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# Constructing a Nonconvex Discrete Function with Positive Semidefinite Discrete Hessian Matrix

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## Abstract

This paper gives an instance of a function defined on integers such that (i) the discrete Hessian matrix is positive semidefinite at each point, but (ii) it cannot be extended to a convex function in continuous variables. The construction is based on a semidefinite programming technique. This example, together with our previous examples, shows that the positive semidefiniteness of the discrete Hessian matrix is independent of convex extensibility of discrete functions.

**Key words:** discrete optimization, discrete convex function, Hessian matrix, semidefinite programming

# 1 Introduction

For functions defined on integer lattice points, discrete versions of the Hessian matrix have been considered in various contexts. In discrete convex analysis [1, 2, 4], for example, certain combinatorial properties of the discrete Hessian matrices are known to characterize  $M^{\natural}$ -convex and  $L^{\natural}$ -convex functions, which can be extended to convex functions in real variables.

For a function  $f : \mathbb{Z}^n \rightarrow \mathbb{R}$  in discrete variables the discrete Hessian matrix  $H_f(x) = (H_{ij}(x))$  of  $f$  at  $x \in \mathbb{Z}^n$  is defined by

$$H_{ij}(x) = f(x + e_i + e_j) - f(x + e_i) - f(x + e_j) + f(x), \quad (1.1)$$

where  $e_i$  denotes the  $i$ th unit vector. The relationship between convex extensibility and discrete Hessian matrices is not fully understood in general, and unfortunately, some vague or imprecise statements have been made in the literature. Recent papers [5, 6, 7] discuss the relationship between convex extensibility of discrete functions and the positive semidefiniteness of their Hessian matrix  $H_f(x)$ . It is certainly true that a univariate discrete function  $f : \mathbb{Z} \rightarrow \mathbb{R}$ , with  $n = 1$ , is convex extensible if and only if the Hessian  $H_f(x)$ , which is actually a real number, is positive semidefinite (i.e., nonnegative). But in case of  $n \geq 2$ , convex extensibility and the positive semidefiniteness of the discrete Hessian matrix is independent of each other. In a previous report [3] we have shown examples to demonstrate that

- Even if  $f : \mathbb{Z}^2 \rightarrow \mathbb{R}$  has a convex extension to a  $C^2$  convex function, its discrete Hessian matrix is not necessarily positive semidefinite.

In the present report we shall construct an example to demonstrate that

- Even if  $f : \mathbb{Z}^2 \rightarrow \mathbb{R}$  has a positive semidefinite discrete Hessian matrix at every point of  $\mathbb{Z}^2$ ,  $f$  is not necessarily convex extensible.

The construction is based on a semidefinite programming technique.

To be specific, we construct a function  $f : \mathbb{Z}^2 \rightarrow \mathbb{R}$  in two variables such that (i) the discrete Hessian matrix  $H_f(x)$  is positive semidefinite at each point  $x \in \mathbb{Z}^2$ , and (ii)  $f(1, 1) - 2f(2, 2) + f(3, 3) < 0$ . The construction of such  $f$  consists of three steps.

1. Construction of such a function  $f$  on a triangular domain with the aid of semidefinite programming.
2. Modification of the function  $f$  to a function defined on the nonnegative orthant  $\mathbb{Z}_+^2$ .
3. Extension of the function defined on  $\mathbb{Z}_+^2$  to a function on  $\mathbb{Z}^2$ .

Our construction proves the following theorem. Note that the second property (ii) above implies that the function  $f$  is not convex extensible, i.e., that no convex function  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfies  $g(x_1, x_2) = f(x_1, x_2)$  for all  $(x_1, x_2) \in \mathbb{Z}^2$ .

**Theorem 1.1.** *There exists a function  $f : \mathbb{Z}^2 \rightarrow \mathbb{R}$  such that  $f$  is not convex extensible and the discrete Hessian matrix  $H_f(x)$  is positive semidefinite for each  $x \in \mathbb{Z}^2$ .*

## 2 Construction by SDP on a Triangular Domain

We denote by  $\mathbb{R}$  the set of reals, by  $\mathbb{Z}$  the set of integers, and by  $\mathbb{Z}_+$  the set of nonnegative integers. For any finite set  $A$ , its cardinality is denoted by  $|A|$ . We use the symbols  $I$  and  $O$  to denote the identity matrix and the zero matrix, respectively.

