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Matching Problems with Delta-Matroid Constraints*

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Abstract

Given an undirected graph $G = (V, E)$ and a delta-matroid $(V, \mathcal{F})$, the *delta-matroid matching problem* is to find a maximum cardinality matching M such that the set of the end vertices of M belongs to $\mathcal{F}$. This problem is a natural generalization of the matroid matching problem to delta-matroids, and thus it cannot be solved in polynomial time in general.

This paper introduces a class of the delta-matroid matching problem, where the given delta-matroid is a projection of a linear delta-matroid. We first show that it can be solved in polynomial time if the given linear delta-matroid is generic. This result enlarges a polynomially solvable class of matching problems with precedence constraints on vertices such as the 2-master/slave matching. In addition, we design a polynomial-time algorithm when the graph is bipartite and the delta-matroid is defined on one vertex side. This result is extended to the case where a linear matroid constraint is additionally imposed on the other vertex side.

Keywords: constrained matching, delta-matroid, polynomial-time algorithm, mixed matrix theory

1 Introduction

For an undirected graph $G = (V, E)$, a subset M of E is called a *matching* if no two edges in M share a common vertex incident to them. The matching problem is a fundamental topic in combinatorial optimization, and many polynomial-time algorithms have been developed (see e.g., [25, 34]). When we apply the matching problem to practical problems such as scheduling, it is often natural to have some additional constraints. In the literature, there are matching problems with a variety of constraints such as matroids [22], trees [9], precedence constraints [1,

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20], and knapsack constraints [2]. These constrained matching problems are known to be NP-hard except for some cases with matroid constraints.

For an undirected graph $G = (V, E)$ and a matroid $\mathcal{M}$ on V, the *matroid matching problem* is a problem of finding a maximum cardinality matching M such that ∂M is independent in $\mathcal{M}$, where ∂M denotes the set of vertices incident to edges in M. This problem, which is equivalent to the matroid parity problem [26], has been investigated in combinatorial optimization as a common generalization of two well-known polynomially solvable problems: the matroid intersection problem and the matching problem. Although it is shown to be intractable in the oracle model [23, 30] and NP-hard for matroids with compact representations [28], Lovász provided a min-max formula [28] for the *linear* matroid matching problem, and a number of polynomial-time algorithms for the linear matroid matching have been developed [11, 15, 27, 29, 33]. On the other hand, when the graph is bipartite and the matroid is defined on each class of the vertex bipartition, the matroid matching problem can be solved in polynomial time for general matroids [22], because this is equivalent to the matroid intersection problem. This problem is also known as the *independent matching problem*.

A *delta-matroid* was introduced by Bouchet [5] as a generalization of matroids, where essentially equivalent combinatorial structures are given independently by [10, 12]. We say that a pair $(V, \mathcal{F})$ of a finite set V and a nonempty family $\mathcal{F}$ of subsets of V is a *delta-matroid* if it satisfies the symmetric exchange axiom:

(DM) For $F, F' \in \mathcal{F}$ and $u \in F \triangle F'$, there exists $v \in F \triangle F'$ such that $F \triangle \{u, v\} \in \mathcal{F}$,

where $I \triangle J$ denotes the symmetric difference, i.e., $I \triangle J = (I \setminus J) \cup (J \setminus I)$. A set $F \in \mathcal{F}$ is called *feasible*. A delta-matroid maintains matroidal properties in the sense that a greedy algorithm is applicable to maximizing linear functions over a delta-matroid [5].

In this paper, we generalize the matroid matching problem in terms of delta-matroids. That is, for an undirected graph $G = (V, E)$ and a delta-matroid $(V, \mathcal{F})$, the *delta-matroid matching problem* is to find a maximum cardinality matching M with $\partial M \in \mathcal{F}$. Since this problem includes the matroid matching problem, the delta-matroid matching problem *cannot* be solved in polynomial time in general.

It should be noted that the feasibility problem for the delta-matroid matching, i.e., finding a matching M such that ∂M is feasible, already generalizes the matroid matching problem. The feasibility problem is reduced to the *delta-covering problem*, posed by Bouchet [7] as a generalization of the matroid parity problem. For this problem, Geelen *et al.* [18] provided a polynomial-time algorithm and a min-max theorem if a given delta-matroid is *linear*, i.e., represented by a skew-symmetric matrix. Thus, for linear delta-matroids, we can find a feasible delta-matroid matching in polynomial time. However, the complexity of the problem to find a maximum delta-matroid matching is still unknown for the linear case.

The main purpose of this paper is to investigate polynomial solvability of the delta-matroid matching problem. We introduce a new class of the delta-matroid matching problem, where the delta-matroid is given by a projection of a linear delta-matroid with a skew-symmetric matrix. We call such problem the *matching problem with a projected linear delta-matroid*.

For this problem, we first show that it can be solved in polynomial time if the given skew-symmetric matrix K is generic, that is, each entry in K is an independent parameter. This can be done by reducing it to the maximum weight matching problem. Let us remark that, although we restrict a delta-matroid to a generic case, our class still includes a variety of constrained matching problems, as presented in Section 3.2. In particular, our class includes polynomially solvable classes of the master/slave matching [20] and its variant [1], arising in a manpower scheduling problem.

In addition, we deal with a bipartite case of our problem. That is, we assume that a graph G is bipartite and that the ground set of a delta-matroid is contained in one class of the vertex bipartition. This problem generalizes the bipartite 2-master/slave matching [19], which is a special case of the master/slave matching. In this setting we prove that the matching problem with a projected linear delta-matroid can be solved in polynomial time. The proof makes use of mixed matrix theory developed by Murota [32]. We further show that our result can be extended to the problem where a linear matroid constraint is additionally imposed on the other vertex side. Note that this problem can be viewed as an intersection problem, i.e., the problem of finding a maximum cardinality set that is both feasible for a delta-matroid and independent for a matroid.

The organization of this paper is as follows. In Section 2, we explain delta-matroid theory and mixed matrix theory. In Section 3 we examine a projection of a linear delta-matroid and provide a variety of examples representable by such delta-matroids. Section 4 presents a polynomial-time algorithm for the generic delta-matroid matching problem. In Section 5, we deal with the bipartite case and its generalization to the problem with additional linear matroid constraint. Finally, Section 6 discusses related intersection problems on matroids and delta-matroids.

2 Matrices and delta-matroids

2.1 Graphs and matrices

Let $G = (V, E)$ be a graph with vertex set V and edge set E. For a subset F of E, we denote by ∂F the set of vertices incident to edges in F. For a subset X of V, the *induced subgraph* on X is a graph $G[X] = (X, E')$, where $E' \subseteq E$ is the set of edges whose both end vertices are included in X. For a bipartite graph $G = (V^+, V^-; E)$ with $E \subseteq V^+ \times V^-$, the induced subgraph on $I \subseteq V^+$ and $J \subseteq V^-$ is denoted by $G[I, J]$. We also define $\partial^+ F = \partial F \cap V^+$ and $\partial^- F = \partial F \cap V^-$ for a subset F of E.

Throughout this paper, we consider a matrix over the real field $\mathbb{R}$[1] unless otherwise specified. For a matrix $K = (K_{ij})$ with row set R and column set C, $K[I, J]$ denotes the submatrix with row set $I \subseteq R$ and column set $J \subseteq C$. For a square matrix K, we denote a principal submatrix with row/column set I by $K[I]$. A matrix K is called *skew-symmetric* if $K_{ij} = -K_{ji}$ for all (i, j) and all diagonal entries of K are zero. For a skew-symmetric matrix $K = (K_{ij})$ with row/column set V such that $|V| = 2n$, the *Pfaffian* of K is defined by

$$\mathrm{pf}\, K = \sum_{P} k_P,$$

[1]In fact, the results of this paper can be applied to a matrix over any ordered field $\mathbf{F}$. That is, the real field $\mathbb{R}$ and the rational field $\mathbb{Q}$ which appear in this paper may be replaced by an ordered field $\mathbf{F}$ and its subfield $\mathbf{K}$.

where the summation is taken over all partitions $P = \{\{i_1, j_1\}, \ldots, \{i_n, j_n\}\}$ of V into unordered pairs and

$$k_P = \operatorname{sgn} \begin{pmatrix} 1 & 2 & \cdots & 2n-1 & 2n \\ i_1 & j_1 & \cdots & i_n & j_n \end{pmatrix} \prod_{l=1}^{n} K_{i_l j_l}.$$

Then it holds that $\det K = (\operatorname{pf} K)^2$.

