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The Double-Exponential Transformation Is Not Always Better Than the Tanh Transformation—Theoretical Convergence Analysis

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Abstract

In this paper, the theoretical convergence rate of the trapezoidal rule combined with the double-exponential (DE) transformation is given for a class of functions for which the tanh transformation is suitable. It is well known that the DE transformation enables the rule to achieve a much higher rate of convergence than the tanh transformation, and the convergence rate has been analyzed and justified theoretically under a proper assumption. Here it should be emphasized that the assumption is more severe than the tanh transformation's one, and there actually exist some examples such that the trapezoidal rule with the tanh transformation achieves its usual rate, whereas the rule with DE does not. Such cases have been observed numerically, but no theoretical analysis has been given so far. This paper reveals the theoretical rate of convergence in such cases, and it turns out that the DE's rate is almost the same, but slightly lower than the tanh's rate.

By using the analysis technique developed here, the theoretical convergence rate of the Sinc approximation/Sinc indefinite integration with the DE transformation is also given for a class of functions for which the tanh transformation is suitable. The results are quite similar to above; the convergence rate in the DE transformation's case is slightly lower than in the tanh transformation's case. Numerical examples which support those three theoretical results are also given.

1 Introduction

The double-exponential formula (DE formula) was proposed by Takahasi–Mori [17] as an *optimal* quadrature formula for the integral with end-point singularity, such as

$$\int_{-1}^{1} \frac{\mathrm{d}t}{(t-2)(1-t)^{1/4}(1+t)^{3/4}}. \tag{1.1}$$

The formula consists of two parts: (i) transformation to the integral over the whole real line, i.e.,

$$\int_a^b f(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} f(\psi_{\mathrm{DE}}(x))\psi_{\mathrm{DE}}'(x)\,\mathrm{d}x, \tag{1.2}$$

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and (ii) application of the (truncated) trapezoidal rule, i.e.,

$$\int_{-\infty}^{\infty} F(x)\,\mathrm{d}x \approx h \sum_{k=-M}^{N} F(kh), \tag{1.3}$$

where $F(x) = f(\psi_{\mathrm{DE}}(x))\psi_{\mathrm{DE}}'(x)$. The variable transformation used in (1.2) is called the "double-exponential transformation" (DE transformation) defined by

$$t = \psi_{\mathrm{DE}}(x) = \frac{b-a}{2}\tanh\left(\frac{\pi}{2}\sinh x\right) + \frac{b+a}{2}, \tag{1.4}$$

which maps the whole real line $\mathbb{R}$ onto the finite interval (a, b). This transformation is crucial in this formula. There are various options for the variable transformation to use with the trapezoidal rule, and in fact, before the DE formula there had already existed several quadrature rules that differ only in the variable transformation. One of such a rule is the tanh rule [2, 8, 9], which is the combination of the trapezoidal rule and the "tanh transformation:"

$$t = \psi_{\mathrm{SE}}(x) = \frac{b-a}{2}\tanh\left(\frac{x}{2}\right) + \frac{b+a}{2}, \tag{1.5}$$

and the rate of convergence is $\mathrm{O}(\exp(-c_0\sqrt{n}))$ (as described later, M and N in (1.3) are defined proportional to n). The transformation is also called the "single-exponential (SE) transformation," and accordingly we call the tanh rule the "SE formula" throughout this paper. Takahasi–Mori [16, 17] argued the difference of the performance between such quadrature rules, and concluded based on intuitive mathematical arguments and numerical experiments that the DE transformation is optimal among those variable transformations. The convergence rate of the DE formula that they have claimed was $\mathrm{O}(\exp(-c_1 n/\log(c_2 n)))$, which actually coincides with their numerical results.

After that, however, a question was raised about the optimality of the DE transformation in Stenger's analysis [10]. A functional analysis approach has been taken in it; he has considered all integrands f that belong to the Hardy space, and shown that under the assumption the optimal rate of convergence should be $\mathrm{O}(\exp(-c_0\sqrt{n}))$, which is the same rate as the SE formula. This result seems to conflict with the claim by Takahasi–Mori that the DE formula is optimal achieving $\mathrm{O}(\exp(-c_1 n/\log(c_2 n)))$.

Sugihara [13] has given an answer to the question by considering another function space. He has clarified the difference between the function spaces that should be assumed in each transformation (SE/DE), and justified theoretically that the DE's rate $\mathrm{O}(\exp(-c_1 n/\log(c_2 n)))$ is actually attainable and optimal in the corresponding function space. The difference of the assumptions between the SE's function space and the DE's one is important. The DE's assumption is stronger than the SE's one, which indicates that there exist some examples such that the SE formula works good while the DE formula does not. Some concrete examples have been considered by Tanaka et al. [18, § 4]. From their numerical results, the DE formula does not seem to achieve its usual rate: $\mathrm{O}(\exp(-c_1 n/\log(c_2 n)))$, whereas $\mathrm{O}(\exp(-c_0\sqrt{n}))$ is consistently observed in the SE formula. It should be noted that even in such a case the DE formula *can* be applied, and it *does* converge.

Here another new question arises: *how fast does the DE formula converge in such a case?* The main purpose of the present paper is to answer this question. Our task here is to estimate the DE formula's error under the same assumption as the SE formula, but the standard argument does not apply as it stands. Therefore we develop a new way to analyze such a case, and show that the desired rate is $\mathrm{O}(\exp(-c_3\sqrt{n}/\log(c_4 n)))$, which is slightly worse than the SE's rate. This in other words suggests from a practical viewpoint that *users can always select the DE* *formula* because there is little difference under the SE's assumption, whereas under the DE's assumption a far better rate can be obtained: $\mathrm{O}(\exp(-c_1 n/\log(c_2 n)))$. Therefore this result is also useful for practical purposes beyond theoretical interests.