For  $c \in \mathbb{Z}_+$ , we define

$$\begin{aligned} P(c) &= \{(x_1, x_2) \in \mathbb{Z}^2 \mid x_1 + x_2 \leq c; x_1, x_2 \geq 0\}, \\ B(c) &= \{(x_1, x_2) \in \mathbb{Z}^2 \mid x_1 + x_2 = c; x_1, x_2 \geq 0\}, \\ S(c) &= \{(x_1, x_2) \in \mathbb{Z}^2 \mid x_1 + x_2 \geq c; x_1, x_2 \geq 0\}. \end{aligned}$$

It is assumed that  $c \geq 8$ .

In this section, we construct a function  $f : P(c) \rightarrow \mathbb{R}$  which satisfies the following conditions:

$$H_f(x_1, x_2) \geq O \quad (\forall (x_1, x_2) \in P(c-2)), \quad (2.1)$$

$$f(1, 1) - 2f(2, 2) + f(3, 3) < 0. \quad (2.2)$$

### 2.1 Construction method

We use a SemiDefinite Programming (SDP) technique to construct a function  $f$  satisfying (2.1) and (2.2) on a triangular domain. We regard (2.1) as a constraint of SDP and want to find  $(f(x_1, x_2) \mid (x_1, x_2) \in P(c))$  which minimizes  $f(1, 1) - 2f(2, 2) + f(3, 3)$  under the constraint (2.1). If the optimal objective function value is negative, its optimal solution is a desired function  $f : P(c) \rightarrow \mathbb{R}$ .

We first consider (2.1) as a constraint of SDP. For  $f : P(c) \rightarrow \mathbb{R}$ , the discrete Hessian matrix at  $(x_1, x_2) \in P(c-2)$  is given by

$$\begin{aligned} H_{11}(x_1, x_2) &= f(x_1 + 2, x_2) - 2f(x_1 + 1, x_2) + f(x_1, x_2), \\ H_{12}(x_1, x_2) &= H_{21}(x_1, x_2) \\ &= f(x_1 + 1, x_2 + 1) - f(x_1 + 1, x_2) - f(x_1, x_2 + 1) + f(x_1, x_2), \\ H_{22}(x_1, x_2) &= f(x_1, x_2 + 2) - 2f(x_1, x_2 + 1) + f(x_1, x_2). \end{aligned}$$

The matrix  $H_f(x_1, x_2)$  can be written as follows:

$$\begin{aligned} H_f(x_1, x_2) &= \\ &f(x_1, x_2) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + f(x_1 + 1, x_2) \begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix} + f(x_1 + 2, x_2) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ &+ f(x_1 + 1, x_2 + 1) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + f(x_1, x_2 + 1) \begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix} + f(x_1, x_2 + 2) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

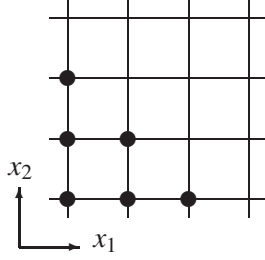


Figure 1: Six points used in the definition of  $H_f(x_1, x_2)$

This shows that the condition  $H_f(x_1, x_2) \geq O$  can be formulated as an SDP constraint on six variables  $f(x_1, x_2), f(x_1+1, x_2), f(x_1+2, x_2), f(x_1+1, x_2+1), f(x_1, x_2+1), f(x_1, x_2+2)$ ; see Fig. 1.

