| \mid (\hat{X}, \hat{Y}) \in \Lambda,\ (X \setminus \hat{X},\ Y \setminus \hat{Y}) \in \Xi,\ \hat{X} \subseteq X,\ \hat{Y} \subseteq Y\}, \quad (19) \\ g(X, Y) &= \max\{|\hat{X}| \mid (\hat{X}, \hat{Y}) \in \Xi,\ (X \setminus \hat{X},\ Y \setminus \hat{Y}) \in \Lambda,\ \hat{X} \subseteq X,\ \hat{Y} \subseteq Y\}. \quad (20) \end{aligned}$$

We then have

$$f(X, Y) \leq |X|, \qquad g(X, Y) \leq |X|. \tag{21}$$

It is also noted that $g_{(\mathbf{A},\mathbf{B})}(X, Y) = f_{(\mathbf{B},\mathbf{A})}(X, Y)$. See Remark 4.1 below for a motivation of the definition (19).

Proposition 4.3. *The functions $f = f_{(\mathbf{A},\mathbf{B})}$ and $g = g_{(\mathbf{A},\mathbf{B})}$ associated with a matroid pencil $(\mathbf{A}, \mathbf{B})$ are valuated bimatroids.*

Proof. Define a function $f_1 : \Lambda \to \mathbb{R}$ by $f_1(X, Y) = |X|$ for $(X, Y) \in \Lambda$, and another function $f_2 : \Xi \to \mathbb{R}$ by $f_2(X, Y) = 0$ for $(X, Y) \in \Xi$. Both f_1 and f_2 are valuated bimatroids. Moreover, the function f in (19) coincides with the union $f_1 \vee f_2$, which is a valuated bimatroid by Proposition 3.3. The assertion for g follows from the above argument, since $g_{(\mathbf{A},\mathbf{B})} = f_{(\mathbf{B},\mathbf{A})}$. □

Denote by $\mathcal{P}$ the set of matroid pencils $(\mathbf{A}, \mathbf{B})$, and by $\mathcal{V}$ the set of pairs (f, g) of valuated bimatroids that are defined on a common bimatroid and satisfy (21). Proposition 4.3 shows that a mapping $V : \mathcal{P} \to \mathcal{V}$ can be defined by $V : (\mathbf{A}, \mathbf{B}) \mapsto (f_{(\mathbf{A},\mathbf{B})}, g_{(\mathbf{A},\mathbf{B})})$.

Conversely, suppose that we are given valuated bimatroids f and g on a common bimatroid, with row set S and column set T, such that (21) holds for every linked pair (X, Y). By defining

$$\Lambda_f = \{(X, Y) \mid f(X, Y) = |X|\}, \tag{22}$$
$$\Xi_g = \{(X, Y) \mid g(X, Y) = |X|\}, \tag{23}$$

we obtain a matroid pencil $(\mathbf{A}, \mathbf{B}) = (S, T; \Lambda_f, \Xi_g)$, as shown in Proposition 4.4 below. This means that a mapping $P : \mathcal{V} \to \mathcal{P}$ can be defined by $P : (f, g) \mapsto (S, T; \Lambda_f, \Xi_g)$.

Proposition 4.4. *$(S, T; \Lambda_f, \Xi_g)$ is a matroid pencil for any $(f, g) \in \mathcal{V}$.*

Proof. By (21) and (22), Λ_f coincides with the set of maximizers of $f(X, Y) - |X|$, which is equal, in the notation of (16), to $f_{pq}(X, Y)$ with $p(x) = 1$ $(x \in S)$ and $q(y) = 0$ $(y \in T)$. Hence (S, T, Λ_f) is a bimatroid by Proposition 3.1. Similarly for Ξ_g. □

The following theorem shows that $P \circ V : \mathcal{P} \to \mathcal{P}$ is the identity mapping. This implies, in particular, that $V : \mathcal{P} \to \mathcal{V}$ is an injection and the representation of $(\mathbf{A}, \mathbf{B}) \in \mathcal{P}$ by $(f, g) \in \mathcal{V}$ is faithful. Note, however, that V is not necessarily a surjection; see Remark 4.2.

Theorem 4.5. *For a matroid pencil $(\mathbf{A}, \mathbf{B}) = (S, T; \Lambda, \Xi)$, we have*

$$\Lambda_{f_{(\mathbf{A},\mathbf{B})}} = \Lambda, \qquad \Xi_{g_{(\mathbf{A},\mathbf{B})}} = \Xi.$$

Proof. Put $f = f_{(\mathbf{A},\mathbf{B})}$. By the definition of f in (19) we see: $f(X, Y) = |X| \Leftrightarrow (X, Y) \in \Lambda$. On the other hand, the definition of Λ_f in (22) shows: $(X, Y) \in \Lambda_f \Leftrightarrow f(X, Y) = |X|$. Hence follows $\Lambda_f = \Lambda$. Similarly for Ξ. □

Remark 4.1. The function f in (19) is a combinatorial counterpart of the degree of subdeterminants. Consider a matrix pencil $D(s) = sA + B$, and an expansion of its determinant:

$$\det D[X, Y] = \sum_{I \subseteq X, J \subseteq Y} \pm s^{|I|} \det A[I, J] \cdot \det B[X \setminus I, Y \setminus J]. \tag{24}$$

If $\mathbf{A} = \mathbf{L}(A)$ and $\mathbf{B} = \mathbf{L}(B)$ are the bimatroids associated with the matrices A and B via nonsingular submatrices (cf., Remark 3.1), $f(X, Y) = f^{\mathrm{b}}(X, Y)$ defined in (19) for $(\mathbf{A},\mathbf{B}) = (\mathbf{L}(A),\mathbf{L}(B))$ coincides with the maximum degree of a term appearing on the right-hand side of (24). If $\mathbf{A} = \mathbf{G}(A)$ and $\mathbf{B} = \mathbf{G}(B)$ are the bimatroids associated with the matrices A and B via term-nonsingular submatrices (cf., Remark 3.2), $f(X,Y) = f^{\mathrm{g}}(X,Y)$ defined in (19) for $(\mathbf{A},\mathbf{B}) = (\mathbf{G}(A),\mathbf{G}(B))$ coincides with the maximum degree of a term appearing in the expansion

$$\det D[X,Y] = \sum_{\sigma}\sum_{I\subseteq X} \pm s^{|I|} \prod_{i\in I} A[i,\sigma(i)] \prod_{i\in X\setminus I} B[i,\sigma(i)], \tag{25}$$

where σ runs over all one-to-one correspondences between X and Y. Therefore,

$$\deg_s \det D[X,Y] \leq f^{\mathrm{b}}(X,Y) \leq f^{\mathrm{g}}(X,Y) \tag{26}$$

in general, and the equality holds if no cancellation occurs among nonzero terms in the summations. It is mentioned that $f^{\mathrm{g}}(X,Y)$ can be evaluated by solving a weighted bipartite matching problem, and $f^{\mathrm{b}}(X,Y)$ by solving a weighted matroid union/intersection problem. ∎

Remark 4.2. Not every member of $\mathcal{V}$ corresponds to a member of $\mathcal{P}$. A necessary condition for (f,g) to be contained in the image of $V : \mathcal{P} \to \mathcal{V}$ is that

$$f(X,Y) + g(X,Y) \geq |X| \tag{27}$$

for all (X,Y). To see this, let $(\hat{X},\hat{Y})$ be a maximizer in (19) with $f(X,Y) = |\hat{X}|$. Since $(X\setminus\hat{X},\ Y\setminus\hat{Y}) \in \Xi$ and $(\hat{X},\hat{Y}) \in \Lambda$, we have $g(X,Y) \geq |X\setminus\hat{X}|$ by (20). Hence follows (27). It is left for the future to identify necessary and sufficient conditions for (f,g) to lie in the image of V. ∎

4.3 Indices of nilpotency

For $k = 0,1,\ldots,r$ we define

$$\delta_k = \delta_k(\mathbf{A},\mathbf{B}) = \max\{f(X,Y) \mid |X| = |Y| = k,\ (X,Y) \in \Lambda \vee \Xi\}, \tag{28}$$

where $f = f_{(\mathbf{A},\mathbf{B})}$ is defined by (19). This serves as a combinatorial counterpart of $\delta_k(sA+B)$ for a matrix pencil $sA+B$ defined in (2); see Remark 4.1.

Proposition 4.6. *The sequence $\delta_k = \delta_k(\mathbf{A},\mathbf{B})$ is concave, i.e., $\delta_{k-1}+\delta_{k+1} \leq 2\delta_k$ for each k with $1 \leq k \leq r-1$.*

Proof. Since f is a valuated bimatroid by Proposition 4.3, the assertion follows from Proposition 3.2. □

The two sequences δ_k and θ_k associated with a matroid pencil $(\mathbf{A}, \mathbf{B})$ are equivalent through the discrete Legendre transformation, which is stated in Theorem 4.7 below. This is a combinatorial counterpart of Proposition 2.2 for a matrix pencil.

