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Abstract

For functions defined on integer lattice points, discrete versions of the Hessian matrix have been considered in various contexts. In discrete convex analysis, for example, certain combinatorial properties of the discrete Hessian matrices are known to characterize $M^\natural$-convex and $L^\natural$-convex functions, which can be extended to convex functions in real variables. The relationship between convex extensibility and discrete Hessian matrices is not fully understood in general, and unfortunately, some vague or imprecise statements have been made in the literature. This note points out that the positive semidefiniteness of the discrete Hessian matrix is not implied by convex extensibility of discrete functions.

Key words: discrete optimization, convex extensibility, Hessian matrix

1 Introduction

For functions defined on integer lattice points, discrete versions of the Hessian matrix have been considered in various contexts. In discrete convex analysis [1, 3, 4], for example, certain combinatorial properties of the discrete Hessian matrices are known to characterize $M^\natural$-convex and $L^\natural$-convex functions, which can be extended to convex functions in real variables. A natural definition of a discrete Hessian matrix $H(x) = (H_{ij}(x))$ of $f : \mathbb{Z}^n \to \mathbb{R}$ at $x \in \mathbb{Z}^n$ is

$$H_{ij}(x) = f(x + e_i + e_j) - f(x + e_i) - f(x + e_j) + f(x),$$

where e_i denotes the ith unit vector. This is used in characterizing M$^\natural$-convex functions, whereas a modified definition is suitable for L$^\natural$-convex functions.

However, the relationship between convex extensibility and discrete Hessian matrices is not fully understood in general, and unfortunately, some vague or imprecise statements have been made in the literature. Recent papers [6, 7, 8] discuss the relationship between convex extensibility of discrete functions and the positive semidefiniteness of their Hessian matrix $H(x)$ at each point x. It is certainly true that a univariate discrete function $f : \mathbb{Z} \to \mathbb{R}$ with $n = 1$ is convex extensible if and only if the Hessian $H(x)$, which is actually a real number, is positive semidefinite (i.e., nonnegative). But this statement is incorrect for $n \geq 2$. We point out the incorrectness of this statement by giving counterexamples in following sections.

This paper is a small contribution toward clarifying the relationship between discrete Hessians and the convex extensibility of discrete functions. To be specific, this paper points out the following facts by examples:

- Even if $f : \mathbb{Z}^2 \to \mathbb{R}$ has a convex extension to a C^2 convex function, its discrete Hessian matrix is not necessarily positive semidefinite.
- Even if $f : \mathbb{R}^2 \to \mathbb{R}$ is C^2 convex, its discrete Hessian matrix is not positive semidefinite.

In addition, we reconsider the previous results [1, 3] for discrete M$^\natural$-convex/L$^\natural$-convex functions to better understand the role of discrete Hessian matrices in discrete convex analysis.

2 Discrete-Variable Functions

In this section we consider functions in discrete variables. We define the discrete Hessian matrix $H(x) = (H_{ij}(x))$ of $f : \mathbb{Z}^n \to \mathbb{R}$ at $x \in \mathbb{Z}^n$ by

$$H_{ij}(x) = f(x + e_i + e_j) - f(x + e_i) - f(x + e_j) + f(x), \tag{2.1}$$

where e_i denotes the ith unit vector.

We first show, by giving counterexamples, that the positive semidefiniteness of the discrete Hessian matrix $H(x)$ in (2.1) is not implied by the convex extensibility of f.