The analysis technique developed here can be applied to other formulas, and as a second contribution of this paper, we also consider the "Sinc approximation" and the "Sinc indefinite integration." These two approximation formulas are frequently used with the SE transformation [11, 12] or the DE transformation [4, 5, 14, 15], and one may face quite the same situation as above there, i.e., the formula with SE works as expected while DE does not. We will analyze the DE's error in such a case, and which will then show the rate of convergence is $\mathrm{O}(\exp(-c_3\sqrt{n}/\log(c_4 n)))$, like as the trapezoidal rule case.

This paper is organized as follows. Sections 2 and 3 are devoted to the arguments of the trapezoidal rule. These are also important for subsequent sections because some key ideas are included. More precisely, in Section 2, the existing theoretical results and a new result for the trapezoidal rule are described, with some illustrative numerical examples. The proof of the new theorem is given in Section 3, with a sketch of the idea to overcome the difficulties in the analysis summarized in the beginning. Then we proceed to Section 4, in which the existing/new theoretical results for the Sinc approximation and the Sinc indefinite integration are described. The new theorems for both formulas are proved in Sections 5 and 6, respectively. In short, the main results of this paper can be found in Sections 2 and 4. Numerical examples that confirm the results are shown in Section 7. Finally in Section 8 we conclude this paper.

2 Existing/new convergence theorems for the SE/DE formula

We firstly review the existing convergence theorems for the SE and DE formulas under the corresponding standard assumptions. Then two kinds of numerical examples follow: (i) the case where the existing error estimates can explain the numerical results of both SE and DE, and (ii) the opposite case where the existing error estimates *cannot* explain the DE's result. After that a new theorem that can explain the result is given.

2.1 Existing convergence theorems under the standard assumptions

We have to introduce the following function space.

Definition 2.1. Let $\mathscr{D}$ be a bounded and simply-connected domain (or Riemann surface) which satisfies $(a, b) \subset \mathscr{D}$, and let α and β be positive constants. Then $\mathbf{L}_{\alpha,\beta}(\mathscr{D})$ denotes the family of all functions f that satisfy the following conditions: (i) f is analytic on $\mathscr{D}$; (ii) there exists a constant K such that for all z in $\mathscr{D}$

$$|f(z)| \leq K|Q_{\alpha,\beta}(z)|, \tag{2.1}$$

where $Q_{\alpha,\beta}(z) = (z-a)^{\alpha}(b-z)^{\beta}$. For simplicity, we write $Q_{1,1}(z)$ as $Q(z)$.

We use the strip region $\mathscr{D}_d$ defined by

$$\mathscr{D}_d := \{\zeta \in \mathbb{C} : |\operatorname{Im}\zeta| < d\} \tag{2.2}$$

for a positive constant d and consider the domains $\psi_{\mathrm{SE}}(\mathscr{D}_d)$ and $\psi_{\mathrm{DE}}(\mathscr{D}_d)$, where

$$\psi_{\mathrm{SE}}(\mathscr{D}_d) = \{z = \psi_{\mathrm{SE}}(\zeta) : \zeta \in \mathscr{D}_d\}, \tag{2.3}$$

$$\psi_{\mathrm{DE}}(\mathscr{D}_d) = \{z = \psi_{\mathrm{DE}}(\zeta) : \zeta \in \mathscr{D}_d\}. \tag{2.4}$$

That is, $\psi_{\mathrm{SE}}(\mathscr{D}_d)$ and $\psi_{\mathrm{DE}}(\mathscr{D}_d)$ are the image of the strip region $\mathscr{D}_d$ by the SE transformation (1.5) and the DE transformation (1.4), respectively, which contain a real interval (a, b). In this paper, we use $\psi_{\mathrm{SE}}(\mathscr{D}_d)$ or $\psi_{\mathrm{DE}}(\mathscr{D}_d)$ as the domain $\mathscr{D}$ in Definition 2.1, and consider the function space $\mathbf{L}_{\alpha,\beta}(\psi_{\mathrm{SE}}(\mathscr{D}_d))$ or $\mathbf{L}_{\alpha,\beta}(\psi_{\mathrm{DE}}(\mathscr{D}_d))$.

As the standard error estimates of the SE formula and the DE formula, the following two theorems have been known.

Theorem 2.2 (Stenger [11, Theorem 4.2.6])**.** Let $fQ \in \mathbf{L}_{\alpha,\beta}(\psi_{\mathrm{SE}}(\mathscr{D}_d))$ for d with $0 < d < \pi$. Let $\mu = \min\{\alpha, \beta\}$, n be a positive integer, and h be selected by the formula

$$h = \sqrt{\frac{2\pi d}{\mu n}}. \tag{2.5}$$

Furthermore, let M and N be positive integers defined by

$$\begin{cases} M = n, \ N = \lceil \alpha n/\beta \rceil & (\text{if } \mu = \alpha) \\ N = n, \ M = \lceil \beta n/\alpha \rceil & (\text{if } \mu = \beta) \end{cases} \tag{2.6}$$

respectively. Then there exists a constant C independent of n such that

$$\left| \int_a^b f(t)\mathrm{d}t - h \sum_{j=-M}^{N} f(\psi_{\mathrm{SE}}(jh))\psi_{\mathrm{SE}}'(jh) \right| \leq C\,\mathrm{e}^{-\sqrt{2\pi d\mu n}}. \tag{2.7}$$

