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Yasuhide NUMATA and Akimichi TAKEMURA

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DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

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Yasuhide NUMATA

Mathematical Informatics, Graduate School of Information Science and Technology, University of Tokyo.
JST CREST.
E-mail: numata@stat.t.u-tokyo.ac.jp

Akimichi TAKEMURA

Mathematical Informatics, Graduate School of Information Science and Technology, University of Tokyo.
JST CREST.
E-mail: takemura@stat.t.u-tokyo.ac.jp

In this article we give a computational study of combinatorics of the discriminantal arrangements. The discriminantal arrangements are parametrized by two positive integers n and k such that $n > k$. The intersection lattice of the discriminantal arrangement with the parameter (n, k) is isomorphic to the intersection lattice of the hyperplane arrangement generated by n generic points in the d-dimensional vector space where $d = n - k - 1$. The combinatorics of the discriminantal arrangements is very hard, except for the special cases of the Boolean arrangements ($k = 0$) and the braid arrangements ($k = 1$). We review some results on the intersection lattices of the arrangements generated by generic points and use them to obtain some computational results on the characteristic polynomials of the discriminantal arrangements.

Keywords: intersection poset; Möbius function; order ideal

1. Introduction

The main subject of this article is the combinatorics of the discriminantal arrangement introduced by Manin and Schechtman.[2] The discriminantal arrangement $\mathcal{B}(n, k)$, $n > k$, is defined as follows (cf. Section 5.6 of Orlik and Terao[5]): Let K be a field of characteristic zero and let $\mathcal{A}_0 = \{ H_1, \dots, H_n \}$ a general position arrangement of n hyperplanes in a k-dimensional vector space over K. Fix a normal vector ϕ_i for each hyperplane H_i. For $a \in K^n$, we define $\mathcal{A}_a$ to be the hyperplane arrangement $\mathcal{A}_a = \{ H_1 + a_1\phi_1, \dots, H_n + a_n\phi_n \}$ obtained from $\mathcal{A}_0$ by parallel translation. The set of the parameter $a \in K^n$ such that the hyperplanes of $\mathcal{A}_a$ are in a general position is a complement $K^n \setminus \mathcal{B}$ of some central hyperplane arrangement $\mathcal{B}$ in K^n.

When $\mathcal{A}_0$ is generic in the sense of of Athanasiadis,[1] the combinatorial structure of $\mathcal{B}$ depends only on n and k, and is called the discriminantal arrangement $\mathcal{B}(n,k)$. Let

$$d = n - k - 1.$$

Bayer and Brandt[4] conjectured that the intersection lattice of $\mathcal{B}(n,k)$ is isomorphic to a lattice $L_{n,d}$ defined in set-theoretical terminology (see Section 2 for the definition of $L_{n,d}$) and Athanasiadis[1] proved the conjecture.

Falk[3] showed the equivalence between the discriminantal arrangements and the arrangements generated by generic points. Regard ϕ_i^t as k-dimensional row vectors and consider the image P of the linear map from K^k to K^n defined by the $n \times k$-matrix

$$\begin{pmatrix} \phi_1^t \\ \vdots \\ \phi_n^t \end{pmatrix}.$$

In other words, P is the subspace of K^n spanned by the column vectors of the matrix. The image of the hyperplane H_i is $\{ x \in P \mid x_i = 0 \}$. Since we assume that $\phi_1, \dots, \phi_n$ are in general position, the dimension of the subspace P is k. Consider the orthogonal complement of P and fix a basis $\{ v_1, \dots, v_{d+1} \}$ of the complement. Let w_i^t be the i-th $(d+1)$-dimensional row vector of the $n \times (d+1)$-matrix $(v_1, \dots, v_{d+1})$. Falk showed that the n points $w_1, \dots, w_n \in K^{d+1}$ are in general position, and that the essential part of the discriminantal arrangement $\mathcal{B}(n,k)$ is equivalent to the central hyperplane arrangement consisting of all hyperplanes in K^{d+1} spanned by subsets of $\{ w_1, \dots, w_n \}$ of size d. It is easy to see that the latter arrangement can be obtained by coning without the additional coordinate hyperplane from the non-central hyperplane arrangement $\mathcal{A}_{n,d}$ generated by generic n points in d-dimensional vector space. (See Section 2 for the definition of $\mathcal{A}_{n,d}$.)

Recently the present authors showed that the order ideals of the intersection lattice of $\mathcal{A}_{n,d}$ can be decomposed into direct products of smaller lattices corresponding to smaller dimensions.[6] The decomposition implies some identities of the Möbius functions and the characteristic polynomials of the intersection lattices. Moreover they give a way to compute the Möbius functions and the characteristic polynomials of the intersection lattices as polynomials in n. We first review this result and then discuss computation of the characteristic polynomials of the discriminantal arrangements.

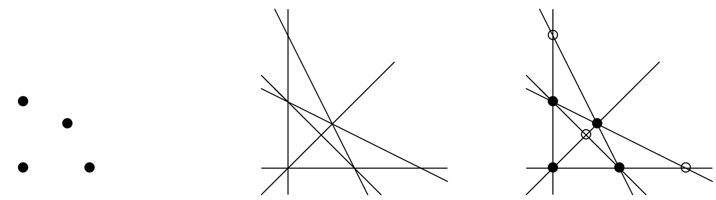
The organization of this article is the following. In Section 2, we set up our definitions and notation. In Section 3, we recall the definition of the Möbius function and the characteristic polynomial of an intersection lattice. These two sections are largely based on our previous paper.[6] Then we give some examples of computational results in Section 4. Fixing k, we have a family of hyperplane arrangement $\mathcal{A}_{d+k+1,d}$ parametrized by d (or equivalently $\mathcal{B}(n,k)$, $n = d+k+1$, parametrized by n). In Section 5, we discuss the structure of the intersection lattices of hyperplane arrangements of the family for $k = 0, 1, 2$. In Section 6, we present some computational results of the characteristic polynomials for $\mathcal{A}_{d+3,d}$.

2. Intersection lattice and order ideal

In this section, we define the hyperplane arrangement $\mathcal{A}_{n,d}$ generated by generic points and recall an interpretation of the intersection lattice of $\mathcal{A}_{n,d}$ in set-theoretical terminology. We also state some fundamental structure of order ideals of the intersection lattice of $\mathcal{A}_{n,d}$.

In our previous paper[6] we mainly considered fixed d and arbitrary $n > d$. For the case of fixed k and arbitrary d, it is more convenient to write n as $d+k+1$. In the following we either write $\mathcal{A}_{n,d}$ or $\mathcal{A}_{d+k+1,d}$ depending on the context.

Let $\mathcal{P} = \{ p_1, \ldots, p_n \}$ be a set of n points in $V = K^d$. We assume that $p_1, \ldots, p_n$ are generic in the sense of Athanasiadis.[1] For $X \subset \mathcal{P}$, define H_X to be the affine hull of X. Let $\mathcal{A} = \{ H_X \mid X \subset \mathcal{P}, |X| = d \}$ be the set of all affine hyperplanes defined by H_X for some $X \subset \mathcal{P}$, $|X| = d$. Since we consider generic points, the combinatorial properties of the arrangement $\mathcal{A}$ do not depend on the points. We define $\mathcal{A}_{n,d}$ to be the arrangement $\mathcal{A}$, and $L(\mathcal{A}_{n,d})$ to be the intersection lattice of $\mathcal{A}_{n,d}$, i.e., the set $\{ H_1 \cap \cdots \cap H_l \mid H_1, \ldots, H_l \in \mathcal{A}_{n,d} \}$, ordered by reverse inclusion. Contrary to the usual convention, here we consider that $\emptyset = \bigcap_{X:\,|X|=d} H_X$ belongs to $L(\mathcal{A}_{n,d})$, so that $L(\mathcal{A}_{n,d})$ is not only a poset but also a lattice.[7] In the usual convention, this corresponds to the coning $c\mathcal{A}_{n,d}$ of $\mathcal{A}_{n,d}$, except that we do not add a coordinate hyperplane. The reason for this unconventional definition is that $\emptyset \in L(\mathcal{A}_{n,d})$ plays an essential role for recursive description of $L(\mathcal{A}_{n,d})$.



(a) $\mathcal{P}$ (b) $\mathcal{A}$ (c) $L(\mathcal{A}_{4,2})$

Fig. 1. Example of $L(\mathcal{A}_{4,2})$

Example 2.1. Consider the two-dimensional vector space $\mathbb{R}^2$. Let $\mathcal{P}$ be the set of points in Figure 1(a). In this case, $\mathcal{A}$ is the set of the lines in Figure 1(b). The set of points, i.e., the elements of codimension two, in the intersection lattice consists of seven points in Figure 1(c). The four black points in Figure 1(c) are original points in $\mathcal{P}$. On the other hand, the three white points in Figure 1(c) are new points described as the intersection of two lines.

For the rest of this section we write n as $d+k+1$. We recall an interpretation of $L(\mathcal{A}_{d+k+1,d})$ in set-theoretical terminology.[1]

Definition 2.1. For a finite set X, we define

$$\operatorname{codim}^k(X) = |X| - k.$$

For distinct finite sets $S_1, \ldots, S_l$, we define

$$\begin{aligned} \rho^k(\{\ S_1, \ldots, S_l\ \}) &= \operatorname{codim}^k(S_1) + \cdots + \operatorname{codim}^k(S_l), \\ D^k(\{\ S_1, \ldots, S_l\ \}) &= \operatorname{codim}^k(\bigcup_i S_i) - \rho^k(\{\ S_1, \ldots, S_l\ \}). \end{aligned}$$

We define $\rho^k(\emptyset) = \rho^k(\{\ \ \}) = 0$.