Example 2.1. The function $f : \mathbb{Z}^2 \to \mathbb{R}$ defined by

$$f(x_1, x_2) = |x_1 - x_2|$$

is extensible to a convex function $\overline{f} : \mathbb{R}^2 \to \mathbb{R}$ given by $\overline{f}(x_1, x_2) = |x_1 - x_2|$. We have

$$
\begin{aligned}
H_{11}(0,0) &= f(2,0) - 2f(1,0) + f(0,0) = 0, \\
H_{12}(0,0) &= H_{21}(0,0) = f(1,1) - f(1,0) - f(0,1) + f(0,0) = -2, \\
H_{22}(0,0) &= f(0,2) - 2f(0,1) + f(0,0) = 0,
\end{aligned}
$$

i.e.,

$$
H(0,0) = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}.
$$

This is not positive semidefinite in spite of the convex extensibility of f. ∎

Whereas the convex extension of the function f in Example 2.1 is piecewise linear, the function f in Example 2.2 below admits a convex extension to a C^2 function. Even in this smooth case, the discrete Hessian matrix is not positive semidefinite.

Example 2.2. Consider a univariate function $\varphi : \mathbb{R} \to \mathbb{R}$ as

$$
\varphi(t) = \begin{cases} -\dfrac{1}{3}|t|^3 + \dfrac{4}{3}|t|^2 & (|t| \leq 1), \\ \dfrac{1}{3}|t|^2 + |t| - \dfrac{1}{3} & (|t| \geq 1), \end{cases} \tag{2.2}
$$

which is a C^2 convex function with $\varphi(0) = 0$, $\varphi(\pm 1) = 1$, and $\varphi(\pm 2) = 3$. Using this φ we define $f : \mathbb{Z}^2 \to \mathbb{R}$ by $f(x_1, x_2) = \varphi(x_1 - x_2)$, which is convex extensible to a C^2 convex function $\overline{f} : \mathbb{R}^2 \to \mathbb{R}$ given by $\overline{f}(x_1, x_2) = \varphi(x_1 - x_2)$. The discrete Hessian matrix of f at $(0, 0)$:

$$
H(0,0) = \begin{bmatrix} \varphi(2) - 2\varphi(1) + \varphi(0) & 2\varphi(0) - \varphi(-1) - \varphi(1) \\ 2\varphi(0) - \varphi(-1) - \varphi(1) & \varphi(-2) - 2\varphi(-1) + \varphi(0) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}
$$

is not positive semidefinite in spite of the extensibility of f to a C^2 convex function. ∎

Naturally, we are also concerned with the converse, that is, whether positive semindefiniteness of the discrete Hessian matrix implies convex extensibility. The following example gives a partial answer to this question that integral convexity, which is much stronger than convex extensibility, is not implied by positive semindefiniteness of the discrete Hessian matrix.

Example 2.3. Consider $f : \mathbb{Z}^2 \to \mathbb{R}$ defined by $f(x_1, x_2) = x_1^2 + 4x_1x_2 + 5x_2^2$. The discrete Hessian matrix at any (x_1, x_2) is given by

$$
H(x_1, x_2) = \begin{bmatrix} 2 & 4 \\ 4 & 10 \end{bmatrix},
$$

which is positive definite. The values of f are given as follows:

	$x_1 = 0$	$x_1 = 1$	$x_1 = 2$	$x_1 = 3$	$x_1 = 4$
$x_2 = 2$	20	29	40	53	68
$x_2 = 1$	5	10	17	26	37
$x_2 = 0$	0	1	4	9	16

Let $\overline{f}$ be the convex closure of f, i.e., the pointwise-largest convex extension of f. It has the triangle with vertices $(x_1, x_2) = (0, 1)$, $(1, 0)$ and $(2, 0)$ as a linearity domain. This shows that f is not integrally convex in the sense of Favati–Tardella; see [4, Section 3.4] for the definition of integral convexity. ■

3 M-convex/L-convex Functions

In this section, we deal with M-convex and L-convex functions, which play central roles in discrete convex analysis [4]. Our objective here is to discuss the significance of the previous results on discrete Hessian matrices of M-convex and L-convex functions in the light of the general phenomena we have seen in Section 2. Both M- and L-convex functions are convex extensible, and their characterizations in terms of discrete Hessians (Theorems 3.1 and 3.5 below) are combinatorial conditions that are stronger than positive semidefiniteness.

