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Hierarchical Network Synthesis for Output Consensus by Eigenvector-based Interlayer Connections

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Abstract

In this paper, we study consensus problem for hierarchical multi-agent dynamical systems with low rank interconnection by eigenvector-based connection. The system considered is more general than the existing one in a sense that the number of agents in each subsystem and the connection structure of each subsystem are allowed to be different from each other. We provide analytical expressions of the eigenvalue sets of the hierarchical interconnection matrices and systematic synthesis procedures for achieving the output consensus with Lyapunov stability. The results are applicable to more general class of hierarchical interconnection structures than previous work, and numerical examples with simulations confirm the effectiveness of the proposed design method.

1 Introduction

As a natural consequence of rapid developments in sensing systems, computing technologies and communication networks, large-scale systems have

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become one of the main focuses in different research areas including control engineering. In particular, control problems of large-scale systems such as consensus and cooperative control problems have been received great interests in recent years. See, for example, [1], [6], [5] and the references therein. It is recognized that, as the scale of the system becomes large and the structure of the system becomes complex, the development of decentralized control scheme by which the large-scale system could reach a desired global behavior also becomes complicated and difficult in practice. On the other hand, the interaction in a large-scale system usually consists of several interactions in some subsystems and weak interactions among these subsystems. This is known as hierarchical structure and has been used as one effective way to analyze large-scale dynamical systems in recent work.

A hierarchical cyclic pursuit scheme was introduced for the consensus problem of multi-agent systems in [9]. It is shown that, comparing to either the traditional cyclic pursuit or an alternate scheme with same number of communication links, the proposed hierarchical scheme increases the rate of convergence of a group of agents to a common point. Motivated by this work but to further investigate the effect of the interactions among subsystems, the concept of low rank interconnection was introduced in [8]. The rank of interconnection matrix which represents the strength of interconnection among subsystems is assumed to be one or two. For these two cases, [8] shows that low rank property leads to more rapid consensus than the case of full rank interconnection as done in [9]. Eigenvalue distribution for low rank interconnection matrix is used to explain the result in [8].

Most recently, [10] showed that the previous result is not only led by the low rank property but also some property related to the eigenvector, which is defined as eigen-connection matrix. As a result, they are able to provide almost explicit formulas for eigenvalue distribution of certain class of hierarchical interconnections. Therefore, it is interesting and important to pay attention to eigenvector-based characterization for hierarchical multi-agent systems, which is a new research line in this field.

However, the hierarchical structure considered in [10] is quite special in a sense that the number of agents in each subsystem is the same, and the interconnection structure in each subsystem is the same as well. This assumption is restrictive from practical point of view, and thus limits the application of the proposed method. Moreover, the work in [10] focuses only on the stabilization problem, and the consensus problem has not been considered. In particular, to stabilize the system, based on the results of systems with generalized frequency variables [3], it is sufficient to locate the eigenvalues of the interconnection within the well defined stability region. This does not guarantee that the agents in the system could reach consensus though.

The purposes of this paper are twofold. The main purpose of this paper is to extend the hierarchical structure considered in [10] to general case in a sense that the number of agents in each subsystem is allowed to be different, and the interconnection structure in each subsystem could be also different. The concept of eigen-connection is extended to this case and it is seen that the eigenvectors of local interconnection matrices play an important role in the definition. We can derive analytical expressions of eigenvalue sets of the hierarchical interconnection matrices with rank one and two eigen-connections, which consist of that of local interconnection matrices and of the matrix determined by the intersubsystem connection matrix and the interlayer connection matrix. The second aimpoint is to study the consensus problem rather than just stabilization. In particular, we provide the essential property to explain how to reach hierarchical consensus by eigenvector-based connection.

After the problem formulation for hierarchical network synthesis for output consensus in Section 2, we provide analytical expressions of the eigenvalue sets of the hierarchical interconnection matrices and systematic synthesis procedures for achieving the output consensus with Lyapunov stability in Sections 3 and 4 for the rank 1 and rank 2 cases, respectively. The results are applicable to more general class of hierarchical interconnection structures than previous work including [10]. Actually, we provide numerical examples with simulations to illustrate the effectiveness of the proposed design method.

2 Problem Formulation

2.1 Output Consensus Problem

Consider a class of multi-agent dynamical systems with N agents depicted in Fig. 1, where the SISO proper transfer function $h(s)$ represents the common dynamics of each agent. The interconnection of N agents is described by

$$\boldsymbol{u} = A\boldsymbol{y} \;\; ; \;\; A \in \mathbb{R}^{N\times N}, \tag{1}$$

where $\boldsymbol{u} := [u_1, \ldots, u_N]^\top, \boldsymbol{y} := [y_1, \ldots, y_N]^\top \in \mathbb{R}^N$. We refer to A as *interconnection matrix* in this paper.

The goal of consensus problem investigated in this paper is to design an interconnection matrix A satisfying the following two requirements:

1. **Lyapunov Stability:** [1] All the poles of the feedback system depicted in Fig. 1 except single pole at the origin lie in the open left half complex plane $\mathbb{C}_-$.

2. **Output Consensus:** For all initial conditions, the output achieves consensus in the following sense:

$$\lim_{t\to\infty} \|y_i(t) - y_j(t)\| = 0, \;\; \forall i, j. \tag{2}$$

[1] For notational simplicity, any pure complex pole is not allowed in our definition.

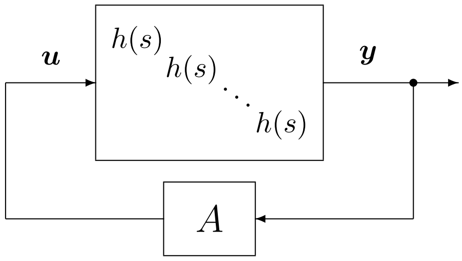

Figure 1: Multi-agent dynamical system.

Suppose the dynamics of the ith agent is represented by a minimal state-space realization

$$\begin{aligned} \dot{x}_i &= A_h x_i + b_h u_i, \\ y_i &= c_h x_i, \qquad\qquad i = 1, \dots, N, \end{aligned} \tag{3}$$

where $A_h \in \mathbb{R}^{m\times m}$ and $b_h, c_h^\top \in \mathbb{R}^m$. In other words, the SISO transfer function $h(s)$ is given by

$$h(s) = c_h(sI_m - A_h)b_h. \tag{4}$$

The total feedback system then can be expressed as

$$\dot{\boldsymbol{x}} = \mathcal{A}\boldsymbol{x}, \tag{5}$$

where $\boldsymbol{x} := [x_1, \dots, x_N]^\top$ and

$$\mathcal{A} = I_N \otimes A_h + A \otimes (b_h c_h). \tag{6}$$

The requirement of the Lyapunov stability above is equivalent to the condition that all the eigenvalues of $\mathcal{A}$ except simple eigenvalue at the origin are contained in $\mathbb{C}_-$. Furthermore, as shown in [7], if A has a simple eigenvalue 0 associated with the right eigenvector $\mathbf{1}_N$, or $A\mathbf{1}_N = 0$ holds, the output consensus is achieved in the sense of (2).