Definition 2.2. For $d > 0$ and $k \geq 0$, we define $L^{d+k+1,d}$ by

$$L^{d+k+1,d} = \left\{ S \subset 2^{[d+k+1]} \;\middle|\; \begin{array}{c} D^k(S') > 0 \text{ for all } S' \subset S \text{ with } |S'| > 1. \\ k+1 \leq |S_i| \leq d+k+1 \text{ for all } S_i \in S. \end{array} \right\}$$

with ordering defined by

$$S < S' \iff \begin{cases} \rho^k(S) < \rho^k(S') & \text{and} \\ \forall S_i \in S,\ \exists S'_j \in S' \text{ such that } S_i \subset S'_j. \end{cases}$$

Remark 2.1. The poset $L^{d+k+1,d}$ is ranked by the rank function ρ^k. The minimum element of $L^{d+k+1,d}$ is $\emptyset$, and the maximum element of $L^{d+k+1,d}$ is $\{\,\{1,\dots,d+k+1\}\,\}$ of rank $d+1$. We use the symbol $\hat{0}$ (resp. $\hat{1}$) to denote the minimum (resp. maximum) element of $L^{d+k+1,d}$. If $d=0$, then the poset $L^{k+1,0}$ consists of two elements $\{\,\hat{0},\hat{1}\,\}$ for any $k \geq 0$.

Fix d and k. Then we have a bijection from $2^{[d+k+1]}$ to $2^{[d+k+1]}$ which maps a set S_l to its complement $T_l = [d+k+1] \setminus S_l$. Hence the definition of codim^k, ρ^k, D^k and $L^{d+k+1,d}$ can be rewritten with the complements as follows:

Definition 2.3. For a finite set X, we define

$$\operatorname{codim}_d(X) = d+1-|X|\,.$$

For distinct finite sets $T_1,\dots,T_l$, we define

$$\begin{aligned}\rho_d(\{\,T_1,\dots,T_l\,\}) &= \operatorname{codim}_d(T_1)+\cdots+\operatorname{codim}_d(T_l),\\ D_d(\{\,T_1,\dots,T_l\,\}) &= \operatorname{codim}_d(T_1\cap\cdots\cap T_l)-\rho_d(\{\,T_1,\dots,T_l\,\}).\end{aligned}$$

We define $\rho_d(\emptyset)=\rho_d(\{\ \})=0$.

This is the definition we employed in our previous paper.[6] In this article we need to use both Definition 2.2 and Definition 2.3. We distinguish them by superscripts and subscripts.

Remark 2.2. Let $T_1,\dots,T_l \subset [d+k+1]$ and $S_i = [d+k+1]\setminus T_i$. Then

$$\begin{aligned}\operatorname{codim}_d(T_i) &= d+1-|T_i|\\ &= d+1+|S_i|-(d+k+1)\\ &= |S_i|-k\\ &= \operatorname{codim}^k(S_i).\end{aligned}$$

We also have

$$\operatorname{codim}_d(\bigcap_i T_i) = \operatorname{codim}^k(\bigcup_i S_i),$$

since $\bigcap_i T_i = [d+k+1]\setminus\bigcup_i S_i$. Hence we have $\rho_d(T)=\rho^k(S)$ and $D_d(T) = D^k(S)$ for $T=\{\,T_1,\dots,T_l\,\}$ and $S=\{\,S_1,\dots,S_l\,\}$.

Definition 2.4. For $d>0$ and $k\geq 0$, we define $L_{d+k+1,d}$ by

$$L_{d+k+1,d} = \left\{ T \subset 2^{\{\,1,\dots,d+k+1\,\}} \;\middle|\; \begin{array}{c} D_d(T')>0 \text{ for all } T'\subset T \text{ with } |T'|>1. \\ 0\leq |T_i|\leq d \text{ for all } T_i\in T. \end{array} \right\}$$

with ordering $<$ defined by

$$T < T' \iff \begin{cases} \rho_d(T) < \rho_d(T') & \text{and} \\ \forall T_i \in T,\ \exists T'_j \in T' \text{ such that } T'_j \subset T_i. \end{cases}$$

The map $L^{d+k+1,d} \ni S \mapsto \{ [d+k+1] \setminus S_i \mid S_i \in S \} \in L_{d+k+1,d}$ is an order-preserving bijection. Hence $L^{d+k+1,d}$ is isomorphic to $L_{d+k+1,d}$ as posets.

We now state some fundamental structures of the intersection lattice $L_{d+k+1,d}$. For $T \in L_{d+k+1,d}$, we define $\mathcal{I}_{d+k+1,d}(T)$ to be the *order ideal* generated by T, that is, $\mathcal{I}_{d+k+1,d}(T) = \{ T' \in L_{d+k+1,d} \mid T' \leq T \}$. Then we have the following theorems.[6]

Theorem 2.1. *For* $\{ T_1 \} \in L_{d+k+1,d}$*,* $\mathcal{I}_{d+k+1,d}(\{T_1\})$ *is isomorphic to* $L_{d+k+1-|T_1|,d-|T_1|} = L_{k+\mathrm{codim}_d(T_1),\mathrm{codim}_d(T_1)-1}$ *as posets.*

Theorem 2.2. *Let* $T \in L_{d+k+1,d}$*,* $T \neq \hat{0}$*. Then the ideal* $\mathcal{I}_{d+k+1,d}(T)$ *is isomorphic to the direct product* $\prod_{T_i \in T} \mathcal{I}_{d+k+1,d}(\{T_i\})$ *as posets. They are also isomorphic to* $\prod_{T_i \in T} L_{k+\mathrm{codim}_d(T_i),\mathrm{codim}_d(T_i)-1}$*.*

3. Computation of the Möbius function and the characteristic polynomial

In this section we apply Theorem 2.2 to the Möbius function and the characteristic polynomial of the intersection lattice $L_{n,d}$.

3.1. *Möbius function and characteristic polynomial*

By Theorem 2.2, the combinatorial structure of $\mathcal{I}_{n,d}(T)$ depends only on $\mathrm{codim}(T_i)$. Therefore we introduce the notion of the type of T.

Let d be a nonnegative integer. We call a weakly-decreasing sequence $\delta = (\delta_1, \delta_2, \ldots)$ of nonnegative integers such that $\sum_i \delta_i = d$ a *partition* of d. We allow one or more zeros to occur at the end, or equivalently, $(\delta_1, \delta_2, \ldots, \delta_l) = (\delta_1, \delta_2, \ldots, \delta_l, 0)$ for any partition $\delta = (\delta_1, \delta_2, \ldots, \delta_l)$. We write $\delta \vdash d$ to say that δ is a partition of d. For example, $\{ \delta \vdash 3 \} = \{ (3), (2,1), (1,1,1) \}$, and $\{ \delta \vdash 0 \}$ is the set consisting of the unique partition of zero, which is denoted by $\varnothing$.

Definition 3.1. For $T = \{ T_1, \ldots, T_l \} \in L_{n,d}$ with $|T_1| \leq \cdots \leq |T_l|$, we call $\gamma_d(T) = (\mathrm{codim}_d(T_1), \ldots, \mathrm{codim}_d(T_l)) \vdash \rho_d(T)$ the *type* of T. We define $\Gamma(n,d) = \{ \gamma_d(T) \mid T \in L_{n,d} \}$.

Example 3.1. For each d, $\gamma_d(\hat{0}) = \varnothing$ and $\gamma_d(\hat{1}) = (d+1)$.

Remark 3.1. If d is fixed and n is sufficiently large, then

$$\Gamma(n,d) = \{\, \gamma \vdash i \mid i = 0, 1, \dots, d \,\} \cup \{\, (d+1) \,\}.$$

The *Möbius function* $\mu_{n,d}$ and the *characteristic polynomial* $\chi_{n,d}(t)$ of the poset $L_{n,d}$ is defined as usual:[7]

$$\mu_{n,d}(T,T) = 1, \qquad \sum_{T'' \colon T \le T'' \le T'} \mu_{n,d}(T,T'') = 0 \quad (T < T'),$$
$$\mu_{n,d}(T) = \mu_{n,d}(\hat{0}, T),$$
$$\chi_{n,d}(t) = \sum_{T \in L_{n,d}} \mu_{n,d}(T) t^{d+1-\rho_d(T)}.$$

The usual characteristic polynomial $\chi(\mathcal{A}_{n,d}, t)$ of the non-central arrangement $\mathcal{A}_{n,d}$ is given as

$$\chi(\mathcal{A}_{n,d}, t) = \sum_{T \in L_{n,d},\, T \neq \hat{1}} \mu_{n,d}(T) t^{d-\rho_d(T)} = \frac{\chi_{n,d}(t) - \mu_{n,d}(\hat{1})}{t}.$$

Since $\chi_{n,d}(1) = 0$,

$$\mu_{n,d}(\hat{1}) = -\sum_{T \in L_{n,d},\, T \neq \hat{1}} \mu_{n,d}(T). \tag{1}$$

Next we evaluate the value of Möbius function of $L_{n,d}$. The Möbius function for the direct product of posets is written as the product of the Möbius functions of posets. Hence Theorem 2.2 implies the following theorem.