A *generic matrix* is a matrix in which each nonzero entry is an independent parameter. More precisely, a matrix is *generic* if the set of nonzero entries is algebraically independent over the rational field $\mathbb{Q}$. A skew-symmetric matrix K is called *generic* if $\{K_{ij} \mid K_{ij} \neq 0, i < j\}$ is algebraically independent over $\mathbb{Q}$.

Let K be a generic skew-symmetric matrix with row/column set V. The *support graph* of K is the undirected graph $H = (V, E_H)$ with $E_H = \{(i,j) \mid K_{ij} \neq 0, i < j\}$. Note that a nonzero term k_P in the Pfaffian corresponds to a perfect matching in the support graph. Hence the rank of a generic skew-symmetric matrix K is twice the size of a maximum matching in its support graph H (see e.g., [31]). By applying this fact to each principal submatrix of K, we obtain the following lemma.

Lemma 2.1. Let K be a generic skew-symmetric matrix with row/column set V, and $H = (V, E_H)$ be its support graph. For a subset $X \subseteq V$, $K[X]$ is nonsingular if and only if $H[X]$ has a perfect matching.

2.2 Delta-matroids and greedy algorithms

We say that a pair $(V, \mathcal{I})$ of a finite set V and a family $\mathcal{I}$ of subsets of V is a *matroid* if it satisfies the following:

(I-1) $\emptyset \in \mathcal{I}$,

(I-2) $I \subseteq J \in \mathcal{I} \Rightarrow I \in \mathcal{I}$,

(I-3) $I, J \in \mathcal{I}, |I| < |J| \Rightarrow I \cup \{v\} \in \mathcal{I}$ for some $v \in J \setminus I$.

A set $I \in \mathcal{I}$ is said to be *independent*, and a maximal independent set is a *base*.

Recall that a *delta-matroid* is a pair $\mathbf{M} = (V, \mathcal{F})$ of a finite set V and a nonempty family $\mathcal{F}$ of subsets of V that satisfies the symmetric exchange axiom (DM). A delta-matroid generalizes a matroid, since the family of independent sets of a matroid forms a delta-matroid. A delta-matroid $(V, \mathcal{F})$ is said to be *even* if $|F \triangle F'|$ is even for all $F, F' \in \mathcal{F}$. A simple example of even delta-matroids is the family of the subsets of even size. The base family of a matroid also forms an even delta-matroid. Another example is the family of vertex sets X with $X = \partial M$ for some matching M in a graph G, called the *matching delta-matroid* [6]. Let K be a skew-symmetric matrix with row/column set V. We denote the family of column indices corresponding to nonsingular principal submatrices by

$$\mathcal{F}(K) = \{X \subseteq V \mid \operatorname{rank} K[X] = |X|\}.$$

Then $\mathbf{M}(K) = (V, \mathcal{F}(K))$ forms an even delta-matroid, where the empty set is feasible [4].

The *twisting* of a delta-matroid $\mathbf{M} = (V, \mathcal{F})$ by $X \subseteq V$ is a delta-matroid defined by $\mathbf{M} \triangle X = (V, \mathcal{F} \triangle X)$, where

$$\mathcal{F} \triangle X = \{F \triangle X \mid F \in \mathcal{F}\}.$$

A delta-matroid $\mathbf{M}'$ is *equivalent to* $\mathbf{M}$ if $\mathbf{M}'$ is a twisting of $\mathbf{M}$ by some $X \subseteq V$. A delta-matroid $\mathbf{M}$ is said to be *linear* if $\mathbf{M}$ is equivalent to $\mathbf{M}(K)$ for some skew-symmetric matrix K. Since $\mathbf{M}(K)$ is an even delta-matroid, a linear delta-matroid is even. We call a linear delta-matroid $\mathbf{M}$ *generic* if $\mathbf{M}$ is equivalent to $\mathbf{M}(K)$ for some generic skew-symmetric matrix K. For $X \subseteq V$, the *projection* of a delta-matroid $\mathbf{M}$ on X is defined by $\mathbf{M}|X = (V \setminus X, \mathcal{F}|X)$, where

$$\mathcal{F}|X = \{F \setminus X \mid F \in \mathcal{F}\}.$$

The projection $\mathbf{M}|X$ is also a delta-matroid. Note that projection does not necessarily preserve evenness. The *deletion* of X from $\mathbf{M}$ is a delta-matroid $\mathbf{M} \setminus X = (V \setminus X, \mathcal{F} \setminus X)$ with

$$\mathcal{F} \setminus X = \{F \in \mathcal{F} \mid F \subseteq V \setminus X\},$$

where $\mathcal{F} \setminus X$ is assumed to be nonempty. The *contraction* of $\mathbf{M}$ by X is defined as $\mathbf{M}/X = (\mathbf{M} \triangle X) \setminus X$.

Given a delta-matroid $\mathbf{M} = (V, \mathcal{F})$ and a weight function $c : V \to \mathbb{R}$, consider the problem to find $F \in \mathcal{F}$ which maximizes $c(F) = \sum_{v \in F} c(v)$. For this problem, Bouchet [5] designed a greedy-type algorithm, and Shioura and Tanaka [35] extended the algorithm to one for *jump systems* [8], which is a generalization of delta-matroids to integral lattice. Their results imply that the greedy algorithm finds an optimal solution in polynomial time under the assumption that a membership oracle for $\mathbf{M}$, which checks whether or not a given set F is in $\mathcal{F}$, is available.

2.3 Mixed skew-symmetric matrices and delta-covering

Mixed matrix theory, developed by Murota [32], is one of the most significant applications of delta-matroid theory. In this section, we explain the concept of mixed skew-symmetric matrices, which will be used to analyze the bipartite delta-matroid matching problem in Section 5.

Let $\mathbf{F}$ be a field and $\mathbf{K}$ be a subfield of $\mathbf{F}$. A typical example is $\mathbf{K} = \mathbb{Q}$ and $\mathbf{F} = \mathbb{R}$. A skew-symmetric matrix A is called a *mixed skew-symmetric matrix* if A is given by $A = Q + T$, where

(MS-Q) $Q = (Q_{ij})$ is a skew-symmetric matrix over $\mathbf{K}$ (i.e., $Q_{ij} \in \mathbf{K}$), and

(MS-T) $T = (T_{ij})$ is a skew-symmetric matrix over $\mathbf{F}$ (i.e., $T_{ij} \in \mathbf{F}$) such that the set $\{T_{ij} \mid T_{ij} \neq 0, i < j\}$ of its nonzero entries in the upper-triangle part is algebraically independent over $\mathbf{K}$.

The problem of computing the rank of a mixed skew-symmetric matrix is discussed in [17, 18]. The rank is expressed by using delta-matroids.

Theorem 2.2 ([32, Theorem 7.3.22])**.** For a mixed skew-symmetric matrix $A = Q + T$ with row/column set V, it holds that

$$\begin{aligned}\operatorname{rank} A &= \max\{\operatorname{rank} Q[I] + \operatorname{rank} T[V \setminus I] \mid I \subseteq V\} \\ &= \max\{|F_Q \triangle F_T| \mid F_Q \in \mathcal{F}(Q), F_T \in \mathcal{F}(T)\},\end{aligned}$$

where $\mathbf{M}(Q) = (V, \mathcal{F}(Q))$ and $\mathbf{M}(T) = (V, \mathcal{F}(T))$.

For a pair of delta-matroids $(V, \mathcal{F}_1)$ and $(V, \mathcal{F}_2)$, the *delta-covering problem* is the problem to find $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$ maximizing $|F_1 \triangle F_2|$. This problem is a generalization of the matroid parity problem, and contains the delta-matroid intersection problem and the delta-matroid partition problem as special cases (see [32]). Since Geelen *et al.* [18] gave a polynomial-time algorithm for the delta-covering problem for linear delta-matroids, it follows from Theorem 2.2 that the rank of a mixed skew-symmetric matrix $A = Q + T$ can be computed in polynomial time. This is rewritten as follows.

Corollary 2.3. Let $A = Q + T$ be a mixed skew-symmetric matrix with row/column set V. Then we can find in polynomial time a maximum cardinality set $X \subseteq V$ such that $A[X]$ is nonsingular. Moreover, we can obtain in polynomial time a set $I \subseteq X$ such that both $Q[I]$ and $T[X \setminus I]$ are nonsingular.