For a vector $x \in \mathbb{Z}^n$ and an element $i \in \{1, 2, \ldots, n\}$, x_i means the component of x with index i. For vectors $x, y \in \mathbb{Z}^n$, we write $x \vee y$ and $x \wedge y$ for their componentwise maximum and minimum. We write the positive and negative supports of a vector x by

$$\mathrm{supp}^+(x) = \{i \in \{1, 2, \ldots, n\} \mid x_i > 0\},$$

$$\mathrm{supp}^-(x) = \{i \in \{1, 2, \ldots, n\} \mid x_i < 0\}.$$

We write $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{Z}^n$ and $\mathbf{0} = (0, 0, \ldots, 0) \in \mathbb{Z}^n$. For a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$, the effective domain, denoted as $\mathrm{dom}\, f$, is defined to be the set of $x \in \mathbb{Z}^n$ for which $f(x)$ is finite.

3.1 M-convex functions

A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be M-convex if it satisfies

(M-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x - y)$, $\exists j \in \mathrm{supp}^-(x - y)$ such that

$$f(x) + f(y) \geq f(x - e_i + e_j) + f(y + e_i - e_j).$$

The effective domain of an M-convex function is contained in $\{x \in \mathbb{Z}^n \mid \sum_{i=1}^n x_i = r\}$ for some $r \in \mathbb{Z}$. In view of this, we say that a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is M$^\natural$-convex if the function $\tilde{f} : \mathbb{Z}^{n+1} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\tilde{f}(x_0, x) = \begin{cases} f(x) & ((x_0, x) \in \mathbb{Z}^{n+1}, x_0 = r - \sum_{i=1}^n x_i), \\ +\infty & \text{(otherwise)} \end{cases}$$

is M-convex. M$^\natural$-convexity is characterized by the following exchange property:

> **(M$^\natural$-EXC)** $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x - y)$, $\exists j \in \mathrm{supp}^-(x - y) \cup \{0\}$ such that
>
> $$f(x) + f(y) \geq f(x - e_i + e_j) + f(y + e_i - e_j),$$
>
> where $e_0 = \mathbf{0}$.

An M$^\natural$-convex function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ can be extended to a convex function $\overline{f} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$.

We now consider the Hessian matrix. M$^\natural$-convex functions can be characterized by a certain combinatorial property of the discrete Hessian matrix, as follows.

Theorem 3.1 ([1]). *A function $f : \mathbb{Z}^n \to \mathbb{R}$ is* M$^\natural$*-convex if and only if the discrete Hessian matrix $H(x) = (H_{ij}(x))$ in* (2.1) *satisfies the following conditions for each $x \in \mathbb{Z}^n$:*

$$H_{ij}(x) \geq \min(H_{ik}(x), H_{jk}(x)) \quad \text{if } \{i, j\} \cap \{k\} = \emptyset, \tag{3.1}$$

$$H_{ij}(x) \geq 0 \quad \text{for any } (i, j). \tag{3.2}$$

It is known that a symmetric matrix satisfying the conditions (3.1) and (3.2) above is necessarily positive semidefinite.

Example 3.2. The function $f : \mathbb{Z}^2 \to \mathbb{R}$ defined as $f(x_1, x_2) = \varphi(x_1 + x_2)$ with a univariate convex function $\varphi : \mathbb{R} \to \mathbb{R}$ is M$^\natural$-convex. The discrete Hessian matrix of f at (x_1, x_2) is given by

$$H(x_1, x_2) = (\varphi(x_1 + x_2 + 2) - 2\varphi(x_1 + x_2 + 1) + \varphi(x_1 + x_2)) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

which is positive semidefinite, since $\varphi(x_1 + x_2 + 2) - 2\varphi(x_1 + x_2 + 1) + \varphi(x_1 + x_2) \geq 0$ by the assumed convexity of φ. For the function $\varphi(t)$ in (2.2), in particular, we have

$$H(0, 0) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Compare this with Example 2.2, which shows that the discrete Hessian matrix of $f(x_1, x_2) = \varphi(x_1 - x_2)$ with $\varphi(t)$ in (2.2) is not positive semidefinite. ■

As we have repeatedly seen, convex extensibility alone does not imply positive semidefiniteness of the discrete Hessian (2.1). On the other hand, M$^\natural$-convexity, which is a combinatorial convexity concept, does imply both convex extensibility and positive semidefiniteness of the discrete Hessian via (3.1) and (3.2).