Theorem 3.1. *For $T \in L_{n,d}$ and $T \neq \hat{0}$, we have*

$$\mu_{n,d}(T) = \prod_{T_i \in T} \mu_{d+k+1-|T_i|, d-|T_i|}(\hat{1}) = \prod_{T_i \in T} \mu_{k+\operatorname{codim}_d(T_i), \operatorname{codim}_d(T_i)-1}(\hat{1}).$$

Remark 3.2. Since $L_{k+1,0} = \{\hat{0}, \hat{1}\}$, we have $\mu_{k+1,0}(\hat{0}) = 1$ and $\mu_{k+1,0}(\hat{1}) = -1$. The equation $\mu_{k+1,0}(\hat{1}) = -1$ implies that $\mu_{k+\operatorname{codim}_d(T_i), \operatorname{codim}_d(T_i)-1}(\hat{1}) = -1$ if $\operatorname{codim}_d(T_i) = 1$.

The value of the Möbius function depends only on the type $\gamma_d(T)$ of T. Therefore, from now on we denote $\mu_{n,d}(T) = \mu_{n,d}(\gamma_d(T))$. Then we can rewrite Theorem 3.1 as follows.

Corollary 3.1. *For $\gamma = (\gamma_1, \dots, \gamma_l) \in \Gamma(n,d) \setminus \{\varnothing\}$,*

$$\mu_{n,d}(\gamma) = \prod_{i=1}^{l} \mu_{n-d+\gamma_i-1, \gamma_i-1}(\hat{1}).$$

Corollary 3.2. *If $d' \leq d$, then*

$$\mu_{d+k+1,d}(\gamma) = \mu_{d'+k+1,d'}(\gamma)$$

for $\gamma \in \Gamma(d'+k+1, d')$.

3.2. *Number of elements of the intersection lattice*

Define $L_{n,d}(\gamma)$ and $\lambda_{n,d}(\gamma)$ by

$$L_{n,d}(\gamma) = \{\, T \in L_{n,d} \,|\, \gamma_d(T) = \gamma \,\}, \quad \lambda_{n,d}(\gamma) = |L_{n,d}(\gamma)|$$

for $\gamma \in \Gamma(n,d)$. Then from the results in the previous sections we have

$$\chi_{n,d}(t) = \mu_{n,d}(\hat{1}) + \sum_{i=0}^{d} \sum_{\gamma \vdash i} \lambda_{n,d}(\gamma)\mu_{n,d}(\gamma)t^{d+1-i},$$

$$\mu_{n,d}(\hat{1}) = -\sum_{i=0}^{d} \sum_{\gamma \vdash i} \lambda_{n,d}(\gamma)\mu_{n,d}(\gamma).$$

Here we discuss how to compute the number $\lambda_{n,d}(\gamma)$.

From the viewpoint of computation, tuples are easier than sets. Therefore we define $\tilde{L}^l_{n,d}$ and $\tilde{L}_{n,d}(\gamma)$ by

$$\tilde{L}^l_{n,d} = \left\{ (T_1, \ldots, T_l) \in \left(2^{\{1,\ldots,n\}}\right)^l \,\middle|\, \begin{array}{c} \{\, T_1, \ldots, T_l \,\} \in L_{n,d}. \\ T_i \neq T_j (i \neq j). \end{array} \right\},$$

$$\tilde{L}_{n,d}(\gamma) = \left\{ (T_1, \ldots, T_l) \in \tilde{L}^l_{n,d} \,\middle|\, \operatorname{codim}_d(T_i) = \gamma_i \text{ for each } i. \right\}$$

for $\gamma = (\gamma_1, \ldots, \gamma_l) \in \Gamma(n,d)$. Obviously we have

$$\left|\tilde{L}_{n,d}(\gamma)\right| = \lambda_{n,d}(\gamma) \cdot \left|\operatorname{Stab}_{\mathfrak{S}_l}(\gamma)\right|,$$

where $\operatorname{Stab}_{\mathfrak{S}_l}(\gamma)$ stands for the stabilizer of the symmetric group $\mathfrak{S}_l$ fixing a partition $\gamma = (\gamma_1, \ldots, \gamma_l)$, i.e., $\{\, \sigma \in \mathfrak{S}_l \,|\, \gamma_i = \gamma_{\sigma(i)} \text{ for each } i \,\}$.

Each element $T = (T_1, \ldots, T_l) \in \tilde{L}^l_{n,d}$ is characterized by information on 'symbols' in each T_i. Conversely, information on indices i such that each 'symbol' belongs to T_i can characterize T. For $T = (T_1, \ldots, T_l) \in \tilde{L}_{n,d}(\gamma)$ and a subset $I \subset \{\, 1, \ldots, l \,\}$, define $\tau_T(I)$ by

$$\tau_T(I) = \{\, t \,|\, t \in T_i \iff i \in I \,\} = \bigcap_{i \in I} T_i \setminus \bigcup_{i \notin I} T_i.$$

For each $T = (T_1, \ldots, T_l) \in \tilde{L}^l_{n,d}$, τ_T is a map from $2^{\{1,\ldots,l\}}$ to $2^{\{1,\ldots,n\}}$. The map $\tilde{L}^l_{n,d} \ni T \mapsto \tau_T \in \left\{\, \pi \colon 2^{\{1,\ldots,l\}} \to 2^{\{1,\ldots,n\}} \,\right\}$ is an injection. Let $\tilde{N}(n,d;\gamma) = \left\{ \tau_T \,\middle|\, T \in \tilde{L}_{n,d}(\gamma) \right\}$. Then $\left|\tilde{N}(n,d;\gamma)\right| = \left|\tilde{L}^l_{n,d}(\gamma)\right|$.

For $\tau \in N(n,d;\gamma)$, define a map

$$\bar{\tau}_T \colon 2^{\{\,1,\dots,l\,\}} \ni I \mapsto |\tau_T(I)| \in \mathbb{N}$$

and let

$$N(n,d;\gamma) = \left\{ \bar{\tau}_T \colon 2^{\{\,1,\dots,l\,\}} \to \mathbb{N} \,\middle|\, \tau_T \in \tilde{N}(n,d;\gamma) \right\}.$$

Moreover, for $\nu \in N(n,d;\gamma)$, define $\tilde{N}(n,d;\gamma,\nu)$ by

$$\tilde{N}(n,d;\gamma,\nu) = \left\{ \tau_T \in \tilde{N}(n,d;\gamma) \,\middle|\, |\tau_T(I)| = \nu(I) \text{ for each } I \in 2^{\{\,1,\dots,l\,\}}. \right\}.$$

Then $\tilde{N}(n,d;\gamma) = \coprod_{\nu \in N(n,d;\gamma)} \tilde{N}(n,d;\gamma,\nu)$, which implies

$$\lambda_{n,d}(\gamma) \cdot |\mathrm{Stab}_{\mathfrak{S}_l}(\gamma)| = \left| \coprod_{\nu \in N(n,d;\gamma)} \tilde{N}(n,d;\gamma,\nu) \right|.$$

It is not difficult to see that

$$\left|\tilde{N}(n,d;\gamma,\nu)\right| = \binom{n}{\nu(I_1),\nu(I_2),\dots,\nu(I_{2^l})},$$

where $2^{\{\,1,\dots,l\,\}} = \{\, I_1,\dots,I_{2^l} \,\}$ and $\binom{n_1+\cdots+n_m}{n_1,n_2,\dots,n_m}$ stands for the multinomial coefficient $(n_1+\cdots+n_m)!/(n_1!\,n_2!\,\cdots n_m!)$. Therefore we have the following proposition.

Proposition 3.1. *For $\gamma \in \Gamma(n,d) \setminus \{\,\varnothing\,\}$, $\lambda_{n,d}(\gamma)$ is given by*

$$\lambda_{n,d}(\gamma) = \frac{1}{\prod_{s=1}^{d} m_s(\gamma)!} \sum_{\nu \in N(n,d;\gamma)} \binom{n}{\nu(I_1),\nu(I_2),\dots,\nu(I_{2^l})}, \tag{2}$$

where $2^{\{\,1,\dots,l\,\}} = \{\, I_1,\dots,I_{2^l} \,\}$ and $m_s(\gamma)$ stands for the multiplicity $|\{\, i \mid \gamma_i = s \,\}|$ of $s \in \{\,1,\dots,d\,\}$ in γ.

Remark 3.3. By interpreting the definition of $L_{n,d}$, we obtain

$$N(n,d;\gamma) = \left\{ \nu \colon 2^{\{\,1,\dots,l\,\}} \to \mathbb{N} \,\middle|\, \begin{array}{l} \displaystyle\sum_{I \in 2^{\{\,1,\dots,l\,\}}} \nu(I) = n. \\ \displaystyle\sum_{I\colon i \in I} \nu(I) = d+1-\gamma_i \qquad (\forall i). \\ \displaystyle\sum_{J\colon I \subset J} \nu(J) < d+1-\sum_{i \in I}\gamma_i \ (\forall I,\ |I| \geq 2). \end{array} \right\}.$$

Although (2) is an explicit formula for $\lambda_{n,d}(\gamma)$, it is actually difficult to perform the summation in (2) due to the large size of $N(n,d;\gamma)$. For example, for $d = 6$ and $\gamma = (1,1,1,1,1,1)$. $|N(n,6;(1,1,1,1,1,1))| = 109719496370$.