Theorem 4.7. *For $\delta_k = \delta_k(\mathbf{A}, \mathbf{B})$ and $\theta_k = \theta_k(\mathbf{A}, \mathbf{B})$ associated with a matroid pencil $(\mathbf{A}, \mathbf{B})$, we have*

$$\delta_k = \min_{\ell \geq 0} (\theta_{\ell+1} - k\ell) \qquad (0 \leq k \leq r), \tag{29}$$

$$\theta_{\ell+1} = \max_{0 \leq k \leq r} (\delta_k + k\ell) \qquad (\ell \geq 0). \tag{30}$$

Proof. Since δ_k is concave by Proposition 4.6 and θ_ℓ is convex by Proposition 4.1, the two expressions (29) and (30) are equivalent; see Remark 2.1 in Section 2.2. The latter expression (30) can be verified as follows.

$$\begin{aligned}
&\max_k (\delta_k + k\ell) \\
&= \max_k \max_{X,Y} \{ f(X, Y) + k\ell \mid |X| = |Y| = k,\ (X, Y) \in \Lambda \vee \Xi \} \\
&= \max_{X,Y} \{ f(X, Y) + \ell|X| \mid (X, Y) \in \Lambda \vee \Xi \} \\
&= \max\{ |\hat{X}| + \ell|\hat{X} \cup \tilde{X}| \mid (\hat{X}, \hat{Y}) \in \Lambda,\ (\tilde{X}, \tilde{Y}) \in \Xi,\ \hat{X} \cap \tilde{X} = \emptyset, \hat{Y} \cap \tilde{Y} = \emptyset \} \\
&= \max\{ (\ell + 1)|\hat{X}| + \ell|\tilde{X}| \mid (\hat{X}, \hat{Y}) \in \Lambda,\ (\tilde{X}, \tilde{Y}) \in \Xi,\ \hat{X} \cap \tilde{X} = \emptyset, \hat{Y} \cap \tilde{Y} = \emptyset \}.
\end{aligned}$$

This is equal to $\theta_{\ell+1}$ by Theorem 4.2. □

Proposition 4.1, Proposition 4.6 and Theorem 4.7 imply the existence of some integers $\mu_1 \geq \cdots \geq \mu_d \geq 1$, uniquely determined, such that

$$\delta_k = k - \sum_{i=r-k+1}^{d} \mu_i \qquad (0 \leq k \leq r), \tag{31}$$

$$\theta_k = kr - \sum_{i=1}^{d} \min(k, \mu_i) \qquad (k \geq 0). \tag{32}$$

The integers $\mu_1, \ldots, \mu_d$ defined above for a matroid pencil $(\mathbf{A}, \mathbf{B})$ are the combinatorial counterpart of the indices of nilpotency for a matrix pencil $sA + B$.

5 Combinatorial Bounds and Their Tightness

Let $D(s) = sA + B$ be a matrix pencil of rank r. We consider a matroid pencil $(\mathbf{L}(A), \mathbf{L}(B))$ defined in terms of nonsingular submatrices of A and B (cf. Remark 3.1) and denote by f^{b} the associated valuated bimatroid in (19). We also consider another matroid pencil $(\mathbf{G}(A), \mathbf{G}(B))$ defined in terms of term-nonsingular submatrices of A and B (cf. Remark 3.2) and denote by f^{g} the associated valuated bimatroid in (19). It is mentioned that "b" and "g" stand for "bimatroid" and "graph," respectively.

In general we have

$$\deg_s \det D[X, Y] \leq f^{\mathrm{b}}(X, Y) \leq f^{\mathrm{g}}(X, Y),$$

as discussed in Remark 4.1, and therefore,

$$\delta_k(sA + B) \leq \delta_k^{\mathrm{b}}(sA + B) \leq \delta_k^{\mathrm{g}}(sA + B), \tag{33}$$

where δ_k^{b} and δ_k^{g} are defined by (28) as

$$\delta_k^{\mathrm{b}}(sA + B) = \delta_k(\mathbf{L}(A), \mathbf{L}(B)), \qquad \delta_k^{\mathrm{g}}(sA + B) = \delta_k(\mathbf{G}(A), \mathbf{G}(B)). \tag{34}$$

In parallel we define

$$\theta_k^{\mathrm{b}}(sA + B) = \theta_k(\mathbf{L}(A), \mathbf{L}(B)), \qquad \theta_k^{\mathrm{g}}(sA + B) = \theta_k(\mathbf{G}(A), \mathbf{G}(B)) \tag{35}$$

to obtain

$$\theta_k(sA + B) \leq \theta_k^{\mathrm{b}}(sA + B) \leq \theta_k^{\mathrm{g}}(sA + B). \tag{36}$$

The inequalities (33) and (36) share a common feature that a linear-algebraic quantity is upper-bounded by combinatorial quantities.

From the general results for matroid pencils we see the following.

Proposition 5.1.
(1) *The sequence* $\delta_k^{\mathrm{b}} = \delta_k^{\mathrm{b}}(sA + B)$ $(0 \leq k \leq r)$ *is concave.*
(2) *The sequence* $\delta_k^{\mathrm{g}} = \delta_k^{\mathrm{g}}(sA + B)$ $(0 \leq k \leq r)$ *is concave.*

Proof. (1) and (2) are special cases of Proposition 4.6. □

Proposition 5.2.
(1) *The sequence* $\theta_k^{\mathrm{b}} = \theta_k^{\mathrm{b}}(sA + B)$ $(k \geq 0)$ *is convex.*
(2) *The sequence* $\theta_k^{\mathrm{g}} = \theta_k^{\mathrm{g}}(sA + B)$ $(k \geq 0)$ *is convex.*

Proof. (1) and (2) are special cases of Proposition 4.1. □

Theorem 5.3.
(1) *For* $\delta_k^{\mathrm{b}} = \delta_k^{\mathrm{b}}(sA + B)$ *and* $\theta_k^{\mathrm{b}} = \theta_k^{\mathrm{b}}(sA + B)$ *we have*

$$\delta_k^{\mathrm{b}} = \min_{\ell \geq 0} \left(\theta_{\ell+1}^{\mathrm{b}} - k\ell \right) \qquad (0 \leq k \leq r), \tag{37}$$

$$\theta_{\ell+1}^{\mathrm{b}} = \max_{0 \leq k \leq r} \left(\delta_k^{\mathrm{b}} + k\ell \right) \qquad (\ell \geq 0). \tag{38}$$

(2) *For* $\delta_k^{\rm g} = \delta_k^{\rm g}(sA+B)$ *and* $\theta_k^{\rm g} = \theta_k^{\rm g}(sA+B)$ *we have*

$$\delta_k^{\rm g} = \min_{\ell \geq 0} \left(\theta_{\ell+1}^{\rm g} - k\ell \right) \qquad (0 \leq k \leq r), \tag{39}$$

$$\theta_{\ell+1}^{\rm g} = \max_{0 \leq k \leq r} \left(\delta_k^{\rm g} + k\ell \right) \qquad (\ell \geq 0). \tag{40}$$

Proof. (1) and (2) are special cases of Theorem 4.7. □

The upper bounds in (33) and (36) are not necessarily tight. For example, for

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad D(s) = \begin{bmatrix} s+1 & s \\ 1 & 1 \end{bmatrix}, \tag{41}$$

we have $\delta_2 = 0$ whereas $\delta_2^{\rm b} = \delta_2^{\rm g} = 1$, and $\theta_2 = 2$ whereas $\theta_2^{\rm b} = \theta_2^{\rm g} = 3$.

Naturally, we are interested in cases where the upper bounds in (33) and (36) are tight for all k. Let us say that a matrix pencil $sA + B$ is $\delta^{\rm b}$*-tight* if it satisfies (42) below, and $\theta^{\rm b}$*-tight* if it satisfies (43) below:

$$\delta_k(sA+B) = \delta_k^{\rm b}(sA+B) \qquad \text{for all } k \text{ with } 0 \leq k \leq r, \tag{42}$$

$$\theta_k(sA+B) = \theta_k^{\rm b}(sA+B) \qquad \text{for all } k \geq 0. \tag{43}$$

Likewise we say that a matrix pencil $sA + B$ is $\delta^{\rm g}$*-tight* if it satisfies (44) below, and $\theta^{\rm g}$*-tight* if it satisfies (45) below:

$$\delta_k(sA+B) = \delta_k^{\rm g}(sA+B) \qquad \text{for all } k \text{ with } 0 \leq k \leq r, \tag{44}$$

$$\theta_k(sA+B) = \theta_k^{\rm g}(sA+B) \qquad \text{for all } k \geq 0. \tag{45}$$

Theorem 5.4.

