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SPERNER PROPERTY AND FINITE-DIMENSIONAL GORENSTEIN ALGEBRAS ASSOCIATED TO MATROIDS

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Abstract. We prove the Lefschetz property for a certain class of finite-dimensional Gorenstein algebras associated to matroids. Our result implies the Sperner property of the vector space lattice. We also discuss the Gröbner fan of the defining ideal of our Gorenstein algebra.

Introduction

The Lefschetz property for Artinian Gorenstein rings is a ring-theoretic abstraction of the Hard Lefschetz Theorem for compact Kähler manifolds. Stanley developed the ideas of applications of the Lefschetz property to combinatorial problems. For example, he showed in [14] the Sperner property of the Bruhat ordering on the Weyl groups based on the Hard Lefschetz Theorem for the flag varieties. One of the main topics of the present paper is an application of the Lefschetz property for a certain kind of finite-dimensional Gorenstein algebras to the *Sperner property* of the vector space lattice $V(q,n)$ consisting of the linear subspaces of the vector space $\mathbb{F}_q^n$. A finite ranked poset $P=\bigcup_{i\geq 0} P_i$ with the level sets P_i is said to have the Sperner property if the maximal cardinality of antichains of P is equal to $\max_i(\#P_i)$.

For a given ranked poset $P=\bigcup_i P_i$, let V_i be the vector space spanned by the elements of P_i. The Sperner property for P can be shown by constructing a sequence $(f_0,f_1,f_2,\ldots)$ of linear maps $f_i: V_i \to V_{i+1}$ satisfying a certain condition. Let $A^{(i)}=(a_{uv}^{(i)})_{u\in P_i, v\in P_{i+1}}$ be the matrix representing f_i, i.e.,

$$f_i(u)=\sum_{v\in P_{i+1}} a_{uv}^{(i)} v, \quad u\in P_i.$$

If every matrix $A^{(i)}$ satisfies the condition $a_{uv}^{(i)}\neq 0 \Rightarrow u<v$, and is of full rank, then P has the Sperner property. (See e.g. [8] for details.)

The Sperner property of the vector space lattice $V(q,n)$ can be deduced from the result on the rank of its incidence matrices due to Kantor [9]. We will give another proof of the Sperner property of $V(q,n)$ by the construction of a finite-dimensional Gorenstein algebra $A_{M(q,n)}$ associated to the matroid $M(q,n)$ on the finite projective space $\mathbb{P}^{n-1}(\mathbb{F}_q)$ and by showing that $A_{M(q,n)}$ has the Lefschetz property.

Our construction can be done for general matroids. For a matroid M and its bases $\mathcal{B}$, we introduce a polynomial $\Phi_M := \sum_{B\in\mathcal{B}} x_B$. The Gorenstein algebra A_M

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will be defined to be the quotient algebra of the ring of the differential polynomials by the annihilator of Φ_M. For a general polynomial F, though F has all the informations on the annihilator $\operatorname{Ann} F$ in principle, the combinatorial structure of $\operatorname{Ann} F$ is quite delicate in general, so it is difficult to describe directly from F. It is remarkable that in our case the Gröbner fan $G(\operatorname{Ann}\Phi_{M(q,n)})$ of the annihilator of $\Phi_{M(q,n)}$ is a refinement of that of the principal ideal generated by $\Phi_{M(q,n)}$, which is also a consequence of our main theorem. As discussed in [1], the Gröbner fan of an ideal is often difficult to compute. We will see that $G(\operatorname{Ann}\Phi_{M(q,n)})$ can be recovered from the tropical hypersurfaces of certain polynomials defined by the bases of the linear subspaces of $\mathbb{P}^{n-1}(\mathbb{F}_q)$.

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1. Finite-dimensional Gorenstein algebras and Lefschetz property

In this section we summarize some fundamental results on the structure of finite-dimensional Gorenstein algebras and on the Lefschetz property, which will be used in the subsequent sections.

Definition 1.1. Let $A = \oplus_{d=0}^{D} A_d$, $A_D \neq 0$, be a graded Artinian algebra. We say that A has *the strong Lefschetz property (in the narrow sense)* if there exists an element $L \in A_1$ such that the multiplication map

$$\times L^{D-2i} : A_i \to A_{D-i}$$

is bijective for $i = 0, \dots, [D/2]$.

In the rest of this paper, we consider the Gorenstein algebras that is finite-dimensional over a field k of characteristic zero.

Definition 1.2. (See [12, Chapter 5, 6.5].) A finite-dimensional graded k-algebra $A = \oplus_{d=0}^{D} A_d$ is called the *Poincaré duality algebra* if $\dim_k A_D = 1$ and the bilinear pairing

$$A_d \times A_{D-d} \to A_D \cong k$$

is non-degenerate for $d = 0, \dots, [D/2]$.

The following is a well-known fact (see e.g. [5], [8], [10]).

Proposition 1.3. *A graded Artinian k-algebra A is a Poincaré duality algebra if and only if A is Gorenstein.*

Let $P = k[x_1, \dots, x_n]$ and $Q = k[X_1, \dots, X_n]$ be polynomial rings over k. We may regard P as a Q-module via the identification $X_i = \partial/\partial x_i$, $i = 1, \dots, n$. For a polynomial $F(x) \in P$, denote by $\operatorname{Ann} F$ the ideal of Q generated by the differential polynomials annihilating F, i.e.,

$$\operatorname{Ann} F := \{\varphi(X) \in Q \mid \varphi(X)F(x) = 0\}.$$

The following is immediate from the theory of the inverse systems (see [2], [4], [6]).