3.2 L-convex functions

A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is called L-convex if it satisfies

$$\begin{aligned} &f(x) + f(y) \geq f(x \vee y) + f(x \wedge y) && (x, y \in \mathbb{Z}^n), \qquad (3.3)\\ &\exists r \in \mathbb{R} \text{ such that } f(x + \mathbf{1}) = f(x) + r && (x \in \mathbb{Z}^n), \end{aligned}$$

where it is understood that the inequality (3.3) is satisfied if $f(x)$ or $f(y)$ is equal to $+\infty$. A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is called L$^\natural$-convex if it is obtained from an L-convex function $\tilde{f}(x_0, x_1, \ldots, x_n)$ by $f(x_1, \ldots, x_n) = \tilde{f}(0, x_1, \ldots, x_n)$. L$^\natural$-convexity is characterized by the following discrete midpoint convexity:

$$f(x) + f(y) \geq f\left(\left\lceil \frac{x+y}{2} \right\rceil\right) + f\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right) \qquad (x, y \in \mathbb{Z}^n),$$

where $\left\lceil \frac{x+y}{2} \right\rceil$ and $\left\lfloor \frac{x+y}{2} \right\rfloor$ denote, respectively, the integer vectors obtained from $\frac{x+y}{2}$ by componentwise round-up and round-down to the nearest integers. An L$^\natural$-convex function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ can be extended to a convex function $\overline{f} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$.

We now consider the Hessian matrix.

Example 3.3. The function $f : \mathbb{Z}^2 \to \mathbb{R}$ defined as $f(x_1, x_2) = \varphi(x_1 - x_2)$ with a univariate convex function $\varphi : \mathbb{R} \to \mathbb{R}$ is L$^\natural$-convex. Recall Example 2.2, which shows that the discrete Hessian matrix (2.1) of $f(x_1, x_2) = \varphi(x_1 - x_2)$ with $\varphi(t)$ in (2.2) is not positive semidefinite at $(x_1, x_2) = (0, 0)$. ■

Example 3.4. The function $f : \mathbb{Z}^4 \to \mathbb{R}$ defined by

$$f(x) = \max(x_1 + x_2, x_3 + x_4, x_1 + x_3 + 1, x_1 + x_4 + 1, x_2 + x_3 + 1, x_2 + x_4 + 1) \qquad (3.4)$$

is L$^\natural$-convex [4]. The discrete Hessian matrix (2.1) at $(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$ is

$$H(0,0,0,0) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

which is not positive semidefinite. ■

The above examples demonstrate that the natural definition (2.1) of the discrete Hessian matrix is not amenable to L$^\natural$-convexity. An alternative possibility is pursued in [3], which turned out to be suitable for L$^\natural$-convexity.

For $f : \mathbb{Z}^n \to \mathbb{R}$, $x \in \mathbb{Z}^n$, and $i, j \in \{1, \ldots, n\}$ with $i \neq j$, we first define

$$\begin{aligned}\eta_{ij}(x) &= -f(x + e_i + e_j) + f(x + e_i) + f(x + e_j) - f(x),\\ \eta_i(x) &= f(x) + f(x + \mathbf{1} + e_i) - f(x + \mathbf{1}) - f(x + e_i).\end{aligned}$$

Then we define a symmetric matrix $\tilde{H}(x) = (\tilde{H}_{ij}(x) \mid i, j = 1, \ldots, n)$ by

$$\tilde{H}_{ij}(x) = -\eta_{ij}(x) \quad (i \neq j), \qquad \tilde{H}_{ii}(x) = \eta_i(x) + \sum_{j \neq i} \eta_{ij}(x) \tag{3.5}$$

as a variant of the discrete Hessian matrix. The modified discrete Hessian matrix gives a characterization of L$^\natural$-convex functions, as follows.