We can also compute in polynomial time the weighted version of finding a nonsingular principal submatrix of A. More generally, we have the following.

Lemma 2.4. Let $A = Q + T$ be a mixed skew-symmetric matrix with row/column set V. For a weight function $c : V \to \mathbb{R}$ and $U \subseteq V$, we can find a set $X \subseteq V$ which maximizes $c(X) = \sum_{v \in X} c(v)$ subject to $U \subseteq X$ and $\operatorname{rank} A[X] = |X|$ in polynomial time, if exists.

Proof. Since A is skew-symmetric, $(V, \mathcal{F}(A))$ is a delta-matroid. We define a weight function $\tilde{c} : V \to \mathbb{R}$ by

$$\tilde{c}(v) = \begin{cases} N & (v \in U) \\ c(v) & (v \in V \setminus U) \end{cases},$$

where N is an integer larger than $c_{\max}|V|$ with $c_{\max} = \max_{v \in V} c(v)$. Let X be a maximum weight feasible set with respect to the delta-matroid $(V, \mathcal{F}(A))$ with weight $\tilde{c}$, which can be obtained by the greedy algorithm. Then if $\tilde{c}(X) < N|U|$ there exists no set $Y \in \mathcal{F}(A)$ with $U \subseteq Y$. If $\tilde{c}(X) \geq N|U|$ then X maximizes $c(X)$ subject to $U \subseteq X$ and $\operatorname{rank} A[X] = |X|$.

In the greedy algorithm, it is necessary to determine whether a given set $Y \subseteq V$ is in $\mathcal{F}(A)$ or not. This is equivalent to computing the rank of $A[Y]$, which can be done in polynomial time by solving the linear delta-covering problem. Thus a membership oracle for $(V, \mathcal{F}(A))$ is available, which implies that we can find a maximum weight feasible set X in polynomial time. □

3 Matching problem with projected linear delta-matroids

Let K be a skew-symmetric matrix with row/column set V_K, and V be a subset of V_K. For a subset S of V, the pair $(V_K, \mathcal{F}(K) \triangle S)$ is a linear delta-matroid. We define

$$\mathcal{F}_K := (\mathcal{F}(K) \triangle S)|R, \tag{1}$$

where $R = V_K \setminus V$. Then $(V, \mathcal{F}_K)$ is a delta-matroid, which we call a *projected linear delta-matroid*.

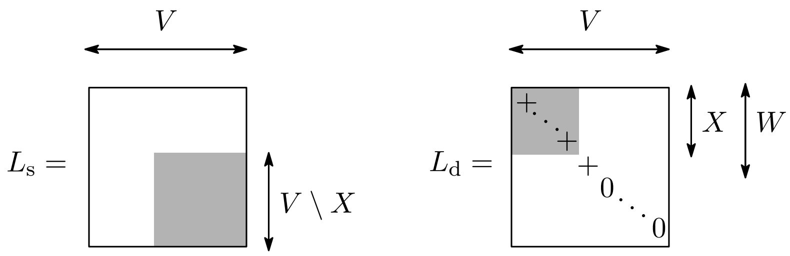

Figure 1: A skew-symmetric matrix L_{s} and a diagonal matrix L_{d}.

We then discuss weighted non-bipartite matching problem with $(V, \mathcal{F}_K)$. That is, given a graph $G = (V, E)$ with edge weight $w : E \to \mathbb{R}_+$, we aim at finding a matching $M \subseteq E$ in G which maximizes $w(M) = \sum_{e \in M} w(e)$ subject to $\partial M \in \mathcal{F}_K$.

We first show in Section 3.1 that the family of nonsingular principal submatrices of a skew-symmetric matrix with nonnegative diagonals can be represented as a projected linear delta-matroid (1). This representation brings a variety of constrained matching problems into our framework, which are provided in Section 3.2.

3.1 Skew-symmetric matrices with nonnegative diagonals

Let L be a *skew-symmetric matrix with nonnegative diagonals*, that is, $L = L_{\mathrm{s}} + L_{\mathrm{d}}$ with a skew-symmetric matrix L_{s} and a nonnegative diagonal matrix L_{d}. We denote the row/column set of L by V, and the row/column set of L_{d} corresponding to positive diagonals by W. A matrix L over $\mathbb{R}$ is called *generic* if $\{L_{ij} \mid L_{ij} \neq 0, i \leq j\}$ is algebraically independent over $\mathbb{Q}$. For a skew-symmetric matrix L with nonnegative diagonals, define $\mathcal{F}(L) = \{X \subseteq V \mid \operatorname{rank} L[X] = |X|\}$.

The main purpose of this section is to prove that $(V, \mathcal{F}(L) \triangle S)$ for $S \subseteq V$ forms a projection of a linear delta-matroid. We first show the following lemma.

Lemma 3.1. Let $L = L_{\mathrm{s}} + L_{\mathrm{d}}$ be a skew-symmetric matrix with nonnegative diagonals, and V be the row/column set of L. Then L is nonsingular if and only if there exists $X \subseteq W$ such that $L_{\mathrm{s}}[V \setminus X]$ is nonsingular.

Proof. Since L_{d} is a diagonal matrix, we have

$$\begin{aligned} \det L &= \sum_{X \subseteq V} \det L_{\mathrm{s}}[V \setminus X] \cdot \det L_{\mathrm{d}}[X] \\ &= \sum_{X \subseteq W} \det L_{\mathrm{s}}[V \setminus X] \cdot \det L_{\mathrm{d}}[X] \qquad (2) \end{aligned}$$

by the definition of W. Figure 1 shows submatrices $L_{\mathrm{s}}[V \setminus X]$ and $L_{\mathrm{d}}[X]$. Since $L_{\mathrm{s}}[V \setminus X]$ is skew-symmetric, $\det L_{\mathrm{s}}[V \setminus X] \geq 0$ holds. Moreover, we have $\det L_{\mathrm{d}}[X] > 0$ for $X \subseteq W$. Hence each term of (2) is nonnegative. Thus, L is nonsingular if and only if there exists $X \subseteq W$ such that $L_{\mathrm{s}}[V \setminus X]$ is nonsingular. $\square$

The copy of W is denoted by W_{c}. For $X \subseteq W$, X_{c} denotes a copy of X included in W_{c}.

We define a skew-symmetric matrix K in the form of

$$K = \begin{array}{c} \begin{array}{ccc} W_{\mathrm{c}} & W & V \setminus W \end{array} \\ \left(\begin{array}{c|cc} O & L_{\mathrm{d}}[W] & O \\ \hline -L_{\mathrm{d}}[W] & \multicolumn{2}{c}{\multirow{2}{*}{L_{s}}} \\ O & & \end{array}\right) \end{array}. \tag{3}$$

The matrices L and K are related as follows.

Lemma 3.2. The following (i) and (ii) hold.

(i) For any $X \subseteq W_{\mathrm{c}} \cup V$ such that $K[X]$ is nonsingular, $L[X \setminus W_{\mathrm{c}}]$ is nonsingular.

(ii) For any $Y \subseteq V$ such that $L[Y]$ is nonsingular, there exists a subset $Z \subseteq W_{\mathrm{c}}$ such that $K[Z \cup Y]$ is nonsingular.

Proof. We first prove (i). Let us denote $X \cap W_{\mathrm{c}}$ by Z_{c} and its copy by $Z \subseteq W$. By the definition of K, it holds that

$$\det K[X] = \det K[Z_{\mathrm{c}}, Z] \cdot \det K[Z, Z_{\mathrm{c}}] \cdot \det K[X \setminus (Z \cup Z_{\mathrm{c}})]. \tag{4}$$

Since $L_{\mathrm{d}}[W]$ is a diagonal matrix with positive entries, both $\det K[Z_{\mathrm{c}}, Z]$ and $\det K[Z, Z_{\mathrm{c}}]$ are nonzero. By $\det K[X] \neq 0$, it holds that $\det K[X \setminus (Z \cup Z_{\mathrm{c}})] = \det L_{\mathrm{s}}[(X \setminus W_{\mathrm{c}}) \setminus Z] \neq 0$. Since $Z \subseteq X \cap W$, this implies by Lemma 3.1 that $L[X \setminus W_{\mathrm{c}}]$ is nonsingular.