Proposition 1.4. *Let I be an ideal of $Q = k[X_1, \dots, X_n]$ and $A = Q/I$ the quotient algebra. Denote by $\mathfrak{m}$ the maximal ideal $(X_1, \dots, X_n)$ of Q. Then $\sqrt{I} = \mathfrak{m}$ and the k-algebra A is Gorenstein if and only if there exists a polynomial $F \in R = k[x_1, \dots, x_n]$ such that $I = \operatorname{Ann}_Q F$.*

Example 1.5. The coinvariant algebra R_W of the finite Coxeter group W is an example of the finite-dimensional Gorenstein algebra with the strong Lefschetz property. The coinvariant algebra R_W is defined to be a quotient of the ring of polynomial functions on the reflection representation V of W by the ideal generated by the fundamental W-invariants. When W is crystallographic (i.e., Weyl group), the Lefschetz property of R_W is a consequence of the Hard Lefschetz Theorem for the corresponding flag variety G/B. Stanley [14] has shown the Sperner property of the strong Bruhat ordering on W from the Lefschetz property of R_W (except for type H_4). The Lefschetz property of R_W of type H_4 has been confirmed in [11]. Since R_W is Gorenstein, it has a presentation as in Proposition 1.4. In fact, R_W is isomorphic to the algebra $\operatorname{Sym} V^*/\operatorname{Ann} F$, where F is the product of the positive roots.

Definition 1.6. Let G be a polynomial in $k[x_1, \dots, x_n]$. When a family $\mathbf{B}_d = \{\alpha_i^{(d)}\}_i$ of homogeneous polynomials of degree $d > 0$ is given, we call the polynomial

$$\det\left((\alpha_i^{(d)}(X)\alpha_j^{(d)}(X)G(x))_{i,j=1}^{\#\mathbf{B}_d} \right) \in k[x_1, \dots, x_n]$$

the *d-th Hessian* of G with respect to $\mathbf{B}_d$, and denote it by $\operatorname{Hess}_{\mathbf{B}_d}^{(d)} G$. We denote the d-th Hessian simply by $\operatorname{Hess}^{(d)} G$ if the choice of $\mathbf{B}_d$ is clear.

When $d = 1$ and $\alpha_j^{(1)}(X) = X_j$, $j = 1, \dots, n$, the first Hessian $\operatorname{Hess}^{(1)} G$ coincides with the usual Hessian:

$$\operatorname{Hess}^{(1)} G = \operatorname{Hess}\ G := \det\left(\frac{\partial^2 G}{\partial x_i \partial x_j} \right)_{ij}.$$

Let a finite-dimensional graded Gorenstein algebra $A = \oplus_d A_d$ have the presentation $A = Q/\operatorname{Ann}_Q F$. The following gives a criterion for an element $L \in A_1$ to be a Lefschetz element.

Proposition 1.7. ([15, Theorem 4]) *Fix an arbitrary k-linear basis $\mathbf{B}_d$ of A_d for $d = 1, \dots, [D/2]$. An element $L = a_1X_1 + \cdots + a_nX_n \in A_1$ is a strong Lefschetz element of $A = Q/\operatorname{Ann}_Q F$ if and only if $F(a_1, \dots, a_n) \neq 0$ and*

$$(\operatorname{Hess}_{\mathbf{B}_d}^{(d)} F)(a_1, \dots, a_n) \neq 0$$

for $d = 1, \dots, [D/2]$.

Corollary 1.8. *If one of the Hessians $\operatorname{Hess}_{\mathbf{B}_d}^{(d)} F$, $d = 1, \dots, [D/2]$, is identically zero, then $A = Q/\operatorname{Ann}_Q F$ does not have the strong Lefschetz property.*

2. Matroids

Definition 2.1. A pair $(E, \mathcal{F})$ of a finite set E and $\mathcal{F} \subset 2^E$ is called a *matroid* if it satisfies the following axioms $(M1), (M2), (M3)$.
$(M1)$ $\emptyset \in \mathcal{F}$.
$(M2)$ If $X \in \mathcal{F}$ and $Y \subset X$, then $Y \in \mathcal{F}$.
$(M3)$ If $X, Y \in \mathcal{F}$ and $\#X > \#Y$, then there exists an element $x \in X \setminus Y$ such that $Y \cup \{x\} \in \mathcal{F}$.
Here, $\mathcal{F}$ is called the *system of independent sets.*

Definition 2.2. Let $M = (E, \mathcal{F})$ be a matroid.
(1) A maximal element $B \in \mathcal{F}$ is called a *basis* of M. We denote by $\mathcal{B} = \mathcal{B}(M) \subset \mathcal{F}$ the set of bases of M.
(2) For a subset $S \subset E$, define $r(S) := \max\{\#F \mid F \in \mathcal{F}, F \subset S\}$. The map $r : 2^E \to \mathbb{Z}$ is called the *rank function* of M.
(3) For a subset $S \subset E$, define the *closure* $\sigma(S)$ of S by

$$\sigma(S) := \{y \in E \mid r(S \cup \{y\}) = r(S)\}.$$

We define an equivalence relation $\sim$ on 2^E by

$$S \sim T \iff \sigma(S) = \sigma(T).$$

Example 2.3. The projective space $\mathbb{P} := \mathbb{P}^{n-1}(\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$ has the structure of a matroid by the usual linear independence. More precisely, if we define the system of independence set $\mathcal{F}$ by

$$\mathcal{F} := \{F \in 2^{\mathbb{P}} \mid F \text{ is linearly independent over } \mathbb{F}_q\},$$

then $(\mathbb{P}, \mathcal{F})$ is a matroid. We denote it by $M(q, n)$. In this case, the closure $\sigma(S)$ of a subset $S \in \mathbb{P}$ coincides with the linear subspace $\langle S \rangle$ of $\mathbb{P}$ spanned by S.