Theorem 3.5 ([3])**.** *A function $f : \mathbb{Z}^n \to \mathbb{R}$ is L$^\natural$-convex if and only if the modified discrete Hessian matrix $\tilde{H}(x) = (\tilde{H}_{ij}(x))$ in* (3.5) *satisfies the following conditions for each $x \in \mathbb{Z}^n$:*

$$\tilde{H}_{ij}(x) \leq 0 \qquad (i, j \in \{1, \ldots, n\}, i \neq j), \tag{3.6}$$

$$\sum_{j=1}^{n} \tilde{H}_{ij}(x) \geq 0 \qquad (i \in \{1, \ldots, n\}). \tag{3.7}$$

It is known that a symmetric matrix satisfying the conditions (3.6) and (3.7) above is necessarily positive semidefinite.

Example 3.6. For the L$^\natural$-convex function $f(x_1, x_2) = \varphi(x_1 - x_2)$ with $\varphi(t)$ in (2.2), treated in Example 3.3, the modified discrete Hessian matrix (3.5) at $(x_1, x_2) = (0, 0)$ is

$$\tilde{H}(0,0) = \begin{bmatrix} -2\varphi(0) + \varphi(-1) + \varphi(1) & 2\varphi(0) - \varphi(-1) - \varphi(1) \\ 2\varphi(0) - \varphi(-1) - \varphi(1) & -2\varphi(0) + \varphi(-1) + \varphi(1) \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}.$$

This is positive semidefinite. ■

Example 3.7. For the L$^\natural$-convex function (3.4) in Example 3.4, the modified discrete Hessian matrix (3.5) at $(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$ is

$$\tilde{H}(0,0,0,0) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

This is positive semidefinite. ■

Convex extensibility alone does not imply positive semidefiniteness of the modified discrete Hessian $\tilde{H}(x) = (\tilde{H}_{ij}(x))$ in (3.5), which is demonstrated by Example 3.8 below. On the other hand, L$^\natural$-convexity, which is a combinatorial convexity concept, does imply both convex extensibility and positive semidefiniteness of $\tilde{H}(x)$ via (3.6) and (3.7).

Example 3.8. We define $f : \mathbb{Z}^2 \to \mathbb{R}$ by $f(x_1, x_2) = \varphi(x_1 + x_2)$ with the univariate convex function

$$\varphi(t) = \begin{cases} |t|^2 & (|t| \leq 1), \\ 2|t| - 1 & (|t| \geq 1). \end{cases}$$

The modified discrete Hessian matrix $\tilde{H}(x_1, x_2)$ of f at $(x_1, x_2) = (0, 0)$ is given by

$$\tilde{H}(0,0) = \begin{bmatrix} \varphi(3) - 2\varphi(2) + \varphi(1) & \varphi(2) - 2\varphi(1) + \varphi(0) \\ \varphi(2) - 2\varphi(1) + \varphi(0) & \varphi(3) - 2\varphi(2) + \varphi(1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This is not positive semidefinite, whereas f is convex extensible (not L$^\natural$-convex). ■