It is difficult to design A directly such that the eigenvalue distribution of $\mathcal{A}$ meets the two requirements, especially when N is very large. Fortunately, the feedback system belongs to a class of LTI systems with generalized frequency variables [2, 3], and hence the assignment of eigenvalues of $\mathcal{A}$ can be reduced to doing that of A (not $\mathcal{A}$) in the associated stability region determined by $h(s)$. For example, when $h(s)$ is given by $h_1(s) = (2s+1)/(s^2+s+1)$ and $h_2(s) = 2/(s^2+3s)$, the corresponding stability regions are the hatched one in left and right of Fig. 2, respectively.

2.2 Hierarchical Output Consensus Problem

As mentioned in the Introduction section, we consider a hierarchical interconnection structure. We divide N agents into M groups which are called

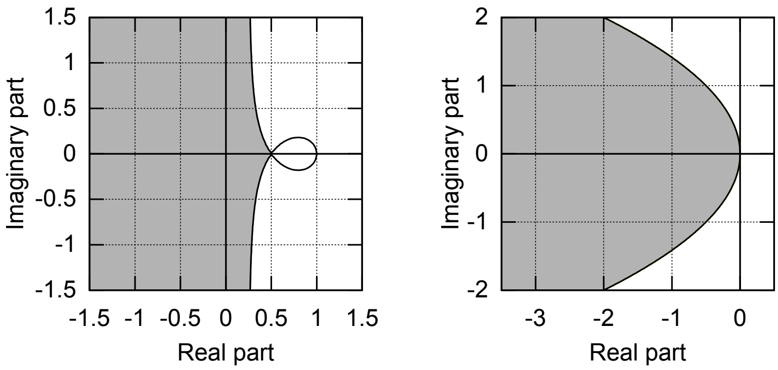

Figure 2: Stability regions for $h_1(s)$ and $h_2(s)$.

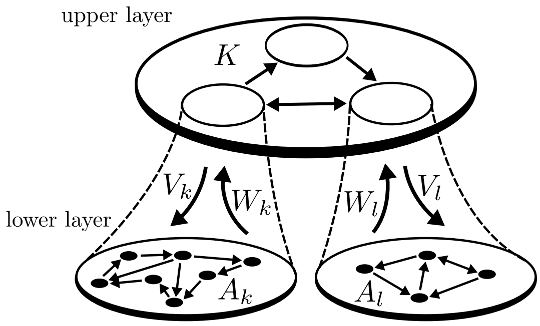

Figure 3: Concept of hierarchical structure.

subsystems in this paper. The concept of hierarchical interconnection is illustrated in Fig. 3. A two-layer hierarchical structure is realized in the sense that the agents in each subsystem exchange information using the local structure in the lower layer and the subsystems do aggregated information in the upper layer. [2]

Consider M subsystems. The kth subsystem contains n_k agents, where n_k $(k = 1, \ldots, M)$ are positive integers satisfying $N = \sum_{k=1}^{M} n_k$. We define the interconnection matrix representing the local connection in the kth subsystem as $A_k \in \mathbb{R}^{n_k \times n_k}$, the intersubsystem connection as $K \in \mathbb{R}^{M \times M}$. The relationship between the I/O of agents and those of subsystems is defined by

$$\begin{aligned} \boldsymbol{u}^k &= A_k \boldsymbol{y}^k + V_k U_k, \\ Y_k &= W_k^\top \boldsymbol{y}^k, \end{aligned} \tag{7}$$

where $\boldsymbol{u}^k := \left[u_1^k, \ldots, u_{n_k}^k\right]^\top, \boldsymbol{y}^k := \left[y_1^k, \ldots, y_{n_k}^k\right]^\top \in \mathbb{R}^{n_k}$ are the input and output of the agents in the kth subsystem, respectively, $U_k, Y_k \in \mathbb{R}^r$ are the

[2] We can naturally extend the concept of hierarchical structure to the case of three or more layers, although we focus on two layer structure for simplicity of discussion in this paper.

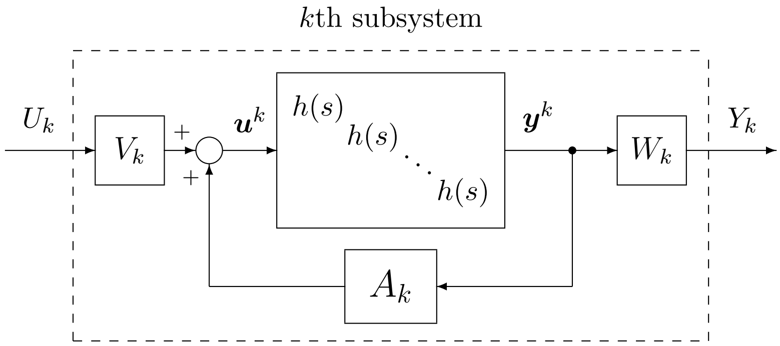

Figure 4: I/O of kth subsystem.

input and output of the kth subsystem, respectively, and $W_k, V_k \in \mathbb{R}^{n_k \times r}$ are the matrices representing aggregation and distribution of information, respectively. The block diagram is depicted in Fig. 4. The I/O dimension r of subsystem represents the level of information aggregation and satisfies $r \leq \min_k\{n_k\}$.

Consider the interconnection between subsystems described by

$$\boldsymbol{U} = (K \otimes I_r)\boldsymbol{Y}, \tag{8}$$

where $\boldsymbol{U} := [U_1^\top, \ldots, U_M^\top]^\top, \boldsymbol{Y} := [Y_1^\top, \ldots, Y_M^\top]^\top \in \mathbb{R}^{rM}$ and $K \in \mathbb{R}^{M \times M}$. Then the interconnection matrix of a two-layer hierarchical interconnection structure is given by

$$A = \mathrm{diag}\{A_k\} + K \odot \Gamma \in \mathbb{M}_\Delta^{M \times M}, \tag{9}$$

where $\odot$ denotes the Khatri-Rao product of two matrices (see Appendix C for the definition). The partition Δ to represent $\mathbb{M}_\Delta^{M \times M}$ satisfies $\rho(\Delta) = N$, and $\delta_i = n_i$. The Khatri-Rao product is a generalization of the Kronecker product, which has been used to represent hierarchical multi-agent systems with common subsystem structures as seen in [9, 8, 10]. The Khatri-Rao product is really required to treat the more general case where the numbers and graph structures of subsystems are different each other. $\Gamma \in \mathbb{M}_\Delta^{M \times M}$ is defined by

$$\Gamma = [\Gamma_{kl}] \quad ; \quad \Gamma_{kl} := V_k W_l^\top \quad (k, l = 1, \ldots, M), \tag{10}$$

and it determines what kind of information is exchanged among the subsystems and which agents receive the effect of the upper-layer interaction. We refer to Γ as an *interlayer connection matrix* in the rest of this paper. Obviously, Γ plays an important role in the hierarchical interconnection structure. For example, the low rank property of Γ_{kl}, or small number of $r = \mathrm{rank}\,\Gamma_{kl}$ is quite effective for achieving the rapid consensus as reported in [8]. Hence, we focus on the rank 1 and rank 2 cases in Sections 3 and 4, respectively.