Let $\tilde{N}'(j,d;\gamma) = \left\{ \tau_T \in \tilde{N}(j,d;\gamma) \,\middle|\, |\tau_T(\emptyset)| = 0 \right\}$ and $c(j,d;\gamma) = \left|\tilde{N}'(j,d;\gamma)\right|$. Since $|\tau_T(\emptyset)|$ is the number of 'symbols' which do not appear in T, $\tilde{N}'(j,d;\gamma)$ can be identified with $\left\{ T \in \tilde{L}(j,d;\gamma) \,\middle|\, \bigcup_i T_i = \{ 1,\dots,j \} \right\}$. Let $M_{j,n}$ be the set of maps $\sigma\colon \{ 1,\dots,j \} \to \{ 1,\dots,n \}$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(j)$. Then there exists a natural bijection between $\tilde{N}'(j,d;\gamma) \times M_{j,n}$ and $\tilde{N}(n,d;\gamma)$. Hence

$$\left|\tilde{N}(n,d;\gamma)\right| = \binom{n}{j} \left|\tilde{N}'(j,d;\gamma)\right| = \binom{n}{j} c(j,d;\gamma).$$

Note that $c(j,d;\gamma)$ does not depend on n. This equation implies the following proposition.

Proposition 3.2. *For $\gamma \in \Gamma(n,d) \setminus \{ \varnothing \}$, $\lambda_{n,d}(\gamma)$ satisfies*

$$\lambda_{n,d}(\gamma) = \frac{1}{\prod_{s=1}^{d} m_s(\gamma)!} \sum_j c(j,d;\gamma) \binom{n}{j}.$$

We can also describe $c(j,d;\gamma)$ as the sum of multinomial coefficients:

$$c(j,d;\gamma) = \sum_{\nu' \in N(j,d;\gamma)\colon\, \nu'(\emptyset)=0} \binom{j}{\nu'(I_1),\dots,\nu'(I_{2^l-1})},$$

where $2^{\{ 1,\dots,l \}} \setminus \{ \emptyset \} = \{ I_1,\dots,I_{2^l-1} \}$.

4. Examples of computations

In this section, we illustrate the computation of the characteristic polynomial $\chi_{n,d}(t)$. We compute the values of $\lambda_{n,d}(\gamma)$ and $c(j,d;\gamma)$ for some $\gamma = (\gamma_1,\dots,\gamma_l)$ and d. If the length l of $\gamma = (\gamma_1,\dots,\gamma_l)$ is small, then the size of $2^{\{1,\dots,l\}}$ is small, so that it is easy to compute $\lambda_{n,d}(\gamma)$ and $c(j,d;\gamma)$.

4.1. *The case when $l = 0$*

In this case $\gamma = \varnothing$ and by definition

$$\lambda_{n,d}(\varnothing) = \left|\{ \hat{0} \}\right| = 1, \quad \mu_{n,d}(\varnothing) = \mu_{n,d}(\hat{0}) = 1.$$

Therefore $\mu_{n,0}(\hat{1}) = -1$ can also be verified as

$$\mu_{n,0}(\hat{1}) = -\sum_{i=0}^{0} \sum_{\gamma \vdash i} \lambda_{n,0}(\gamma)\mu_{n,0}(\gamma) = -\lambda_{n,0}(\varnothing)\mu_{n,0}(\varnothing) = -1.$$

This equation implies

$$\mu_{n,d}((\underbrace{1,\ldots,1}_{m})) = \prod_{i=0}^{m} \mu_{n-d,0}(\hat{1}) = (-1)^m.$$

We also have

$$\chi(\mathcal{A}_{n,0},t) = \sum_{i=0}^{0}\sum_{\gamma\vdash i} \lambda_{n,0}(\gamma)\mu_{n,0}(\gamma)t^{0-i} = \lambda_{n,0}(\varnothing)\mu_{n,0}(\varnothing) = 1.$$

4.2. *The case when* $l = 1$

In this case $\gamma = (\gamma_1)$. Then $c(j,d;(\gamma_1)) = \delta_{j,d+1-\gamma_1}$ and

$$\lambda_{n,d}((\gamma_1)) = \binom{n}{d+1-\gamma_1}.$$

For example, $\lambda_{n,1}((1)) = \binom{n}{1+1-1} = n$. Hence we have

$$\begin{aligned}
\mu_{n,1}(\hat{1}) &= -\sum_{i=0}^{1}\sum_{\gamma\vdash i} \lambda_{n,1}(\gamma)\mu_{n,1}(\gamma)\\
&= -\lambda_{n,1}(\varnothing)\mu_{n,1}(\varnothing) - \lambda_{n,1}((1))\mu_{n,1}((1))\\
&= -1\cdot 1 - n\cdot(-1)^1\\
&= n-1.
\end{aligned}$$

This implies

$$\mu_{n,d}((\underbrace{2,\ldots,2}_{m})) = \prod_{i=0}^{m} \mu_{n+1-d,1}(\hat{1}) = (n-d)^m.$$

Hence

$$\begin{aligned}
\mu_{n,d}((\underbrace{2,\ldots,2}_{m_2},\underbrace{1,\ldots,1}_{m_1})) &= \mu_{n,d}((\underbrace{2,\ldots,2}_{m_2}))\mu_{n,d}((\underbrace{1,\ldots,1}_{m_1}))\\
&= (-1)^{m_1}(n-d)^{m_2}.
\end{aligned}$$

We also have

$$\chi(\mathcal{A}_{n,1},t) = \lambda_{n,1}((1))\mu_{n,1}((1)) + \lambda_{n,1}(\varnothing)\mu_{n,1}(\varnothing)t = -n+t.$$

4.3. *The case when $l = 2$*

In this case $\gamma = (\gamma_1, \gamma_2)$. Let $\nu \in N(j, d; \gamma)$ with $\nu(\emptyset) = 0$ and $\nu(\{1, 2\}) = v$. Then $\nu(\{1\}) = d + 1 - \gamma_1 - v$ and $\nu(\{2\}) = d + 1 - \gamma_2 - v$. Since $\nu(\emptyset) = 0$, $j = \nu(\{1\}) + \nu(\{2\}) + \nu(\{1, 2\}) = 2d + 2 - \gamma_1 - \gamma_2 - v$. Hence $\nu(\{1, 2\}) = v = 2d+2-\gamma_1-\gamma_2-j$, $\nu(\{1\}) = d+1-\gamma_1-v = \gamma_2+j-d-1$, and $\nu(\{2\}) = d+1-\gamma_2-v = \gamma_1+j-d-1$. Since ν satisfies $\nu(\{1, 2\}) < d+1-\gamma_1-\gamma_2$, we also have $j > d+1$, which implies $\nu(\{1\}), \nu(\{2\}) \geq 0$. Since $\gamma(\{1, 2\})$ should be nonnegative, we have $2d+2-\gamma_1-\gamma_2 \geq j$. Hence

$$c(j, d; (\gamma_1, \gamma_2)) = \binom{j}{\gamma_1 + j - d - 1, \gamma_2 + j - d - 1, 2d + 2 - \gamma_1 - \gamma_2 - j}$$

if $2d+2-\gamma_1-\gamma_2 \geq j > d+1$. Let $j' = j - d - 1$. Then we can rewrite the equation as

$$\begin{aligned} c(d + 1 + j', d; (\gamma_1, \gamma_2)) &= \binom{d + 1 + j'}{\gamma_1 + j', \gamma_2 + j', d + 1 - \gamma_1 - \gamma_2 - j'} \\ &= \binom{d + 1 + j'}{\gamma_1 + j'}\binom{d + 1 - \gamma_1}{\gamma_2 + j'} \end{aligned}$$

if $d + 1 - \gamma_1 - \gamma_2 \geq j' > 0$. Hence

$$\lambda_{n,d}((\gamma_1, \gamma_2)) = \sum_{j'=1}^{d+1-\gamma_1-\gamma_2} \binom{d + 1 + j'}{\gamma_1 + j'}\binom{d + 1 - \gamma_1}{\gamma_2 + j'}\binom{n}{d + 1 + j'}$$

if $\gamma_1 \neq \gamma_2$. We can also obtain

$$\lambda_{n,d}((\gamma_1, \gamma_2)) = \frac{1}{2}\sum_{j'=1}^{d+1-\gamma_1-\gamma_2} \binom{d + 1 + j'}{\gamma_1 + j'}\binom{d + 1 - \gamma_1}{\gamma_2 + j'}\binom{n}{d + 1 + j'}$$

if $\gamma_1 = \gamma_2$. For example,

$$\lambda_{n,2}((1, 1)) = \frac{1}{2}\sum_{j'=1}^{2+1-2} \binom{3 + j'}{1 + j'}\binom{3 - 1}{1 + j'}\binom{n}{3 + j'} = 3\binom{n}{4}.$$

We also have

$$\begin{aligned} \mu_{n,2}(\hat{1}) =& - \lambda_{n,2}(\varnothing)\mu_{n,2}(\varnothing) - \lambda_{n,2}((1))\mu_{n,2}((1)) \\ & - \lambda_{n,2}((2))\mu_{n,2}((2)) - \lambda_{n,2}((1, 1))\mu_{n,2}((1, 1)) \\ =& - 1 \cdot 1 - \binom{n}{2} \cdot (-1)^1 - \binom{n}{1} \cdot (n - 2)^1 - 3\binom{n}{4} \cdot (-1)^2 \\ =& - 1 + \frac{-n(n - 3)}{2} - 3\binom{n}{4} = -1 + n - \binom{n}{2} - 3\binom{n}{4} \\ =& \frac{-(n - 2)(n - 1)(n^2 - 3n + 4)}{8}. \end{aligned}$$