Lemma 2.4. *Let $S, T \in \mathcal{F}$. Then we have*

$$S \sim T \Leftrightarrow \{U \in \mathcal{F} \mid U \cap S = \emptyset, U \cup S \in \mathcal{F}\} = \{U \in \mathcal{F} \mid U \cap T = \emptyset, U \cup T \in \mathcal{F}\}.$$

Proof. Let S, U be independent sets. If $U \cap S = \emptyset$ and $S \cup U \in \mathcal{F}$, then $r(S \cup \{y\}) = r(S) + 1$ for all $y \in U$, and we have $U \cap \sigma(S) = \emptyset$. If $U \cap S = \emptyset$ and $S \cup U \notin \mathcal{F}$, then there exists an element $y \in U$ such that $r(S \cup \{y\}) = r(S)$. So we have $U \cap \sigma(S) \neq \emptyset$. Hence $\sigma(S)$ determines the set $\{U \in \mathcal{F} \mid U \cap S = \emptyset, U \cup S \in \mathcal{F}\}$, and vice versa. □

Definition 2.5. For a given matroid $M = (E, \mathcal{F})$, the *matroid polytope* P_M is defined by the following system of inequalities:

$$x_e \geq 0 \ \ (e \in E), \qquad \sum_{e \in A} x_e \leq r(A) \ \ (A \in 2^E).$$

For each independent set $F \in \mathcal{F}$, we define the *incidence vector* $\vec{v}_F = (v_{F,e})_{e \in E} \in \mathbb{R}^E$ as follows:

$$v_{F,e} := \begin{cases} 1, & \text{if } e \in F, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.6. (Edmonds [3]) *The matroid polytope P_M coincides with the convex hull of $\vec{0}$ and the* incidence vectors *of $\mathcal{F}$:*

$$P_M = \operatorname{conv}(\{\vec{0}\} \cup \{\vec{v}_F \mid F \in \mathcal{F}\}).$$

Let Δ_M be the face of P_M defined by the equation $\sum_{e \in E} x_e = r(E)$, which is also obtained as the convex hull of the incidence vectors corresponding to the bases of M.

Example 2.7. Let M be a matroid defined by the following vectors.

v_1	v_2	v_3	v_4	v_5
1	0	0	1	0
0	1	0	1	1
0	0	1	0	1

Then $\mathcal{B} = \{\{1,2,3\},\{1,2,5\},\{1,3,4\},\{1,3,5\},\{1,4,5\},\{2,3,4\},\{2,4,5\},\{3,4,5\}\}$. The polytope Δ_M is the convex hull of the following points in $\mathbb{R}^5$:

$$(1,1,1,0,0),(1,1,0,0,1),(1,0,1,1,0),(1,0,1,0,1),$$
$$(1,0,0,1,1),(0,1,1,1,0),(0,1,0,1,1),(0,0,1,1,1).$$

3. Gorenstein algebras associated to matroids

For a matroid $M = (E, \mathcal{F})$, we define a polynomial $\Phi_M \in k[x_e|e \in E]$ by

$$\Phi_M := \sum_{B\in\mathcal{B}} x_B,$$

where $x_B := \prod_{b\in B} x_b$. Note that the Newton polytope of Φ_M coincides with Δ_M in $\mathbb{R}^E$. In the subsequent part of this paper, we discuss the structure of the Gorenstein ring $A_M := Q/\operatorname{Ann}_Q \Phi_M$.

Proposition 3.1. *The ideal* $\operatorname{Ann}\Phi_M$ *contains*

$$\Lambda_M := \{x_e^2|e \in E\} \cup \{x_S|S \notin \mathcal{F}\} \cup \{x_A - x_{A'}|A, A' \in \mathcal{F}, A \sim A'\}.$$

Proof. Since Φ_M is square-free and does not contain the monomials of form x_S, $S \notin \mathcal{F}$, the ideal $\operatorname{Ann}\Phi_M$ contains $\{x_e^2|e \in E\}$ and $\{x_S|S \notin \mathcal{F}\}$. If $A, A' \in \mathcal{F}$ are equivalent, then we have $\partial_A \Phi_M = \partial_{A'}\Phi_M$ from Lemma 2.4. □

We denote by $J_M \subset Q$ the ideal generated by the set Λ_M. Let $M = (E, \mathcal{F})$ be a matroid, and $\mathcal{F}_i \subset \mathcal{F}$ for $i = 1, \dots, r(E)$, the set of independent sets of cardinality i, i.e.,

$$\mathcal{F}_i := \{F \in \mathcal{F} \mid \#F = i\}.$$

Let $\Omega := 2^E/\sim$, $\overline{\mathcal{F}}_l := \mathcal{F}_l/\sim$ and $m_l := \#\overline{\mathcal{F}}_l$. We can identify Ω with the set of subsets S of E such that $S = \sigma(S)$. Under this identification, we define the subset $\Omega(l)$, $l = 1, \dots, r(E)$, of Ω by

$$\Omega(l) := \{S \in 2^E \mid S = \sigma(S), r(S) = l\}.$$

For an equivalence class $\tau \in \Omega$, consider a polynomial f_τ given by

$$f_\tau := \sum_{F\in\mathcal{F}\cap\tau} x_F.$$

Proposition 3.2. *We have*

$$J_M = \bigcap_{\tau\in\Omega} \operatorname{Ann} f_\tau.$$

Proof. It is easy to see that Λ_M is contained in $\cap_{\tau\in\Omega}\operatorname{Ann} f_\tau$. It is enough to show that a polynomial $p \in \cap_{\tau\in\Omega}\operatorname{Ann} f_\tau$ of form

$$p = \sum_{\tau\in\Omega}\sum_{F\in\mathcal{F}\cap\tau} a_F x_F, \quad a_F \in k,$$

is a linear combination of polynomials of Λ_M. Put $p_\tau := \sum_{F\in\mathcal{F}\cap\tau} a_F x_F$ and consider the polynomial

$$p' := \sum_{\tau\in\Omega, p_\tau\notin\Lambda_M} p_\tau.$$

Choose $\tau_0 \in \Omega$ with $p_\tau \neq 0$ of minimum rank. Then

$$p(\partial) f_{\tau_0} = p_{\tau_0}(\partial) f_{\tau_0} = \sum_{F\in\mathcal{F}\cap\tau_0} a_F = 0.$$