2.3 Hierarchical Network Synthesis for Output Consensus

Our main concern in this paper for hierarchical network synthesis to achieve the output consensus is to propose a systematic way for the following problem:

For given $h(s)$ and A_k $(k = 1, \dots, M)$, the problem is to find matrices K and $\Gamma_{kl} = V_k W_l^\top$ $(k, l = 1, \dots, M)$ with low rank so that A is Lyapunov stable and $A\mathbf{1}_N = 0$ holds.

The key idea is focusing on eigenvector of A_k for the selection of Γ_{kl}, which makes us to derive an analytic expression of eigenvalue distribution of A and to assure the output consensus condition or $A\mathbf{1}_N = 0$. To this end, we assume the followings throughout this paper:

- $h(s)$ has a simple pole at the origin of the complex plane and all other poles are located in $\mathbb{C}_-$.
- $-A_k$ $(k = 1, \dots, M)$ are all graph Laplacian, and hence $-A_k\mathbf{1}_{n_k} = 0$ hold for $\forall\, k = 1, \dots, M$.

Since $\mathcal{A}$ in (6) should have a simple eigenvalue at 0, all the poles of $h(s)$ are included in the set of all closed-loop poles. In other words, the first assumption is a necessary condition for achieving the output consensus with Lyapunov stability, and hence it is not restrictive. The second assumption assures the output consensus inside each subsystem, and hence it is reasonable when we consider the hierarchical structure. However, it should be emphasized that the second assumption does not guarantees the Lyapunov stability even if we have no interlayer connection, or $K = 0$. The reason is that the stability region for LTI systems with generalized frequency variables is not $\mathbb{C}_-$ as seen in Fig. 2. It is characterized as the complement of Ω_+ in $\mathbb{C}$, or $\Omega_+^c = \mathbb{C} \setminus \Omega_+$, where

$$\Omega_+ := \phi(\mathbb{C}_+) = \{\lambda \in \mathbb{C} \mid \exists s \in \mathbb{C}_+,\, \phi(s) = \lambda\}$$

with $\phi(s) := 1/h(s)$ and $\mathbb{C}_+ = \{s \in \mathbb{C} \mid \mathrm{Re}\,(s) \geq 0\}$ [3].

3 Design of rank one interconnection

3.1 Eigenvalue distribution

In this section, we consider the case where $r = 1$, i.e., V_k and W_k in (7) are represented by

$$V_k = v_k \in \mathbb{R}^{n_k},\ W_k = w_k \in \mathbb{R}^{n_k}. \tag{11}$$

Then, the input and output of the kth subsystem are interpreted as the proportional distribution with the weight vector v_k and the weighted sum with the weight vector w_k, respectively. It is clear that Γ satisfies

$$\mathrm{rank}\,\Gamma_{kl} = \mathrm{rank}\, v_k w_l^\top = 1,\ \forall k, l, \tag{12}$$

which means the number of independent exchanged information in upper layer is just one.

We now extend the concept of the eigen-connection introduced in [10] for $A_k = A_0, \forall k$ to more general cases with having different A_k to capture many practical situations.

Definition 1. *For $k = 1, \ldots, M$, let A_k be $n_k \times n_k$ matrix and λ_{k1} be one of eigenvalues of A_k. An interlayer connection matrix defined by (10) is called a* left (resp., right) eigen-connection matrix *of $\{A_k\}$ associated with the eigenvalue $\{\lambda_{k1}\}$, if w_k (resp., v_k) in (11) is a left (resp., right) eigenvector of A_k associated with eigenvalue λ_{k1}.*

The following theorem shows that the eigenvalue set of the interconnection matrix of the left eigen-connected system consists of that of interconnection matrices in lower layer and that of the matrix determined by the interconnection matrix in upper layer and the interlayer connection matrix.

Theorem 2. *For each $k = 1, \ldots, M$, let A_k be an $n_k \times n_k$ matrix which has at least one simple eigenvalue λ_{k1} and let $v_k, w_k \in \mathbb{R}^{n_k}$. If Γ is a left eigen-connection matrix of $\{A_k\}$ associated with $\{\lambda_{k1}\}$, then, for any matrix $K \in \mathbb{R}^{M \times M}$, the set of all the eigenvalues of A defined by (9) is given by*

$$\sigma(A) = \bigcup_{k=1}^{M} \Big(\sigma(A_k) \setminus \{\lambda_{k1}\}\Big) \cup \sigma\Big(DK + \Lambda\Big), \tag{13}$$

where $D = \operatorname{diag}\left\{v_k^\top w_k\right\}$ and $\Lambda = \operatorname{diag}\{\lambda_{k1}\}$.

Proof. See Appendix A. □

A similar result can be derived for right eigen-connection matrices and is given as a corollary of Theorem 2.

Corollary 3. *In the same setting of Theorem 2, if Γ is a right eigen-connection matrix of $\{A_k\}$ associated with $\{\lambda_{k1}\}$, then the set of all the eigenvalues of A is given by*

$$\sigma(A) = \bigcup_{k=1}^{M} \Big(\sigma(A_k) \setminus \{\lambda_{k1}\}\Big) \cup \sigma\Big(KD + \Lambda\Big), \tag{14}$$

where D and Λ are defined in Theorem 2.

Remark 4. *Even when A_k, K and Γ are complex matrices, Theorem 2 and Corollary 3 still hold if the transpose is replaced by the complex conjugate.*

Remark 5. *Theorem 2 is an extension of Theorem 2 in [10]. If we take $A_k = A_0, v_k = v, w_k = w$ and $\lambda_k = \lambda, \forall k$, (13) can be written as*

$$\begin{aligned} \sigma(A) &= \Big(\sigma(A_0) \setminus \{\lambda\}\Big) \cup \sigma\Big(v^\top w K + \lambda I_M\Big) \\ &= \Big(\sigma(A_0) \setminus \{\lambda\}\Big) \cup \Big\{v^\top w\eta + \lambda \mid \eta \in \sigma(K)\Big\}, \end{aligned}$$

which is the same as the result in [10].

Theorem 2 tells us that an interlayer eigen-connection matrix of rank one affects only eigenvalues of the local interconnection matrix A_k used for the eigen-connection. If we have unstable subsystems, there are two approaches to make the whole system stable. One is to adjust the local interconnection from which unstable eigenvalues are derived. The other is to connect subsystems using the eigen-connection associated with unstable eigenvalues and to design K, v_k and w_k so as to shift unstable eigenvalues into the stability region.