This implies

$$
\begin{aligned}
\mu_{n,d}((\underbrace{3,\dots,3}_{m})) &= \prod_{i=0}^{m} \mu_{n+2-d,2}(\hat{1}) \\
&= \left(\frac{-(n-d)(n-d+1)(d^2-2dn+n^2+n+2-d)}{8} \right)^{m}.
\end{aligned}
$$

If $1 \leq \gamma_i \leq 3$ for all i, we can obtain an explicit formula for $\mu_{n,d}(\gamma)$. We also have

$$
\begin{aligned}
\chi(\mathcal{A}_{n,2}, t) =& \lambda_{n,2}((1,1))\mu_{n,2}((1,1)) + \lambda_{n,2}((2))\mu_{n,2}((2)) \\
&+ \lambda_{n,2}((1))\mu_{n,2}((1))t + \lambda_{n,2}(\varnothing)\mu_{n,2}(\varnothing)t^2 \\
=& 3\binom{n}{4} + (n-2)n - \binom{n}{2}t + t^2 \\
=& -n + 2\binom{n}{2} + 3\binom{n}{4} - \binom{n}{2}t + t^2.
\end{aligned}
$$

4.4. *The case when $l \geq 3$*

Finally we consider the case when $l \geq 3$. In this case, it seems hard to compute $c(j,d;\gamma)$ by hand, because $\tilde{N}'(j,d;\gamma)$ is complicated due to the size of $2^{\{1,\dots,l\}}$. We can, however, enumerate $c(j,d;\gamma)$ for a particular j, d and γ by a computer. For example, Program 4.1 is a program for the computer algebra system `Sage`[8] to define the function `getC` to compute the value of $c(j,d;\gamma)$.

Program 4.1.

```
def condition2(n,d,gamma,nu):
  l=len(gamma)
  for i in range(l):
    lhs=sum(nu[I] for I in nu.keys() if i in I)
    rhs=d+1-gamma[i]
    if not lhs == rhs:
      return False
  return True

def condition3(n,d,gamma,nu):
  for I in nu.keys():
    if len(I)>1:
      rhs=sum(nu[J] for J in nu.keys() if I.issubset(J))
      lhs=d+1-sum(gamma[i] for i in I)
      if not rhs < lhs:
        return False
  return True
```

```
def xLatticePoints(n,total):
  if n==1:
    yield [total]
    return
  for i in (0..total):
    for v in xLatticePoints(n-1,total-i):
      yield [i]+v

def xN(j,d,gamma):
  l=len(gamma)
  index=[frozenset(I) for I in powerset((0..l-1)) if I != []]
  for v in xLatticePoints(2^l-1,j):
    nu=dict(zip(index,v))
    if condition2(j,d,gamma,nu):
      if condition3(j,d,gamma,nu):
        yield nu

def getC(j,d,gamma):
  return sum(multinomial(nu.values()) for nu in xN(j,d,gamma))
```

Running the program, we can obtain the table of $c(j,d;\gamma)$. However computation is heavy if j, d or the length of γ is large. For example, the following shows the values of $c(3d,d;(1,1,1))$ and its processing time:

$$\begin{aligned}
c(9,3;(1,1,1)) &= 1680 && 0.44S\\
c(12,4;(1,1,1)) &= 34650 && 1.36S\\
c(15,5;(1,1,1)) &= 756756 && 3.50S\\
c(18,6;(1,1,1)) &= 17153136 && 7.87S\\
c(21,7;(1,1,1)) &= 399072960 && 15.96S\\
c(24,8;(1,1,1)) &= 9465511770 && 30.04S\\
c(27,9;(1,1,1)) &= 227873431500 && 53.10S.
\end{aligned}$$

The length of γ is more critical than j and d. For example, we enumerated $c(8,4;(1,1,1)) = 68040$ in 0.3 seconds and $c(8,4;(1,1,1,1)) = 2900520$ in 1 minute. Other computational data by the program are available from the authors.

5. Combinatorics of discriminantal arrangements

We now consider the arrangement $\mathcal{B}(d+k+1,k) = \mathcal{A}_{d+k+1,d}$, parametrized by d for fixed k. The combinatorics is already very hard for $k=2$. Therefore in this section we discuss the class of arrangements $\mathcal{A}_{d+k+1,d}$ for $k=0,1,2$.

We first show an inequality for general $k \geq 0$. Each element $S \in L^{d+k+1,d}$ satisfies the condition

$$D^k(S') > 0$$

for $S' \subset S$ with $|S'| > 1$. For $S_i, S_j \in S$ with $S_i \neq S_j$, the condition means

$$\begin{aligned} D^k(\{ S_i, S_j \}) &= |S_i \cup S_j| - k - (|S_i| - k) - (|S_j| - k) \\ &= |S_i \cup S_j| - |S_i| - |S_j| + k \\ &= -|S_i \cap S_j| + k > 0. \end{aligned}$$

Hence we have the inequality

$$k > |S_i \cap S_j| \geq 0. \qquad (3)$$

5.1. *The case when* $k = 0$

Here we consider $L^{d+1,d}$. We show that $L^{d+1,d}$ is isomorphic to the intersection lattice of the Boolean arrangement, i.e., the set of the hyperplanes $\{ (x_1, \ldots, x_{d+1}) \mid x_i = 0 \}$ for $i = 1, \ldots, d+1$, as posets.

For $k = 0$ the inequality (3) is

$$0 > |S_i \cap S_j| \geq 0,$$

which is always false. Hence $|S| \leq 1$ for $S \in L^{d+1,d}$. Therefore we have

$$L^{d+1,d} = \{ \{S_1\} \mid S_1 \subset [d+1], |S_1| > 0 \} \cup \{ \emptyset \}.$$

The type of any element in $L^{d+1,d}$ has at most one row, namely,

$$\Gamma(d+1, d) = \{ \varnothing, (1), (2), \ldots, (d+1) \}.$$

For $\{ S_1 \}$ and $\{ S'_1 \} \in L^{d+1,d}$, $\{ S_1 \} \leq \{ S'_1 \}$ if and only if $S_1 \subset S'_1$. Hence the map from $L^{d+1,d}$ to the Boolean lattice $2^{[d+1]}$ defined by

$$\begin{aligned} \{S_1\} &\mapsto S_1, \\ \emptyset &\mapsto \emptyset \end{aligned}$$

is an order-preserving bijection. Via this bijection, $L^{d+1,d}$ is isomorphic to the Boolean lattice $2^{[d+1]}$ as posets.

5.2. *The case when $k = 1$*

Here we consider $L^{d+2,d}$. It is known that $L^{d+2,d}$ is isomorphic to intersection poset of the braid arrangement (e.g. Section 1 of Falk,[3] Section 5.6 of Orlik and Terao[5]). We summarize the correspondence here.

In this case, (3) implies $|S_i \cap S_j| = 0$ for $S_i, S_j \in S \in L^{d+2,d}$ with $S_i \neq S_j$. Hence

$$L^{d+2,d} = \left\{ S \subset 2^{[d+2]} \;\middle|\; \begin{array}{c} S_i \cap S_j = \emptyset \text{ for } S_i, S_j \in S \text{ with } S_i \neq S_j. \\ 1 < |S_i| < d+2 \text{ for } S_i \in S. \end{array} \right\}.$$

Since $S_i \cap S_j = \emptyset$ for $S_i \neq S_j \in S$, we have $\sum_{S_i \in S} |S_i| = \left|\bigcup_{S_i \in S} S_i\right|$ for $S \in L^{d+2,d}$. Hence, for $S \in L^{d+2,d}$ of type $(\gamma_1, \dots, \gamma_l)$, we have

$$\sum_i \gamma_i = \sum_{S_i \in S} \operatorname{codim}^1(S_i) = \sum_{S_i \in S} (|S_i| - 1) = -l + \left| \bigcup_{S_i \in S} S_i \right|.$$

Since $\bigcup_{S_i \in S} S_i \subset [d+2]$, we obtain

$$\sum_i \gamma_i = -l + \left| \bigcup_{S_i \in S} S_i \right| \leq -l + d + 2.$$

Therefore, if $\sum_{i=1}^{l} \gamma_i > d + 2 - l$, then there does not exist an element of $L^{d+2,d}$ of type $(\gamma_1, \dots, \gamma_l)$. Moreover, if $\sum_i \gamma_i \leq d + 2 - l$, then the number of elements of $L^{d+2,d}$ whose type is $(\gamma_1, \dots, \gamma_l)$ is

$$\frac{1}{\prod_s m_s(\gamma)!} \binom{d+2}{\gamma_1, \dots, \gamma_l, d + 2 - \sum_i \gamma_i}.$$

Now we discuss the structure of the poset $L^{d+2,d}$. Let $H_{j,j'} = \{ (x_1, \dots, x_{d+2}) \in \mathbb{R}^{d+2} \mid x_j = x_{j'} \}$. The arrangement $\{ H_{j,j'} \mid j \neq j' \}$ is called the braid arrangement $\mathcal{A}_{d+1}$. For $S \in L^{d+2,d}$, define

$$\varphi(S) = \{ (x_1, \dots, x_{d+2}) \in \mathbb{R}^{d+2} \mid x_j = x_{j'} \text{ for all } j, j' \in S_i \in S \}.$$