Theorem 2.3 (Okayama et al. [6, Theorem 2.11])**.** Let $fQ \in \mathbf{L}_{\alpha,\beta}(\psi_{\mathrm{DE}}(\mathscr{D}_d))$ for d with $0 < d < \pi/2$. Let $\mu = \min\{\alpha, \beta\}$, $\nu = \max\{\alpha, \beta\}$, n be a positive integer with $n > \nu/(4d)$, and h be selected by the formula

$$h = \frac{\log(4dn/\mu)}{n}. \tag{2.8}$$

Furthermore, let M and N be positive integers defined by

$$\begin{cases} M = n, \ N = n - \lfloor \log(\beta/\alpha)/h \rfloor & (\text{if } \mu = \alpha) \\ N = n, \ M = n - \lfloor \log(\alpha/\beta)/h \rfloor & (\text{if } \mu = \beta) \end{cases} \tag{2.9}$$

respectively. Then there exists a constant C independent of n such that

$$\left| \int_a^b f(t)\mathrm{d}t - h \sum_{j=-M}^{N} f(\psi_{\mathrm{DE}}(jh))\psi_{\mathrm{DE}}'(jh) \right| \leq C\,\mathrm{e}^{-2\pi dn/\log(4dn/\mu)}. \tag{2.10}$$

Remark 2.4. The condition $n > \nu/(4d)$ is derived from the following requirements: (i) $n > \mu/(4d)$ is needed for $h > 0$, and (ii) $n > \nu/(4d)$ is needed for $M, N > 0$. For this reason, such a condition is also assumed on n in all of the subsequent DE's theorems.

2.2 Numerical example that can be explained by the existing theorems

In order to confirm the theorems above numerically, let us consider the following example.

Example 1. Consider the function

$$f_1(t) = \frac{2(1-t^2)}{\tan^2(1/2) + t^2} \tag{2.11}$$

and its definite integral on $(-1,\,1)$:

$$\int_{-1}^{1} f_1(t)\,\mathrm{d}t = \frac{4(\pi - 1 - \sin(1))}{\sin(1)}. \tag{2.12}$$

The function f_1 satisfies $f_1 Q \in \mathbf{L}_{2,2}(\psi_{\mathrm{SE}}(\mathscr{D}_{1-\epsilon}))$ and $f_1 Q \in \mathbf{L}_{2,2}(\psi_{\mathrm{DE}}(\mathscr{D}_{\arcsin\{(1-\epsilon)/\pi\}}))$ for any ϵ with $0 < \epsilon < 1$.

The domains above: $\psi_{\mathrm{SE}}(\mathscr{D}_{1-\epsilon})$ and $\psi_{\mathrm{DE}}(\mathscr{D}_{\arcsin\{(1-\epsilon)/\pi\}})$ are determined as follows. Let us recall that the SE formula is a combination of the SE transformation

$$\int_a^b f(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} f(\psi_{\mathrm{SE}}(x))\psi_{\mathrm{SE}}'(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} f(\psi_{\mathrm{SE}}(x))\frac{Q(\psi_{\mathrm{SE}}(x))}{b-a}\,\mathrm{d}x \tag{2.13}$$

and the trapezoidal formula (1.3). We have to determine the regularity of the transformed integrand: $F(\zeta) = f_1(\psi_{\mathrm{SE}}(\zeta))Q(\psi_{\mathrm{SE}}(\zeta))/2$. Easily one may find $Q(\psi_{\mathrm{SE}}(\cdot))$ is analytic in $\mathscr{D}_\pi$. As for $f_1(\psi_{\mathrm{SE}}(\cdot))$, there are two poles at $\zeta = \pm\mathrm{i}$ (see Figure 1). Therefore $f_1(\psi_{\mathrm{SE}}(\cdot))$ and accordingly F are analytic in $\mathscr{D}_1$. The reason for setting $\mathscr{D}_{1-\epsilon}$, not $\mathscr{D}_1$, is due to (2.1), which requires boundedness of $\sup_{\zeta\in\mathscr{D}_d}|F(z)|$. In the same manner, one can find in the DE case that the transformed integrand has two poles at $\zeta = \pm\mathrm{i}\,\arcsin(1/\pi)$ (see Figure 2).

The numerical errors[1] of the SE formula and the DE formula are plotted in Figure 5. From the graph, we can observe the expected rates for both formulas, $\mathrm{O}(\exp(-c_0\sqrt{n}))$ and $\mathrm{O}(\exp(-c_1 n/\log(c_2 n)))$, as predicted in Theorems 2.2 and 2.3.

2.3 Numerical example that cannot be explained by the existing theorem for the DE formula

Let us now turn to the next example, which is an unfavorable case for the DE formula.

Example 2. Consider the function [18, § 4.2]

$$f_2(t) = \frac{2(1-t^2)}{\cos(4\,\mathrm{arctanh}\,t) + \cosh(2)} \tag{2.14}$$

and its definite integral on $(-1,\,1)$:[2]

$$\int_{-1}^{1} f_2(t)\,\mathrm{d}t = 0.7119438\cdots. \tag{2.15}$$

The function f_2 satisfies $f_2 Q \in \mathbf{L}_{2,2}(\psi_{\mathrm{SE}}(\mathscr{D}_{1-\epsilon}))$ for any ϵ with $0 < \epsilon < 1$, but does not satisfy $f_2 Q \in \mathbf{L}_{2,2}(\psi_{\mathrm{DE}}(\mathscr{D}_d))$ for any $d > 0$.