(1) *A matrix pencil* $sA + B$ *is* $\delta^{\rm b}$*-tight if and only if it is* $\theta^{\rm b}$*-tight.*

(2) *A matrix pencil* $sA + B$ *is* $\delta^{\rm g}$*-tight if and only if it is* $\theta^{\rm g}$*-tight.*

Proof. By Proposition 2.2 the sequence $(\delta_k \mid k = 0, 1, \ldots, r)$ uniquely determines the sequence $(\theta_k \mid k = 0, 1, 2, \ldots)$, and vice versa, by the discrete Legendre transformation (8) and (9). The same is true for $(\delta_k^{\rm b} \mid k = 0, 1, \ldots, r)$ and $(\theta_k^{\rm b} \mid k = 0, 1, 2, \ldots)$ by Theorem 5.3(1). Hence follows the equivalence of (42) and (43). Similarly for $(\delta_k^{\rm g} \mid k = 0, 1, \ldots, r)$ and $(\theta_k^{\rm g} \mid k = 0, 1, 2, \ldots)$ by Theorem 5.3(2). □

A matrix is called a *generic matrix* if its nonzero entries are independent parameters. A *generic matrix pencil* means a pencil $sA + B$ with generic matrices A and B, where the independent parameters in A and B are all distinct.

It is well known that (44) holds for a generic matrix pencil; see, e.g., [19, Theorem 6.2.2]. It is also shown in [11] that (45) holds for a generic matrix pencil. In other words, a generic matrix pencil is both $\delta^{\rm g}$-tight and $\theta^{\rm g}$-tight. Theorem 5.4(2) above says that these two properties of a generic matrix pencil are in fact equivalent.

As an application of Theorem 5.4(1) we can show the following.

Theorem 5.5. *A matrix pencil $sA + B$ with generic A or B is δ^{b}-tight and θ^{b}-tight.*

Proof. In the expansion (24) in Remark 4.1, no distinct terms cancel one another by the assumed genericity of A or B. Therefore, we have $\deg_s \det D[X,Y] = f^{\mathrm{b}}(X,Y)$, which implies (42). Then (43) also holds by Theorem 5.4(1). □

Applications of Theorem 5.5 to electric networks are discussed in the next section.

Remark 5.1. A generic matrix pencil is obviously δ^{g}-tight; see, e.g., [19, Theorem 6.2.2]. Then Theorem 5.4(2) gives an alternative proof of its θ^{g}-tightness, which was established in [11] by way of "periodic matching." ■

Remark 5.2. We have $\delta_k^{\mathrm{b}}(sA+B) = \delta_k^{\mathrm{g}}(sA+B)$ for all k with $0 \le k \le r$ if and only if $\theta_k^{\mathrm{b}}(sA+B) = \theta_k^{\mathrm{g}}(sA+B)$ for all $k \ge 0$. The proof is similar to that of Theorem 5.4. ■

6 Applications to Electric Networks

6.1 RLC networks

A matrix pencil $sA + B$ arises from frequency-domain descriptions of electric networks. First we consider RLC networks, which, by definition, consist of resistors (R), inductors (L), and capacitors (C).

As a concrete instance, let us consider the simple RLC network in Fig. 1 with a current source I_0. The network can be described in terms of branch currents i_C, i_R, i_L and branch voltages v_C, v_R, v_L as

$$i_C + i_R + i_L = I_0, \quad v_C = v_R = v_L, \quad i_C = sC\, v_C, \quad v_R = R\, i_R, \quad v_L = sL\, i_L,$$

where s is a variable to represent the Laplace transformation. Accordingly we obtain

$$sA + B = \begin{array}{c} \begin{array}{cccccc} i_C & i_R & i_L & v_C & v_R & v_L \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & & & \\ \hline & & & 1 & -1 & 0 \\ & & & 0 & 1 & -1 \\ \hline -1 & 0 & 0 & sC & 0 & 0 \\ 0 & R & 0 & 0 & -1 & 0 \\ 0 & 0 & sL & 0 & 0 & -1 \end{array}\right] \end{array} \tag{46}$$

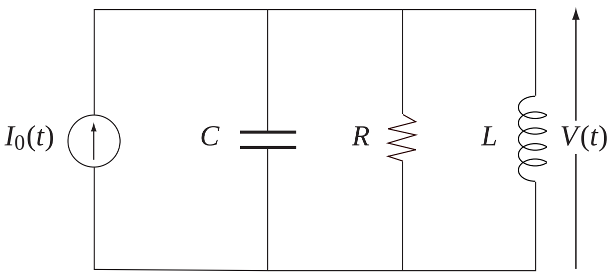

Figure 1: A simple RLC network

with

$$A = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & & & \\ \hline & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & L & 0 & 0 & 0 \end{array}\right], \quad B = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & & & \\ \hline & & & 1 & -1 & 0 \\ & & & 0 & 1 & -1 \\ \hline -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & R & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right]. \tag{47}$$

The matrix A consists of two nonzero entries, C and L, whereas the matrix B comprises R and constant nonzero entries, $+1$ and -1. If R, L, and C are independent parameters, A is a generic matrix and B is a mixed matrix (see Section 7.1 for the definition of a mixed matrix).

In RLC networks in general, we may reasonably assume that the physical characteristics of resistors, inductors, and capacitors are mutually independent parameters. Then the nonzero entries of the matrix A in $sA + B$, representing capacitances and inductances, are independent parameters. This means that A is a generic matrix, and the assumption of Theorem 5.5 is satisfied. Therefore, the matrix pencil $sA + B$ arising from an RLC network as above is δ^{b}-tight and θ^{b}-tight.

Moreover, both $\mathbf{L}(A)$ and $\mathbf{L}(B)$ are computationally tractable objects. By genericity of A, the bimatroid $\mathbf{L}(A)$ is equal to $\mathbf{G}(A)$, and hence it can be represented by bipartite matchings. On the other hand, B is a mixed matrix, and the structure of $\mathbf{L}(B)$ can be represented by independent matchings (or matroid intersection) for a graphic matroid and a transversal matroid; see [19, Chapter 4] for details.

6.2 Networks containing gyrators and transformers

An ideal element called a gyrator is commonly employed in electric network theory. It is a two-port element, the element characteristic of which is represented as

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

for the current-voltage pairs (i_1, v_1) and (i_2, v_2) at the ports. Another common element is an ideal transformer with the characteristic given by

$$\begin{bmatrix} v_2 \\ i_2 \end{bmatrix} = \begin{bmatrix} t & 0 \\ 0 & -1/t \end{bmatrix} \begin{bmatrix} v_1 \\ i_1 \end{bmatrix}.$$

Since the element characteristics of gyrators and transformers are free from the variable s, the matrix A in $sA + B$ remains generic even when gyrators and transformers are contained in addition to resistors, inductors, and capacitors. This means that the matrix pencil $sA + B$ remains δ^{b}-tight and θ^{b}-tight by Theorem 5.5.

It is worth mentioning in passing that any passive network is known to be “equivalent” to an RCG network, which is, by definition, a network consisting of resistors, capacitors, and gyrators (and possibly, sources), although the transformation to an equivalent RCG network is not always compatible with the independence of parameters. In RCG networks, the matrix B is no longer a mixed matrix and consequently the representation of $\mathbf{L}(B)$ by independent matchings is no longer valid.

7 Mixed Matrix Pencils

Efficient combinatorial algorithms have been developed for computing δ_k and θ_k for mixed matrix pencils. They are based on combinatorial characterizations that are similar to, but somewhat different from, those discussed in Section 5. We describe in this section the methods for mixed matrix pencils developed by Murota [18] and Iwata–Takamatsu [12], as a preliminary to an extension of the Legendre duality to be presented in Section 8.

7.1 Mixed matrices and mixed matrix pencils

A matrix A is called a *mixed matrix* [19, 21] if it is a sum of a constant matrix Q and a generic matrix T:

$$A = Q + T. \tag{48}$$

For instance, the matrix B in (47) is a mixed matrix. It is easy to see that $\mathbf{L}(A) = \mathbf{L}(Q) \vee \mathbf{L}(T)$; see Remark 3.3 and [19, Theorem 4.2.9].

A matrix pencil $D(s) = sA + B$ is called a *mixed matrix pencil* if $A = Q_A + T_A$ and $B = Q_B + T_B$ are mixed matrices such that the independent parameters in T_A and T_B are all distinct.

For a mixed matrix pencil, the combinatorial bounds discussed in Section 5 can be nontight. That is, it may be that $\delta_k(sA + B) \neq \delta_k^{\mathrm{b}}(sA + B)$ or $\theta_k(sA + B) \neq \theta_k^{\mathrm{b}}(sA + B)$ in (33) and (36). In fact, (41) is a counterexample with $T_A = T_B = O$. In contrast, the bounds are tight if $Q_A = O$ or $Q_B = O$ by Theorem 5.5.