If $S = \emptyset$, then $\varphi(S) = \mathbb{R}^{d+2}$. If $S = \{ S_1 \}$ and $|S_1| \geq 2$, then

$$\begin{aligned} \varphi(S) &= \{ (x_1, \dots, x_{d+2}) \in \mathbb{R}^{d+2} \mid x_j = x_{j'} \ (j, j' \in S_1) \} \\ &= \bigcap_{j, j' \in S_1} H_{j,j'} \in \mathcal{A}_{d+1}. \end{aligned}$$

If $S = \{ S_1, \dots, S_l \}$, then

$$\varphi(S) = \bigcap_{S_i \in S} \varphi(\{ S_i \}) \in \mathcal{A}_{d+1}$$

since $S_i \cap S_{i'} = \emptyset$ for $S_i \neq S_{i'} \in S$. Hence φ is a map from $L^{d+2,d}$ to the intersection lattice of the braid arrangement $\mathcal{A}_{d+1}$. Conversely, for $H_{i_1,j_1} \cap \cdots \cap H_{i_m,j_m}$, there exist a set partition $S = \{ S_1, \ldots, S_l \}$ of $[d+2]$ such that

$$H_{i_1,j_1} \cap \cdots \cap H_{i_m,j_m} = \left\{ (x_1, \ldots, x_{d+2}) \in \mathbb{R}^{d+2} \,\middle|\, x_j = x_{j'} \ (j, j' \in S_i \in S) \right\}.$$

Let $\psi(H_{i_1,j_1} \cap \cdots \cap H_{i_m,j_m}) = \{ S_i \in S \mid |S_i| \geq 2 \}$. Then, since $S_i \cap S_{i'} = \emptyset$ for $S_i \neq S_{i'} \in S$ and $2 \leq |S_i| \leq d+2$ for $S_i \in \psi(H_{i_1,j_1} \cap \cdots \cap H_{i_m,j_m})$, $\psi(H_{i_1,j_1} \cap \cdots \cap H_{i_m,j_m})$ is an element in $L^{d+2,d}$. Since the map ψ provides the inverse map of φ, the map φ is a bijection. For $S_1 \subset S_1' \subset [d+2]$ with $|S_1| > 1$, we have

$$\varphi(\{ S_1 \}) = \bigcap_{j,j' \in S_1} H_{j,j'} \supset \bigcap_{j,j' \in S_1} H_{j,j'} \cap \bigcap_{j,j' \in S_1' \setminus S_1} H_{j,j'} = \varphi(\{ S_1' \}).$$

Hence, for $S = \{ S_1, \ldots, S_l \} < S' = \{ S_1', \ldots, S_{l'}' \} \in L^{d+2,d}$, we have

$$\varphi(S) = \bigcap_{S_i \in S} \varphi(\{ S_i \}) \supset \bigcap_{S_i' \in S'} \varphi(\{ S_i' \}) = \varphi(S'),$$

and φ is an order-preserving bijection. Therefore $L^{d+2,d}$ is isomorphic to the intersection lattice of the braid arrangement $\mathcal{A}_{d+1}$ as posets.

5.3. *The case when $k = 2$.*

Here we consider $L^{d+3,d}$. Although $L^{d+3,d}$ is an analogue of the Boolean arrangement and the braid arrangement, the structure of $L^{d+3,d}$ is complicated and it is difficult to compute the characteristic polynomials for large d. For $S = \{ S_1, \ldots, S_l \} \in L^{d+k+1,d}$, the set

$$\Delta(S) = \left\{ I \subset [l] \,\middle|\, \bigcap_{i \in I} S_i \neq \emptyset \right\}$$

is a simplicial complex, or equivalently, $\Delta(S)$ satisfies

$$F \in \Delta(S),\ F' \subset F \implies F' \in \Delta(S).$$

For $k = 2$ the inequality (3) implies $|S_i \cap S_j| \leq 1$. Hence any two facets of the simplicial complex $\Delta(S)$ share at most one vertex. Moreover each face in the simplicial complex with positive dimension corresponds to only one symbol in $[d+3]$. Here we discuss a way to enumerate $L^{d+3,d}$ from the information of the simplicial complex $\Delta(S)$.

First we define notation for simplicial complexes. Let $\mathcal{V}_0(l)$ be the set of simplicial complexes $\Delta \subset 2^{[l]}$ such that $\{ i \} \in \Delta$ for all i. For $\Delta \in \mathcal{V}_0(l)$,

Fig. 2. Example of $\mathcal{F}(\Delta)$ for $\Delta \in \mathcal{V}_2(3)$

we define $\mathcal{F}(\Delta)$ to be the set of facets with positive dimension. Let $\mathcal{V}_1(l)$ be the set of simplicial complexes $\Delta \subset 2^{[l]}$ such that the dimension of the face $F \cap F'$ is at most one for any distinct facets $F, F' \in \mathcal{F}(\Delta)$. For $I \subset [l]$ and $\Delta \in \mathcal{V}_1(l)$, define $\Delta|_I = \{ F \cap I \mid F \in \Delta \}$. Let

$$\mathcal{V}_2(l) = \left\{ \Delta \in \mathcal{V}_1(l) \,\middle|\, 2(|I| - 1) > \sum_{F \in \mathcal{F}(\Delta|_I)} \dim F \quad (\forall I \subset [l], |I| > 1). \right\}.$$

For $\Delta \in \mathcal{V}_2(l)$, we define $\alpha(\Delta) = (\alpha_1, \dots, \alpha_l)$ with $\alpha_i = |\{ F \in \mathcal{F}(\Delta) \mid i \in F \}|$.

Example 5.1. For $\Delta \in \mathcal{V}_2(2)$, $\mathcal{F}(\Delta) = \{ \{1, 2\} \}$ or $\emptyset$.

Example 5.2. For $\Delta \in \mathcal{V}_2(3)$, $\mathcal{F}(\Delta)$ is a subset of one of the following:

$$\{ \{1, 2, 3\} \}, \{ \{1, 2\}, \{2, 3\}, \{1, 3\} \}.$$

Figure 2(a) and 2(b) show $\{ \{1, 2, 3\} \}$ and $\{ \{1, 2\}, \{2, 3\}, \{1, 3\} \}$ as hypergraphs, respectively, where a hyperedge e with $|e| = 2$ is written as an ordinary edge.

Example 5.3. For $\Delta \in \mathcal{V}_2(4)$, $\mathcal{F}(\Delta)$ is a subset of one of the following:

$$\begin{aligned} &\{ \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\} \} && \text{(Figure 3(a))}, \\ &\{ \{\sigma_1, \sigma_2, \sigma_3\}, \{\sigma_1, \sigma_4\}, \{\sigma_2, \sigma_4\}, \{\sigma_3, \sigma_4\} \} && \text{(Figure 3(b))}, \\ &\{ \{\sigma_1, \sigma_2\}, \{\sigma_2, \sigma_3\}, \{\sigma_1, \sigma_3\}, \{\sigma_4, \sigma_2\}, \{\sigma_2, \sigma_3\}, \{\sigma_4, \sigma_3\} \} && \text{(Figure 3(c))}, \end{aligned}$$

where $\sigma \in \mathfrak{S}_4$.

Example 5.4. For $\Delta \in \mathcal{V}_2(5)$, the hyper graph $\mathcal{F}(\Delta)$ is a subset of one of (a)–(i) in Figure 4.

Remark 5.1. It follows by definition that

$$\mathcal{V}_2(l - 1) = \left\{ \Delta|_{[l-1]} \,\middle|\, \Delta \in \mathcal{V}_2(l) \right\}.$$

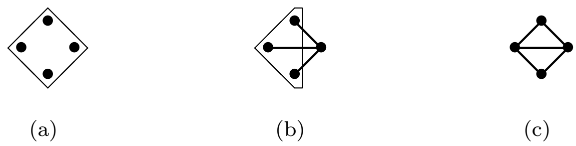


Fig. 3. Example of $\mathcal{F}(\Delta)$ for $\Delta \in \mathcal{V}_2(4)$

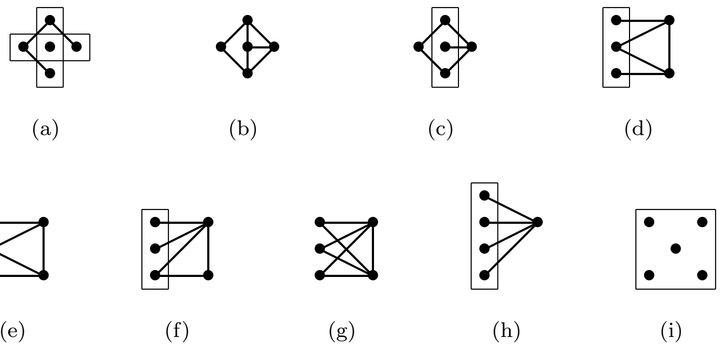


Fig. 4. Example of $\mathcal{F}(\Delta)$ for $\Delta \in \mathcal{V}_2(5)$

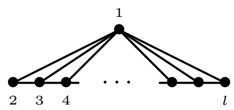


Fig. 5. Δ in Remark 5.2

Remark 5.2. Consider a graph Δ in $\mathcal{V}_2(l)$, namely, $\Delta \in \mathcal{V}_2(l)$ such that $\dim F = 1$ for $F \in \mathcal{F}(\Delta)$. Since $\sum_{F \in \mathcal{F}(\Delta)} \dim F = |\mathcal{F}(\Delta)|$, the number of edges in Δ is less than or equal to $2l - 3$. Let Δ be the graph with edges

$$\begin{cases} \{ 1, j \} & (j = 2, \ldots, l) \\ \{ j, j+1 \} & (j = 2, \ldots, l-1). \end{cases}$$