To find  $(f(x_1, x_2) \mid (x_1, x_2) \in P(c))$  which minimizes  $f(1, 1) - 2f(2, 2) + f(3, 3)$  under (2.1), we consider the following SDP:

$$\mathcal{P} : \begin{cases} \text{minimize} & \sum_{(x_1, x_2) \in P(c)} c(x_1, x_2) f(x_1, x_2) \\ \text{subject to} & X = \sum_{(x_1, x_2) \in P(c)} F(x_1, x_2) f(x_1, x_2), \\ & X \geq O, \end{cases}$$

where  $(f(x_1, x_2) \mid (x_1, x_2) \in P(c))$  is the vector of decision variables,

$$c(x_1, x_2) = \begin{cases} 1 & ((x_1, x_2) = (1, 1), (3, 3)), \\ -2 & ((x_1, x_2) = (2, 2)), \\ 0 & (\text{otherwise}), \end{cases}$$

and  $F(x_1, x_2)$  is a sparse block-diagonal  $(2|P(c-2)|) \times (2|P(c-2)|)$  matrix with  $|P(c-2)|$  diagonal blocks corresponding to  $(x_1, x_2) \in P(c-2)$ .

The problem  $\mathcal{P}$  is homogeneous. This means that, if there exists a feasible solution of  $\mathcal{P}$  with a negative objective value, the optimal objective value is  $-\infty$ , that is, the problem  $\mathcal{P}$  is unbounded.

To obtain a bounded SDP, we add the following constraints for normalization:

$$f(0, 0) = 1, \tag{2.3}$$

$$l \leq f(x_1, x_2) \leq u \quad (\forall (x_1, x_2) \in P(c)) \tag{2.4}$$

with  $l < 1 < u$ .

**Remark 2.1.** When we solve SDP by a computer software, we cannot avoid round-off errors. In practice, it is useful to replace the constraint  $X \geq O$  by  $X - \sigma I \geq O$  with  $\sigma > 0$ , and to round the approximate optimal variable vector computed by the software to an integer vector.

**Remark 2.2.** Here is a remark related to our choice  $f(1, 1) - 2f(2, 2) + f(3, 3)$  in (2.2). The positive semidefiniteness of the discrete Hessian matrix  $H_f(x_1, x_2)$  implies (discrete) convexity along a line with  $x_1 + x_2$  constant:

$$f(x_1 + 2, x_2) - 2f(x_1 + 1, x_2 + 1) + f(x_1, x_2 + 2) \geq 0, \quad (2.5)$$

which can be shown as follows. By  $H_f(x_1, x_2) \succeq O$  we have  $H_{11} \geq 0$  and  $H_{22} \geq 0$ . We also have

$$\det H_f(x_1, x_2) = H_{11}H_{22} - H_{12}^2 \geq 0,$$

from which follows that

$$H_{12} \leq |H_{12}| \leq \sqrt{H_{11}H_{22}} \leq \frac{1}{2}(H_{11} + H_{22}).$$

By substituting the expressions of  $H_{12}$ ,  $H_{11}$  and  $H_{22}$ , we obtain (2.5). In view of (2.5) we search for the failure of convexity in the direction of constant difference in (2.2).

**Remark 2.3.** Our construction method by SDP is valid not only for the case  $n = 2$ , but also for an arbitrary dimension  $n$ .

## 2.2 Formulation in SDPA format

We solve SDP by SDPA [8], which is a computer software package for solving SDP while extensively utilizing the sparseness of the matrices. We here show a transformation of SDP to use SDPA.

We define the variable vector  $f \in \mathbb{R}^{|P(c)|-1}$  as

$$f_1 = f(1, 0), f_2 = f(0, 1), f_3 = f(2, 0), f_4 = f(1, 1), \dots, f_i = f(x_1, x_2), \dots,$$

i.e.,

$$f_i = f(x_1, x_2) \quad (\forall (x_1, x_2) \in P(c) \setminus \{(0, 0)\}),$$

where the index  $i$  is given by

$$i = r(x_1, x_2) := \frac{1}{2}(x_1 + x_2)(x_1 + x_2 + 1) + x_1. \quad (2.6)$$

It is noted that  $r(c, 0) = |P(c)| - 1$ .

Now, the SDP  $\mathcal{P}$  is transformed as follows:

$$\begin{cases} \text{minimize} & \sum_{i=1}^m c_i f_i \\ \text{subject to} & X = \sum_{i=1}^m F_i f_i - F_0, \\ & X \succeq O, \end{cases}$$

where  $m = |P(c)| - 1$ ,  $F_i (i = 1, \dots, m)$  are sparse block-diagonal  $(2|P(c-2)|) \times (2|P(c-2)|)$  matrices with  $|P(c-2)|$  diagonal blocks, and

$$\begin{cases} c_4 = 1, & c_{12} = -2, & c_{24} = 1, \\ & c_i = 0 & \quad (i \neq 4, 12, 24). \end{cases}$$

Note that  $r(1, 1) = 4$ ,  $r(2, 2) = 12$ , and  $r(3, 3) = 24$  from (2.6).