Let us first examine the poles of the integrand in the SE case. The function $f_2(\psi_{\mathrm{SE}}(\cdot))$ has infinite number of poles at

$$\zeta = \left(\frac{\pi}{2} + m\pi\right) \pm \mathrm{i} \quad (m \in \mathbb{Z}), \tag{2.16}$$

as shown in Figure 3. From this we can see the transformed integrand is analytic in $\mathscr{D}_1$, same as the previous example. Therefore there is no problem for the SE formula.

[1] Computation programs in this section were written in C with quadruple-precision floating-point arithmetic, which is available on PowerPC CPU by using `long double` type. And we set $\epsilon = 0.001$ for the computation.

[2] We used the value `0.71194382297059827888000405031539396435` as an answer, which was calculated by Mathematica 7 with sufficient accuracy. This is the same manner as Tanaka et al. [18, § 4.2].

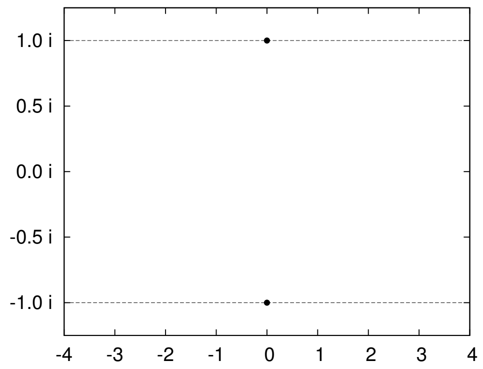

Figure 1. Two poles of $f_1(\psi_{\mathrm{SE}}(\cdot))$ ($\zeta = \pm \mathrm{i}$).

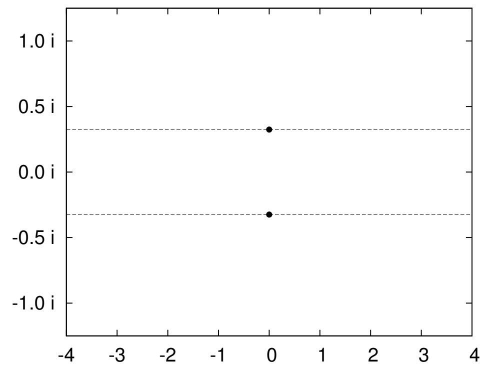

Figure 2. Two poles of $f_1(\psi_{\mathrm{DE}}(\cdot))$ ($\zeta = \pm\mathrm{i}\arcsin(1/\pi)$).

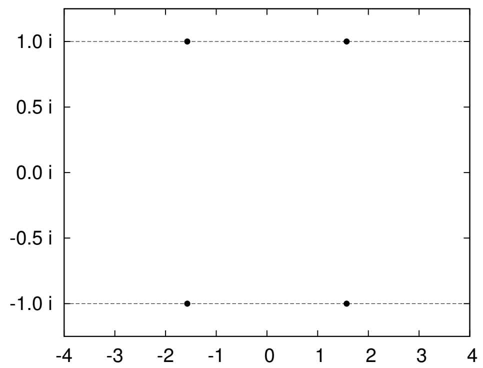

Figure 3. Infinite number of poles of $f_2(\psi_{\mathrm{SE}}(\cdot))$ (defined by (2.16)).

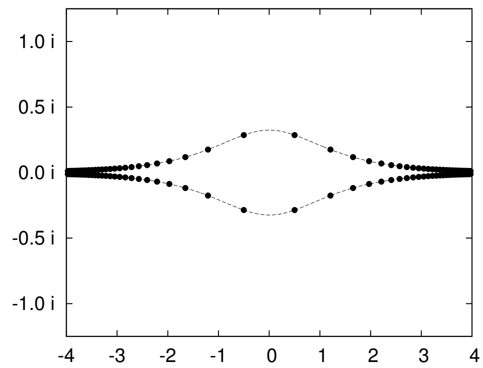

Figure 4. Infinite number of poles of $f_2(\psi_{\mathrm{DE}}(\cdot))$ (defined by (2.17)).

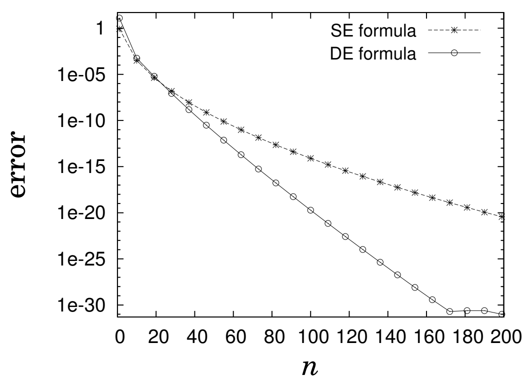

Figure 5. Error of the SE formula and the DE formula in Example 1.

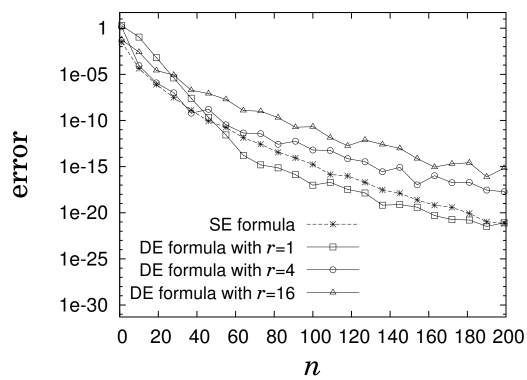

Figure 6. Error of the SE formula and the DE formula (h is selected as (2.18)) in Example 2.