We next prove (ii). Since $L[Y]$ is nonsingular, Lemma 3.1 implies that there exists a subset $Z \subseteq W \cap Y$ such that $L_{\mathrm{s}}[Y \setminus Z]$ is nonsingular. Letting $X = Z_{\mathrm{c}} \cup Y$, we know that $K[X]$ is nonsingular from (4), which proves (ii). $\square$

Let us define $\mathcal{F}_S(K) = \mathcal{F}(K) \triangle (S \cup W_{\mathrm{c}})$. Since K is a skew-symmetric matrix, $(W_{\mathrm{c}} \cup V, \mathcal{F}_S(K))$ is a linear delta-matroid. Lemma 3.2 leads to the following theorem.

Theorem 3.3. Let L be a skew-symmetric matrix with nonnegative diagonals, V be the row/column set of L, and K be a skew-symmetric matrix defined by (3). For a subset S of V, the pair $(V, \mathcal{F}(L) \triangle S)$ is a projection of the linear delta-matroid $(W_{\mathrm{c}} \cup V, \mathcal{F}_S(K))$ on W_{c}. Moreover, if L is generic, then $(W_{\mathrm{c}} \cup V, \mathcal{F}_S(K))$ is a generic linear delta-matroid.

Proof. The projection of $\mathcal{F}_S(K)$ on W_{c} is equal to

$$\begin{aligned} \mathcal{F}_S(K)|W_{\mathrm{c}} &= \{F \setminus W_{\mathrm{c}} \mid F \in \mathcal{F}_S(K)\} \\ &= \{(X \triangle (S \cup W_{\mathrm{c}})) \setminus W_{\mathrm{c}} \mid \operatorname{rank} K[X] = |X|, X \subseteq W_{\mathrm{c}} \cup V\} \\ &= \{(X \setminus W_{\mathrm{c}}) \triangle S \mid \operatorname{rank} K[X] = |X|, X \subseteq W_{\mathrm{c}} \cup V\}. \end{aligned}$$

If $X \subseteq W_{\mathrm{c}} \cup V$ satisfies $\operatorname{rank} K[X] = |X|$, then $L[X \setminus W_{\mathrm{c}}]$ is nonsingular by (i) in Lemma 3.2. This implies $\mathcal{F}_S(K)|W_{\mathrm{c}} \subseteq \mathcal{F}(L) \triangle S$. Conversely, if $Y \subseteq V$ satisfies $\operatorname{rank} L[Y] = |Y|$, it follows from (ii) in Lemma 3.2 that there exists a subset $Z \subseteq W_{\mathrm{c}}$ such that $K[Z \cup Y]$ is nonsingular. Hence $Y \triangle S = ((Z \cup Y) \setminus W_{\mathrm{c}}) \triangle S \in \mathcal{F}_S(K)|W_{\mathrm{c}}$. Thus we obtain $\mathcal{F}(L) \triangle S \subseteq \mathcal{F}_S(K)|W_{\mathrm{c}}$.

If L is generic, K defined by (3) is a generic skew-symmetric matrix, which implies that $(W_{\mathrm{c}} \cup V, \mathcal{F}_S(K))$ is a generic linear delta-matroid. $\square$

The following corollary immediately follows from Theorem 3.3, because a projection of a delta-matroid is also a delta-matroid. Note that the resulting delta-matroid is not necessarily linear.

Corollary 3.4. Let L be a skew-symmetric matrix with nonnegative diagonals, and V be the row/column set of L. For a subset $S \subseteq V$, the pair $(V, \mathcal{F}(L) \triangle S)$ is a delta-matroid.

3.2 Constrained matching problems with delta-matroids

In this section, we provide a variety of constrained matching problems that are included in the matching problem with a projected linear delta-matroid. We first remark that the identity matrix is a skew-symmetric matrix with nonnegative diagonals, which represents free constraints, and hence the standard matching problem is contained in our problem by Theorem 3.3.

Example 3.1. (2-master/slave-constraint) The master/slave matching problem, introduced by Hefner and Kleinschmidt [20], arises in manpower scheduling problem in printing plants. Let $G = (V, E)$ be an undirected graph and $D = (V, A)$ be a directed graph with the same vertex set as G. For $(u_{\mathrm{s}}, u_{\mathrm{m}}) \in A$, we say that u_{s} is a *slave* of u_{m} and u_{m} is a *master* of u_{s}. A *master/slave matching* (*MS-matching* for short) is a matching M in G which satisfies the following *master/slave constraint* (*MS-constraint*):

$$(u_{\mathrm{s}}, u_{\mathrm{m}}) \in A, u_{\mathrm{s}} \in \partial M \Rightarrow u_{\mathrm{m}} \in \partial M. \tag{5}$$

The *MS-matching problem* (*MSMP*) is to find an MS-matching of maximum cardinality. Given a weight function $w : E \to \mathbb{R}_+$, the *weighted MSMP* is defined similarly. Hefner and Kleinschmidt [20] proved that it is NP-hard in general, while it can be solved in $O(|V|^3)$ time under the assumption that $k(D) \leq 2$, where $k(D)$ denotes the maximum size of a connected component of D. Such MSMP is called the *2-MSMP*.

Let $D = (V, A)$ be a directed graph with $k(D) \leq 2$. Then D consists of the following three kinds of components:

$$\text{(a)} \quad \circ \qquad \text{(b)} \quad \circ \longleftarrow \circ \qquad \text{(c)} \quad \circ \longleftrightarrow \circ$$

MS-constraints (5) arising from (a), (b), and (c) correspond to $\mathcal{F}(L^{\mathrm{a}})$, $\mathcal{F}(L^{\mathrm{b}})$, and $\mathcal{F}(L^{\mathrm{c}})$, respectively, where

$$L^{\mathrm{a}} = \begin{pmatrix} d \end{pmatrix}, \quad L^{\mathrm{b}} = \begin{pmatrix} d & k \\ -k & 0 \end{pmatrix}, \quad \text{and} \quad L^{\mathrm{c}} = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}$$

with constants $d > 0$ and $k \neq 0$. Here, L^{a}, L^{b}, and L^{c} are generic skew-symmetric matrices with nonnegative diagonals. Thus the 2-MSMP is included in the matching problem with projected linear delta-matroids. It should be noted that MS-constraints with $k(D) \geq 3$ are not necessarily contained in our framework. $\square$

Example 3.2. (multi-master/single-slave constraint) Amanuma and Shigeno [1] introduced the following variant of the MSMP, motivated by the observation of staff scheduling with new staffs. Given an undirected graph $G = (V, E)$ and a directed graph $D = (V, A)$, we say that a *multi-master single-slave* matching (*MMSS-matching*) is a matching M in G satisfying

$$u \in \partial M,\ N^+(u) \neq \emptyset \Rightarrow N^+(u) \cap \partial M \neq \emptyset,$$

where $N^+(u) = \{v \in V \mid (u,v) \in A\}$. Note that if $k(D) \leq 2$, then the family of MMSS-matchings coincides with that of MS-matchings. Amanuma and Shigeno showed that this problem can be solved in polynomial time if each connected component of D has (i) exactly one slave vertex u_{s}, or (ii) exactly two slave vertices $u_{\mathrm{s}}^1, u_{\mathrm{s}}^2$ with $(u_{\mathrm{s}}^1, u_{\mathrm{s}}^2), (u_{\mathrm{s}}^2, u_{\mathrm{s}}^1) \in A$. MMSS-constraints arising from (i) and (ii) correspond to $\mathcal{F}(L^{\mathrm{i}})$ and $\mathcal{F}(L^{\mathrm{ii}})$ determined by

$$L^{\mathrm{i}} = \begin{array}{c} \begin{array}{cccc} u_{\mathrm{s}} & u_{\mathrm{m}}^1 & \cdots & u_{\mathrm{m}}^n \end{array} \\ \begin{pmatrix} 0 & k_1 & \cdots & k_n \\ -k_1 & & & \\ \vdots & & D & \\ -k_n & & & \end{pmatrix} \end{array} \quad \text{and} \quad L^{\mathrm{ii}} = \begin{array}{c} \begin{array}{ccccc} u_{\mathrm{s}}^1 & u_{\mathrm{s}}^2 & U_{\mathrm{m}}^1 & U_{\mathrm{m}}^2 & U_{\mathrm{m}}^{12} \end{array} \\ \begin{pmatrix} 0 & k_1 & \boldsymbol{k}_1^\top & \boldsymbol{0}^\top & \boldsymbol{k}_2^\top \\ -k_1 & 0 & \boldsymbol{0}^\top & \boldsymbol{k}_3^\top & \boldsymbol{k}_4^\top \\ -\boldsymbol{k}_1 & \boldsymbol{0} & & & \\ \boldsymbol{0} & -\boldsymbol{k}_3 & & D & \\ -\boldsymbol{k}_2 & -\boldsymbol{k}_4 & & & \end{pmatrix} \end{array},$$