Though the normalization (2.3) decreases the number of variables, it does not change the number of blocks of the constraint matrices because the positive semidefiniteness of  $H_f(0, 0)$  remains unaffected. We, expediently, assign  $H_f(0, 0) \succeq O$  to the last block. Now, we have

$$F_0 = \begin{bmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & 1 & 1 & \\ & & & 1 & 1 & \end{bmatrix}.$$

In order to add upper and lower bound constraints (2.4), we introduce slack variables  $(t^1, \dots, t^m, s^1, \dots, s^m)$ , i.e.,

$$t^i = -f_i + u \geq 0, \quad s^i = f_i - l \geq 0 \quad (i = 1, \dots, m).$$

Hence we can reduce the above problem to the following standard form SDP:

$$\mathcal{P}' : \begin{cases} \text{minimize} & \sum_{i=1}^m c_i f_i \\ \text{subject to} & \tilde{X} = \sum_{i=1}^m \tilde{F}_i f_i - \tilde{F}_0, \\ & \tilde{X} \succeq O, \end{cases}$$

where

$$\tilde{F}_i = \begin{bmatrix} F_i & & & & & & & & & & \\ & 0 & & & & & & & & & \\ & & \ddots & & & & & & & & \\ & & & 0 & & & & & & & \\ & & & & -1 & & & & & & \\ & & & & & 0 & & & & & \\ & & & & & & \ddots & & & & \\ & & & & & & & 0 & & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & & 0 & 1 \\ & & & & & & & & & & 0 \\ & & & & & & & & & & \ddots \\ & & & & & & & & & & & 0 \end{bmatrix} \quad (i = 1, \dots, m),$$



$$\tilde{F}_0 = \begin{bmatrix} F_0 & & & & & & \\ & -u & & & & & \\ & & \ddots & & & & \\ & & & -u & & & \\ & & & & l & & \\ & & & & & \ddots & \\ & & & & & & l \end{bmatrix}, \tilde{X} = \begin{bmatrix} X & & & & & & \\ & t^1 & & & & & \\ & & \ddots & & & & \\ & & & t^m & & & \\ & & & & s^1 & & \\ & & & & & \ddots & \\ & & & & & & s^m \end{bmatrix}.$$

### 2.3 Numerical result

We here show the concrete numerical result of  $f : P(c) \rightarrow \mathbb{R}$  satisfying (2.1) and (2.2) for  $c = 10$ . We solve SDP  $\mathcal{P}'$  with  $l = -100, u = 100$  by SDPA. We also put  $\sigma = 3$  in  $X - \sigma I \geq O$  to construct an integer-valued  $f$  from the output of SDPA; see Remark 2.1. By rounding an approximate optimal variable vector in the output of SDPA to an integer vector, we get  $f(x_1, x_2)$  ( $(x_1, x_2) \in P(10)$ ) as follows:

	10	100											
	9	73 45											
	8	50	21 -4										
	7	29	1	-25 -47									
$x_2$	6	11	-17	-43	-64 -56								
	5	-3	-32	-57	-79	-71 -59							
	4	-15	-44	-69	-91	-82	-71 -56						
	3	-24	-52	-78	-100	-91	-79	-64 -47					
	2	-23	-48	-66	-78	-69	-57	-43	-25 -4				
	1	-15	-35	-48	-52	-44	-32	-17	1	21	45		
	0	0	-15	-23	-24	-15	-3	11	29	50	73	100	
			0	1	2	3	4	5	6	7	8	9	10
			$x_1$										

This function  $f : P(c) \rightarrow \mathbb{R}$  serves as an example that the positive semidefiniteness of the discrete Hessian matrix does not imply convex extensibility.