In the case of the DE formula, however, the situation is different. The function $f_2(\psi_{\mathrm{DE}}(\cdot))$ has infinite number of poles at

$$\operatorname{arcsinh}\left[\frac{1}{\pi}\left\{\left(\frac{\pi}{2}+m\pi\right)\pm \mathrm{i}\right\}\right] \quad (m\in\mathbb{Z}), \tag{2.17}$$

which approach the real axis as $|m|\to\infty$ (see Figure 4). This causes a problem; we cannot take any $\mathscr{D}_d$ as a domain in which $f_2(\psi_{\mathrm{DE}}(\cdot))$ is analytic. Therefore Theorem 2.3 cannot be used in this case for the DE formula.

Furthermore, there is a problem in the implementation. A mesh size h should be selected by the formula (2.8), but in this case we have no clue how to choose d (we easily see $\mu = 2$ though). Here let us try

$$h = \frac{\log(rdn/2)}{n}, \tag{2.18}$$

where $d = \arcsin((1-\epsilon)/\pi)$, and r is either 4^0, 4^1, 4^2.

The results are shown in Figure 6. In the case of the SE formula, we can observe a quite similar result to the previous example (Figure 5), and it coincides with the claim of Theorem 2.2. On the other hand, in the case of the DE formula, the predicted rate: $\mathrm{O}(\exp(-c_1 n/\log(c_2 n)))$ does not seem to be attained anymore. However, even in this case the DE formula *does* converge, and as an important observation, it seems to converge at a rate quite similar to the SE case. A similar situation has been also observed in Tanaka et al. [18, § 4.2], but no theoretical explanation has been given so far. The main purpose of this paper is to explain such a numerical observation in a theoretical way.

2.4 New convergence theorem for the DE formula under the same assumption as the SE formula

Here we show a new theorem giving theoretical explanation for the DE formula's convergence in Example 2. Notice that the assumption on the integrand f is the same as Theorem 2.2.

Theorem 2.5. Let $fQ \in \mathbf{L}_{\alpha,\beta}(\psi_{\mathrm{SE}}(\mathscr{D}_d))$ for d with $0 < d < \pi$, and put $d' = \arcsin(d/\pi)$. Let $\mu = \min\{\alpha,\beta\}$, $\nu = \max\{\alpha,\beta\}$, and let c be a positive number. For a positive integer n with $n > \nu/(c\mu)$, define h as

$$h = \frac{\log(cn)}{n}, \tag{2.19}$$

and define M and N as (2.9). Then there exists a constant $C_{\alpha,\beta,c,d'}$ depending only on α, β, c, d' such that

$$\left|\int_a^b f(t)\mathrm{d}t - h\sum_{j=-M}^{N} f(\psi_{\mathrm{DE}}(jh))\psi'_{\mathrm{DE}}(jh)\right| \leq K(b-a)^{\alpha+\beta-1}C_{\alpha,\beta,c,d'}\exp\left(-\frac{2\pi d'}{\sqrt{c}}\frac{\sqrt{n}}{\log(cn)}\right), \tag{2.20}$$

where K is the constant in (2.1).

This theorem convinces us that in this case the convergence rate of the DE formula is $\mathrm{O}(\exp(-c_3\sqrt{n}/\log(c_4 n)))$. This rate is slightly worse than the SE's rate: $\mathrm{O}(\exp(-c_0\sqrt{n}))$, but quite similar.

Remark 2.6. In Theorem 2.3, a mesh size h is determined explicitly by using μ and d as (2.8). In contrast, in Theorem 2.5, an undetermined constant c is used in h as (2.19). This is because h varied as (2.18) in Example 2 (in this case $c = rd/\mu$).

Remark 2.7. The estimate (2.20) may give an impression that the convergence rate can be improved by taking c smaller. But a smaller c makes $C_{\alpha,\beta,c,d'}$ larger. A practical choice is $c = 4d'/\mu$ (see Remark 3.6 for the reason).

Remark 2.8. Unlike usual error estimates such as (2.7) and (2.10), the constant in (2.20) is more explicitly given. This explicit form is needed for another study: convergence analysis of the scheme for weakly singular Volterra integral equations (see the comments on the future works in Section 8). Only to grasp the main claim of this theorem (convergence rate), one can just put $C = K(b-a)^{\alpha+\beta-1}C_{\alpha,\beta,c,d'}$ as a constant independent of n.

3 Proof of Theorem 2.5 (for the DE formula)

To show the idea for the proof of Theorem 2.5, we begin with the review of the proof of the standard theorem: Theorem 2.3. In this review, we notice a difficulty arises, and in order to settle it, two new steps should be introduced in the proof. They are explained in detail in Sections 3.2 and 3.3, respectively. The two steps are also keys to analyze the other approximation formulas described in the subsequent sections. Finally we prove Theorem 2.5 in Section 3.4.