where D is a diagonal matrix with positive entries and $\boldsymbol{k}_j = (k_1^j, \ldots, k_{n_j}^j)^\top$ is a vector of size n_j with constants $k_1^j, \ldots, k_{n_j}^j$ for $j = 1, \ldots, 4$. Indeed, if a feasible set F of $\mathcal{F}(L^{\mathrm{i}})$ contains u_{s}, F contains at least one u_{m}^i. Next, consider the case of $\mathcal{F}(L^{\mathrm{ii}})$. If a feasible set F of $\mathcal{F}(L^{\mathrm{ii}})$ contains both u_{s}^1 and u_{s}^2, then F can contain any subset of $U_{\mathrm{m}}^1 \cup U_{\mathrm{m}}^2 \cup U_{\mathrm{m}}^{12}$. This is because u_{s}^1 is a master of u_{s}^2 and vice versa. If F contains u_{s}^1 and not u_{s}^2, then F has to contain at least one element of $U_{\mathrm{m}}^1 \cup U_{\mathrm{m}}^{12}$, which is a master of u_{s}^1. Here, L^{i} and L^{ii} are generic skew-symmetric matrices with nonnegative diagonals. □

Example 3.3. (matching constraint) Let $G = (V, E_G)$ be an undirected graph and $(V, \mathcal{F}_H)$ be the matching delta-matroid of an undirected graph $H = (V, E_H)$, i.e., $F \in \mathcal{F}_H$ if H has a matching M with $\partial M = F$. Since $(V, \mathcal{F}_H)$ is represented by a generic skew-symmetric matrix whose support graph coincides with H, the matching problem with $(V, \mathcal{F}_H)$ is contained in our framework with no projection.

This constraint arises in the maximum cycle subgraph problem in a 2-edge-colored multigraph [3]. A 2-edge-colored multigraph is a graph which has red edges and blue edges. A cycle subgraph is a union of disjoint cycles whose successive edges differ in color. It is known that a maximum cycle subgraph can be found in polynomial time by the maximum matching problem. Let G_{r} and G_{b} be the subgraphs consisting of all the red edges and all the blue edges, respectively. We know that the maximum cycle subgraph problem is equivalent to finding a maximum cardinality matching M in G_{r} subject to ∂M is feasible for the matching delta-matroid of G_{b}. □

Example 3.4. (upper/lower size constraint) Let $G = (V, E)$ be an undirected graph. Define $V_1, \ldots, V_m$ to be a partition of V and $r_1, \ldots, r_m$ to be positive integers. In this setting we aim at finding a maximum cardinality matching M with $|\partial M \cap V_i| \leq r_i$ for any i.

This problem can be viewed as the matching problem with a projection of a generic even delta-matroid. Indeed, define a bipartite graph $H = (V, W; E_H)$ as follows:

$$W = \bigcup_{i=1}^{m} W_i, \text{ where } W_i = \{w_1^i, \dots, w_{r_i}^i\} \quad (i = 1, \dots, m),$$
$$E_H = \bigcup_{i=1}^{m} E_i, \text{ where } E_i = \{(v, w) \mid v \in V_i, w \in W_i\} \quad (i = 1, \dots, m).$$

Thus H is the disjoint union of complete bipartite graphs with vertex sets V_i and W_i for all i. Consider the matching matroid of H, denoted by $\mathcal{F}_H$. Then the size constraint is represented by using $(V, \mathcal{F}_H|W)$.

Note that $X \in \mathcal{F}_H \triangle V|W$ if and only if $|X \cap V_i| \geq |V_i| - r_i$ for any i. Hence by using the delta-matroid $(V, \mathcal{F}_H \triangle V|W)$, we can also deal with finding a maximum cardinality matching M with $|\partial M \cap V_i| \geq r_i'$ for any i, where $r_i' = |V_i| - r_i$. □

Example 3.5. (linear matroid constraint) Let $G = (V, E)$ be an undirected graph and $(V, \mathcal{I})$ be a linear matroid. Then we discuss finding a maximum cardinality matching M with $\partial M \in \mathcal{I}$.

We denote a representation of $(V, \mathcal{I})$ by a matrix A with row set W and column set V. Define a skew-symmetric matrix K to be

$$K = \begin{pmatrix} O & A \\ -A^\top & O \end{pmatrix}.$$

The column set of K is denoted by $W \cup V$. Then it is not difficult to see that $\mathcal{F}(K)|W = \mathcal{I}$. Thus the matching problem with linear matroid constraint can be reduced to one with projected linear delta-matroids. □

We conclude this section with a relation between our delta-matroid and a simultaneous delta-matroid. A pair $(V, \mathcal{F})$ is called a *simultaneous delta-matroid* if it satisfies the following simultaneous exchange property:

(SDM) For $F, F' \in \mathcal{F}$ and $u \in F \triangle F'$, there exists $v \in F \triangle F'$ such that $F \triangle \{u, v\} \in \mathcal{F}$ and $F' \triangle \{u, v\} \in \mathcal{F}$.

It is shown in [13, 37] that an even delta-matroid satisfies (SDM). In fact, a delta-matroid is simultaneous if and only if it is a projection of some even delta-matroid on a singleton [13]. Simultaneous delta-matroids are also discussed in Takazawa [36], where he showed that delta-matroids arising from branchings and matching forests both satisfy (SDM). We prove that our delta-matroid is a simultaneous delta-matroid as follows.

Theorem 3.5. A delta-matroid $(V, \mathcal{F}_K)$ given by (1) is a simultaneous delta-matroid.

Proof. We denote $\mathcal{F}(K) \triangle S$ by $\tilde{\mathcal{F}}$. Then $(V_K, \tilde{\mathcal{F}})$ is an even delta-matroid. We prove that $\mathcal{F}_K = \tilde{\mathcal{F}}|R$ satisfies the simultaneous exchange axiom (SDM). Let $F, F' \in \mathcal{F}_K$ and $u \in F \triangle F'$. Then there exist $Y, Y' \subseteq R$ such that $F \cup Y, F' \cup Y' \in \tilde{\mathcal{F}}$. Since $\tilde{\mathcal{F}}$ satisfies the simultaneous exchange axiom (SDM), there exists $\tilde{v} \in (F \cup Y) \triangle (F' \cup Y') = (F \triangle F') \cup (Y \triangle Y')$ such that

$$(F \cup Y) \triangle \{u, \tilde{v}\} \in \tilde{\mathcal{F}} \quad \text{and} \quad (F' \cup Y') \triangle \{u, \tilde{v}\} \in \tilde{\mathcal{F}}. \tag{6}$$

If $\tilde{v} \in F \triangle F'$, (6) is rewritten as $(F \triangle \{u, \tilde{v}\}) \cup Y \in \tilde{\mathcal{F}}$ and $(F' \triangle \{u, \tilde{v}\}) \cup Y' \in \tilde{\mathcal{F}}$. Hence we have $F \triangle \{u, \tilde{v}\} \in \mathcal{F}_K$ and $F' \triangle \{u, \tilde{v}\} \in \mathcal{F}_K$. Thus (SDM) with $v = \tilde{v}$ holds. If $\tilde{v} \in Y \triangle Y'$, it follows from (6) that $F \triangle \{u\} \in \mathcal{F}_K$ and $F' \triangle \{u\} \in \mathcal{F}_K$. This implies that (SDM) with $v = u$ holds. $\square$

4 Generic delta-matroid matching

In this section, we discuss weighted non-bipartite matching problem with a projected linear delta-matroid, and show that it can be solved in polynomial time if a given skew-symmetric matrix K is generic. Note that all the delta-matroids given in Examples 3.1–3.4 are generic. Thus weighted matching problems with these constraints can be solved in polynomial time.

Let $G = (V, E_G)$ be a graph, $w : E_G \to \mathbb{R}_+$ be an edge weight, and K be a skew-symmetric matrix with row/column set $V_K \supseteq V$. We denote $V_K \setminus V$ by R. For a subset S of V, let $\mathcal{F}_K$ be given by (1). The main theorem of this section is the following.