For a mixed matrix pencil $D(s) = sA + B$ it is convenient to use an expression

$$D(s) = Q(s) + T(s) \tag{49}$$

with

$$Q(s) = sQ_A + Q_B, \qquad T(s) = sT_A + T_B. \tag{50}$$

This is the splitting of $D(s)$ into the constant-coefficient part $Q(s)$ and the generic-coefficient part $T(s)$.

7.2 Characterization of δ_k

We first consider $\delta_k(sA + B)$. It follows from the expansion

$$\det D[X, Y] = \sum_{|I|=|J|} \pm \det Q[I, J] \cdot \det T[X \setminus I, Y \setminus J]$$

that

$$\deg_s \det D[X, Y] \le \max_{|I|=|J|} \{\deg_s \det Q[I, J] + \deg_s \det T[X \setminus I, Y \setminus J]\}, \tag{51}$$

where the inequality is in fact an equality by virtue of the genericity of T. This identity, observed first in [15, Proposition 5.3], can be formulated in terms of valuated bimatroids defined by

$$f_D(X, Y) = \deg_s \det D[X, Y] \qquad ((X, Y) \in \mathbf{L}(D(s))), \tag{52}$$

$$f_Q(X, Y) = \deg_s \det Q[X, Y] \qquad ((X, Y) \in \mathbf{L}(Q(s))), \tag{53}$$

$$f_T(X, Y) = \deg_s \det T[X, Y] \qquad ((X, Y) \in \mathbf{L}(T(s))) \tag{54}$$

as follows.

Theorem 7.1 ([19, Theorem 6.2.4])**.** *For a mixed matrix pencil $D(s) = Q(s)+T(s)$, we have*

$$f_D = f_Q \vee f_T,$$

where $\vee$ means the union of valuated bimatroids. ∎

The function f_T associated with $T(s) = sT_A + T_B$ admits a further decomposition $f_T = f_{T1} \vee f_{T0}$ with valuated bimatroids f_{T1} and f_{T0} on $\mathbf{L}(T_A)$ and $\mathbf{L}(T_B)$ defined by

$$f_{T1}(X, Y) = |X|, \qquad f_{T0}(X, Y) = 0. \tag{55}$$

In other words, we have $f_T = f_{(\mathbf{L}(T_A),\mathbf{L}(T_B))}$, where the right-hand side means the function in (19) associated with the matroid pencil $(\mathbf{L}(T_A), \mathbf{L}(T_B))$.

We now define

$$\begin{aligned} &\delta_k^{\mathrm{m}}(sA + B) \\ &= \max\{(f_Q \vee f_{T1} \vee f_{T0})(Z, W) \mid |Z| = k,\ (Z, W) \in \mathbf{L}(Q(s)) \vee \mathbf{L}(T_A) \vee \mathbf{L}(T_B)\} \end{aligned} \tag{56}$$

$$= \max\{\deg_s \det Q[I, J] + |X| \mid |I| + |X| + |\tilde{X}| = k,\ (I, X, \tilde{X}; J, Y, \tilde{Y}) \in \mathbf{L}^{\vee}\}, \tag{57}$$

where "m" in δ_k^{m} stands for "mixed" and $\mathbf{L}^{\vee}$ denotes the set of tuples $(I, X, \tilde{X}; J, Y, \tilde{Y})$ such that

$$
\begin{array}{l}
(I, J) \in \mathbf{L}(Q(s)), \quad (X, Y) \in \mathbf{L}(T_A), \quad (\tilde{X}, \tilde{Y}) \in \mathbf{L}(T_B) \\
I \cap X = I \cap \tilde{X} = X \cap \tilde{X} = \emptyset, \quad J \cap Y = J \cap \tilde{Y} = Y \cap \tilde{Y} = \emptyset.
\end{array}
$$

It then follows from (51) that

$$
\delta_k(sA + B) \leq \delta_k^{\mathrm{m}}(sA + B) \tag{58}
$$

for a matrix pencil $sA + B$ represented as (49) with (50), irrespective of the genericity of the T-part. For a mixed matrix pencil the bound is tight indeed.

Theorem 7.2 ([18]). *For a mixed matrix pencil, we have equality in* (58). ∎

An efficient algorithm for mixed matrix pencils is designed in [18] that computes $\delta_k(sA + B)$ on the basis of Theorem 7.2. The function f_Q can be computed as the degree of a subdeterminant of the constant-coefficient matrix pencil $Q(s) = sQ_A + Q_B$; see, e.g., [8, 16] (and references therein) for algorithms. The function $f_T = f_{T1} \vee f_{T0}$ can be evaluated by solving a weighted bipartite matching problem. Finally, the union of f_Q and f_T can be computed by the valuated matroid intersection algorithm. See [18, 19] for details.

7.3 Characterization of θ_k

Next we turn to $\theta_k(sA + B)$. Recall the definitions of expanded matrix $\Theta_k(sA + B)$ in (5) and its bimatroid version $\Theta_k(\mathbf{A}, \mathbf{B})$ in (18). It follows from the expression

$$
sA + B = (sQ_A + Q_B) + (sT_A + T_B)
$$

and the inequality (15) in Remark 3.3 that

$$
\begin{aligned}
\theta_k(sA + B) &= \operatorname{rank} \Theta_k(sA + B) \\
&= \operatorname{rank} (\Theta_k(sQ_A + Q_B) + \Theta_k(sT_A + T_B)) \\
&\leq \operatorname{rank} \left(\mathbf{L}(\Theta_k(sQ_A + Q_B)) \vee \mathbf{L}(\Theta_k(sT_A + T_B))\right),
\end{aligned} \tag{59}
$$

which is true, irrespective of the genericity of the T-part.

To obtain a more tractable expression by taking advantage of the genericity, we replace the second term $\mathbf{L}(\Theta_k(sT_A + T_B))$ by $\Theta_k(\mathbf{L}(T_A), \mathbf{L}(T_B))$. This yields another upper bound, since $\mathbf{L}(\Theta_k(sT_A + T_B)) \subseteq \Theta_k(\mathbf{L}(T_A), \mathbf{L}(T_B))$ in the notation (14) in Remark 3.3. By defining

$$
\theta_k^{\mathrm{m}}(sA + B) = \operatorname{rank} \left(\mathbf{L}(\Theta_k(sQ_A + Q_B)) \vee \Theta_k(\mathbf{L}(T_A), \mathbf{L}(T_B))\right), \tag{60}
$$

we thus arrive at another combinatorial upper bound

$$
\theta_k(sA + B) \leq \theta_k^{\mathrm{m}}(sA + B). \tag{61}
$$

It is noted in this connection that $\mathbf{L}(\Theta_k(sT_A + T_B))$ and $\Theta_k(\mathbf{L}(T_A), \mathbf{L}(T_B))$ are different in general, but they share the same rank [11].

It is shown by Iwata–Takamatsu [12] that the upper bound in (61) is indeed tight for mixed matrix pencils.

Theorem 7.3 ([12])**.** *For a mixed matrix pencil, we have equality in* (61).

Proof. This is a reformulation of Theorem 6.2 in [12]. The right-hand side of (60) coincides with the optimal value of the independent matching problem $\mathrm{IMP}(\Theta_k(D))$ considered in [12]. This is explained in Appendix A. □

An efficient algorithm for mixed matrix pencils is constructed in [12] that computes $\theta_k(sA + B)$ on the basis of Theorem 7.3. Both $\mathbf{L}(\Theta_k(sQ_A + Q_B))$ and $\Theta_k(\mathbf{L}(T_A), \mathbf{L}(T_B))$ are computationally tractable objects; the former is represented by a constant matrix, of size $km \times kn$, and the latter by an unweighted bipartite matching problem. Then the combination of the two components is carried out by the (unweighted) matroid intersection (or union) algorithm. See [12] for details.

Remark 7.1. We may be tempted to replace $\mathbf{L}(\Theta_k(sQ_A + Q_B))$ in (61) with $\Theta_k(\mathbf{L}(Q_A), \mathbf{L}(Q_B))$. Since $\mathbf{L}(\Theta_k(sQ_A + Q_B)) \subseteq \Theta_k(\mathbf{L}(Q_A), \mathbf{L}(Q_B))$, we obtain

$$\begin{aligned} \theta_k(sA + B) &\leq \mathrm{rank}\ (\Theta_k(\mathbf{L}(Q_A), \mathbf{L}(Q_B)) \vee \Theta_k(\mathbf{L}(T_A), \mathbf{L}(T_B))) \\ &= \mathrm{rank}\ (\Theta_k(\mathbf{L}(Q_A) \vee \mathbf{L}(T_A), \mathbf{L}(Q_B) \vee \mathbf{L}(T_B))) \end{aligned} \tag{62}$$

as a third upper bound. This bound is also derived from (36) as follow:

$$\begin{aligned} \theta_k(sA + B) &\leq \theta_k(\mathbf{L}(A), \mathbf{L}(B)) \\ &= \mathrm{rank}\ (\Theta_k(\mathbf{L}(A), \mathbf{L}(B))) \\ &\leq \mathrm{rank}\ (\Theta_k(\mathbf{L}(Q_A) \vee \mathbf{L}(T_A), \mathbf{L}(Q_B) \vee \mathbf{L}(T_B))), \end{aligned}$$

where $\mathbf{L}(A) \subseteq \mathbf{L}(Q_A) \vee \mathbf{L}(T_A)$ and $\mathbf{L}(B) \subseteq \mathbf{L}(Q_B) \vee \mathbf{L}(T_B)$ are used. Unfortunately, however, the upper bound (62) is not tight for mixed matrix pencils, as is seen for the mixed matrix pencil $D(s)$ in (41) with $T_A = T_B = O$. ∎

8 Bounds for Formal Mixed Matrix Pencils

8.1 Mixed-type bounds and their tightness

We have seen that the inequalities (58) and (61) are valid for a matrix pencil $D(s)$ expressed as

$$D(s) = sA + B = (sQ_A + Q_B) + (sT_A + T_B) = Q(s) + T(s). \tag{63}$$

The genericity of the T-part is a sufficient condition for (58) and (61) to hold with equalities, but it is not a necessary condition.