3.1 Review of the standard proof and emerging difficulties

In the proof of Theorem 2.3, the following orthodox technique is used to estimate the error:

$$\begin{aligned}&\left|\int_a^b f(t)\,\mathrm{d}t - h\sum_{j=-M}^{N} f(\psi_{\mathrm{DE}}(jh))\psi'_{\mathrm{DE}}(jh)\right| \\ &= \left|\int_{-\infty}^{\infty} F(x)\,\mathrm{d}x - h\sum_{j=-M}^{N} F(jh)\right| \leq E'_1(F,n) + E'_2(F,n), \end{aligned} \tag{3.1}$$

where $F(x) = f(\psi_{\mathrm{DE}}(x))\psi'_{\mathrm{DE}}(x)$ and

$$E'_1(F,n) = \left|\int_{-\infty}^{\infty} F(x)\,\mathrm{d}x - h\sum_{j=-\infty}^{\infty} F(jh)\right|, \tag{3.2}$$

$$E'_2(F,n) = h\sum_{j=-\infty}^{-M-1} |F(jh)| + h\sum_{j=N+1}^{\infty} |F(jh)|. \tag{3.3}$$

The quantities E'_1 and E'_2 are referred to as a discretization error and a truncation error, respectively. For them, we have

$$E'_1(F,n) = \mathrm{O}(\mathrm{e}^{-2\pi d/h}), \tag{3.4}$$

$$E'_2(F,n) = \mathrm{O}(\mathrm{e}^{-\frac{\pi}{2}\mu\exp(nh)}). \tag{3.5}$$

Finally, substituting $h = \log(4dn/\mu)/n$ into these expressions, we obtain the conclusion.

The most critical part of the above proof is the derivation of (3.4), where the key is rewriting the quadrature rule by a complex contour integral:

$$h\sum_{j=-\infty}^{\infty} F(jh) = \lim_{k\to\infty}\left\{\frac{1}{2\mathrm{i}}\oint_{\Gamma_k}\frac{F(\zeta)}{\tan(\pi\zeta/h)}\,\mathrm{d}\zeta\right\}. \tag{3.6}$$

The contour Γ_k is illustrated in Figure 7. The horizontal paths of the contour are taken as close as possible to the boundary of $\mathscr{D}_d$. If the transformed function F is analytic on $\mathscr{D}_d$, the expression (3.6) holds from the residue theorem.

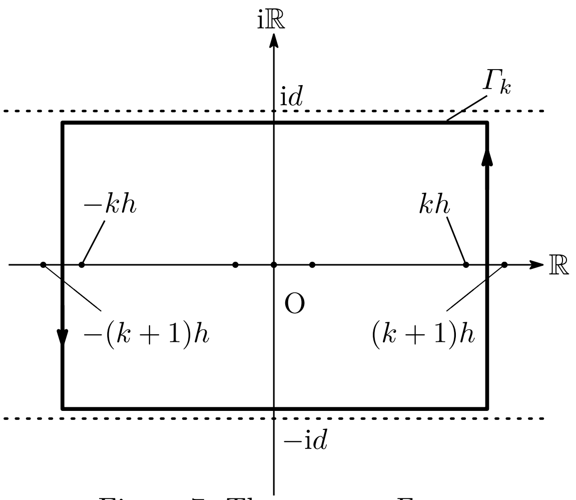

Figure 7. The contour $\varGamma_k$.

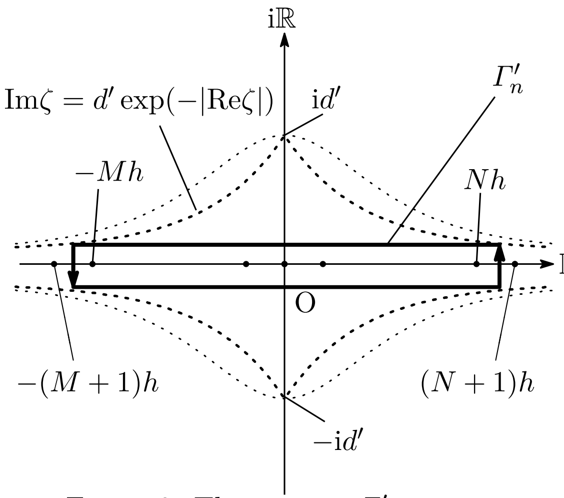

Figure 8. The contour $\varGamma'_n$.

Under the assumption of Theorem 2.5, however, the transformed function F is not always analytic on $\mathscr{D}_d$. See also Figure 4; in this case we cannot take any positive constant d for a strip domain $\mathscr{D}_d$. Therefore the expression (3.6) is no longer possible. This is the first difficulty.

An immediate way to revive the expression is to delete the limiting process from both sides in (3.6). This in fact holds if we choose the contour $\varGamma_k$ in the domain in which the transformed function F is analytic. To this end, we have to take the following two additional steps:

- determine the explicit form of the domain in which $F(\zeta) = f(\psi_{\mathrm{DE}}(\zeta))\psi'_{\mathrm{DE}}(\zeta)$ is analytic,
- choose a proper contour in the domain so that a sharp error estimate can be given.

In what follows we explain these two steps one by one. In the latter step, the second difficulty arises; a certain natural choice of the contour leads to a meaningless error estimate. The difficulty and its remedy are described in Section 3.3.

3.2 Determining the explicit form of the domain to be considered

The transformed integrand F is analytic on the domain $\psi_{\mathrm{DE}}^{-1}(\psi_{\mathrm{SE}}(\mathscr{D}_d))$, and we have to clarify the expression of the domain.