3.2 Design procedure with numerical examples

We here show a design procedure of a hierarchical interconnection structure for dynamical multi-agent systems based on the rank 1 eigen-connection investigated in the previous subsection through numerical examples. The dynamics of each agent is given by

$$h(s) = \frac{b}{s(s+a)} e^{-\tau s}\ ; \quad a = \pi,\ b = \frac{\pi^2}{2}, \tag{15}$$

where τ represents the time delay for exchanging information. The number of agents is $N = 7$.

The 7 agents are divided into 2 subsystems which consist of 4 and 3 agents, respectively. Assume the local interconnection matrices A_1 and A_2 are given by

$$A_1 = \begin{bmatrix} -2.8 & 1.8 & 0 & 1 \\ 0 & -0.5 & 0.5 & 0 \\ 0.1 & 0 & -0.6 & 0.5 \\ 0.5 & 0 & 0 & -0.5 \end{bmatrix}, A_2 = \begin{bmatrix} -1.5 & 1.5 & 0 \\ 0.5 & -1 & 0.5 \\ 0.5 & 0 & -0.5 \end{bmatrix}. \tag{16}$$

The eigenvalues of A_1 and A_2 are given by

$$\sigma(A_1) = \left\{0, -\frac{7 \pm \sqrt{14}\,j}{10}, -3\right\}, \quad \sigma(A_2) = \{0, -1, -2\}.$$

Case 1) $\tau = 0$

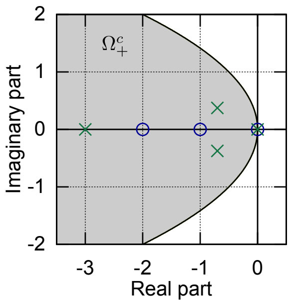

Figure 5: Eigenvalue distribution of A_1('×'), A_2('◯') and stability region Ω_+^c (Case1)

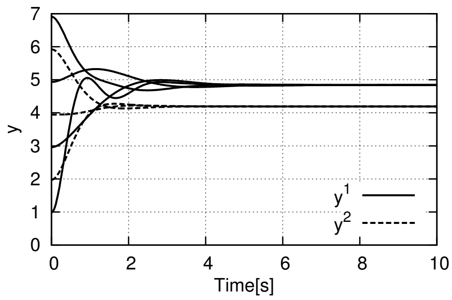

Figure 6: Output of each subsystem without interconnection (Case 1).

The region Ω_+^c and the eigenvalues of A_1 and A_2 are shown in Fig. 5. Both matrices have an eigenvalue 0 associated with a right eigenvector $\mathbf{1}_4$ and $\mathbf{1}_3$, respectively. Since other eigenvalues lie in Ω_+^c, each subsystem achieves consensus. However, all the agents in the whole system do not reach consensus as shown in Fig. 6.

To achieve consensus of all the agents, we need to design the eigen-connection between two subsystems. The eigenvalue which we have to change is one of two 0s, and preserve the right eigenvectors of them, that is, $\mathbf{1}_4$ and $\mathbf{1}_3$. Let Γ in (9) be right eigen-connection matrix of $\{A_1, A_2\}$ associated with eigenvalues $\{0, 0\}$, that is, $v_1 = \mathbf{1}_4,\ v_2 = \mathbf{1}_3$. Furthermore, we need to set $\mathbf{1}_7$ as the right eigenvector of A associated with 0 eigenvalue which has not been changed, i.e., $A\mathbf{1}_7 = 0$. This can be achieved if and only if matrix $KD + \Lambda$ in (14) has an eigenvalue 0 associated with the right eigenvector $\mathbf{1}_2$. Choose w_1 and w_2 such that D is an identity matrix to facilitate the design of K. Hence, we may set w_1 and w_2 as $w_1 = (1/4)\mathbf{1}_4,\ w_2 = (1/3)\mathbf{1}_3$.

This means that the aggregated information is an arithmetic average of outputs of all the agents in each subsystem. Then, we get $KD + \Lambda = K$ and K must have an eigenvalue 0 associated with the right eigenvector $\mathbf{1}_2$ and the other eigenvalue in Ω_+^c. In this example, we employ

$$K = \begin{bmatrix} -3/4 & 3/4 \\ 3/4 & -3/4 \end{bmatrix},$$

which satisfies $\sigma(K) = \{0, -3/2\}$. The eigenvalues of the whole interconnection matrix A are plotted in Fig. 7 as '□', and thus the consensus is achieved as shown in Fig. 8.

Case 2) $\tau = 0.25$

The region Ω_+^c for the delay case and the eigenvalues of A_1 and A_2 are illustrated in Fig. 9. Since one eigenvalue of A_1, $\lambda_1 = -3$, lies outside of Ω_+^c, subsystem 1 is unstable as seen in Fig. 10.

To stabilize the system and achieve consensus of each subsystem, we

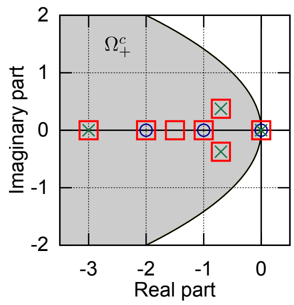

Figure 7: Eigenvalue distribution of A_1('×'), A_2('◯') and A('□') (Case1).

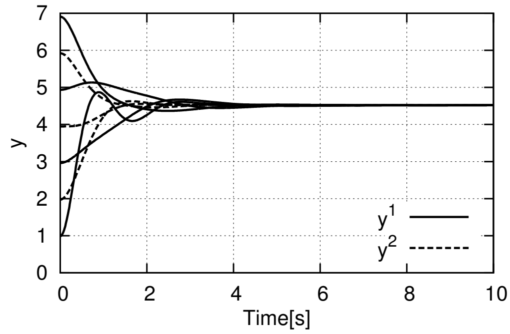

Figure 8: Outputs of the agents connected by eigen-connection (Case 1).

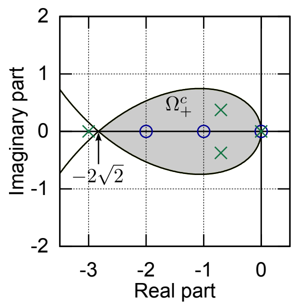

Figure 9: Eigenvalue distribution of A_1('×'), A_2('◯') and stability region Ω_+^c (Case2)

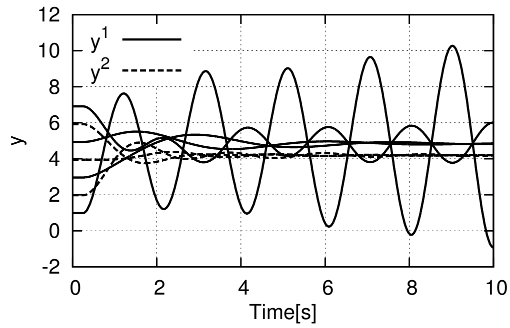

Figure 10: Output of each subsystem without interconnection (Case 2).