Let $\mathcal{F} \cap \tau = \{F_1, \dots, F_s\}$. Then we have

$$p_\tau = a_{F_1}(x_{F_1} - x_{F_2}) + (a_{F_1} + a_{F_2})(x_{F_2} - x_{F_3}) + \cdots + (a_{F_1} + \cdots + a_{F_{s-1}})(x_{F_{s-1}} - x_{F_s}).$$

□

Proposition 3.3. *The subset Λ_M of Q is a universal Gröbner basis of J_M.*

Proof. The proof is based on Buchberger's criterion. Fix a monomial ordering $\leq$ on the polynomial ring Q. For non-zero monic polynomials $f, g \in Q$, the S-polynomial $S(f, g)$ is given as follows:

$$S(f,g) := -\frac{\Gamma(f,g)}{\mathrm{in}_{\leq}(f)} f + \frac{\Gamma(f,g)}{\mathrm{in}_{\leq}(g)} g, \quad \Gamma(f,g) := \mathrm{L.C.M}(\mathrm{in}_{\leq}(f), \mathrm{in}_{\leq}(g)).$$

Let $\Lambda_1 := \{x_A - x_{A'} \mid A, A' \in \mathcal{F}, A \sim A'\}$, $\Lambda_2 := \{x_e^2 | e \in E\}$ and $\Lambda_3 := \{x_S | S \notin \mathcal{F}\}$. We will show that the S-polynomials $S(f, g)$ are reduced to zero by the division algorithm with respect to $\Lambda_M \setminus \{f, g\}$ for cases:
(i) $f, g \in \Lambda_1$, (ii) $f \in \Lambda_1$, $g \in \Lambda_2$, (iii) $f \in \Lambda_1$, $g \in \Lambda_3$, (iv) $f, g \in \Lambda_2 \cup \Lambda_3$.

Case (i): Take polynomials $f := x_A - x_{A'}$, $g := x_B - x_{B'} \in \Lambda_1$ with $x_A > x_{A'}$ and $x_B > x_{B'}$. If $A \cap B = \emptyset$, it is easy to see that $S(f, g)$ is reduced to zero. Assume that $A \cap B \neq \emptyset$. Let $C := A \cap B$, $\hat{A} = A \setminus C$ and $\hat{B} = B \setminus C$. Then we have $S(f, g) = x_{A'} x_{\hat{B}} - x_{B'} x_{\hat{A}}$. Note that we have

$$r(A' \cup \hat{B}) = r(A \cup \hat{B}) = r(\hat{A} \cup C \cup \hat{B}),$$
$$r(B' \cup \hat{A}) = r(B \cup \hat{A}) = r(\hat{A} \cup C \cup \hat{B}),$$

so $r(A' \cup \hat{B}) = r(B' \cup \hat{A})$.
(i-1) If $A' \cap \hat{B} \neq \emptyset$, then $x_{A'} x_{\hat{B}} \in \Lambda_2$. In this case, we have

$$(*) \quad r(\hat{A} \cup B') = r(A' \cup \hat{B}) < r(A') + r(\hat{B}) = \#A' + \#\hat{B} = \#\hat{A} + \#B',$$

which means that $\hat{A} \cap B' \neq \emptyset$ or $\hat{A} \cup B' \notin \mathcal{F}$. Hence we also have $x_{\hat{A}} x_{B'} \in \Lambda_2 \cup \Lambda_3$.
(i-2) Assume that $A' \cap \hat{B} = \emptyset$. If $A' \cup \hat{B} \notin \mathcal{F}$, then we have $x_{A'} x_{\hat{B}} \in \Lambda_3$. Moreover, again from the inequality $(*)$, we see that $x_{\hat{A}} x_{B'} \in \Lambda_2 \cup \Lambda_3$. If $A' \cup \hat{B} \in \mathcal{F}$, we have

$$r(\hat{A} \cup B') = r(A' \cup \hat{B}) = r(A') + r(\hat{B}) = \#A' + \#\hat{B} = \#\hat{A} + \#B',$$

which means that $\hat{A} \cup B' \in \mathcal{F}$. Hence we have $S(f, g) = x_{A'} x_{\hat{B}} - x_{B'} x_{\hat{A}} \in \Lambda_1$.
Case (ii): Take polynomials $f := x_A - x_{A'} \in \Lambda_1$ and $g := x_e^2 \in \Lambda_2$ with $x_A > x_{A'}$. If $e \notin A$, then $S(f, g) = x_e^2 x_{A'}$ is reduced to zero. If $e \in A$, then $S(f, g) = x_e x_{A'}$. Since $r(A' \cup \{e\}) = r(A \cup \{e\}) = r(A)$, we have $x_e x_{A'} \in \Lambda_2 \cup \Lambda_3$.
Case (iii): Take polynomials $f := x_A - x_{A'} \in \Lambda_1$ and $g := x_B \in \Lambda_3$ with $x_A > x_{A'}$. If $A \cap B = \emptyset$, then $S(f, g) = x_{A'} x_B$ is reduced to zero. If $A \cap B \neq \emptyset$, then $S(f, g) = x_{A'} x_{B \setminus A}$. The inequality

$$r(A' \cup (B \setminus A)) = r(A \cup (B \setminus A)) = r(A \cup B) < \#(A \cup B) = \#(A' \cup (B \setminus A))$$

implies that $x_{A'} x_{B \setminus A} \in \Lambda_2 \cup \Lambda_3$.
Case (iv): This case is easy because Λ_2 and Λ_3 are consisting of monomials. □