Theorem 4.1. If a skew-symmetric matrix K is generic, the weighted delta-matroid matching problem with $(V, \mathcal{F}_K)$ can be solved in $O(|V|^3)$ time.

By the definition, we have

$$\mathcal{F}_K = \{(X \setminus R) \triangle S \mid \operatorname{rank} K[X] = |X|, X \subseteq R \cup V\},$$

because S and R are disjoint. Let $H = (R \cup V, E_H)$ be the support graph of K. Since K is generic, we obtain

$$\mathcal{F}_K = \{(\partial M_H \setminus R) \triangle S \mid M_H\text{: matching in } H\}$$

by Lemma 2.1. Then, the given problem is rewritten as a problem to find a matching M_G in G which maximizes $w(M_G)$ such that there exists a matching M_H in H with $(\partial M_H \setminus R) \triangle S = \partial M_G$.

The above problem is reduced to the maximum weight matching problem as follows. Let V_G and V_H be copies of V, and V_S be a copy of $S \subseteq V$. The copies of $i \in V$ are denoted by $i_G \in V_G$, $i_H \in V_H$, and $i_S \in V_S$, respectively. We set $U = V_G \cup V_H \cup V_S$ for convenience. For $G = (V_G, E_G)$ and $H = (R \cup V_H, E_H)$, we construct a graph $\Gamma = (U \cup R, E_G \cup E_H \cup \tilde{E})$ with

$$\tilde{E} = \{(i_G, i_H) \mid i \in V \setminus S\} \cup \{(i_G, i_S), (i_S, i_H) \mid i \in S\}.$$

The weight $\gamma(e)$ of an edge $e \in E_G \cup E_H \cup \tilde{E}$ is defined to be

$$\gamma(e) = \begin{cases} w(e) & (e \in E_G) \\ 0 & (e \in E_H \cup \tilde{E}) \end{cases}.$$

Then the following lemma holds.

Lemma 4.2. The following (i) and (ii) hold.

(i) For a matching M in Γ with $U \subseteq \partial M$, the set $M_G = M \cap E_G$ is a matching in G with $\partial M_G \in \mathcal{F}_K$ and $w(M_G) = \gamma(M)$.

(ii) For a matching M_G in G with $\partial M_G \in \mathcal{F}_K$, there exists a matching M in Γ with $U \subseteq \partial M$ and $\gamma(M) = w(M_G)$.

Proof. Define $M_H = M \cap E_H$. Since $U \subseteq \partial M$, it holds that $i_G \in \partial M_G$ if and only if $i_H \in \partial M_H$ for any $i \in V \setminus S$, and $i_G \in \partial M_G$ if and only if $i_H \notin \partial M_H$ for any $i \in S$. Hence $(\partial M_H \setminus R) \triangle S$ corresponds to ∂M_G. This means that M_G is a matching satisfying $\partial M_G \in \mathcal{F}_K$ and $w(M_G) = \gamma(M)$.

We next prove (ii). By the definition of $\mathcal{F}_K$ where K is generic, there exists a matching M_H in H with $(\partial M_H \setminus R) \triangle S = \partial M_G$. We define

$$M = M_G \cup M_H \cup \{(i_S, i_H) \mid i \in S \cap \partial M_G\} \\ \cup \{(i_G, i_S) \mid i \in S \cap \partial M_H\} \cup \{(i_G, i_H) \mid i \notin \partial M_G \cup \partial M_H)\}.$$

Then M is a matching in Γ satisfying $U \subseteq \partial M$ and $\gamma(M) = w(M_G)$. □

By Lemma 4.2, a matching M in Γ has a corresponding matching M_G in G, and vice versa. Thus, we reduce our problem to maximum weight matching problem in Γ, and solve this problem in $O(|V|^3)$ time [16, 26]. This completes the proof of Theorem 4.1. □

5 Bipartite delta-matroid matching

Let $G = (V^+, V^-; E)$ be a bipartite graph, and $(V^-, \mathcal{F}^-)$ be a delta-matroid on V^-. The *bipartite delta-matroid matching* is to find a maximum cardinality matching M in G satisfying $\partial^- M \in \mathcal{F}^-$. Although Section 4 assumes that a matrix K is generic, this section deals with K over the rational field $\mathbb{Q}$. This problem is a special case of the delta-matroid matching problem, where G is restricted to be bipartite and the delta-matroid is defined on V^-. The general case is discussed in Section 6.

Let K be an $n \times n$ skew-symmetric matrix with row/column set V_K with $V^- \subseteq V_K$. Define a delta-matroid $(V^-, \mathcal{F}_K^-)$ to be

$$\mathcal{F}_K^- = \mathcal{F}(K)|R \tag{7}$$

with $R = V_K \setminus V^-$. In this section, we mainly focus on the bipartite delta-matroid matching with $(V^-, \mathcal{F}_K^-)$. We remark that K is not assumed to be generic, which is different from Section 4, and instead, we assume that $\mathcal{F}_K$ is not twisted in this section.

In Section 5.1, we describe a polynomial-time algorithm for this problem, which is based on mixed matrix theory described in Section 2.3. For that purpose, we first define operations induced by a bipartite graph for a delta-matroid, and show that this operation preserves linear delta-matroidness. We further impose linear matroid constraints on V^+ in Section 5.2.

5.1 Algorithm via mixed skew-symmetric matrices

Let $G = (V^+, V^-; E)$ be a bipartite graph. For a delta-matroid $(V^-, \mathcal{F}^-)$ on the one vertex side of G, we define

$$\mathcal{F}^+ = \{J^+ \subseteq V^+ \mid G[J^+, J^-] \text{ has a perfect matching for some } J^- \in \mathcal{F}^-\}.$$

Then it is known that $(V^+, \mathcal{F}^+)$ forms a delta-matroid [5], called the delta-matroid induced by G. We will further show that for a delta-matroid $(V^-, \mathcal{F}_K^-)$ given by (7), the delta-matroid $(V^+, \mathcal{F}_K^+)$ induced by G is also represented by a skew-symmetric matrix.

Define $T(G) = (T_{ij})$ to be a generic matrix with row set V^+ and column set V^- provided that $T_{ij} \neq 0$ if $(i,j) \in E$ and $T_{ij} = 0$ otherwise. Note that for vertex sets $J^+ \subseteq V^+$ and $J^- \subseteq V^-$, $G[J^+, J^-]$ has a perfect matching if and only if $T(G)[J^+, J^-]$ is nonsingular. We denote by K' a mixed skew-symmetric matrix in the form of

$$K' = \begin{array}{c} \begin{array}{cccc} V^+ & V_{\mathrm{c}}^- & V^- & R \end{array} \\ \left(\begin{array}{cc|c} O & T(G) & O \quad O \\ -T(G)^\top & O & D \quad O \\ \hline O & -D & \multirow{2}{*}{K} \\ O & O & \end{array}\right) \end{array}, \tag{8}$$

where $D = (d_{ii})$ denotes a generic diagonal matrix all of whose diagonal entries are nonzero. We identify each element of V^- with that of V_{c}^- in correspondence with D. Let V' be the whole column set of K' and $U' = V_{\mathrm{c}}^- \cup V^-$.

Lemma 5.1. Let $G = (V^+, V^-; E)$ be a bipartite graph and $(V^-, \mathcal{F}_K^-)$ be a delta-matroid given by (7). Then the delta-matroid $(V^+, \mathcal{F}_K^+)$ induced by G is equal to $\mathcal{F}(K')/U'|R$, where K' is defined by (8). In particular, if $R = \emptyset$ then $(V^+, \mathcal{F}_K^+)$ is a linear delta-matroid.

Proof. Let $J^+ \subseteq V^+$ be a feasible set in $\mathcal{F}(K')/U'|R$. Then there exists $Y \subseteq R$ such that $K'[J^+ \cup U' \cup Y]$ is nonsingular. This submatrix is skew-symmetric, and its Pfaffian is equal to

$$\operatorname{pf} K'[J^+ \cup U' \cup Y] = \sum_{X \subseteq V^-} \pm \det T(G)[J^+, X_{\mathrm{c}}] \cdot \prod_{i \in V^- \setminus X} d_{ii} \cdot \operatorname{pf} K[X \cup Y]. \tag{9}$$

Since the left-hand side is nonzero, there exists $X \subseteq V^-$ such that both of $T(G)[J^+, X_{\mathrm{c}}]$ and $K[X \cup Y]$ are nonsingular. This implies that $X \in \mathcal{F}_K^-$ and hence $J^+ \in \mathcal{F}_K^+$.