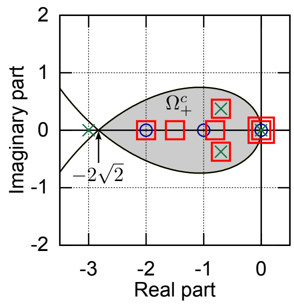

Figure 11: Eigenvalue distribution of A_1('×'), A_2('◯') and A('□') (Case2).

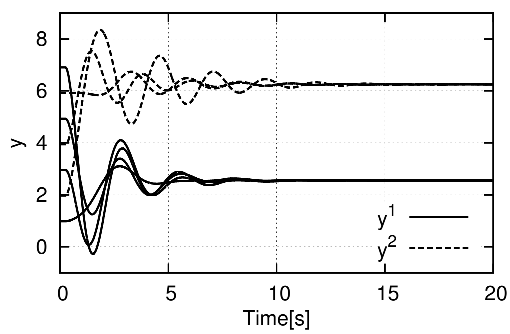

Figure 12: Outputs of the agents connected by eigen-connection (Case 2).

need to design an eigen-connection between two subsystems appropriately. In particular, we have to change the unstable eigenvalue of A_1, $\lambda_1 = -3$, and preserve the right eigenvectors of two 0 eigenvalues. Therefore, let Γ in (9) be the left eigen-connection matrix of $\{A_1, A_2\}$ associated with the eigenvalues $\{\lambda_1, \lambda_2\}$, where $\lambda_2 = -1$ is the eigenvalue of A_2 used for the eigen-connection. The left eigenvectors of A_1 and A_2 associated with λ_1 and λ_2 are given by

$$w_1 = \begin{bmatrix}1 & -0.72 & 0.15 & -0.43\end{bmatrix}^\top, \ \ w_2 = \begin{bmatrix}0 & 1 & -1\end{bmatrix}^\top,$$

respectively. We set w_1 and w_2 as the output weight vectors. The input weight vectors v_1 and v_2 are chosen to make D an identity matrix to facilitate the design of K. Hence, we may set v_1 and v_2 as

$$v_1 = \begin{bmatrix}2 & 1 & 1 & 1\end{bmatrix}^\top, \ \ v_2 = \begin{bmatrix}1 & 2 & 1\end{bmatrix}^\top,$$

respectively. Then λ_1 and λ_2 are shifted to the eigenvalues of the matrix $K_\Lambda = K + \mathrm{diag}\,\{\lambda_1, \lambda_2\}$. Let $K = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, then the eigen equation of K_Λ is given by

$$s^2 - (\lambda_1 + \lambda_2 + a + d)s + \lambda_1\lambda_2 + a\lambda_2 + d\lambda_1 + ad - bc = 0. \qquad (17)$$

Thus, we should choose a, b, c and d such that all the roots of (17) are contained in Ω_+^c. For example, assigning the roots at -0.8 and -1.5 leads to

$$\lambda_1 + \lambda_2 + a + d = -2.3 \ \Leftrightarrow \ a + d = 1.7.$$

Employing $a = 1$ and $d = 0.7$ yields

$$\lambda_1\lambda_2 + a\lambda_2 + d\lambda_1 + ad - bc = 1.2 \ \Leftrightarrow \ bc = -0.6.$$

We fix $b = 0.6$ and $c = -1$. All the eigenvalues of the whole interconnection matrix A, which are plotted in Fig. 11 as '□', lie in Ω_+^c. As a result, each subsystem achieves consensus as shown in Fig. 12.

Note that we can only move at most one eigenvalue per subsystem with rank one eigen-connection. It is therefore impossible to achieve consensus and stabilization simultaneously in the Case 2. There is another problem that we cannot always choose w_k from the eigenvectors of A_k. Furthermore, the corresponding eigenvector w_k must be a real vector for practical reason, but it may be a complex vector. We will introduce the rank two eigen-connection in the next section to overcome these problems.

4 Design of rank two interconnection

4.1 Eigenvalue distribution

In this section, we consider $r = 2$ case. Both V_k and W_k in (7) are then $n_k \times 2$ matrices. Let

$$V_k = [v_{k1}\ v_{k2}] \in \mathbb{R}^{n_k \times 2}, \ W_k = [w_{k1}\ w_{k2}] \in \mathbb{R}^{n_k \times 2}. \qquad (18)$$

If (v_{k1}, v_{k2}) and (w_{k1}, w_{k2}) are the pairs of linearly independent vectors, Γ satisfies

$$\operatorname{rank} \Gamma_{kl} = \operatorname{rank} \begin{bmatrix} v_{k1} & v_{k2} \end{bmatrix} \begin{bmatrix} w_{l1}^\top \\ w_{l2}^\top \end{bmatrix} = 2, \ \forall k, l, \tag{19}$$

which means the number of information exchanged independently in upper layer is two.

We now define the rank two eigen-connection in the similar manner as Definition 1. This is an extension of the rank two eigen-connection defined in [10].

Definition 6. *For $k = 1, \ldots, M$, let A_k be $n_k \times n_k$ matrix and $\lambda_{k1}, \lambda_{k2}$ be two eigenvalues of A_k. An interlayer connection matrix Γ defined by (10) is a* left (resp., right) eigen-connection matrix *of $\{A_k\}$ associated with the eigenvalues $\{\lambda_{k1}\}, \{\lambda_{k2}\}$, if w_{k1} and w_{k2} (resp., v_{k1} and v_{k2}) belong to the linear subspace spanned by the left (resp., right) eigen-vectors of A_k associated with the eigenvalues λ_{k1} and λ_{k2}.*

We state the main result for the hierarchical interconnection with rank two eigen-connection in the following theorem.

Theorem 7. *For each $k = 1, \ldots, M$, let A_k be an $n_k \times n_k$ matrix which has at least two simple eigenvalues λ_{k1} and λ_{k2} and let v_{k1}, v_{k2}, w_{k1} and w_{k2} be n_k-dimensional column vectors. If Γ given by (18) is a left eigen-connection matrix of $\{A_k\}$ associated with $\{\lambda_{k1}\}, \{\lambda_{k2}\}$, then, for any $M \times M$ matrix K, the set of all the eigenvalues of A defined by (9) is given by*

$$\sigma(A) = \bigcup_{k=1}^{M} \Big(\sigma(A_k) \setminus \{\lambda_{k1}, \lambda_{k2}\} \Big) \cup \sigma\Big(\Xi (K \otimes I_2) T^\top + \Lambda \Big). \tag{20}$$

Here, $\Xi = \operatorname{diag}\{\Xi_k\}, T = \operatorname{diag}\{T_k\}$ and $\Lambda = \operatorname{diag}\{\Lambda_k\}$, where Ξ_k and Λ_k are 2×2 matrices satisfying

$$\Xi_k = \begin{bmatrix} \mu_{k1}^\top \\ \mu_{k2}^\top \end{bmatrix} \begin{bmatrix} v_{k1} & v_{k2} \end{bmatrix}, \ \Lambda_k = \begin{bmatrix} \lambda_{k1} & 0 \\ 0 & \lambda_{k2} \end{bmatrix}.$$

μ_{k1} and μ_{k2} are the left eigenvectors of A_k associated with λ_{k1} and λ_{k2}, respectively, and T_k is a 2×2 matrix satisfying

$$\begin{bmatrix} w_{k1} & w_{k2} \end{bmatrix} = \begin{bmatrix} \mu_{k1} & \mu_{k2} \end{bmatrix} T_k.$$

Proof. See Appendix B. □

As in the rank one case, we can show that Theorem 7 is an extension of Theorem 6 in [10], and we can obtain an analogous result for right eigen-connection matrices.