Corollary 3.4. *The Hilbert polynomial of Q/J_M is given by*

$$\mathrm{Hilb}(Q/J_M, t) = \sum_{i=0}^{r(E)} (\#\bar{\mathcal{F}}_i) t^i.$$

Example 3.5. Let M be the matroid as defined in Example 2.7. Then the ideal $\operatorname{Ann}\Phi_M$ contains an additional generator other than Λ_M. In fact, we have

$$\operatorname{Ann}\Phi_M = J_M + (x_{13} + x_{45} - x_{15} - x_{34}).$$

The Hilbert series of $Q/\operatorname{Ann}\Phi_M$ is $(1,5,5,1)$ and that of Q/J_M is $(1,5,6,1)$. In particular, Q/J_M is not Gorenstein. By direct computation, we get

$$\operatorname{Hess}\Phi_M = 8(x_1 + x_4)(x_3 + x_5)\Phi_M.$$

This implies that $Q/\operatorname{Ann}\Phi_M$ has the Lefschetz property.

4. Vector space lattice

In this section we treat the matroid $M = M(q,n)$ defined in Example 2.3. We define polynomials $\Phi_M^{(i)} := \sum_{G\in\mathcal{F}_i} x_G$ for $i = 1,\ldots,n$. Note that $\Phi_M^{(n)} = \Phi_M$.

Lemma 4.1. *For $M = M(q,n)$ and $l \le [n/2]$, the polynomials $\partial^F\Phi_M^{(2l)}$, $F \in \bar{\mathcal{F}}_l$, are linearly independent over k.*

Proof. In the following, $\langle S\rangle$ stands for a linear subspace in $\mathbb{F}_q^n$ spanned by a subset $S \subset \mathbb{P}^{n-1}(\mathbb{F}_q)$. For $B \in \mathcal{F}_l$ and $0 \le i \le l$, define

$$\mathcal{F}_l(B,i) := \{A \in \mathcal{F}_l \mid \dim(\langle A\rangle \cap \langle B\rangle) = i\}.$$

Then we have $\mathcal{F}_l(B,l) = \{A \in \mathcal{F}_l \mid A \sim B\}$ and

$$\mathcal{F}_l = \bigcup_{i=0}^{l} \mathcal{F}_l(B,i).$$

For $A, B \in \mathcal{F}_l$, we also define

$$\begin{aligned}\mathcal{F}_l^A(B,i) &:= \{A' \in \mathcal{F}_l(B,i) \mid \langle A\rangle \cap \langle A'\rangle = \{\vec{0}\}\}\\ &= \{A' \in \mathcal{F}_l(B,i) \mid A \cup A' \in \mathcal{F}_{2l}\}.\end{aligned}$$

For $B \in \mathcal{F}_l$, consider a polynomial $\Phi(B,i) := \sum_{A\in\mathcal{F}_l(B,i)} x_A$ and a differential polynomial $P(B,i) := \sum_{A\in\mathcal{F}_l(B,i)} \partial^A$. We have

$$\begin{aligned}P(B,i)\Phi_M^{(2l)} &= \sum_{A\in\mathcal{F}_l(B,i)} \partial^A\Phi_M^{(2l)}\\ &= \sum_{A\in\mathcal{F}_l(B,i)} \sum_{\substack{A'\in\mathcal{F}_l\\ A\cup A'\in\mathcal{F}_{2l}}} x_{A'}\\ &= \sum_{A'\in\mathcal{F}_l} \sum_{\substack{A\in\mathcal{F}_l(B,i)\\ A\cup A'\in\mathcal{F}_{2l}}} x_{A'}\\ &= \sum_{j=0}^{l} \sum_{A'\in\mathcal{F}_l(B,j)} \#\{A \in \mathcal{F}_l(B,i) | A\cup A' \in \mathcal{F}_{2l}\} x_{A'}\\ &= \sum_{j=0}^{l} \sum_{A'\in\mathcal{F}_l(B,j)} \#\mathcal{F}_l^{A'}(B,i) x_{A'}.\end{aligned}$$

Here, $\#\mathcal{F}_l^{A'}(B,i)$ is independent of the choice of $A' \in \mathcal{F}_l(B,j)$ for $M = M(q,n)$. Put $a_{ij}^B := \#\mathcal{F}_l^{A'}(B,i)$ for $B \in \mathcal{F}_l$ and $A' \in \mathcal{F}_l(B,j)$. Now we have

$$P(B,i)\Phi_M^{(2l)} = \sum_{j=1}^{l} a_{ij}^B \sum_{A' \in \mathcal{F}_l(B,j)} x_{A'} = \sum_{j=1}^{l} a_{ij}^B \Phi(B,j).$$

If $i+j > l$, then $\dim(\langle A\rangle \cap \langle B\rangle) + \dim(\langle A'\rangle \cap \langle B\rangle) = i+j > l$. Hence, we have $\dim(\langle A\rangle \cap \langle A'\rangle \cap \langle B\rangle) > 0$ and $\langle A\rangle \cap \langle A'\rangle \neq \{\vec{0}\}$. This means that $a_{ij}^B = \#\mathcal{F}_l^{A'}(B,i) = 0$.

Assume that $i+j = l$. For $A \in \mathcal{F}_l(B,j)$, take an element $A_1 \in \mathcal{F}_j$ such that $\langle A_1\rangle = \langle A\rangle \cap \langle B\rangle$. We also take an element $A_2 \in \mathcal{F}_{l-j} = \mathcal{F}_i$ such that $\langle A_1 \cup A_2\rangle = \langle B\rangle$, and $A_3 \in \mathcal{F}_{n-l}$ such that $\langle B \cup A_3\rangle = \mathbb{F}_q^n$. Put $A^* := A_2 \cup A_3$. Since $\dim\langle A^*\rangle = n-j \geq n-l \geq l$, there exists an element $A' \in \mathcal{F}_l$ such that $\langle A^*\rangle \cap \langle B\rangle \subset \langle A'\rangle \subset \langle A^*\rangle$. Since $\langle A'\rangle \cap \langle B\rangle = \langle A^*\rangle \cap \langle B\rangle = \langle A_2\rangle$, we can see that $A' \in \mathcal{F}_l^A(B,i)$. Hence we have $a_{ij}^B > 0$ in this case.