Conversely, let $J^+ \subseteq V^+$ be a feasible set in $\mathcal{F}_K^+$. Then there exists $J^- \subseteq V^-$ such that $T(G)[J^+, J^-]$ is nonsingular and $J^- \in \mathcal{F}_K^-$. It follows from $J^- \in \mathcal{F}_K^-$ that there exists $Y \subseteq R$ such that $K[J^- \cup Y]$ is nonsingular. Consider the principal submatrix $K'[J^+ \cup U' \cup Y]$ of K'. Its Pfaffian is equal to (9). The right-hand side has at least one nonzero term when $X = J^-$. Since each nonzero term has distinct d_{ii}'s, we conclude that $\operatorname{pf} K'[J^+ \cup U' \cup Y] \neq 0$ by genericity of d_{ii}'s. Thus $J^+ \cup U' \cup Y$ is feasible for $\mathcal{F}(K')$, and hence $J^+ \in \mathcal{F}(K')/U'|R$.

Therefore, the first part of the statement holds. Note that, since $K'[U']$ is nonsingular, $\mathcal{F}(K')/U'$ can be represented by some skew-symmetric matrix by pivoting operation. Hence if $R = \emptyset$ then $(V^+, \mathcal{F}_K^+)$ is a linear delta-matroid. □

We prove the following theorem using Lemma 5.1.

Theorem 5.2. The bipartite delta-matroid matching with $(V^-, \mathcal{F}_K^-)$ given by (7) can be solved in polynomial time.

Proof. We show that this problem can be solved by a greedy algorithm for a delta-matroid. By the definition of $\mathcal{F}_K^+$, for a set $X \subseteq V^+$, there exists a matching M in G with $X = \partial^+ M$

and $\partial^- M \in \mathcal{F}_K^-$ if and only if $X \in \mathcal{F}_K^+$. Hence the maximum size of delta-matroid matchings is equal to the maximum size of feasible sets in $\mathcal{F}_K^+$. Since $(V^+, \mathcal{F}_K^+)$ is a delta-matroid, we can find a maximum feasible set in $\mathcal{F}_K^+$ by a greedy algorithm. In fact, for a delta-matroid $\mathcal{F}(K')$, define a weight function $c : V' \to \mathbb{R}_+$ as follows:

$$c(v) = \begin{cases} 1 & (v \in V^+) \\ 0 & (v \notin V^+) \end{cases}.$$

Then, by Lemma 5.1, finding a maximum feasible set in $\mathcal{F}_K^+$ is equivalent to finding a feasible set X in $\mathcal{F}(K')$ with $U' \subseteq X$ that maximizes $c(X)$. Such X can be found in polynomial time by Lemma 2.4. Note that a maximum matching corresponding to X can also be computed by Corollary 2.3. □

5.2 Bipartite delta-matroid matching with linear matroid constraints

In this section, we deal with the bipartite delta-matroid matching with linear matroid constraints on the other vertex side. That is, for the bipartite delta-matroid matching with $(V^-, \mathcal{F}_K^-)$ given by (7), we are additionally given a linear matroid $(V^+, \mathcal{I})$, and our task is to find a maximum cardinality matching M in G subject to $\partial^+ M \in \mathcal{I}$ and $\partial^- M \in \mathcal{F}_K^-$.

Let A be an $m \times n$ matrix representing $(V^+, \mathcal{I})$. The column set corresponds to V^+ and the row set is denoted by W. We now construct a mixed skew-symmetric matrix

$$\check{K} = \begin{array}{c} \begin{array}{cccccc} W & V_{\mathrm{c}}^+ & V^+ & V_{\mathrm{c}}^- & V^- & R \end{array} \\ \left(\begin{array}{cc|cccc} O & A & O & O & O & O \\ -A^\top & O & D' & O & O & O \\ \hline O & -D' & & & & \\ O & O & & \multicolumn{2}{c}{K'} & \\ O & O & & & & \\ O & O & & & & \end{array}\right) \end{array},$$

where K' is defined in (8) and $D' = (d'_{ii})$ is a generic diagonal matrix all of whose diagonal entries are nonzero. Let $\check{V}$ be the whole column set of $\check{K}$ and $\check{U} = V_{\mathrm{c}}^+ \cup V^+ \cup V_{\mathrm{c}}^- \cup V^-$. We consider a delta-matroid $(W, \mathcal{F}(\check{K})/\check{U}|R)$. Then we can prove the following lemma in a similar way to Section 5.1.

Lemma 5.3. *Let $X \subseteq W$. Then there exists a matching M in G such that $A[X, \partial^+ M]$ is nonsingular and $\partial^- M \in \mathcal{F}_K^-$ if and only if $X \in \mathcal{F}(\check{K})/\check{U}|R$.*

Proof. Assume that X is a feasible set in $\mathcal{F}(\check{K})/\check{U}|R$. Then there exists $Y \subseteq R$ such that $\check{K}[X \cup \check{U} \cup Y]$ is nonsingular. The Pfaffian of this submatrix is equal to

$$\operatorname{pf} \check{K}[X \cup \check{U} \cup Y] = \sum_{J \subseteq V^+} \pm \det A[X, J_{\mathrm{c}}] \cdot \prod_{i \in V^+ \setminus J} d'_{ii} \cdot \operatorname{pf} K'[J \cup V_{\mathrm{c}}^- \cup V^- \cup Y]. \tag{10}$$

Since the left-hand side is nonzero, there exists $J \subseteq V^+$ such that both of $A[X, J_{\mathrm{c}}]$ and $K'[J \cup V_{\mathrm{c}}^- \cup V^- \cup Y]$ are nonsingular. Moreover, the nonsingularity of the latter matrix implies that

$J \in \mathcal{F}_K^+$ by Lemma 5.1, which means that there exists a matching M in G with $\partial^+ M = J$ and $\partial^- M \in \mathcal{F}_K^-$. Thus the sufficiency of the statement holds.

Conversely, let M be a matching in G such that $A[X, \partial^+ M]$ is nonsingular and $\partial^- M \in \mathcal{F}_K^-$. We set $J^+ = \partial^+ M$ and $J^- = \partial^- M$. Then $J^+ \in \mathcal{F}_K^+$ follows from the definition of $\mathcal{F}_K^+$. By Lemma 5.1, there exists $Y \subseteq R$ such that $K'[J^+ \cup V_c^- \cup V^- \cup Y]$ is nonsingular. Consider the principal submatrix $\check{K}[X \cup \check{U} \cup Y]$ of $\check{K}$. Its Pfaffian is equal to (10), whose right-hand side has at least one nonzero term when $J = J^+$. Since each nonzero term has distinct d'_{ii}'s, we conclude that $\operatorname{pf} \check{K}[X \cup \check{U} \cup Y] \neq 0$ by genericity of d'_{ii}'s. Thus $X \in \mathcal{F}(\check{K})/\check{U}|R$ holds, and hence we obtain the necessity of the statement. □

Therefore, in order to find a maximum matching M satisfying given delta-matroid constraints, it suffices to find a maximum feasible set $X \subseteq W$ in $\mathcal{F}(\check{K})/\check{U}|R$. Such X can be found in polynomial time by applying Lemma 2.4 to a linear delta-matroid $(\check{V}, \mathcal{F}(\check{K}'))$, a set $\check{U}$ required to be contained, and a weight function $c : \check{V} \to \mathbb{R}_+$ defined by

$$c(v) = \begin{cases} 1 & (v \in W) \\ 0 & (v \notin W) \end{cases}.$$

We can construct from X a desired matching M by Corollary 2.3.

Thus we obtain the following theorem.

Theorem 5.4. For a delta-matroid $(V^-, \mathcal{F}_K^-)$ given by (7), the bipartite delta-matroid matching with linear matroid constraints on V^+ can be solved in polynomial time.