We can see a failure of midpoint convexity in

$$f(1, 1) - 2f(2, 2) + f(3, 3) = -35 + 2 \times 66 - 100 = -3. \quad (2.7)$$

The discrete Hessian matrices are as follows:

$$\begin{array}{l}
H(0, 8) = \\
\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \\
\det = 15 \\
\hline
H(0, 7) = H(1, 7) = \\
\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \\
\det = 3 \quad \det = 15 \\
\hline
H(0, 6) = H(1, 6) = H(2, 6) = \\
\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 29 & -1 \\ -1 & 3 \end{bmatrix} \\
\det = 6 \quad \det = 10 \quad \det = 86 \\
\hline
H(0, 5) = H(1, 5) = H(2, 5) = H(3, 5) = \\
\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \quad \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \quad \begin{bmatrix} 30 & 1 \\ 1 & 4 \end{bmatrix} \quad \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \\
\det = 15 \quad \det = 8 \quad \det = 119 \quad \det = 8 \\
\hline
H(0, 4) = H(1, 4) = H(2, 4) = H(3, 4) = H(4, 4) = \\
\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 31 & 0 \\ 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \quad \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \\
\det = 8 \quad \det = 9 \quad \det = 62 \quad \det = 5 \quad \det = 15 \\
\hline
H(0, 3) = H(1, 3) = H(2, 3) = H(3, 3) = H(4, 3) = H(5, 3) = \\
\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \quad \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \quad \begin{bmatrix} 31 & 0 \\ 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \\
\det = 5 \quad \det = 15 \quad \det = 93 \quad \det = 9 \quad \det = 5 \quad \det = 8 \\
\hline
H(0, 2) = H(1, 2) = H(2, 2) = H(3, 2) = H(4, 2) = H(5, 2) = H(6, 2) = \\
\begin{bmatrix} 7 & -3 \\ -3 & 10 \end{bmatrix} \quad \begin{bmatrix} 6 & -8 \\ -8 & 12 \end{bmatrix} \quad \begin{bmatrix} 21 & -10 \\ -10 & 21 \end{bmatrix} \quad \begin{bmatrix} 3 & 0 \\ 0 & 31 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 \\ 0 & 31 \end{bmatrix} \quad \begin{bmatrix} 4 & 1 \\ 1 & 30 \end{bmatrix} \quad \begin{bmatrix} 3 & -1 \\ -1 & 29 \end{bmatrix} \\
\det = 61 \quad \det = 8 \quad \det = 341 \quad \det = 93 \quad \det = 62 \quad \det = 119 \quad \det = 86 \\
\hline
H(0, 1) = H(1, 1) = H(2, 1) = H(3, 1) = H(4, 1) = H(5, 1) = H(6, 1) = H(7, 1) = \\
\begin{bmatrix} 7 & -5 \\ -5 & 7 \end{bmatrix} \quad \begin{bmatrix} 9 & -5 \\ -5 & 9 \end{bmatrix} \quad \begin{bmatrix} 12 & -8 \\ -8 & 6 \end{bmatrix} \quad \begin{bmatrix} 4 & 1 \\ 1 & 14 \end{bmatrix} \quad \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \quad \begin{bmatrix} 4 & 1 \\ 1 & 14 \end{bmatrix} \\
\det = 24 \quad \det = 56 \quad \det = 8 \quad \det = 15 \quad \det = 9 \quad \det = 8 \quad \det = 10 \quad \det = 15 \\
\hline
H(0, 0) = H(1, 0) = H(2, 0) = H(3, 0) = H(4, 0) = H(5, 0) = H(6, 0) = H(7, 0) = H(8, 0) = \\
\begin{bmatrix} 7 & -5 \\ -5 & 7 \end{bmatrix} \quad \begin{bmatrix} 7 & -5 \\ -5 & 7 \end{bmatrix} \quad \begin{bmatrix} 10 & -3 \\ -3 & 7 \end{bmatrix} \quad \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \quad \begin{bmatrix} 4 & 1 \\ 1 & 14 \end{bmatrix} \quad \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \begin{bmatrix} 4 & 1 \\ 1 & 14 \end{bmatrix} \\
\det = 24 \quad \det = 24 \quad \det = 61 \quad \det = 5 \quad \det = 8 \quad \det = 15 \quad \det = 6 \quad \det = 3 \quad \det = 15
\end{array}$$