We have seen that the matrix $(a_{i,l-j}^B)_{i,j=0}^l$ is upper-triangular, so

$$\det(a_{i,l-j}^B)_{ij} = \prod_{i=0}^{l} a_{i,l-i}^B > 0.$$

Since the matrix $(a_{i,l-j})_{ij}$ is invertible, $\Phi_M(B,l)$ is written as a linear combination of $P(B,0)\Phi_M^{(2l)}, P(B,1)\Phi_M^{(2l)}, \ldots, P(B,l)\Phi_M^{(2l)}$, and hence it is a linear combination of the polynomials $\partial^F \Phi_M^{(2l)}$, $F \in \bar{\mathcal{F}}_l$. On the other hand, it is easy to see the linear-independency of the polynomials $\Phi_M(B,l)$, $B \in \bar{\mathcal{F}}_l$. Therefore the polynomials $\partial^F \Phi_M^{(2l)}$, $F \in \bar{\mathcal{F}}_l$, are linearly independent. ☐

Theorem 4.2. *Let $M = M(q,n)$. Take a representative $F_1, \ldots, F_{m_l} \in \mathcal{F}_l$ of $\overline{\mathcal{F}}_l$. Then the determinant of the matrix*

$$\left(\partial^{F_i}\partial^{F_j}\Phi_M\right)_{i,j=1}^{m_l}$$

is not identically zero.

Proof. For $F \in \mathcal{F}_j$, define $c(F,i) := \#\{F' \in \mathcal{F}_i \mid F \cup F' \in \mathcal{F}_{i+j}\}$. Then the equality $c(F_1,i) = c(F_2,i)$ holds for any $F_1, F_2 \in \mathcal{F}_j$ and for $j = 1, \ldots, r(E)-1$. It is easy to see that

$$\det\left(\partial^{F_i}\partial^{F_j}\Phi_M\right)_{i,j=1}^{m_l}\Big|_{x=1} = \gamma \cdot \det\left(\delta_{\sigma(F_i),\sigma(F_j)}\right)_{i,j},$$

where $\gamma = c(F,l)^{m_l} \neq 0$ for any $F \in \mathcal{F}_l$, and δ_{τ_1,τ_2}, $\tau_1, \tau_2 \in \Omega(l)$, is defined by

$$\delta_{\tau_1,\tau_2} := \begin{cases} 1, & \text{if } \tau_1 \cap \tau_2 = \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

At the same time, we have

$$\det\left(\partial^{F_i}\partial^{F_j}\Phi_M^{(2l)}\right)_{i,j} = \det\left(\delta_{\sigma(F_i),\sigma(F_j)}\right)_{i,j}.$$

Note that the algebra $B^{(2l)} := Q/\operatorname{Ann}\Phi_M^{(2l)}$ is also Gorenstein, and the natural pairings

$$\langle\ ,\ \rangle : B_i^{(2l)} \times B_{2l-i}^{(2l)} \to B_{2l}^{(2l)} \cong k$$

are non-degenerate for $i = 0, \ldots, l$. From Lemma 4.1, we see that $\{x_{F_i} | i = 1, \ldots, m_l\}$ gives a basis of $B_l^{(2l)}$. Since the matrix $\left(\partial^{F_i}\partial^{F_j}\Phi_M^{(2l)}\right)_{i,j}$ represents the pairing $\langle\ ,\ \rangle$

at the intermediate part $B_l^{(2l)} \times B_l^{(2l)} \to k$, we see that its determinant is non-zero. Therefore, $\det\left(\partial^{F_i}\partial^{F_j}\Phi_M\right)\Big|_{x=1}$ is non-zero, and hence it cannot be identically zero. $\square$

Corollary 4.3. (1) *The algebra* $A_{M(q,n)}$ *has the strong Lefschetz property.*
(2) *The ideal* $\operatorname{Ann}\Phi_{M(q,n)}$ *is generated by* $\Lambda_{M(q,n)}$*, i.e.,* $\operatorname{Ann}\Phi_{M(q,n)} = J_{M(q,n)}$*. In particular, it is a binomial ideal.*
(3) *We have*

$$\operatorname{Hilb}(Q/\operatorname{Ann}\Phi_{M(q,n)}, t) = \sum_{i=0}^{n} t^i \binom{n}{i}_q,$$

where $\binom{n}{i}_q$*,* $0 \le i \le n$*, are* q*-binomial coefficients.*
(4) *The vector space lattice* $V(q,n)$ *consisting of the linear subspaces of* $\mathbb{F}_q^n$ *has the Sperner property.*

Remark 4.4. For $i \le n$, let $M^{(i)}(q,n)$ be a matroid structure on $\mathbb{P}^{n-1}(\mathbb{F}_q)$ obtained by regarding $\mathcal{F}_i$ as a system of bases. We see that $\Phi_{M^{(i)}(q,n)} = \Phi^{(i)}_{M(q,n)}$. It can be shown by a similar manner as the proof of Theorem 4.2 that $Q/\operatorname{Ann}\Phi_{M^{(i)}(q,n)}$ has the Lefschetz property, and $\operatorname{Ann}\Phi_{M^{(i)}(q,n)} = J_{M^{(i)}(q,n)}$.