We refer to a matrix pencil as a *formal mixed matrix pencil* if it is represented as (63), where the genericity of the T-part is not assumed. To be precise, a formal mixed matrix pencil does not mean a matrix pencil itself, but it denotes a representation $Q(s) + T(s)$ with two matrix pencils $Q(s)$ and $T(s)$. For example, both

$$\begin{bmatrix} s+1 & s \\ 1 & \alpha \end{bmatrix} = \begin{bmatrix} s & s \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}$$

and

$$\begin{bmatrix} s+1 & s \\ 1 & \alpha \end{bmatrix} = \begin{bmatrix} s+1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & s \\ 1 & \alpha \end{bmatrix}$$

are formal mixed matrix pencils, and we distinguish between the two.

For a formal mixed matrix pencil we define $\delta_k^{\mathrm{m}}(sA+B)$ by (57) and $\theta_k^{\mathrm{m}}(sA+B)$ by (60), i.e.,

$$\begin{aligned} \delta_k^{\mathrm{m}}(sA+B) &= \max\{\deg_s \det Q[I,J] + |X| \mid \\ &\qquad |I| + |X| + |\tilde{X}| = k,\ (I,X,\tilde{X};J,Y,\tilde{Y}) \in \mathbf{L}^{\vee}\}, && (64) \\ \theta_k^{\mathrm{m}}(sA+B) &= \operatorname{rank}\,(\mathbf{L}(\Theta_k(sQ_A+Q_B)) \vee \Theta_k(\mathbf{L}(T_A),\mathbf{L}(T_B))) && (65) \end{aligned}$$

with reference to its representation (63). Then we have

$$\begin{aligned} \delta_k(sA+B) &\leq \delta_k^{\mathrm{m}}(sA+B), && (66) \\ \theta_k(sA+B) &\leq \theta_k^{\mathrm{m}}(sA+B). && (67) \end{aligned}$$

The meanings of δ_k^{m} and θ_k^{m} can be rephrased as follows. For a formal mixed matrix pencil $D(s) = Q(s) + T(s)$ we consider a (genuine) mixed matrix pencil $\tilde{D}(s) = Q(s) + \tilde{T}(s)$ by changing the nonzero coefficients of $T(s)$ to independent parameters; see Example 8.1 below for a concrete example. Then we have

$$\begin{aligned} \delta_k^{\mathrm{m}}(D) = \delta_k^{\mathrm{m}}(\tilde{D}) = \delta_k(\tilde{D}), && (68) \\ \theta_k^{\mathrm{m}}(D) = \theta_k^{\mathrm{m}}(\tilde{D}) = \theta_k(\tilde{D}), && (69) \end{aligned}$$

where the first equalities in (68) and (69) are obvious from the definitions, and the second equalities are due to Theorems 7.2 and 7.3, respectively.

Example 8.1. Consider a formal mixed matrix pencil

$$D(s) = \begin{bmatrix} \alpha s+1 & \beta s \\ \beta & s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} \alpha s & \beta s \\ \beta & 0 \end{bmatrix}$$

with α and β being independent parameters. By changing βs to γs with a new parameter γ, we obtain a (genuine) mixed matrix pencil

$$\tilde{D}(s) = \begin{bmatrix} \alpha s+1 & \gamma s \\ \beta & s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} \alpha s & \gamma s \\ \beta & 0 \end{bmatrix}.$$

Then we have $\delta_k^{\mathrm{m}}(D) = \delta_k^{\mathrm{m}}(\tilde{D})$ by the definition (64), and $\delta_k^{\mathrm{m}}(\tilde{D}) = \delta_k(\tilde{D})$ by Theorem 7.2. To explain θ_k^{m} we choose $k = 3$ to obtain

$$\theta_3^{\mathrm{m}}(D) = \theta_3^{\mathrm{m}}(\tilde{D}) = \mathrm{rank} \left[\begin{array}{cc|cc|cc} \alpha_1 & \gamma_1 & & & & \\ 0 & 1 & & & & \\ \hline 1 & 0 & \alpha_2 & \gamma_2 & & \\ \beta_1 & 0 & 0 & 1 & & \\ \hline & & 1 & 0 & \alpha_3 & \gamma_3 \\ & & \beta_2 & 0 & 0 & 1 \end{array} \right]$$

by the definition (65), where $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3$ are independent parameters. On the other hand, $\theta_3^{\mathrm{m}}(\tilde{D}) = \theta_3(\tilde{D})$ by Theorem 7.3. ∎

Proposition 8.1. *The sequence* $\delta_k^{\mathrm{m}} = \delta_k^{\mathrm{m}}(sA+B)$ *is concave, i.e.,* $\delta_{k-1}^{\mathrm{m}} + \delta_{k+1}^{\mathrm{m}} \leq 2\delta_k^{\mathrm{m}}$ *for each* k *with* $1 \leq k \leq r-1$.

Proof. Let $\tilde{D}(s)$ be the mixed matrix pencil associated with $D(s) = sA + B$. The concavity of $\delta_k^{\mathrm{m}}(D)$ follows from $\delta_k^{\mathrm{m}}(D) = \delta_k(\tilde{D})$ in (68) and the concavity of $\delta_k(\tilde{D})$ in (4). □

Proposition 8.2. *The sequence* $\theta_k^{\mathrm{m}} = \theta_k^{\mathrm{m}}(sA+B)$ *is convex, i.e.,* $\theta_{k-1}^{\mathrm{m}} + \theta_{k+1}^{\mathrm{m}} \geq 2\theta_k^{\mathrm{m}}$ *for every* $k \geq 1$.

Proof. Let $\tilde{D}(s)$ be the mixed matrix pencil associated with $D(s) = sA + B$. The convexity of $\theta_k^{\mathrm{m}}(D)$ follows from $\theta_k^{\mathrm{m}}(D) = \theta_k(\tilde{D})$ in (69) and the convexity of $\theta_k(\tilde{D})$ in (7). □

Theorem 8.3. *For* $\delta_k^{\mathrm{m}} = \delta_k^{\mathrm{m}}(sA+B)$ *and* $\theta_k^{\mathrm{m}} = \theta_k^{\mathrm{m}}(sA+B)$ *associated with a formal mixed matrix pencil* $sA+B$ *in* (63), *we have*

$$\delta_k^{\mathrm{m}} = \min_{\ell \geq 0} \left(\theta_{\ell+1}^{\mathrm{m}} - k\ell \right) \qquad (0 \leq k \leq r), \tag{70}$$

$$\theta_{\ell+1}^{\mathrm{m}} = \max_{0 \leq k \leq r} \left(\delta_k^{\mathrm{m}} + k\ell \right) \qquad (\ell \geq 0). \tag{71}$$

Proof. Let $\tilde{D}(s)$ be the mixed matrix pencil associated with $D(s) = sA + B$. Then the assertions follow from $\delta_k^{\mathrm{m}}(D) = \delta_k(\tilde{D})$ in (68), $\theta_{\ell+1}^{\mathrm{m}}(D) = \theta_{\ell+1}(\tilde{D})$ in (69), and the Legendre duality between $\delta_k(\tilde{D})$ and $\theta_{\ell+1}(\tilde{D})$ given in Proposition 2.2. □

Let us say that a formal mixed matrix pencil $sA + B$ in (63) is δ^{m}*-tight* if it satisfies (72) below, and θ^{m}*-tight* if it satisfies (73) below:

$$\delta_k(sA+B) = \delta_k^{\mathrm{m}}(sA+B) \qquad \text{for all } k \text{ with } 0 \leq k \leq r, \tag{72}$$

$$\theta_k(sA+B) = \theta_k^{\mathrm{m}}(sA+B) \qquad \text{for all } k \geq 0. \tag{73}$$

Theorem 8.4. *A formal mixed matrix pencil $sA + B$ in* (63) *is δ^{m}-tight if and only if it is θ^{m}-tight.*

Proof. By Proposition 2.2 the sequence (δ_k) uniquely determines the sequence (θ_k), and vice versa, through the discrete Legendre transformation (8) and (9) . The same is true for (δ_k^{m}) and (θ_k^{m}) by (70) and (71) in Theorem 8.3. Hence follows the equivalence of (72) and (73). □

A mixed matrix pencil is δ^{m}-tight by Theorem 7.2 and θ^{m}-tight by Theorem 7.3. Theorem 8.4 above shows that these two statements are equivalent. It may be emphasized that, by considering formal mixed matrix pencils without involving the assumption of genericity, we can reveal the essence in the relationship between δ_k^{m} and θ_k^{m}.