### 3 Extending the Domain of Definition

#### 3.1 Extension to a function on the nonnegative orthant $\mathbb{Z}_+^2$

In this section, we construct a function  $\tilde{f} : \mathbb{Z}_+^2 \rightarrow \mathbb{R}$  which satisfies the following conditions:

$$H_{\tilde{f}}(x_1, x_2) \geq 0 \quad (\forall (x_1, x_2) \in \mathbb{Z}_+^2), \quad (3.1)$$

$$\tilde{f}(1, 1) - 2\tilde{f}(2, 2) + \tilde{f}(3, 3) < 0. \quad (3.2)$$

Let  $f$  be the function on  $P(c)$  constructed in Section 2.3. To extend it to  $f :$

$\mathbb{Z}_+^2 \rightarrow \mathbb{R}$ , we define the function value for  $(x_1, x_2) \in S(c+1)$  by

$$f(x_1, x_2) = \alpha(x_1 + x_2) + \beta \quad (3.3)$$

with arbitrary  $\alpha$  and  $\beta$ . Then we have

$$H_f(x_1, x_2) = O \quad (\forall (x_1, x_2) \in S(c+1))$$

and  $H_f(x_1, x_2)$  is known to be positive semidefinite unless  $(x_1, x_2) \in B(c-1) \cup B(c)$ .

To take care of the case  $(x_1, x_2) \in B(c-1) \cup B(c)$ , we now define the following functions  $g$  and  $h$  with  $a > 0$ :

$$g(x_1, x_2) = a(x_1 - x_2)^2 \quad ((x_1, x_2) \in \mathbb{Z}_+^2),$$

$$h(x_1, x_2) = \begin{cases} 0 & ((x_1, x_2) \in P(c)), \\ a(x_1 + x_2 - c)^2 & ((x_1, x_2) \in S(c+1)). \end{cases}$$

It is noted that

$$g(1, 1) = g(2, 2) = g(3, 3) = 0, \quad (3.4)$$

$$h(1, 1) = h(2, 2) = h(3, 3) = 0, \quad (3.5)$$

and the discrete Hessian matrices of  $g$  and  $h$  are given as follows:

$$H_g(x_1, x_2) = a \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \quad ((x_1, x_2) \in \mathbb{Z}_+^2),$$

$$H_h(x_1, x_2) = \begin{cases} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & ((x_1, x_2) \in P(c-2)), \\ a \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & ((x_1, x_2) \in B(c-1)), \\ a \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} & ((x_1, x_2) \in S(c)). \end{cases}$$

**Lemma 3.1.** For  $\tilde{f} = f + g + h$ , conditions (3.1) and (3.2) hold.

**Proof.** We can see the failure of midpoint convexity (3.2) easily from (2.7), (3.4) and (3.5). We here prove the positive semidefiniteness of the discrete Hessian matrix (3.1). Note first that

$$H_{\tilde{f}} = H_f + H_g + H_h.$$

We consider the following four cases:

(i) For  $(x_1, x_2) \in P(c-2)$ : From  $H_f \geq O$  and  $H_h = O$ , we have

$$H_{\tilde{f}} = H_f + a \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \geq O.$$

- (ii) For  $(x_1, x_2) \in B(c-1)$ : Although  $H_f$  is not necessarily positive semidefinite, using a sufficiently large  $a$ , the positive semidefiniteness

$$H_{\tilde{f}} = H_f + a \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \geq O$$

holds for all  $(x_1, x_2) \in B(c-1)$ . It is noted that such  $a$  exists since  $B(c-1)$  is a finite set.

- (iii) For  $(x_1, x_2) \in B(c)$ : Although  $H_f$  is not necessarily positive semidefinite, using a sufficiently large  $a$ , the positive semidefiniteness

$$H_{\tilde{f}} = H_f + a \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \geq O$$

holds for any  $(x_1, x_2) \in B(c)$ . It is noted that such  $a$  exists since  $B(c)$  is a finite set.