4 Mixed-Variable Functions

Convexity concepts have also been discussed for functions $f : \mathbb{Z}^n \times \mathbb{R}^m \to \mathbb{R}$ with discrete and continuous variables [2, 5, 6, 7, 8]. In [6, 7, 8], to be specific, "mixed convexity" is discussed for functions $f : \mathbb{Z}^n \times \mathbb{R}^m \to \mathbb{R}$ with particular reference to the mixed Hessian matrix, which is defined as follows:

$$H(x,y) = \begin{bmatrix} [\nabla_{ij}(f(x,y))]_{n\times n} & [\nabla_i(\frac{\partial}{\partial y_l} f(x,y))]_{n\times m} \\ \left[\frac{\partial}{\partial y_k}(\nabla_j f(x,y))\right]_{m\times n} & \left[\frac{\partial^2}{\partial y_k \partial y_l} f(x,y)\right]_{m\times m} \end{bmatrix}, \tag{4.1}$$

where

$$\begin{aligned} \nabla_{ij}(f(x,y)) &= f(x+e_i+e_j,y) - f(x+e_i,y) - f(x+e_j,y) + f(x,y), \\ \nabla_i(\frac{\partial}{\partial y_l} f(x,y)) &= \frac{\partial f}{\partial y_l}(x+e_i,y) - \frac{\partial f}{\partial y_l}(x,y), \\ \frac{\partial}{\partial y_k}(\nabla_j f(x,y)) &= \frac{\partial f}{\partial y_k}(x+e_j,y) - \frac{\partial f}{\partial y_k}(x,y), \\ \frac{\partial^2}{\partial y_k \partial y_l} f(x,y) &= \frac{\partial^2 f}{\partial y_k \partial y_l}(x,y); \end{aligned}$$

and $[\nabla_{ij}(f(x,y))]_{n\times n}$ means the $n \times n$ matrix that has $\nabla_{ij}(f(x,y))$ as its (i, j) entry, and similarly for $[\nabla_i(\frac{\partial}{\partial y_l} f(x,y))]_{n\times m}$, $\left[\frac{\partial}{\partial y_k}(\nabla_j f(x,y))\right]_{m\times n}$, and $\left[\frac{\partial^2}{\partial y_k \partial y_l} f(x,y)\right]_{m\times m}$. See Remark 4.3 for some definitions and propositions in [8].

In this section we point out, by way of examples, that the positive semidefiniteness of the mixed Hessian matrix of f is not implied by the convex extensibility of f.

Let $\varphi(t)$ be a univariate C^2 convex function $\varphi : \mathbb{R} \to \mathbb{R}$. We define $f : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ as $f(x, y) = \varphi(x - y)$. Such function f is convex extensible, with a convex extension $\overline{f}$ given by $\overline{f}(x, y) = \varphi(x - y)$. According to the definition (4.1) we have

$$H(0,0) = \begin{bmatrix} \varphi(2) - 2\varphi(1) + \varphi(0) & -\varphi'(1) + \varphi'(0) \\ -\varphi'(1) + \varphi'(0) & \varphi''(0) \end{bmatrix}.$$

Example 4.1. In the case of $\varphi(t) = t^4$, we have $\varphi(0) = \varphi'(0) = \varphi''(0) = 0$; $\varphi(1) = 1$, $\varphi'(1) = 4$; $\varphi(2) = 16$ and the Hessian matrix (4.1) of f is

$$H(0,0) = \begin{bmatrix} 14 & -4 \\ -4 & 0 \end{bmatrix}.$$

This is not positive semidefinite in spite of the convex extensibility of f. ■

Example 4.2. In the case of f is defined by $\varphi(t)$ of (2.2), the Hessian matrix (4.1) at $(0, 0)$ is

$$H(0,0) = \frac{1}{3}\begin{bmatrix} 3 & -5 \\ -5 & 8 \end{bmatrix}.$$

This is not positive semidefinite in spite of the convex extensibility of f. ■

The above examples show that the mixed Hessian matrix is not necessarily positive semindefinite even when f is convex extensible.

Remark 4.3. Some definitions and propositions in [8] are reproduced here for critical comments.