Remark 8.1. Alternative proofs of Propositions 8.1 and 8.2 and Theorem 8.3 are mentioned here. Since $f = f_Q \vee f_{T1} \vee f_{T0}$ in (56) is a valuated bimatroid by Proposition 3.3, Proposition 3.2 shows the concavity of δ_k^{m} claimed in Proposition 8.1. The identity (71) can be proved by slightly modifying the arguments in [12], which is explained in Appendix A. The other identity (70) follows from (71); see Remark 2.1 in Section 2.2. Convexity of θ_k^{m}, claimed in Proposition 8.2, also follows from (71). ∎

9 Conclusion

We may summarize the results of this paper as follows:

linear algebra		matroid		graph
$\delta_k(D)$	$\leq$	$\delta_k^{\mathrm{b}}(D)$	$\leq$	$\delta_k^{\mathrm{g}}(D)$
$\Updownarrow$	$\updownarrow$	$\Updownarrow$	$\updownarrow$	$\Updownarrow$
$\theta_k(D)$	$\leq$	$\theta_k^{\mathrm{b}}(D)$	$\leq$	$\theta_k^{\mathrm{g}}(D)$

Here $\Updownarrow$ denotes the duality with respect to the discrete Legendre transformation. If any one of the inequalities above is tight for all k, then the corresponding inequality, indicated by $\updownarrow$, is also tight for all k. A similar diagram holds for formal mixed matrices:

linear algebra		valuated matroid
$\delta_k(D)$	$\leq$	$\delta_k^{\mathrm{m}}(D)$
$\Updownarrow$	$\updownarrow$	$\Updownarrow$
$\theta_k(D)$	$\leq$	$\theta_k^{\mathrm{m}}(D)$

It is hoped that the Legendre duality as well as its consequences discussed in this paper sheds a new light on the combinatorial study of matrix pencils.

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A Proof of the identity (71) in Theorem 8.3

We give an alternative proof to the identity (71) in Theorem 8.3. Instead of using the statement of Theorem 7.3, we extract essential ingredients from the proof of Theorem 7.3 in [12] and make some additional observations. In so doing we intend to understand the combinatorial essence in the proof of Theorem 7.3.

To be specific, we prove

$$\theta_k^{\mathrm{m}} = \max_{0 \le \ell \le r} \left(\delta_\ell^{\mathrm{m}} + (k-1)\ell \right) \qquad (k \ge 1), \tag{A.1}$$

which is equivalent to (71) in Theorem 8.3. The proof consists of three stages: (i) to reduce the proof to the layered case, where the nonzero rows of $Q(s)$ and $T(s)$ in the formal mixed matrix pencil $D(s) = Q(s) + T(s)$ are disjoint, (ii) to confirm that a combinatorial identity established in [12] for mixed matrix pencils remains valid for formal mixed matrix pencils, and (iii) to translate that identity to the discrete Legendre transformation in (A.1).

A.1 Reduction to the layered case

A *layered mixed matrix pencil* (or an *LM-matrix pencil*) is defined to be a mixed matrix pencil such that the Q-part and the T-part have disjoint nonzero rows. A *formal LM-matrix pencil* is defined similarly.

Let $D_{\mathrm{M}}(s) = s(Q_A + T_A) + (Q_B + T_B)$ be an $m \times n$ formal mixed matrix pencil of rank r. We associate a $(2m) \times (m+n)$ formal LM-matrix pencil

$$D(s) = \left[\begin{array}{c} Q(s) \\ \hline T(s) \end{array} \right] = s \left[\begin{array}{cc} I_m & Q_A \\ \hline -I_m & T_A \end{array} \right] + \left[\begin{array}{cc} O & Q_B \\ \hline O & T_B \end{array} \right] = \left[\begin{array}{cc} sI_m & sQ_A + Q_B \\ \hline -sI_m & sT_A + T_B \end{array} \right], \tag{A.2}$$

the rank of which is equal to $m + r$. As proved below, we have

$$\delta_k^{\mathrm{m}}(D) = \begin{cases} k & (0 \le k \le m), \\ \delta_{k-m}^{\mathrm{m}}(D_{\mathrm{M}}) + m & (m \le k \le m + r), \end{cases} \tag{A.3}$$

$$\theta_k^{\mathrm{m}}(D) = \theta_k^{\mathrm{m}}(D_{\mathrm{M}}) + km. \tag{A.4}$$

With these formulas we can derive (A.1) for D_{M} from (A.1) for D as follows:

$$\begin{aligned} &\max_{0 \le \ell \le r} \left(\delta_\ell^{\mathrm{m}}(D_{\mathrm{M}}) + (k-1)\ell \right) = \max_{0 \le \ell \le r} \left(\delta_{\ell+m}^{\mathrm{m}}(D) - m + (k-1)\ell \right) \\ &= \max_{m \le \ell \le m+r} \left(\delta_\ell^{\mathrm{m}}(D) + (k-1)\ell \right) - km = \max_{0 \le \ell \le m+r} \left(\delta_\ell^{\mathrm{m}}(D) + (k-1)\ell \right) - km \\ &= \theta_k^{\mathrm{m}}(D) - km = \theta_k^{\mathrm{m}}(D_{\mathrm{M}}), \end{aligned}$$

where it is noted that $\delta_\ell^{\mathrm{m}}(D) + (k-1)\ell$ is increasing in ℓ for $0 \le \ell \le m$.

A.1.1 Proof of (A.3) for $\delta_k^{\rm m}$:

We denote the row set of $D_{\rm M}(s)$ by R and the column set by C. Letting R^Q and R^T be copies of R, we denote the row set of $D(s)$ as $R^Q \cup R^T$ and the column set as $R \cup C$.

Obviously, $\delta_k^{\rm m}(D) = k$ for $0 \leq k \leq m$. Consider the expression (56) of $\delta_k^{\rm m}(D)$ for $D(s)$ in (A.2), where f_Q, f_{T1}, and f_{T0} are valuated bimatroids associated with $[sI_m \;\; sQ_A + Q_B]$, $[-sI_m \;\; sT_A]$, and $[O \;\; T_B]$ by (53) and (55). For $k = m$, the maximum on the right-hand side of (56) for $D(s)$ is attained by $(Z, W) = (R^Q, R)$. For $k \geq m$ we may restrict ourselves to $(Z, W) \in \mathbf{L}(D(s))$ with $Z \supseteq R^Q$ and $W \supseteq R$ by [17, Theorem 2] (see also [19, Theorem 5.2.12]). This means that $Z = R^Q \cup I^Q \cup X^Q \cup \tilde{X}^Q$ and $W = R \cup J \cup Y \cup \tilde{Y}$ for some $(I, J) \in \mathbf{L}(sQ_A + Q_B)$, $(X, Y) \in \mathbf{L}(T_A)$, and $(\tilde{X}, \tilde{Y}) \in \mathbf{L}(T_B)$ with $I \cap X = I \cap \tilde{X} = X \cap \tilde{X} = \emptyset$ and $J \cap Y = J \cap \tilde{Y} = Y \cap \tilde{Y} = \emptyset$, where $I, X, \tilde{X} \subseteq R$ and their copies in R^Q are denoted as $I^Q, X^Q, \tilde{X}^Q$. Then we have

$$(f_Q \vee f_{T1} \vee f_{T0})(Z, W) = m + \deg_s \det((sQ_A + Q_B)[I, J]) + |X|,$$

where $|I| + |X| + |\tilde{X}| = k - m$ since $|Z| = k$ and $|R^Q| = m$. Taking the maximum over all (Z, W) we obtain $\delta_k^{\rm m}(D) = \delta_{k-m}^{\rm m}(D_{\rm M}) + m$.