Example 4.5. Let $[n] := \{1, 2, \ldots, n\}$ be an n-element set. The set $2^{[n]}$ of the subsets of $[n]$ has a natural lattice structure induced by the operations $\cup$ and $\cap$. The obtained lattice is called the Boolean lattice. Sperner's theory originates his work [13] on the maximal cardinality of the antichains of the Boolean lattice. On the other hand, $M([n]) := ([n], 2^{[n]})$ satisfies the axioms of the matroid. The matroid $M([n])$ has the unique basis $[n]$, so the corresponding Gorenstein algebra is given by

$$A_{M([n])} = k[X_1, \ldots, X_n]/\operatorname{Ann}(x_1 \cdots x_n).$$

In [7], it has been proved that $M([n])$ is another example of matroids for which Theorem 4.2 holds. As a consequence, we obtain $\operatorname{Ann}\Phi_{M([n])} = J_{M([n])}$ and the Lefschetz property for $A_{M([n])}$, which gives another proof of the Sperner property for the Boolean lattice.

Conjecture. The algebra A_M has the strong Lefschetz property for an arbitrary matroid M.

5. Gröbner fan of J_M

In this section, we discuss the Gröbner fan of the ideals J_M and $\operatorname{Ann}\Phi_{M(q,n)}$. The initial ideal $\operatorname{in}_{\vec\omega}(I)$ of an ideal $I \subset Q$ with respect to the weight vector $\vec\omega \in \mathbb{R}^E$ is given by

$$\operatorname{in}_{\vec\omega}(I) := (\operatorname{in}_{\vec\omega}(f) \mid f \in I, f \neq 0).$$

For a weight vector $\vec\omega$, the set $C(\vec\omega) := \operatorname{closure}\{\vec\lambda \in \mathbb{R}^E \mid \operatorname{in}_{\vec\lambda}(I) = \operatorname{in}_{\vec\omega}(I)\}$ is a polyhedral cone in $\mathbb{R}^E$. The set of cones $\{C(\vec\omega) \mid \vec\omega \in \mathbb{R}^E \setminus \{\vec0\}\}$ forms a fan $G(I)$. The fan $G(I)$ is called the *Gröbner fan* of I. Denote by $G^d(I)$ the set of d-dimensional cones in $G(I)$. The Gröbner fan $G(I)$ of a homogeneous ideal I has the translation invariance in the direction of $\vec n := (1, \ldots, 1) \in \mathbb{R}^E$. Let H be the hyperplane in $\mathbb{R}^E$ defined by the equation $\sum_{e\in E} x_e = 0$. Denote by $\bar G(I)$ the restriction of $G(I)$ to H.

For two distinct independent sets $F, F' \in \mathcal{F}$ with $F \sim F'$, define a cone $W_{F,F'}$ by the condition

$$\sum_{e\in F} x_e = \sum_{e\in F'} x_e, \quad \sum_{e\in F} x_e \le \sum_{e\in F''} x_e \ \ (\forall F'' \in \mathcal{F},\ F'' \sim F).$$

Let $C_1, \dots, C_p$ be the closures of the connected components of

$$\mathbb{R}^E \setminus \bigcup_{\substack{F,F'\in\mathcal{F}\\ F\sim F', F\neq F'}} W_{F,F'}.$$

Proposition 5.1. *The maximal cones of $G(J_M)$ are given by $C_1, \dots, C_p$, i.e., $G^{\#E}(J_M) = \{C_1, \dots, C_p\}$.*

Proof. Since Λ_M is a universal Gröbner basis of J_M, $\mathrm{in}_{\vec{\omega}}(J_M)$ is not a monomial ideal if and only if $\mathrm{in}_{\vec{\omega}}(J_M)$ contains $x_F - x_{F'}$ for two distinct independent sets F, F' with $F \sim F'$ and does not contain x_F or $x_{F'}$. This is the case when $\vec{\omega} \in W_{F,F'}$. □

The *tropical hypersurface* $V_{\mathrm{trop}}(\Phi_M) \subset \mathbb{R}^E$ is defined as the locus in $\mathbb{R}^E$ where the piecewise linear function

$$\mathrm{trop}(\Phi_M) = \max\left(\sum_{e\in B} x_e \mid B \in \mathcal{B}\right)$$

is not smooth. The tropical hypersurface $V_{\mathrm{trop}}(\Phi_M)$ can be considered as a subcomplex of $G(\Phi_M)$ (see [1]). Since Φ_M is homogeneous, the corresponding tropical hypersurface $V_{\mathrm{trop}}(\Phi_M)$ has the translation invariance in the direction of the vector $\vec{n}$. Denote by $\bar{V}_{\mathrm{trop}}(\Phi_M)$ the restriction of $V_{\mathrm{trop}}(\Phi_M)$ to H. In our case, $\bar{V}_{\mathrm{trop}}(\Phi_M)$ is also regarded as a fan. The following proposition shows that the tropical variety $\bar{V}_{\mathrm{trop}}(\Phi_M)$ is directly obtained from the matroid polytope of M.

Proposition 5.2. *The piecewise linear function* $\mathrm{trop}(\Phi_M)|_H$ *is a support function for the polytope* $\Delta_M^0 := \Delta_M - r(E)(\#E)^{-1} \cdot \vec{n} \subset H$.

Proof. The polytope Δ_M^0 is spanned by the vectors $\vec{u}_B := \vec{v}_B - r(E)(\#E)^{-1} \cdot \vec{n}$, $B \in \mathcal{B}$, by Proposition 2.6. We also have the inequality

$$\langle \vec{u}_B, \vec{y} \rangle = \sum_{b\in B} y_b \le \mathrm{trop}(\Phi_M)(\vec{y}), \ \ \forall \vec{y} = (y_e)_{e\in E} \in H,$$

and for $\vec{y} = \vec{u}_B$,

$$\langle \vec{u}_B, \vec{u}_B \rangle = r(E) - \frac{r(E)^2}{\#E} = \mathrm{trop}(\Phi_M)(\vec{u}_B).$$

Hence, the polytope Δ_M^0 is described as

$$\Delta_M^0 = \{\vec{x} \in H \mid \langle \vec{x}, \vec{y} \rangle \le \mathrm{trop}(\Phi_M)(\vec{y}),\ \forall \vec{y} \in H\}.$$

□

For a fan Σ, define $-\Sigma := \{-\sigma | \sigma \in \Sigma\}$.