See Figure 5. Then $|\Delta| = 2l - 3$. If a subset $I \subset [l]$ does not contain 1, then $\mathcal{F}(\Delta|_I) = \{ \{i, i+1\} \,|\, i, i+1 \in I \}$, which implies $|\mathcal{F}(\Delta|_I)| \leq |I| - 1$. If a subset $I \subset [l]$ contains 1, then $\mathcal{F}(\Delta|_I) = \{ \{i, i+1\} \,|\, i, i+1 \in I \setminus \{ 1 \} \} \cup \{ \{i, 1\} \,|\, i \in I \setminus \{ 1 \} \}$, which implies $|\mathcal{F}(\Delta|_I)| \leq (|I| - 2) + (|I| - 1) = 2\,|I| - 3$. Hence $\Delta \in \mathcal{V}_2(l)$. Therefore the inequality $|\mathcal{F}(\Delta)| \leq 2l - 3$ for $\Delta \in \mathcal{V}_2(l)$ is the best possible.

Remark 5.3. Let $\Delta \in \mathcal{V}_2(l)$ and $F = \{ v_1, \ldots, v_m \} \in \mathcal{F}(\Delta)$. Consider the simplicial complex Δ' such that

$$\mathcal{F}(\Delta') = (\mathcal{F}(\Delta) \setminus F) \cup \{ \{v_i, v_{i+1}\} \mid i = 1, \ldots, m \} .$$

Since $\dim F|_I \geq \sum_{v_i, v_{i+1} \in I} \dim(\{v_i, v_{i+1}\})$, the simplicial complex Δ' is in $\mathcal{V}_2(l)$. It also follows that $|\mathcal{F}(\Delta)| \leq |\mathcal{F}(\Delta')|$. Hence there exists a graph $\Delta \in \mathcal{V}_2(l)$ such that $|\mathcal{F}(\Delta)| = \max\{ |\mathcal{F}(\Delta')| \mid \Delta' \in \mathcal{V}_2(l) \}$. Therefore we have

$$\max\{ |\mathcal{F}(\Delta')| \mid \Delta' \in \mathcal{V}_2(l) \} = 2l - 3.$$

Now we continue the investigation of $L^{d+3,d}$. For the computational reason, we consider the tuples of subsets in $[d+3]$. Let

$$\begin{aligned} \tilde{L}^{d+3,d} &= \left\{ (S_1, \ldots, S_l) \,\middle|\, \{ S_1, \ldots, S_l \} \in L^{d+3,d} \right\}, \\ \tilde{L}^{d+3,d;\gamma} &= \left\{ (S_1, \ldots, S_l) \in \tilde{L}^{d+3,d} \,\middle|\, |S_i| = \gamma_i + 2 \right\}. \end{aligned}$$

For $S = (S_1, \ldots, S_l) \in \tilde{L}^{d+3,d}$ and $t \in [d+3]$, we define

$$X(S, t) = \{ i \mid t \in S_i \} .$$

For each facet F of $\Delta(S)$, there exists a symbol $t \in [d+3]$ such that $F = X(S, t)$. The inequality (3) implies

$$2 > |S_i \cap S_j| \geq 0$$

for $S_i \neq S_j \in S \in L^{d+3,d}$. Hence $|S_i \cap S_j|$ is at most one, which implies $|X(S,t) \cap X(S,t')| \leq 1$ for $t \neq t'$. In other words, the dimension of the face $X(S,t) \cap X(S,t')$ of $\Delta(S)$ is at most one. Hence we have $\Delta(S) \in \mathcal{V}_1(l)$. Moreover, there exists one-to-one correspondence between $\mathcal{F}(\Delta(S))$ and $\{ t \in [d+3] \mid |X(S,t)| > 1 \}$.

The number of symbols used in $S \in \tilde{L}^{d+3,d}$ is $|\bigcup_i S_i|$. On the other hand, since the number of times the symbol t appears in S is $|X(S,t)|$, the number of symbols used in S is $\sum_i |S_i| - \sum_t (|X(S,t)| - 1)$. Hence we have $|\bigcup_i S_i| = \sum_i |S_i| - \sum_t (|X(S,t)| - 1)$. Therefore for any $k \geq 0$

$$\begin{aligned} D^k(\{ S_1, \ldots, S_l \}) &= \left| \bigcup_{i \in [l]} S_i \right| - k - \sum_{i \in [l]} (|S_i| - k) \\ &= \sum_{i \in [l]} |S_i| - \sum_{t \in [d+3]} (|X(S,t)| - 1) - k - \sum_{i \in [l]} (|S_i| - k) \\ &= k(l-1) - \sum_t (|X(S,t)| - 1). \end{aligned}$$

Hence the inequality $D^k(\{ S_1, \ldots, S_l \}) > 0$ for $k = 2$ is rewritten as

$$2(l-1) > \sum_{t \in [d+3]} \dim X(S,t),$$

where $\dim X(S,t)$ stands for the dimension $|X(S,t)| - 1$ of $X(S,t)$ as a simplex. Since there exists a one-to-one correspondence between $\mathcal{F}(\Delta(S))$ and $\{ t \in [d+3] \,|\, \dim X(S,t) > 0 \}$, the inequality means

$$2(l-1) > \sum_{F \in \mathcal{F}(\Delta(S))} \dim F.$$

Similarly we have inequalities

$$2(|I|-1) > \sum_{F \in \mathcal{F}(\Delta(S)|_I)} \dim F$$

for $I \subset [l]$ with $|I| > 1$. Hence the simplicial complex $\Delta(S)$ is in $\mathcal{V}_2(l)$.

Conversely, let $\Delta \in \mathcal{V}_2(l)$ with $\alpha(\Delta) = (\alpha_1, \ldots, \alpha_l)$ and $S_1, \ldots, S_l \subset [d+3]$. Assume that

$$\begin{aligned} &S_i \cap S_j = \emptyset \quad (i \neq j), \\ &|\mathcal{F}(\Delta)| + \sum_{i=1}^{l} |S_i| \leq d+3, \\ &3 \leq \alpha_i + |S_i| \leq d+3. \end{aligned}$$

Fix an injection τ from $\mathcal{F}(\Delta)$ to $[d+3] \setminus \bigcup_i S_i$, and let $S_i^\tau = \{ \tau(F) \,|\, i \in F \in \mathcal{F}(\Delta) \} \cup S_i$. From these assumptions we have $3 \leq |S_i^\tau| \leq d+3$. Moreover, since $\Delta \in \mathcal{V}_2(l)$, we have $D^2(\left\{ S^\tau_{i_1}, \ldots, S^\tau_{i_{l'}} \right\}) > 0$. Hence $(S_1, \ldots, S_l) \in \tilde{L}^{d+3,d}$. By this construction, we can enumerate the number of elements in $\tilde{L}^{d+3,d}$ of type γ as follows.

Proposition 5.1. *For $\gamma = (\gamma_1, \ldots, \gamma_l)$,*

$$\left|\tilde{L}^{d+3,d;\gamma}\right| = \sum_{\alpha'} \sum_{\Delta} \binom{d+3}{\alpha'_1, \ldots, \alpha'_l, d+3-\sum_i \alpha'_i} \frac{(d+3-\sum_i \alpha'_i)!}{(d+3-\sum_i \alpha'_i - |\mathcal{F}(\Delta)|)!}, \tag{4}$$

where the first sum is over $\alpha' = (\alpha'_1, \ldots, \alpha'_l)$ such that $0 \leq \alpha'_i \leq \gamma_i + 2$ for all i, and the second sum is over $\Delta \in \mathcal{V}_3(l)$ such that $\alpha(\Delta) = (\gamma_1 + 2 - \alpha'_1, \ldots, \gamma_l + 2 - \alpha'_l)$ and $|\mathcal{F}(\Delta)| \leq d+3-\sum_i \alpha'_i$.