A.1.2 Proof of (A.4) for $\theta_k^{\rm m}$:

Consider $\Theta_k(D)$ and $\Theta_k(D_{\rm M})$. For $k = 3$, e.g., we have

$$\Theta_3(D) = \left[\begin{array}{cccccc} I_m & O & O & Q_A & O & O \\ O & I_m & O & Q_B & Q_A & O \\ O & O & I_m & O & Q_B & Q_A \\ \hline -I_m & O & O & T_A & O & O \\ O & -I_m & O & T_B & T_A & O \\ O & O & -I_m & O & T_B & T_A \end{array}\right], \quad \Theta_3(D_{\rm M}) = \left[\begin{array}{ccc} Q_A & O & O \\ Q_B & Q_A & O \\ O & Q_B & Q_A \\ \hline T_A & O & O \\ T_B & T_A & O \\ O & T_B & T_A \end{array}\right],$$

where the rows are permuted. To consider the rank of $\Theta_k(D)$ we may restrict ourselves to submatrices that contain the identity matrix of order km in the upper-left position. Then we can see the relation (A.4) easily from the definition of $\theta_k^{\rm m}$.

Remark A.1. Alternative proofs of (A.3) and (A.4) are given here. First note their linear algebraic counterparts:

$$\delta_k(D) = \begin{cases} k & (0 \leq k \leq m), \\ \delta_{k-m}(D_{\rm M}) + m & (m \leq k \leq m + r), \end{cases} \tag{A.5}$$

$$\theta_k(D) = \theta_k(D_{\rm M}) + km, \tag{A.6}$$

which can be proved easily by elimination arguments. By (68) and (69) we have $\delta_k^{\rm m}(D) = \delta_k(\tilde{D})$ and $\theta_k^{\rm m}(D) = \theta_k(\tilde{D})$ for the LM-matrix pencil $\tilde{D}(s)$ associated with $D(s)$, and also $\delta_k^{\rm m}(D_{\rm M}) = \delta_k(\tilde{D}_{\rm M})$ and $\theta_k^{\rm m}(D_{\rm M}) = \theta_k(\tilde{D}_{\rm M})$ for the mixed matrix pencil $\tilde{D}_{\mathrm{M}}(s)$ associated with $D_{\mathrm{M}}(s)$. By applying (A.5) and (A.6) to $\tilde{D}$ and $\tilde{D}_{\mathrm{M}}$ we obtain (A.3) and (A.4). ∎

Remark A.2. The definition of $D(s)$ here is different from that in [12, (4.1)], where the lower-left I_m is modified to a diagonal matrix with independent parameters. This modification to a generic diagonal matrix is a standard technique in mixed matrix theory, necessary to obtain an LM-matrix pencil. But this does not seem adequate in discussing formal LM-matrix pencils. With this modification the relation in (A.5) is no longer true. For instance, for

$$D_{\mathrm{M}}(s) = \begin{bmatrix} \alpha s + 1 & s \\ \alpha & 1 \end{bmatrix} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \alpha s & 0 \\ \alpha & 0 \end{bmatrix}$$

with an independent parameter α, we have $\delta_2(D_{\mathrm{M}}) = 0$, whereas

$$\det \left[\begin{array}{cc|cc} s & & 1 & s \\ & s & 0 & 1 \\ \hline -s & & \alpha s & 0 \\ & -s & \alpha & 0 \end{array} \right] = s^2, \quad \det \left[\begin{array}{cc|cc} s & & 1 & s \\ & s & 0 & 1 \\ \hline -t_1 s & & \alpha s & 0 \\ & -t_2 s & \alpha & 0 \end{array} \right] = s^2(\alpha(t_2 - t_1)s + t_1 t_2)$$

with $\delta_4 = 2$ and $\delta_4 = 3$, respectively. ∎

A.2 A combinatorial identity

We basically follow the notation in [12], except that we denote $sQ_A + Q_B$ and $sT_A + T_B$ instead of $sX_Q + Y_Q$ and $sX_T + Y_T$ in [12].

Four combinatorial optimization problems are defined in [12] for an LM-matrix pencil $D(s) = \begin{bmatrix} Q(s) \\ T(s) \end{bmatrix}$ and a positive integer k:

- $\mathrm{IMP}(\Theta_k(D))$: an independent matching problem (maximization),
- $\mathrm{DIMP}(\Theta_k(D))$: the dual of $\mathrm{IMP}(\Theta_k(D))$ (minimization),
- $\mathrm{VIAP}_k(D)$: a valuated independent assignment problem (maximization),
- $\mathrm{DVIAP}_k(D)$: the dual of $\mathrm{VIAP}_k(D)$ (minimization),

where $\mathrm{VIAP}_k(D)$ and $\mathrm{DVIAP}_k(D)$ designate $\mathrm{VIAP}(D)$ and $\mathrm{DVIAP}(D)$ in [12], respectively; showing the dependence on k explicitly. It is shown in [12] that

$$\mathrm{opt}\,(\mathrm{VIAP}_k(D)) = \mathrm{opt}\,(\mathrm{IMP}(\Theta_k(D))) = \mathrm{opt}\,(\mathrm{DIMP}(\Theta_k(D))) = \mathrm{opt}\,(\mathrm{DVIAP}_k(D)), \tag{A.7}$$

where $\mathrm{opt}\,(\cdot)$ denotes the optimal value of an optimization problem.

These optimization problems, defined without reference to the genericity of the T-part, can be considered also for a formal LM-matrix pencil; the details are given in Section A.3. Moreover, the relation (A.7) is maintained, as follows.

Lemma A.1 (cf. [12, Lemma 6.4])**.** *For a formal LM-matrix pencil $D(s)$ we have*

$$\mathrm{opt}\,(\mathrm{VIAP}_k(D)) \le \mathrm{opt}\,(\mathrm{IMP}(\Theta_k(D))).$$

Proof. The proof of [12, Lemma 6.4], stated for an LM-matrix pencil, works for a formal LM-matrix pencil. □

Lemma A.2 (cf. [12, Lemma 6.5])**.** *For a formal LM-matrix pencil $D(s)$ we have*

$$\mathrm{opt}\,(\mathrm{DIMP}(\Theta_k(D))) \le \mathrm{opt}\,(\mathrm{DVIAP}_k(D)).$$

Proof. The proof of [12, Lemma 6.5], stated for an LM-matrix pencil, works for a formal LM-matrix pencil. □

In addition to the key facts given in Lemmas A.1 and A.2, which are specific to our problem, we know two general principles, the weak duality for the independent matching problem:

$$\mathrm{opt}\,(\mathrm{IMP}(\Theta_k(D))) \le \mathrm{opt}\,(\mathrm{DIMP}(\Theta_k(D)))$$

and the strong duality [19, Theorem 5.2.39] for the valuated independent assignment problem:

$$\mathrm{opt}\,(\mathrm{VIAP}_k(D)) = \mathrm{opt}\,(\mathrm{DVIAP}_k(D)).$$

Combining the above facts, we obtain

$$\mathrm{opt}\,(\mathrm{VIAP}_k(D)) \le \mathrm{opt}\,(\mathrm{IMP}(\Theta_k(D))) \le \mathrm{opt}\,(\mathrm{DIMP}(\Theta_k(D))) \le \mathrm{opt}\,(\mathrm{DVIAP}_k(D)),$$

where all the inequalities are in fact equalities. We single out the following identity as the combinatorial essence of our problem.

Lemma A.3 (cf. [12, (6.14)])**.** *For a formal LM-matrix pencil $D(s)$ we have*

$$\mathrm{opt}\,(\mathrm{VIAP}_k(D)) = \mathrm{opt}\,(\mathrm{IMP}(\Theta_k(D))). \tag{A.8}$$

■

In Section A.3 we will show

$$\begin{aligned} \mathrm{opt}\,(\mathrm{VIAP}_k(D)) &= \max_{0\le\ell\le r}\left(\delta_\ell^{\mathrm{m}} + (k-1)\ell\right), && \text{(A.9)} \\ \mathrm{opt}\,(\mathrm{IMP}(\Theta_k(D))) &= \theta_k^{\mathrm{m}}. && \text{(A.10)} \end{aligned}$$

Substitution of (A.9) and (A.10) into (A.8) yields (A.1).

A.3 Translation to the Legendre duality

Let

$$D(s) = \begin{bmatrix} Q(s) \\ T(s) \end{bmatrix} = \begin{bmatrix} sQ_A + Q_B \\ sT_A + T_B \end{bmatrix} \tag{A.11}$$

be a formal LM-matrix pencil, and denote the column set of $D(s)$ by C and the row sets of $Q(s)$ and $T(s)$ by R^Q and R^T, respectively.