Lemma 3.1. Let d be a constant with $0 < d \leq \pi$ and put $\mathscr{D} = \psi_{\mathrm{SE}}(\mathscr{D}_d)$. Then we have

$$\psi_{\mathrm{DE}}^{-1}(\mathscr{D}) = \left\{ \zeta \in \mathbb{C} : |\operatorname{Im}\zeta| < \arcsin\left[\frac{d}{\pi\cosh(\operatorname{Re}\zeta)}\right] \right\}. \tag{3.7}$$

Proof. The inverse of the DE transformation is expressed as $\psi_{\mathrm{DE}}^{-1}(z) = \operatorname{arcsinh}[\psi_{\mathrm{SE}}^{-1}(z)/\pi]$. Therefore we have $\psi_{\mathrm{DE}}^{-1}(\mathscr{D}) = \operatorname{arcsinh}[\mathscr{D}_d/\pi]$. Using the expression $\xi = \sinh(x)\sqrt{\pi^2 - (d/\cosh x)^2} \pm \mathrm{i}\, d$ of the boundary of the domain $\mathscr{D}_d$, we can show that $\zeta = x \pm \mathrm{i}\, \arcsin(d/(\pi\cosh x))$ iff ζ belongs to the boundary of $\operatorname{arcsinh}[\mathscr{D}_d/\pi]$, which proves the lemma. ■

But this domain is not very convenient to handle. Thus we define another domain contained in it with a simpler expression. For a nonnegative constant δ, let us set

$$\varDelta_d^{\delta} = \{\zeta \in \mathbb{C} : |\operatorname{Im}\zeta| < d\exp(-\delta|\operatorname{Re}\zeta|)\} \tag{3.8}$$

and consider $\varDelta^1_{\arcsin(d/\pi)}$. Then, as shown below, $\varDelta^1_{\arcsin(d/\pi)}$ is contained in the domain expressed by (3.7).

Lemma 3.2. Let d be a constant with $0 < d \leq \pi$ and put $\mathscr{D} = \psi_{\mathrm{SE}}(\mathscr{D}_d)$. Then we have

$$\varDelta^1_{\arcsin(d/\pi)} \subseteq \psi_{\mathrm{DE}}^{-1}(\mathscr{D}). \tag{3.9}$$

Proof. Since the inequality

$$\sin[\arcsin(d/\pi)y] \leq \frac{d}{\pi}\frac{2y}{1+y^2} \tag{3.10}$$

holds for y with $0 \leq y \leq 1$, by putting $y = \mathrm{e}^{-|x|}$ we have

$$\arcsin(d/\pi)\,\mathrm{e}^{-|x|} \leq \arcsin\left(\frac{d}{\pi}\frac{1}{\cosh x}\right) \tag{3.11}$$

for arbitrary $x \in \mathbb{R}$. Thus the lemma is proven. ■

Immediately from Lemma 3.2, we can rewrite the assumption in Theorem 2.2 as follows.

Lemma 3.3. $fQ \in \mathbf{L}_{\alpha,\beta}(\psi_{\mathrm{SE}}(\mathscr{D}_d))$ implies $fQ \in \mathbf{L}_{\alpha,\beta}(\psi_{\mathrm{DE}}(\varDelta^1_{\arcsin(d/\pi)}))$.

Thanks to this lemma, we may set $d' = \arcsin(d/\pi)$ and consider the error estimate of the DE formula under the assumption $fQ \in \mathbf{L}_{\alpha,\beta}(\psi_{\mathrm{DE}}(\varDelta^1_{d'}))$.

3.3 Choosing a proper contour: difficulty and its remedy

Now let us return to the expression (3.6). In the domain $\varDelta^1_{d'}$, we can take the contour $\varGamma'_n$ illustrated in Figure 8 (recall that M, N, and h are determined from n, through the definitions (2.9) and (2.19)). Observe how it differs from $\varGamma_k$. The height of $\varGamma'_n$ must tend to zero as $n \to \infty$, whereas the height of $\varGamma_k$ can be fixed. For this reason we cannot obtain the same expression as (3.6), but it holds without the limiting process, i.e.,

$$h\sum_{j=-M}^{N} F(jh) = \frac{1}{2\mathrm{i}}\oint_{\varGamma'_n}\frac{F(\zeta)}{\tan(\pi\zeta/h)}\,\mathrm{d}\zeta. \tag{3.12}$$

In order to use this expression (3.12), one may naturally modify the splitting of the quadrature error (3.1) as follows:

$$\left|\int_{-\infty}^{\infty} F(x)\,\mathrm{d}x - h\sum_{j=-M}^{N} F(jh)\right| \leq E''_1(F,n) + E''_2(F,n), \tag{3.13}$$

where

$$E''_1(F,n) = \left|\int_{-(M+1/2)h}^{(N+1/2)h} F(x)\,\mathrm{d}x - h\sum_{j=-M}^{N} F(jh)\right|, \tag{3.14}$$

$$E''_2(F,n) = \left|\int_{-\infty}^{-(M+1/2)h} F(x)\,\mathrm{d}x\right| + \left|\int_{(N+1/2)h}^{\infty} F(x)\,\mathrm{d}x\right|. \tag{3.15}$$

Notice the difference between $E'_1(F,n)$ and $E''_1(F,n)$. In order to use a complex contour integral for estimating the error $E''_1(F,n)$, any limiting process should not appear in it. This is because the transformed integrand F is analytic not on $\mathscr{D}_d$, but on $\varDelta^1_{d'}$.