> **Definition 2.** A mixed function $\Psi : \mathbb{Z}^n\times\mathbb{R}^m \to \mathbb{R}$ is called mixed convex if it is discretely convex with respect to its integer variables and convex with respect to its continuous variables.
>
> **Definition 3.** A function is called strictly mixed convex if it is strictly discrete convex and strictly convex with respect to the continuous variables, simultaneously.
>
> **Definition 5.** A mixed function is k-smooth if it is k-times differentiable (i.e. C^k) with respect to the real variables and if it can be differenced k-times with respect to the integer variables.
>
> **Theorem 4.4.** A function $\Psi : \mathbb{Z}^n \times \mathbb{R}^m \to \mathbb{R}$ is 2-smooth strictly mixed convex if and only if the mixed Hessian matrix for Ψ is strictly positive.[1]

[1] An obvious typo "$\mathbb{R}^n$" in [8] is corrected to "$\mathbb{R}^m$" here.

Corollary 5.1. $\Theta : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is a mixed convex function if and only if $\Theta|_{\mathbb{Z}}$ is an integer convex function and $\Theta|_{\mathbb{R}}$ is a real convex function.

Corollary 5.3. A function $\Theta : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is 2-smooth strictly mixed convex if and only if Θ has a positive Hessian matrix.

As far as the present authors understand from the above definitions and Corollary 5.1, the function $\Theta : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ defined by $\Theta(x, y) = x^2 y^2$ for $x \in \mathbb{Z}$ and $y \in \mathbb{R}$ is mixed convex. The mixed Hessian matrix (4.1) at (x, y) is given as

$$H(x,y) = \begin{bmatrix} 2y^2 & 2(2x+1)y \\ 2(2x+1)y & 2x^2 \end{bmatrix}.$$

We note that the diagonal entries are positive, but the matrix H is not positive semidefinite since its determinant $\det H(x, y) = -4y^2(3x^2 + 4x + 1)$ can be negative. It is also noted that the off-diagonal entries of H are positive or negative depending on (x, y). This seems to contradict Corollary 5.3 above. ■

5 Conclusion

We may summarize our observations as follows.

- Convex extensibility alone does not imply positive semidefiniteness of the discrete Hessian matrix (2.1). Counterexamples for $f : \mathbb{Z}^2 \to \mathbb{R}$ have been given in Examples 2.1 and 2.2.
- Positive semidefiniteness of the discrete Hessian matrix (2.1) does not imply integral convexity. A counterexample for $f : \mathbb{Z}^2 \to \mathbb{R}$ has been given in Example 2.3. It is left unanswered whether positive semidefiniteness of the discrete Hessian matrix implies convex extensibility or not.
- $\mathrm{M}^\natural$-convexity, which is a combinatorial convexity concept, does imply both convex extensibility and positive semidefiniteness of the discrete Hessian matrix in (2.1). Conversely, a certain combinatorial property of the discrete Hessian matrix, which is stronger than positive semidefiniteness, implies convex extensibility via $\mathrm{M}^\natural$-convexity (Theorem 3.1).
- $\mathrm{L}^\natural$-convexity, which is another combinatorial convexity concept, is not compatible with the discrete Hessian in (2.1), as shown in Examples 3.3 and 3.4. With the modified version of the discrete Hessian matrix in (3.5), $\mathrm{L}^\natural$-convexity does imply both convex extensibility and positive semidefiniteness

of the (modified) discrete Hessian. Conversely, a certain combinatorial property of the (modified) discrete Hessian matrix, which is stronger than positive semidefiniteness, implies convex extensibility via $\mathrm{L}^{\natural}$-convexity (Theorem 3.5).

- Convex extensibility of a mixed function $f : \mathbb{Z}^n \times \mathbb{R}^m \to \mathbb{R}$ does not imply positive semidefiniteness of the discrete Hessian matrix (4.1). Counterexamples for $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ have been given in Examples 4.1 and 4.2.

The concepts of $\mathrm{M}^{\natural}$-convexity and $\mathrm{L}^{\natural}$-convexity for mixed functions $f : \mathbb{Z}^n \times \mathbb{R}^m \to \mathbb{R}$ are introduced in [2] and [5], respectively. The Hessian matrices of such functions are yet to be investigated.

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