Proposition 5.3. (1) *For an equivalence class $\tau \in \Omega(l)$ with $l \ge 2$, we have*

$$G^{\#E-1}(f_\tau) = \{-W_{F,F'} | F, F' \in \mathcal{F} \cap \tau, F \neq F'\}.$$

(2)
$$V_{\mathrm{trop}}(f_\tau) = \bigcup_{\sigma \in G^{\#E-1}(f_\tau)} \sigma = \bigcup_{\substack{F,F' \in \mathcal{F}\cap\tau \\ F \neq F'}} -W_{F,F'}.$$

(3)
$$\bigcup_{\sigma \in G^{\#E-1}(J_M)} -\sigma = \bigcup_{\tau \in \Omega} V_{\mathrm{trop}}(f_\tau).$$

Proof. Since the Newton polytope of f_τ does not contain interior lattice points, every monomial x_F, $F \in \mathcal{F} \cap \tau$, appearing in f_τ can be the initial monomial for a choice of monomial ordering. Hence, $\mathrm{in}_{\vec{\omega}}(f_\tau)$ is not a monomial ideal if $\vec{\omega}$ belongs to $-W_{F,F'}$ for a pair $F, F' \in \mathcal{F} \cap \tau$, $F \neq F'$. This shows (1). The second claim (2) follows from the definition of the tropical hypersurface $V_{\mathrm{trop}}(f_\tau)$. The claim (3) is a consequences of (2) and Proposition 5.1. □

Corollary 5.4. *The tropical hypersurface* $V_{\mathrm{trop}}(\Phi_M)$ *is a subcomplex of the fan* $-G(J_M)$.

For $M = M(q,n)$, we have $G(\mathrm{Ann}\,\Phi_{M(q,n)}) = G(J_{M(q,n)})$ from Corollary 4.3 (2). By Proposition 5.3, the Gröbner fan $G(\mathrm{Ann}\,\Phi_{M(q,n)})$ can be computed from the tropical hypersurfaces $V_{\mathrm{trop}}(f_\tau)$.

Example 5.5. The matroid $M(2,2)$ is defined by the following 3 vectors,

v_1	v_2	v_3
1	0	1
0	1	1

so we have

$$\Phi_{M(2,2)} = x_1x_2 + x_1x_3 + x_2x_3,$$
$$\mathrm{Ann}\,\Phi_{M(2,2)} = (x_1^2, x_2^2, x_3^2, x_1x_2 - x_1x_3, x_1x_2 - x_2x_3, x_1x_3 - x_2x_3).$$

In this case, the Gröbner fans $G(\mathrm{Ann}\,\Phi_{M(2,2)})$, $G(J_{M(2,2)})$ and $-G(\Phi_{M(2,2)})$ are same. Their restrictions $\bar{G}(\mathrm{Ann}\,\Phi_{M(2,2)})$, $\bar{G}(J_{M(2,2)})$, $-\bar{G}(\Phi_{M(2,2)})$ to the plane H are determined by 3 rays:

$$R_1 := \mathbb{R}_{\geq 0}(-2,1,1),\ R_2 := \mathbb{R}_{\geq 0}(1,-2,1),\ R_3 := \mathbb{R}_{\geq 0}(1,1,-2).$$

Moreover, $\bar{V}_{\mathrm{trop}}(\Phi_{M(2,2)}) = (-R_1) \cup (-R_2) \cup (-R_3)$.

Example 5.6. The Gröbner fan $\bar{G}(\mathrm{Ann}\,\Phi_{M(2,3)}) = \bar{G}(J_{M(2,3)})$ contains 420 cones of maximal dimension 6 and 49 rays. The fan $\bar{G}(\Phi_{M(2,3)})$ contains 28 maximal cones and 21 rays.

Example 5.7. Let M be the matroid from Example 2.7. The fan $\bar{G}(J_M)$ contains 12 cones of maximal dimension 4 and 7 rays:

$$\mathbb{R}_{\geq 0}(-4,1,1,1,1), \mathbb{R}_{\geq 0}(-2,-2,3,-2,3), \mathbb{R}_{\geq 0}(-1,4,-1,-1,-1), \mathbb{R}_{\geq 0}(1,1,-4,1,1),$$
$$\mathbb{R}_{\geq 0}(1,1,1,-4,1),\ \mathbb{R}_{\geq 0}(1,1,1,1,-4),\ \mathbb{R}_{\geq 0}(3,-2,-2,3,-2).$$

The fan $\bar{G}(\Phi_M)$ contains 8 maximal cones, and $\bar{G}^1(\Phi_M) = -\bar{G}^1(J_M)$. In this case, $\bar{G}(\mathrm{Ann}\,\Phi_M)$ is a refinement of $\bar{G}(J_M)$. The fan $\bar{G}(\mathrm{Ann}\,\Phi_M)$ contains 20 maximal cones and 9 rays:

$$\mathbb{R}_{\geq 0}(-4,1,1,1,1),\ \mathbb{R}_{\geq 0}(-3,2,2,-3,2),\ \mathbb{R}_{\geq 0}(-2,-2,3,-2,3),$$
$$\mathbb{R}_{\geq 0}(-1,4,-1,-1,-1),\ \mathbb{R}_{\geq 0}(1,1,-4,1,1),\ \mathbb{R}_{\geq 0}(1,1,1,-4,1),$$
$$\mathbb{R}_{\geq 0}(1,1,1,1,-4),\ \mathbb{R}_{\geq 0}(2,2,-3,2,-3),\ \mathbb{R}_{\geq 0}(3,-2,-2,3,-2).$$

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