Corollary 8. *In the same setting in Theorem 7, if* Γ *is a right eigen-connection matrix of* $\{A_k\}$ *associated with* $\{\lambda_{k1}\}, \{\lambda_{k2}\}$*, then for any* $M \times M$ *matrix* K*, the set of all the eigenvalues of* A *defined by (9) is given by*

$$\sigma(A) = \bigcup_{k=1}^{M} \Big(\sigma(A_k) \setminus \{\lambda_{k1}, \lambda_{k2}\}\Big) \cup \sigma\Big(S(K \otimes I_2)\Phi + \Lambda\Big). \tag{21}$$

Here, $\Phi = \operatorname{diag}\{\Phi_k\}, S = \operatorname{diag}\{S_k\}$ *and* $\Lambda = \operatorname{diag}\{\Lambda_k\}$*, where* Φ_k *and* Λ_k *are* 2×2 *matrices satisfying*

$$\Phi_k = \begin{bmatrix} w_{k1}^\top \\ w_{k2}^\top \end{bmatrix} \begin{bmatrix} \gamma_{k1} & \gamma_{k2} \end{bmatrix}, \quad \Lambda_k = \begin{bmatrix} \lambda_{k1} & 0 \\ 0 & \lambda_{k2} \end{bmatrix}.$$

γ_{k1} *and* γ_{k2} *are the right eigenvectors of* A_k *associated with* λ_{k1} *and* λ_{k2}*, respectively, and* S_k *is a* 2×2 *matrix satisfying*

$$\begin{bmatrix} v_{k1} & v_{k2} \end{bmatrix} = \begin{bmatrix} \gamma_{k1} & \gamma_{k2} \end{bmatrix} S_k.$$

Remark 9. *Theorem 7 is an extension of Theorem 6 in [10]. If we take* $A_k = A_0, v_{k1} = v_1, v_{k2} = v_2, w_{k1} = w_1, w_{k2} = w_2, \lambda_{k1} = \lambda_1, \lambda_{k2} = \lambda_2, \Lambda_k = \Lambda_0, \Xi_k = \Xi_0, T_k = T_0, \forall k$*, (20) can be written as*

$$\begin{aligned}\sigma(A) &= \Big(\sigma(A_0) \setminus \{\lambda_1, \lambda_2\}\Big) \cup \sigma\Big(K \otimes \Xi_0 T_0 + I_M \otimes \Lambda_0\Big) \\ &= \Big(\sigma(A_0) \setminus \{\lambda_1, \lambda_2\}\Big) \cup \left\{ \bigcup_{\lambda_K \in \sigma(K)} \sigma\Big(\Lambda_0 + \lambda_K \Xi_0 T_0\Big) \right\},\end{aligned}$$

which is the same as the result in [10].

4.2 Design procedure with numerical example

We here show a design procedure of a hierarchical interconnection structure based on rank two eigen-connection through a numerical example.

Consider the case where the dynamics of each agent is given by (15) and $N = 7$. The 7 agents are divided into 2 subsystems which consists of 4 and 3 agents, respectively. Assume that local interconnection structure A_1 and A_2 is given by (16). To achieve consensus and stabilization simultaneously, we need to design the rank two eigen-connection between two subsystems. We have to change the unstable eigenvalue of A_1, $\lambda_1 = -3$ and one of two 0 eigenvalues, and preserve the right eigenvectors of 0s, that is, $\mathbf{1}_4$ and $\mathbf{1}_3$. Let Γ in (9) be rank two right eigen-connection matrix of $\{A_1, A_2\}$ associated with eigenvalues $\{0, 0\}$ and $\{\lambda_1, \lambda_2\}$, where $\lambda_2 = -1$ is the eigenvalue of A_2 used for eigen-connection. The right eigenvectors of A_1 and A_2 associated with eigenvalues 0 are given by $\gamma_{11} = \mathbf{1}_4,\ \gamma_{21} = \mathbf{1}_3$, respectively. Let right eigenvectors associated with λ_1 and λ_2 be

$$\gamma_{12} = \begin{bmatrix}1 & 0 & 0 & -1/5\end{bmatrix}^\top, \gamma_{22} = \begin{bmatrix}1 & 1/3 & 1\end{bmatrix}^\top,$$

respectively. To our end, we need to set $\mathbf{1}_7$ as the right eigenvector associated with 0 eigenvalue which has not been changed and to let the other eigenvalues lie in the stability region Ω_+^c. This is the case if and only if matrix $S(K \otimes I_2)\Phi + \Lambda$ in (21) has an eigenvalue 0 associated with the right eigenvector $c = [1, 0, 1, 0]^\top$ and the other three eigenvalues in Ω_+^c. This comes from the fact that the right eigenvector of A associated with 0 eigenvalue is given by $\mathrm{diag}\left\{\begin{bmatrix}\gamma_{k1} & \gamma_{k2}\end{bmatrix}\right\} c$. We choose V_k to make S be an identity matrix to facilitate the design of K. Hence, we may set $V_k, k = 1, 2$ as

$$V_1 = \begin{bmatrix}\gamma_{11} & \gamma_{12}\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1/5\end{bmatrix}^\top,$$
$$V_2 = \begin{bmatrix}\gamma_{21} & \gamma_{22}\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 \\ 1 & 1/3 & 1\end{bmatrix}^\top.$$

Furthermore, $W_k, k = 1, 2$ is chosen to make Φ be an diagonal matrix. To this end, set

$$W_1 = \begin{pmatrix}\alpha\begin{bmatrix}1/5 & -1/5 & 0 & 1\end{bmatrix} \\ \beta\begin{bmatrix}1 & -1/2 & -1/2 & 0\end{bmatrix}\end{pmatrix}^\top,$$
$$W_2 = \begin{pmatrix}\gamma\begin{bmatrix}1/2 & 0 & 1/2\end{bmatrix} \\ \delta\begin{bmatrix}1/2 & 0 & -1/2\end{bmatrix}\end{pmatrix}^\top.$$

We then get $\Phi = \mathrm{diag}\{\alpha, \beta, \gamma, \delta\}$. One may think that it is difficult to design W_k in this manner. However, by letting the lth column of W_k be chosen from linear subspace of left eigenvectors of A_k associated with eigenvalues except λ_{kl}, we can see that Φ is diagonal from Lemma 10 in Appendix B. Letting $K = \begin{bmatrix}a & b \\ c & d\end{bmatrix}$ gives

$$S(K \otimes I_2)\Phi + \Lambda = \begin{bmatrix}\alpha a & 0 & \gamma b & 0 \\ 0 & \beta a - 3 & 0 & \delta b \\ \alpha c & 0 & \gamma d & 0 \\ 0 & \beta c & 0 & \delta d - 1\end{bmatrix}.$$

The eigenvalue distribution of this matrix is decomposed to those of two matrices $K_1 = \begin{bmatrix}\alpha a & \gamma b \\ \alpha c & \gamma d\end{bmatrix}$ and $K_2 = \begin{bmatrix}\beta a - 3 & \delta b \\ \beta c & \delta d - 1\end{bmatrix}$. To achieve consensus, K_1 needs an eigenvalue 0 associated with right eigenvector $\mathbf{1}_2$, the other eigenvalue of K_1 and both eigenvalues of K_2 have to lie in Ω_+^c while to achieve stabilization. Set $b = -a, c = -d$ to guarantee the first condition.