A.3.1 Proof of (A.9) for $\mathrm{VIAP}_k(D)$:

To define the problem $\mathrm{VIAP}_k(D)$ for a formal LM-matrix pencil $D(s)$ in (A.11), we consider a polynomial matrix

$$Z_Q(s) = \begin{bmatrix} I & s^{k-1}Q(s) \end{bmatrix} = \begin{bmatrix} I & s^k Q_A + s^{k-1}Q_B \end{bmatrix}.$$

The the row set of Z_Q is identified with R^Q, and the column set with $R^Q \cup C$. The underlying bipartite graph $G = (V^+, V^-; E)$ of the valuated independent assignment problem $\mathrm{VIAP}_k(D)$ is defined as

$$V^+ = R^Q \cup C^Q \cup R^T, \quad V^- = R \cup C, \quad E = E^Q \cup E^A \cup E^B,$$

where $C^Q = \{j^Q \mid j \in C\}$ is a copy of C, R is a copy of R^Q, or $R^Q = \{i^Q \mid i \in R\}$, and

$$\begin{aligned} E^Q &= \{(j^Q, j) \mid j \in R \cup C\}, \\ E^A &= \{(i, j) \mid i \in R^T,\ j \in C,\ (i, j)\text{-entry of } T_A \text{ is nonzero}\}, \\ E^B &= \{(i, j) \mid i \in R^T,\ j \in C,\ (i, j)\text{-entry of } T_B \text{ is nonzero}\}. \end{aligned}$$

For $F \subseteq E$ in general, $\partial^+ F$ means the set of vertices in V^+ that are incident to some edge in F; and similarly for $\partial^- F$. The weight $w(e)$ of an edge $e \in E$ is specified as

$$w(e) = \begin{cases} 0 & (e \in E^Q), \\ k & (e \in E^A), \\ k-1 & (e \in E^B). \end{cases}$$

Denote by $\tilde{\mathcal{B}}$ the family of subsets $\tilde{B} \subseteq R^Q \cup C^Q$ that correspond to a column basis of the matrix Z_Q, and define a function $\tilde{\omega} : \tilde{\mathcal{B}} \to \mathbb{Z}$ as

$$\tilde{\omega}(\tilde{B}) = \deg_s \det Z_Q[R^Q, \tilde{B}].$$

Then the problem $\mathrm{VIAP}_k(D)$ reads as follows:

> On the bipartite graph $G = (V^+, V^-; E)$, find a pair $(M, \tilde{B})$ of a matching $M \subseteq E$ and a base $\tilde{B} \in \tilde{\mathcal{B}}$ that maximizes $w(M) + \tilde{\omega}(\tilde{B})$ subject to the condition that $\partial^+ M \cap (R^Q \cup C^Q) = \tilde{B}$.

Suppose we are given a matching M and a base $\tilde{B}$. For $I = R^Q \setminus \tilde{B}$ and $J = C^Q \cap \tilde{B}$ we have

$$\tilde{\omega}(\tilde{B}) = \deg_s \det Z_Q[R^Q, \tilde{B}] = \deg_s \det Q[I, J] + (k-1)|I|.$$

In particular, (I, J) is a linked pair in $\mathbf{L}(Q(s))$. The matching M determines a linked pair $(X, Y) = (\partial^+(M \cap E^A), \partial^-(M \cap E^A))$ in $\mathbf{L}(T_A)$ and a linked pair $(\tilde{X}, \tilde{Y}) = (\partial^+(M \cap E^B), \partial^-(M \cap E^B))$ in $\mathbf{L}(T_B)$. The associated weight is given by

$$\begin{aligned} w(M) + \tilde{\omega}(\tilde{B}) &= k|X| + (k-1)|\tilde{X}| + \deg_s \det Q[I, J] + (k-1)|I| \\ &= \deg_s \det Q[I, J] + |X| + (k-1)(|I| + |X| + |\tilde{X}|). \end{aligned}$$

In this way $(M, \tilde{B})$ induces $(I, X, \tilde{X}; J, Y, \tilde{Y}) \in \mathbf{L}^\vee$, and vice versa. For the weight we have

$$\begin{aligned} &\mathrm{opt}\,(\mathrm{VIAP}_k(D)) \\ &= \max\{w(M) + \tilde{\omega}(\tilde{B}) \mid M, \tilde{B}\} \\ &= \max_\ell \Big((k-1)\ell \\ &\quad + \max\{\deg_s \det Q[I, J] + |X| \mid |I| + |X| + |\tilde{X}| = \ell, (I, X, \tilde{X}; J, Y, \tilde{Y}) \in \mathbf{L}^\vee\}\Big) \\ &= \max_\ell \Big((k-1)\ell + \delta_\ell^{\mathrm{m}}\Big), \end{aligned}$$

which establishes (A.9).

A.3.2 Proof of (A.10) for IMP($\Theta_k(D)$):

The identity (A.10) is nothing but a straightforward translation of the definition of θ_k^{m}, which is explained here for completeness.

To define the problem IMP($\Theta_k(D)$) we consider $\Theta_k(D)$ with rows permuted:

$$\Theta_k(D) = \begin{bmatrix} \Theta_k(sQ_A + Q_B) \\ \Theta_k(sT_A + T_B) \end{bmatrix} = \begin{bmatrix} \overline{Q} \\ \overline{T} \end{bmatrix}.$$

For $k = 3$, for example, we have

$$\Theta_3(D) = \left[\begin{array}{ccc} Q_A & O & O \\ Q_B & Q_A & O \\ O & Q_B & Q_A \\ \hline T_A & O & O \\ T_B & T_A & O \\ O & T_B & T_A \end{array}\right], \quad \overline{Q} = \begin{bmatrix} Q_A & O & O \\ Q_B & Q_A & O \\ O & Q_B & Q_A \end{bmatrix}, \quad \overline{T} = \begin{bmatrix} T_A & O & O \\ T_B & T_A & O \\ O & T_B & T_A \end{bmatrix}.$$

We denote the hth column set of $\Theta_k(D)$ by C_h and the hth row set of $\overline{T}$ by R_h^T for $h = 1, \dots, k$. Then $C_h = \{j_h \mid j \in C\}$ is a copy of C, and $R_h^T = \{i_h \mid i \in R^T\}$ is a copy of R^T; we put $\overline{R}^T = \bigcup_{h=1}^k R_h^T$. Let $C_h^Q = \{j_h^Q \mid j \in C\}$ be a copy of C for

$h = 1, \ldots, k$; we put $\overline{C}^Q = \bigcup_{h=1}^k C_h^Q$, which is identified with the column set of $\overline{Q}$. The row set of $\overline{Q}$ is denoted by $\overline{R}^Q$.

The underlying bipartite graph $G_k = (\overline{V}^+, \overline{V}^-; \overline{E})$ of the independent matching problem IMP($\Theta_k(D)$) is defined as[1]

$$\overline{V}^+ = \bigcup_{h=1}^k C_h^Q \cup \bigcup_{h=1}^k R_h^T, \quad \overline{V}^- = \bigcup_{h=1}^k C_h, \quad \overline{E} = \bigcup_{h=1}^k E_h^Q \cup \bigcup_{h=1}^k E_h^A \cup \bigcup_{h=1}^{k-1} E_h^B,$$

where

$$\begin{aligned} E_h^Q &= \{(j_h^Q, j_h) \mid j_h^Q \in C_h^Q,\ j_h \in C_h\}, \\ E_h^A &= \{(i_h, j_h) \mid i \in R^T,\ j \in C,\ (i,j)\text{-entry of } T_A \text{ is nonzero}\}, \\ E_h^B &= \{(i_{h+1}, j_h) \mid i \in R^T,\ j \in C,\ (i,j)\text{-entry of } T_B \text{ is nonzero}\}. \end{aligned}$$

The problem IMP($\Theta_k(D)$) reads as follows:.

> On the bipartite graph $G_k = (\overline{V}^+, \overline{V}^-; \overline{E})$, find a matching $M \subseteq \overline{E}$ that maximizes $|M|$ subject to the condition that the submatrix $\overline{Q}[\overline{R}^Q, \partial^+ M \cap \overline{C}^Q]$ is of column-full rank.

Suppose we are given a matching M such that $\operatorname{rank} \overline{Q}[\overline{R}^Q, J] = |J|$ for $J = \partial^+ M \cap \overline{C}^Q$. Then there exists $I \subseteq \overline{R}^Q$ such that $\overline{Q}[I, J]$ is nonsingular, which means that (I, J) is a linked pair in $\mathbf{L}(\overline{Q}) = \mathbf{L}(\Theta_k(sQ_A + Q_B))$. For each h, $M \cap E_h^A$ determines a linked pair in $\mathbf{L}(T_A)$ with row set R_h^T and column set C_h. Similarly, for each h, $M \cap E_h^B$ determines a linked pair in $\mathbf{L}(T_B)$ with row set R_{h+1}^T and column set C_h. In this way an independent matching M induces a linked pair in $\mathbf{L}(\Theta_k(sQ_A + Q_B)) \vee \Theta_k(\mathbf{L}(T_A), \mathbf{L}(T_B))$, and vice versa. This shows that $\operatorname{opt}(\text{IMP}(\Theta_k(D)))$ is equal to the rank of $\mathbf{L}(\Theta_k(sQ_A + Q_B)) \vee \Theta_k(\mathbf{L}(T_A), \mathbf{L}(T_B))$, which is denoted as θ_k^{m} in (65). This establishes (A.10).

[1] See [12, Fig. 6.2] for an illustration of this graph.