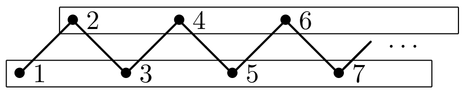


Fig. 6. Facets of Δ in Remark 5.4

Remark 5.4. In the case when $k \leq 1$, there exists $(\gamma_1, \ldots, \gamma_l)$ which does not appear as the type of any element of $L^{d+k+1,d}$. In the case when $k = 2$, all $(\gamma_1, \ldots, \gamma_l) \vdash d' \leq d$ appear as the type of some element of $L^{d+k+1,d}$. This can be seen as follows. Consider the simplicial complex Δ with facets

$$\{\, i \in [l] \,|\, i \text{ is odd} \,\}, \quad \{\, i \in [l] \,|\, i \text{ is even} \,\}, \quad \{\, j, j+1 \,\} \text{ for } j = 1, \ldots, l-1.$$

See Figure 6. Then $\sum_{F \in \mathcal{F}(\Delta)} \dim F = 2l - 3 < 2(l-1)$ since $\dim \{\, i \in [l] \,|\, i \text{ is odd} \,\} + \dim \{\, i \in [l] \,|\, i \text{ is even} \,\} = l - 2$. Let $I \subset [l]$ with $|I| \geq 2$. Then facets of $\Delta|_I$ with positive dimension are

$$\{\, i \in I \,|\, i \text{ is odd} \,\}, \quad \{\, i \in I \,|\, i \text{ is even} \,\}, \quad \{\, j, j+1 \,\} \text{ for } j, j+1 \in I.$$

Hence $\sum_{F \in \mathcal{F}(\Delta|_I)} \dim F \leq |I| - 2 + |I| - 1 = 2\,|I| - 3$. Therefore Δ is in $\mathcal{V}_2(l)$. It follows from the direct calculation that $\alpha_1 = \alpha_l = 2$ and $\alpha_2 = \cdots = \alpha_{l-1} = 3$ for $\alpha(\Delta) = (\alpha_1, \ldots, \alpha_l)$. Since $|\mathcal{F}(\Delta)| = 2 + l - 1 = l + 1$ and $\sum_i \alpha_i = 3l - 2$, we have

$$|\mathcal{F}(\Delta)| + \sum_{i=1}^{l} (\gamma_i + 2 - \alpha_i) = 3 + \sum_{i=1}^{l} \gamma_i$$

for $\gamma = (\gamma_1, \ldots, \gamma_l)$. If $\sum_i \gamma_i \geq d$, then we have $|\mathcal{F}(\Delta)| + \sum_{i=1}^{l} (\gamma_i + 2 - \alpha_i) \leq d + 3$. Since $\alpha_i \leq 3$ for each i, we can construct an element in $L^{d+3,d}$ of type $(\gamma_1, \ldots, \gamma_l) \vdash d' \leq d$.

6. Computation of $\mathcal{A}_{d+3,d}$

Here we give some computational data of $\Delta \in \mathcal{V}_2(l)$ and $\mathcal{A}_{d+3,d}$. More comprehensive computational data are available from the authors.

The summand in the equation (4) is written as

$$\begin{aligned}
&\binom{d+3}{\alpha'_1, \ldots, \alpha'_l, d + 3 - \sum_i \alpha'_i} \frac{(d + 3 - \sum_i \alpha'_i)!}{(d + 3 - \sum_i \alpha'_i - |\mathcal{F}(\Delta)|)!} \\
=&\binom{d+3}{\alpha'_1, \ldots, \alpha'_l, |\mathcal{F}(\Delta)|\,, d + 3 - \sum_i \alpha'_i - |\mathcal{F}(\Delta)|} |\mathcal{F}(\Delta)|! \\
=&\binom{d+3}{|\mathcal{F}(\Delta)| + \sum_i \alpha'_i} \binom{|\mathcal{F}(\Delta)| + \sum_i \alpha'_i}{\alpha'_1, \ldots, \alpha'_l, |\mathcal{F}(\Delta)|} |\mathcal{F}(\Delta)|!.
\end{aligned}$$

For $(\alpha'_1, \dots, \alpha'_l)$, $(\alpha_1, \dots, \alpha_l)$ and t, define

$$P(\alpha'_1, \dots, \alpha'_l; t) = \binom{d+3}{t + \sum_i \alpha'_i} \binom{t + \sum_i \alpha'_i}{\alpha'_1, \dots, \alpha'_l, t} t!,$$

$$\mathcal{V}_2((\alpha_1, \dots, \alpha_l), t) = \left\{ \Delta \in \mathcal{V}_2(l) \;\middle|\; \begin{array}{c} \alpha(\Delta) = (\alpha_1, \dots, \alpha_l) \\ |\mathcal{F}(\Delta)| = t \end{array} \right\}.$$

Since $\alpha_i = \gamma_i + 2 - \alpha'_i$ holds in the equation (4), we rewrite the equation as

$$\begin{aligned} \left|\tilde{L}^{d+3,d;\gamma}\right| &= \sum_{\alpha} \sum_{t} \sum_{\Delta \in \mathcal{V}_2(\alpha,t)} P(\gamma_1 + 2 - \alpha_1, \dots, \gamma_l + 2 - \alpha_l; |\mathcal{F}(\Delta)|) \\ &= \sum_{\alpha} \sum_{t} |\mathcal{V}_2(\alpha, t)|\, P(\gamma_1 + 2 - \alpha_1, \dots, \gamma_l + 2 - \alpha_l; t), \end{aligned}$$

where the sum is over all $\alpha = (\alpha_1, \dots, \alpha_l)$ such that $0 \le \alpha_i \le \gamma_i + 2$ for all i. Since $\left|L^{d+3,d;\gamma}\right| = \left|\tilde{L}^{d+3,d;\gamma}\right| / \prod_s m_s(\gamma)!$, by computing $|\mathcal{V}_2(\alpha, t)|$, we obtain Table 1 of $\lambda_{d+3,d}(\gamma) = \left|L^{d+3,d;\gamma}\right|$. The table of $|\mathcal{V}_2(\alpha, t)|$ is available from the authors. From Table 1 we obtain the value of Möbius function

$$\begin{aligned} &\mu_{1+3,1}(\hat{1}) = 3,\ \mu_{2+3,2}(\hat{1}) = -21,\ \mu_{3+3,3}(\hat{1}) = 300,\ \mu_{4+3,4}(\hat{1}) = -7890, \\ &\mu_{5+3,5}(\hat{1}) = 349650,\ \mu_{6+3,6}(\hat{1}) = -24188850,\ \mu_{7+3,7}(\hat{1}) = 2449878480, \end{aligned}$$

and characteristic polynomials

$$\begin{aligned} \chi_{1+3,1} =& t^2 - 4t + 3, \\ \chi_{2+3,2} =& t^3 - 10t^2 + 30t - 21, \\ \chi_{3+3,3} =& t^4 - 20t^3 + 145t^2 - 426t + 300, \\ \chi_{4+3,4} =& t^5 - 35t^4 + 490t^3 - 3381t^2 + 10815t - 7890, \\ \chi_{5+3,5} =& t^6 - 56t^5 + 1330t^4 - 17136t^3 + 124971t^2 - 458760t + 349650, \\ \chi_{6+3,6} =& t^7 - 84t^6 + 3108t^5 - 65926t^4 + 868014t^3 - 7039908t^2 \\ & + 30423645t - 24188850, \\ \chi_{7+3,7} =& t^8 - 120t^7 + 6510t^6 - 209692t^5 + 4414095t^4 - 62509140t^3 \\ & + 578334366t^2 - 2969914500t + 2449878. \end{aligned}$$

We can also compute the number of elements in $\mathcal{V}_2(l)$:

$$\begin{aligned} &|\mathcal{V}_2(1)| = 1,\ |\mathcal{V}_2(2)| = 2,\ |\mathcal{V}_2(3)| = 9,\ |\mathcal{V}_2(4)| = 96, \\ &|\mathcal{V}_2(5)| = 2419,\ |\mathcal{V}_2(6)| = 133787,\ |\mathcal{V}_2(7)| = 14377347. \end{aligned}$$

Table 1. $\lambda_{d+3,d}(\gamma)$

γ	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
(1)	4	10	20	35	56	84	120
(2)	1	5	15	35	70	126	210
(1^2)	0	15	100	385	1120	2730	5880
(3)	0	1	6	21	56	126	252
(21)	0	0	60	455	1960	6300	16800
(1^3)	0	0	120	1575	10080	44380	154000
(4)	0	0	1	7	28	84	210
(31)	0	0	0	105	896	4284	15120
(2^2)	0	0	0	70	595	2835	9975
(21^2)	0	0	0	1260	16800	112770	525000
(1^4)	0	0	0	2100	42000	389025	2368800
(5)	0	0	0	1	8	36	120
(41)	0	0	0	0	168	1596	8400
(32)	0	0	0	0	280	2646	13860
(31^2)	0	0	0	0	3360	47250	337680
(2^21)	0	0	0	0	5040	68040	474600
(21^3)	0	0	0	0	47040	892080	8240400
(1^5)	0	0	0	0	70560	1829520	22085280
(6)	0	0	0	0	1	9	45
(51)	0	0	0	0	0	252	2640
(42)	0	0	0	0	0	504	5250
(41^2)	0	0	0	0	0	7560	113400
(3^2)	0	0	0	0	0	315	3276
(321)	0	0	0	0	0	30240	428400
(31^3)	0	0	0	0	0	181440	3477600
(2^3)	0	0	0	0	0	7560	103600
(2^21^2)	0	0	0	0	0	430920	7862400
(21^4)	0	0	0	0	0	2872800	68443200
(1^6)	0	0	0	0	0	4011840	122673600
(7)	0	0	0	0	0	1	10
(61)	0	0	0	0	0	0	360
(52)	0	0	0	0	0	0	840
(51^2)	0	0	0	0	0	0	15120
(43)	0	0	0	0	0	0	1260
(421)	0	0	0	0	0	0	75600
(41^3)	0	0	0	0	0	0	554400
(3^21)	0	0	0	0	0	0	50400
(32^2)	0	0	0	0	0	0	75600
(321^2)	0	0	0	0	0	0	3628800
(31^4)	0	0	0	0	0	0	15120000
(2^31)	0	0	0	0	0	0	1890000
(2^21^3)	0	0	0	0	0	0	49140000
(21^5)	0	0	0	0	0	0	264751200
(1^7)	0	0	0	0	0	0	350481600

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