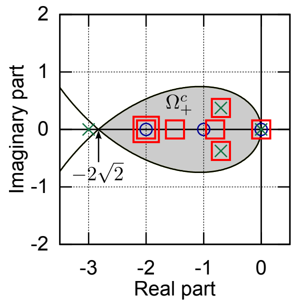

Figure 13: Eigenvalue distribution of A_1('×'), A_2('◯') and A('□') (rank 2)

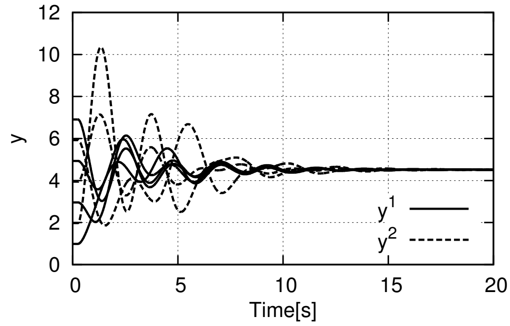

Figure 14: Outputs of the agents connected by rank two eigen-connection.

Then the eigenvalues of K_1 are 0 and $a+d$. We see from the second condition that

$$-2\sqrt{2} < a + d < 0 \tag{22}$$

holds. The eigen equation of K_2 is given by

$$s^2 - (\beta a + \delta d - 4)s + 3 - \beta a - 3\delta d = 0. \tag{23}$$

We now choose β, a, δ and d such that all the roots of (23) lie in Ω_+^c. For example, distributing the roots at -0.8 and -1.5 gives

$$\begin{cases} \beta a + \delta d - 4 = -2.3 \\ 3 - \beta a - 3\delta d = 1.2 \end{cases} \Leftrightarrow \begin{cases} \beta a + \delta d = 1.7 \\ \beta a + 3\delta d = 1.8 \end{cases},$$

which yields $\beta a = 1.65$ and $\delta d = 0.05$. Then let $a = d = -1$ such that (22) is satisfied. Finally we can get $\beta = -1.65, \delta = -0.05$. All the eigenvalues of the whole interconnection matrix A, which is plotted in Fig. 13 as '□', lie in Ω_+^c except one 0 eigenvalue. As a result the system achieves stabilization and consensus simultaneously as shown in Fig. 14, since $A\mathbf{1}_7 = 0$ holds.

5 Conclusion

In this paper, we have investigated two-layer hierarchical multi-agent dynamical systems with rank one and two interlayer connection and provided a solution to hierarchical consensus problem by eigenvector-based connection. In particular, the concept of eigen-connection was extended to the case where the number of agents in each subsystem and the connection structure of each subsystem are different from each other. We provided analytical expressions of the eigenvalue sets of the hierarchical interconnection matrices and systematic synthesis procedures for achieving the output consensus with Lyapunov stability, and numerical examples with simulations confirmed the effectiveness of the proposed design method.

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A Proof of Theorem 2

Let us first introduce the following lemma.

Lemma 10. *Left (generalized) eigenvectors of $N \times N$ matrix A are orthogonal to right (generallized) eigenvectors except the one associated with the same eigenvalue.*

Proof. We omit the proof, since it is well-known fact. □

Now we are ready to prove Theorem 2.

Let γ_k be a right eigenvector of A_k associated with an eigenvalue $\lambda_k \in \sigma(A_k) \setminus \{\lambda_{k1}\}$, and $\boldsymbol{\gamma}_k^e \in \mathbb{M}_\Delta^M$ be M block column vector whose kth block is γ_k and other blocks are all zero. Right multiplying A by $\boldsymbol{\gamma}_k^e$, we obtain the following relation:

$$A\boldsymbol{\gamma}_k^e = \lambda_k \boldsymbol{\gamma}_k^e + (K \odot \Gamma)\boldsymbol{\gamma}_k^e.$$

The lth block of second term in right side is $a_{lk} v_l w_k^\top \gamma_k$. Since w_k is a left eigenvector of A_k, we get $w_l^\top \gamma_k = 0$ by Lemma 10, which further implies $A\boldsymbol{\gamma}_k^e = \lambda_k \boldsymbol{\gamma}_k^e$. This means λ_k is also an eigenvalue of A. This corresponds to the first term in right side of (13).

Next, let $c = [c_1, \ldots, c_M]^\top \in \mathbb{R}^M$ be a left eigenvector of $DK + \Lambda$ associated with an eigenvalue η. That is, $c^\top (DK + \Lambda) = \eta c^\top$. Comparing the kth entry of both sides implies

$$\sum_{l=1}^{M} c_l v_l^\top w_l a_{lk} + c_k \lambda_k = \eta c_k. \tag{24}$$

Define $w := \mathrm{diag}\left\{w_k^\top\right\} \in \mathbb{M}^{M \times M}(\Delta, \Delta_1)$, where partition Δ_1 satisfies $\delta_1^i = 1, \forall i = 1, \ldots, M$. Using (24), the kth block of $(wc)^\top A$ can be written as

$$c_k \lambda_k w_k^\top + \sum_{l=1}^{M} c_l a_{lk} w_l^\top v_l w_k^\top = \left(\sum_{l=1}^{M} c_l v_l^\top w_l a_{lk} + c_k \lambda_k \right) w_k^\top = \eta c_k w_k^\top$$

This yields $(wc)^\top A = \eta (wc)^\top$, which means η is an eigenvalue of A. This corresponds to the second term in right side of (13). The proof is thus completed.