- (iv) For  $(x_1, x_2) \in S(c+1)$ : From  $H_f = O$ , we have

$$H_{\tilde{f}} = H_g + H_h = a \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \geq O.$$

This completes the proof of Lemma 3.1. □

### 3.2 Extension to a function on $\mathbb{Z}^2$

In this section, we construct a function  $\hat{f} : \mathbb{Z}^2 \rightarrow \mathbb{R}$  which satisfies the following conditions:

$$H_{\hat{f}}(x_1, x_2) \geq O \quad (\forall (x_1, x_2) \in \mathbb{Z}^2), \quad (3.6)$$

$$\hat{f}(1, 1) - 2\hat{f}(2, 2) + \hat{f}(3, 3) < 0. \quad (3.7)$$

**Lemma 3.2.** *Suppose that a function  $f : \mathbb{Z}_+^2 \rightarrow \mathbb{R}$  satisfies the following conditions:*

$$H_f(x_1, x_2) \geq O \quad (\forall (x_1, x_2) \in \mathbb{Z}_+^2), \quad (3.8)$$

$$\sup_{x_1 \in \mathbb{Z}_+} (f(x_1, 0) - f(x_1, 1)) < +\infty, \quad \sup_{x_2 \in \mathbb{Z}_+} (f(0, x_2) - f(1, x_2)) < +\infty. \quad (3.9)$$

Choose a constant  $b$  as

$$b \geq \max \left( \sup_{x_1 \in \mathbb{Z}_+} (f(x_1, 0) - f(x_1, 1)), \sup_{x_2 \in \mathbb{Z}_+} (f(0, x_2) - f(1, x_2)) \right) \quad (3.10)$$

and define a function  $\hat{f} : \mathbb{Z}^2 \rightarrow \mathbb{R}$  as

$$\hat{f}(x_1, x_2) = \begin{cases} f(x_1, x_2) & (x_1 \geq 0, x_2 \geq 0), \\ f(x_1, 0) - bx_2 & (x_1 \geq 0, x_2 \leq 0), \\ f(0, x_2) - bx_1 & (x_1 \leq 0, x_2 \geq 0), \\ f(0, 0) - b(x_1 + x_2) & (x_1 \leq 0, x_2 \leq 0). \end{cases} \quad (3.11)$$

Then the condition (3.6) is satisfied.

**Proof.** From condition (3.8), we have

$$\begin{aligned} f(x_1 + 2, 0) - 2f(x_1 + 1, 0) + f(x_1, 0) &\geq 0 & (x_1 \geq 0), \\ f(0, x_2 + 2) - 2f(0, x_2 + 1) + f(0, x_2) &\geq 0 & (x_2 \geq 0). \end{aligned}$$

We check the positive semidefiniteness (3.6) in the following cases:

(i) For  $x_1 \geq 0, x_2 \geq 0$ : We have  $H_{\hat{f}}(x_1, x_2) = H_f(x_1, x_2) \geq O$ .

(ii) For  $x_1 \geq 0, x_2 = -1$ : We have

$$H_{\hat{f}}(x_1, -1) = \begin{bmatrix} f(x_1 + 2, 0) - 2f(x_1 + 1, 0) + f(x_1, 0) & 0 \\ 0 & f(x_1, 1) - f(x_1, 0) + b \end{bmatrix} \geq O.$$

(iii) For  $x_1 \geq 0, x_2 \leq -2$ : We have

$$H_{\hat{f}}(x_1, x_2) = \begin{bmatrix} f(x_1 + 2, 0) - 2f(x_1 + 1, 0) + f(x_1, 0) & 0 \\ 0 & 0 \end{bmatrix} \geq O.$$

(iv) For  $x_1 = -1, x_2 = -1$ : We have

$$H_{\hat{f}}(-1, -1) = \begin{bmatrix} f(1, 0) - f(0, 0) + b & 0 \\ 0 & f(0, 1) - f(0, 0) + b \end{bmatrix} \geq O.$$

(v) For  $x_1 = -1, x_2 \leq -2$ : We have

$$H_{\hat{f}}(-1, x_2) = \begin{bmatrix} f(1, 0) - f(0, 0) + b & 0 \\ 0 & 0 \end{bmatrix} \geq O.$$

(vi) For  $x_1 \leq -2, x_2 \leq -2$ : We have  $H_{\hat{f}}(x_1, x_2) = O$ .

(vii) In the above we have dealt with the case where  $x_1 \geq x_2$ . The remaining case with  $x_1 \leq x_2$  follows from symmetry.