Remark 5.5. Hefner and Kleinschmidt [21] extended the 2-MSMP to one with capacity constraints. That is, given a bipartite graph $G = (V^+, V^-; E)$, a directed graph $D = (V^+ \cup V^-, A)$ with $k(D) \leq 2$, a partition $(V_1^+, \dots, V_m^+)$ of V^+, and positive integers $r_1, \dots, r_m$, we aim at finding a maximum cardinality MS-matching M in G subject to $|\partial M \cap V_i^+| \leq r_i$ for $1 \leq i \leq m$. They showed that this problem is NP-hard in general.

Theorem 5.4 indicates that if 2-MS-constraints are imposed only on V^-, the 2-MSMP with capacity constraints can be solved in polynomial time, because the capacity constraints on V^+ can be represented by a linear matroid.

6 Related intersection-type problem

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two families of subsets in a finite set V. The *intersection problem with respect to* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ is the problem of finding a maximum cardinality set in $\mathcal{S}_1 \cap \mathcal{S}_2$. A well-known example of this kind of problem is the case where both $\mathcal{S}_1$ and $\mathcal{S}_2$ are matroids, called the *matroid intersection problem*, which can be solved in polynomial time (see e.g., [34]). In this section, we summarize our results from the viewpoint of the intersection problem with respect to a matroid and a delta-matroid. For a matroid $(V, \mathcal{I})$ and a delta-matroid $(V, \mathcal{F})$, we classify related intersection problems with respect to $\mathcal{I}$ and $\mathcal{F}$ as Table 1. Note that an untwisted linear delta-matroid in Table 1 means a delta-matroid $\mathcal{F}(K)$ for some skew-symmetric matrix K.

Let us first discuss the case where $(V, \mathcal{I})$ is a transversal matroid, that is, given a bipartite graph $G = (U, V; E)$, $I \in \mathcal{I}$ holds if and only if G has a matching covering I. Hence the intersection problem with a transversal matroid $(V, \mathcal{I})$ coincides with the bipartite delta-matroid

Table 1: Intersection problems with a matroid and a delta-matroid

		delta-matroid $(V, \mathcal{F})$		
		matching delta-matroid	untwisted linear delta-matroid	general delta-matroid
matroid $(V, \mathcal{I})$	transversal matroid	poly. time (Theorem 4.1)	poly. time (Theorem 5.2)	exp. time (Theorem 6.1)
	linear matroid	poly. time [28] (linear matroid matching)	poly. time (Theorem 5.4)	exp. time
	general matroid	exp. time [23] (matroid matching)	exp. time	exp. time

matching. Therefore, it follows from Theorems 4.1 and 5.2 that we can solve the intersection problems in polynomial time if $(V, \mathcal{F})$ is a matching delta-matroid and an untwisted linear delta-matroid, respectively. However, it turns out to be intractable if $(V, \mathcal{F})$ is not linear.

Theorem 6.1. The bipartite delta-matroid matching problem, i.e., the intersection problem with respect to a transversal matroid and a delta-matroid, cannot be solved in polynomial time in the oracle model.

Proof. Let $G = (V, E)$ be a graph with $|V| = n$ and $(V, \mathcal{I})$ be a matroid with rank r. We consider to find a matching M_G of size r with $\partial M_G \in \mathcal{I}$, which is known to require exponential time in the oracle model [34, §43.9].

We define two delta-matroids $(V, \mathcal{F}_{\rm b})$ and $(V, \mathcal{F}_G)$ to be the base family of the given matroid and the matching delta-matroid of G, respectively. We denote by V_1 and V_2 the copies of V, and by $v_i \in V_i$ $(i = 1, 2)$ the copies of $v \in V$. Construct a bipartite graph $H = (V, V_1 \cup V_2; E)$ with $E = \{(v, v_i) \mid v \in V, i = 1, 2\}$. Let $(V_1 \cup V_2, \mathcal{F})$ be a delta-matroid defined by the direct sum of $(V_1, \mathcal{F}_{\rm b})$ and $(V_2, \mathcal{F}_G \triangle V_2)$. Then H has a matching M_H of size n with $\partial M_H \cap (V_1 \cap V_2) \in \mathcal{F}$ if and only if G has a matching M_G of size r with $\partial M_G \in \mathcal{I}$. Indeed, if such M_G exists, by letting J_i be the copies of ∂M_G in V_i $(i = 1, 2)$, we know that $J_1 \in \mathcal{F}_{\rm b}$ and $J_2 \triangle V_2 = V_2 \setminus J_2 \in \mathcal{F}_G \triangle V_2$, and hence H has the matching M_H covering $J_1 \cup (V_2 \setminus J_2)$, which satisfies the constraints. The converse also holds by the same argument. Therefore, we can reduce finding a matching M_G of size r with $\partial M_G \in \mathcal{I}$ to the bipartite delta-matroid matching, and thus the statement holds. $\square$

Let us remark that the master/slave matching problem in Example 3.1 is shown to be NP-hard even if the graph G is bipartite and $k(D) = 3$ [1, 20]. Since the MS-constraint with $k(D) = 3$ can be represented by a (non-linear) delta-matroid, except for the case where the directed graph is complete, this fact implies that the bipartite delta-matroid matching is NP-hard even when an explicit representation of the delta-matroid is given.

Next assume that $(V, \mathcal{F})$ is a matching delta-matroid of a graph H. Then $(V, \mathcal{F})$ is represented by a generic slew-symmetric matrix whose support graph coincides with H. This means that the intersection problem with a matching delta-matroid $(V, \mathcal{F})$ is equivalent to the matroid matching. Therefore, if $(V, \mathcal{I})$ is linear then it can be solved in polynomial time, while

Table 2: Intersection problems with two delta-matroids

		delta-matroid $(V, \mathcal{F}_2)$		
		matching	linear	general
delta-matroid $(V, \mathcal{F}_1)$	matching	poly. time (matching)	open (linear delta-matroid matching)	exp. time (matroid matching)
	linear		open (linear delta-matroid matching)	exp. time

it requires exponential time if $(V, \mathcal{I})$ is general. Hence, for general $(V, \mathcal{I})$, the intersection problem requires exponential time, because a matching delta-matroid is a special case of (linear) delta-matroids.

The remaining case is that for a linear matroid and an untwisted delta-matroid. This problem can be solved in polynomial time by Theorem 5.4. Indeed, for a linear matroid $(V, \mathcal{I})$ and an untwisted linear delta-matroid $(V, \mathcal{F})$, construct a bipartite graph $G = (V, V'; E)$, where V' is a copy of V and E consists only of edges connecting a vertex in V and its copy. Then the intersection problem with respect to $\mathcal{I}$ and $\mathcal{F}$ is equivalent to finding a maximum matching M in G with $\partial^+ M \in \mathcal{I}$ and $\partial^- M \in \mathcal{F}$.

It should be noted that Fleiner *et al.* [14] deal with the intersection problem with respect to a matroid and the 2-MS-constraint in Example 3.1, which is a special case of a matching delta-matroid. They showed that this problem can be solved in polynomial time with the aid of the matroid intersection.

We conclude this section with remarks on the intersection problems with respect to two delta-matroids $\mathbf{M}_i = (V, \mathcal{F}_i)\,(i = 1, 2)$, which is summarized in Table 2. If $\mathbf{M}_1$ is a matching delta-matroid of a graph H, the problem is equivalent to the delta-matroid matching. This implies that if $\mathbf{M}_2$ is also a matching delta-matroid then we can solve it in polynomial time by Theorem 4.1, and if $\mathbf{M}_2$ is general then it requires exponential time. Assume that $\mathbf{M}_2$ is linear in addition. This case, which is equivalent to the matching problem with linear delta-matroids, is an important broader class of the delta-matroid matching, because it includes the linear matroid matching and the linear delta-covering as a special case. Let us remark that this problem is equivalent to that with two linear delta-matroids.

Theorem 6.2. The intersection problem with respect to two linear delta-matroids is reduced to the linear delta-matroid matching, i.e., the intersection problem with respect to a matching delta-matroid and a linear delta-matroid.

Proof. Let $(V, \mathcal{F}_i)\,(i = 1, 2)$ be two linear delta-matroids. Then construct a bipartite graph $G = (V, V'; E)$, where V' is a copy of V and E consists only of parallel edges connecting a vertex in V and its copy. Then the intersection problem with respect to $\mathcal{F}_1, \mathcal{F}_2$ is reduced to the bipartite delta-matroid matching with $(V \cup V', \mathcal{F})$, where $\mathcal{F}$ is a direct sum of $\mathcal{F}_1$ and $\mathcal{F}_2$. □

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