B Proof of Theorem 7

Let γ_k be a right eigenvector of A_k associated with an eigenvalue $\lambda_k \in \sigma(A_k) \setminus \{\lambda_{k1}, \lambda_{k2}\}$, and $\boldsymbol{\gamma}_k^e \in \mathbb{M}_\Delta^M$ be M block column vector whose kth block is γ_k and other blocks are all zero. Right multiplying A by $\boldsymbol{\gamma}_k^e$, we obtain the following relation:

$$A\boldsymbol{\gamma}_k^e = \lambda_k \boldsymbol{\gamma}_k^e + (K \odot \Gamma)\boldsymbol{\gamma}_k^e.$$

The lth block of second term in right side is $a_{lk}[v_{k1}\ v_{k2}][w_{l1}\ w_{l2}]^\top \gamma_k$. Since w_{k1} and w_{k2} is a left eigenvector of A_k, we get $[w_{l1}\ w_{l2}]^\top \gamma_k = [0\ 0]^\top$ by Lemma 10, which further implies $A\boldsymbol{\gamma}_k^e = \lambda_k \boldsymbol{\gamma}_k^e$. This means λ_k is also an eigenvalue of A. This corresponds to the first term in right side of (20).

Next, let $c = [c_{11}, c_{12} \mid \cdots \mid c_{M1}, c_{M2}]^\top \in \mathbb{M}_{\Delta_2}^M$ be a left eigenvector of $\Xi(K \otimes I_2)T^\top + \Lambda$ associated with an eigenvalue η, where partition Δ_2 satisfies $\delta_2^i = 2, \forall i = 1, \ldots, M$. That is,

$$c^\top \Big(\Xi(K \otimes I_2)T^\top + \Lambda \Big) = \eta c^\top.$$

Comparing the kth entry of both sides implies

$$\sum_{l=1}^M a_{lk} \begin{bmatrix} c_{l1} & c_{l2} \end{bmatrix} \begin{bmatrix} \mu_{l1}^\top \\ \mu_{l2}^\top \end{bmatrix} \begin{bmatrix} v_{l1} & v_{l2} \end{bmatrix} T_k^\top + \begin{bmatrix} c_{k1} & c_{k2} \end{bmatrix} \begin{bmatrix} \lambda_{k1} & 0 \\ 0 & \lambda_{k2} \end{bmatrix} = \eta \begin{bmatrix} c_{k1} & c_{k2} \end{bmatrix}. \tag{25}$$

Define

$$\mu = \mathrm{diag}\left\{ \begin{bmatrix} \mu_{k1} & \mu_{k2} \end{bmatrix} \right\} \in \mathbb{M}^{M \times M}(\Delta, \Delta_2),$$

then the kth block of $\mu c \in \mathbb{M}_\Delta^M$ is $c_{k1}\mu_{k1} + c_{k2}\mu_{k2}$. Using (25), the kth block of $(\mu c)^\top A$ can be written as

$$\begin{aligned}
& c_{k1}\lambda_{k1}\mu_{k1}^\top + c_{k2}\lambda_{k1}\mu_{k2}^\top + \sum_{l=1}^M a_{lk} \begin{bmatrix} c_{l1} & c_{l2} \end{bmatrix} \begin{bmatrix} \mu_{l1}^\top \\ \mu_{l2}^\top \end{bmatrix} \begin{bmatrix} v_{l1} & v_{l2} \end{bmatrix} \begin{bmatrix} w_{k1}^\top \\ w_{k2}^\top \end{bmatrix} \\
&= \left(\begin{bmatrix} c_{k1} & c_{k2} \end{bmatrix} \begin{bmatrix} \lambda_{k1} & 0 \\ 0 & \lambda_{k2} \end{bmatrix} + \sum_{l=1}^M a_{lk} \begin{bmatrix} c_{l1} & c_{l2} \end{bmatrix} \begin{bmatrix} \mu_{l1}^\top \\ \mu_{l2}^\top \end{bmatrix} \begin{bmatrix} v_{l1} & v_{l2} \end{bmatrix} T_k^\top \right) \begin{bmatrix} \mu_{k1}^\top \\ \mu_{k2}^\top \end{bmatrix} \\
&= \eta \begin{bmatrix} c_{k1} & c_{k2} \end{bmatrix} \begin{bmatrix} \mu_{k1}^\top \\ \mu_{k2}^\top \end{bmatrix}
\end{aligned}$$

This yields $(\mu c)^\top A = \eta(\mu c)^\top$, which means η is an eigenvalue of A. This corresponds to the second term in right side of (13). The proof is thus completed.

C Khatri-Rao Product

Consider an interval $I = [0, N]$, where N is a positive integer. Take finite number of integer points $0 = \zeta_0 < \zeta_1 < \cdots < \zeta_n = N$ in I, and define the *partition* Δ which divides I into $n(\leq N)$ small intervals as $\Delta = \{\zeta_1, \ldots, \zeta_n\}$. We denote the length of interval N as $\rho(\Delta)$, the number of partition n as $\#(\Delta)$. Let the length of each interval be $\delta^i := \zeta_i - \zeta_{i-1}$. Then, define the set of $m \times n$ block matrices as follow:

$$\mathbb{M}^{m\times n}(\Delta_1, \Delta_2) := \{A \in \mathbb{R}^{\rho(\Delta_1)\times\rho(\Delta_2)} | A_{ij} \in \mathbb{R}^{\delta_1^i \times \delta_2^j}\}$$

where partitions Δ_1 and Δ_2 satisfy $\#(\Delta_1) = m$ and $\#(\Delta_2) = n$, respectively, and A_{ij} represents the (i, j) block matrix of A.

We now define *Khatri-Rao product*.

Definition 11. *A binary operator* Khatri-Rao product

$$\odot : \mathbb{R}^{m\times n} \times \mathbb{M}^{m\times n}(\Delta_1, \Delta_2) \to \mathbb{M}^{m\times n}(\Delta_1, \Delta_2)$$

is defined as follow:

$$\begin{aligned} A = [a_{ij}] \in \mathbb{R}^{m\times n}, B = [B_{ij}] \in \mathbb{M}^{m\times n}(\Delta_1, \Delta_2) \\ A \odot B := C \in \mathbb{M}^{m\times n}(\Delta_1, \Delta_2),\ C_{ij} = a_{ij} B_{ij}. \end{aligned}$$

Khatri-Rao product defined in [4] is the binary operator which acts on two block matrices, and is defined as the Kronecker product for each block entry. We assume that all the block entries of A is scalar in this paper.

If $\forall i, j, \delta_1^i = \delta_2^j = 1$, that is, $B_{ij} \in \mathbb{R}$, then $A \odot B$ is equivalent to the Hadamard product of A and B. If $\forall i, j, B_{ij} = B_0$, then we can rewrite $A \odot B = A \otimes B_0$ using Kronecker product.

For simplicity of notation, we write $\mathbb{M}_\Delta^{m\times n}$ instead of $\mathbb{M}^{m\times n}(\Delta, \Delta)$ in the case where $\Delta_1 = \Delta_2 = \Delta$. Moreover, considering the set of $n \times 1$ and $1\times n$ block matrices, Δ_2 and Δ_1 are uniquely determined by the length of the interval. We let $\mathbb{M}_{\Delta_1}^n$ and $\mathbb{M}_{\Delta_2}^{1\times n}$ stand for $\mathbb{M}^{n\times 1}(\Delta_1, \Delta_2)$ and $\mathbb{M}^{1\times n}(\Delta_1, \Delta_2)$, respectively.