□

**Lemma 3.3.** For the function  $\tilde{f} : \mathbb{Z}_+^2 \rightarrow \mathbb{R}$  constructed in Lemma 3.1, we have

$$\begin{aligned}\tilde{f}(x_1, 0) - \tilde{f}(x_1, 1) &= 2a(c - 1) - \alpha \quad (x_1 \geq c), \\ \tilde{f}(0, x_2) - \tilde{f}(1, x_2) &= 2a(c - 1) - \alpha \quad (x_2 \geq c).\end{aligned}$$

Therefore, the function  $\tilde{f}$  satisfies the condition (3.9) and, the condition (3.10) holds for  $b$  with

$$b \geq \max \left( \max_{0 \leq x_1 \leq c} (\tilde{f}(x_1, 0) - \tilde{f}(x_1, 1)), \max_{0 \leq x_2 \leq c} (\tilde{f}(0, x_2) - \tilde{f}(1, x_2)), 2a(c - 1) - \alpha \right).$$

**Proof.** For  $x_1 \geq c$ , we have

$$\begin{aligned}\tilde{f}(x_1, 0) - \tilde{f}(x_1, 1) &= [(\alpha x_1 + \beta) - (\alpha(x_1 + 1) + \beta)] + [g(x_1, 0) - g(x_1, 1)] + [h(x_1, 0) - h(x_1, 1)] \\ &= -\alpha + a[x_1^2 - (x_1 - 1)^2] + a[(x_1 - c)^2 - (x_1 + 1 - c)^2] \\ &= 2a(c - 1) - \alpha.\end{aligned}$$

We can calculate  $\tilde{f}(0, x_2) - \tilde{f}(1, x_2)$  similarly. □

The function  $\hat{f} : \mathbb{Z}^2 \rightarrow \mathbb{R}$  constructed in Lemma 3.2 with  $f = \tilde{f}$  in Lemma 3.1 satisfies (3.6) and (3.7). The positive semidefiniteness (3.6) follows from Lemmas 3.1, 3.2 and 3.3, whereas the failure of midpoint convexity (3.7) follows easily from the definition of  $\hat{f}$  and Lemma 3.1. Hence this function  $\hat{f}$  serves as the function  $f$  in Theorem 1.1.

The construction method presented in the above is summarized as follows:

1. We construct a function  $f : P(c) \rightarrow \mathbb{R}$  on the triangular domain  $P(c)$  with the aid of SDP. The positive semidefiniteness of the discrete Hessian matrix holds for  $(x_1, x_2) \in P(c - 2)$ . The midpoint convexity fails at  $(x_1, x_2) = (1, 1), (2, 2), (3, 3)$ .
2. We extend  $f$  to a function  $f : \mathbb{Z}_+^2 \rightarrow \mathbb{R}$  on the nonnegative orthant by (3.3). The positive semidefiniteness of the discrete Hessian matrix holds except at  $(x_1, x_2) \in B(c - 1) \cup B(c)$ .
3. We define  $\tilde{f} : \mathbb{Z}_+^2 \rightarrow \mathbb{R}$  by  $\tilde{f} = f + g + h$ . The positive semidefiniteness of the discrete Hessian matrix holds for  $(x_1, x_2) \in \mathbb{Z}_+^2$ . The midpoint convexity fails at  $(x_1, x_2) = (1, 1), (2, 2), (3, 3)$ .
4. We extend  $\tilde{f}$  to a function  $\hat{f} : \mathbb{Z}^2 \rightarrow \mathbb{R}$  by (3.11). The positive semidefiniteness of the discrete Hessian matrix holds for  $(x_1, x_2) \in \mathbb{Z}^2$ . The midpoint convexity fails at  $(x_1, x_2) = (1, 1), (2, 2), (3, 3)$ .

**Remark 3.1.** In Theorem 1.1 the function  $f$  can be chosen to be integer-valued. To see this note that the function  $f$  on  $P(c)$  satisfying (2.1) and (2.2) computed by SDP is integer-valued and the parameters  $\alpha$ ,  $\beta$ ,  $a$ , and  $b$  can be chosen to be integers. Then the resulting function  $\hat{f}$  is integer-valued.

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