The splitting (3.13) is, however, unsuitable for our analysis. If we estimate E_1'' in a similar manner to E_1', we find that d in (3.4) should be replaced with $d'\exp(-(n+1/2)h)$, which is the height of $\varGamma_n'$ from the real axis. From this and (2.19), we have

$$\begin{aligned} E_1''(F,n) &\approx \mathrm{O}\left(\exp\left(-\frac{2\pi d'}{h}\exp(-nh)\right)\right) \\ &= \mathrm{O}\left(\exp\left(-\frac{2\pi d' n}{\log(cn)}\exp(-\log(cn))\right)\right) \\ &= \mathrm{O}\left(\exp\left(-\frac{2\pi d'}{c\log(cn)}\right)\right) \\ &\to \mathrm{O}(1) \quad (n\to\infty), \end{aligned} \tag{3.16}$$

which is meaningless as an error estimate. This is the second difficulty mentioned at the end of Section 3.1.

In order to remedy the issue, we change the splitting as follows:

$$\left|\int_{-\infty}^{\infty} F(x)\,\mathrm{d}x - h\sum_{j=-M}^{N} F(jh)\right| \leq E_1(F,n) + E_2(F,n), \tag{3.17}$$

where

$$E_1(F,n) = \left|\int_{-(\lceil\frac{M}{2}\rceil+\frac{1}{4})h}^{(\lceil\frac{N}{2}\rceil+\frac{1}{4})h} F(x)\,\mathrm{d}x - h\sum_{j=-\lceil\frac{M}{2}\rceil}^{\lceil\frac{N}{2}\rceil} F(jh)\right|, \tag{3.18}$$

$$E_2(F,n) = \int_{-\infty}^{-(\lceil\frac{M}{2}\rceil+\frac{1}{4})h} |F(x)|\,\mathrm{d}x + \int_{(\lceil\frac{N}{2}\rceil+\frac{1}{4})h}^{\infty} |F(x)|\,\mathrm{d}x + h\sum_{j=-M}^{-\lceil\frac{M}{2}\rceil-1} |F(jh)| + h\sum_{j=\lceil\frac{N}{2}\rceil+1}^{N} |F(jh)|. \tag{3.19}$$

Roughly speaking, M and N in (3.14) are replaced with $M/2$ and $N/2$, respectively. Accordingly the contour $\varGamma_n'$ is replaced with $\varGamma_{n/2}'$. By this modification, the estimate (3.16) is improved as

$$\begin{aligned} E_1(F,n) &\approx \mathrm{O}\left(\exp\left(-\frac{2\pi d'}{h}\exp(-(n/2)h)\right)\right) \\ &= \mathrm{O}\left(\exp\left(-\frac{2\pi d' n}{\log(cn)}\exp(-\log(cn)/2)\right)\right) \\ &= \mathrm{O}\left(\exp\left(-\frac{2\pi d'\sqrt{n}}{\sqrt{c}\log(cn)}\right)\right), \end{aligned} \tag{3.20}$$

which gives a meaningful estimate. And for E_2 we can show

$$E_2(F,n) = \mathrm{O}\left(\exp\left(-\frac{\pi\mu}{2}\sqrt{cn}\right)\right). \tag{3.21}$$

Thus we obtain the desired conclusion.

3.4 Proofs

For simplicity, we write d' as d and assume that $fQ \in \mathbf{L}_{\alpha,\beta}(\psi_{\mathrm{DE}}(\varDelta_d^1))$ throughout the following proof. As described above, this proof consists of the following two estimates.

Lemma 3.4. Under the assumptions of Theorem 2.5, for $F(x) = f(\psi_{\mathrm{DE}}(x))\psi'_{\mathrm{DE}}(x)$, there exists a constant $\tilde{C}_{\alpha,\beta,c,d}$ depending only on α, β, c, d such that

$$E_1(F,n) \leq K(b-a)^{\alpha+\beta-1}\tilde{C}_{\alpha,\beta,c,d}\exp\left(-\frac{2\pi d}{\sqrt{c}}\frac{\sqrt{n}}{\log(cn)}\right). \tag{3.22}$$

Lemma 3.5. Under the assumptions of Theorem 2.5, for $F(x) = f(\psi_{\mathrm{DE}}(x))\psi'_{\mathrm{DE}}(x)$, there exists a constant $\tilde{C}_{\alpha,\beta,c}$ depending only on α, β, c such that

$$E_2(F,n) \leq K(b-a)^{\alpha+\beta-1}\tilde{C}_{\alpha,\beta,c}\exp\left(-\frac{\pi\mu}{2}\sqrt{cn}\right). \tag{3.23}$$

Remark 3.6. As mentioned in Remark 2.7, a nearly optimal c is $c = 4d/\mu$. This is because this almost equates the $\exp(\,\cdot\,)$ parts of (3.22) and (3.23).

To prove these lemmas, the next inequality is useful. It immediately follows from the existing lemma [6, Lemma 4.21], and we omit the proof.

Lemma 3.7. Let $fQ \in \mathbf{L}_{\alpha,\beta}(\psi_{\mathrm{DE}}(\mathscr{D}_d^1))$ for d with $0 < d < \pi/2$. Let $F(\zeta) = f(\psi_{\mathrm{DE}}(\zeta))\psi'_{\mathrm{DE}}(\zeta)$, and let x and y be arbitrary real numbers with $x + \mathrm{i}\,y \in \mathscr{D}_d^1$. Then it holds that

$$|F(x+\mathrm{i}\,y)| \leq K(b-a)^{\alpha+\beta-1}G_{\alpha,\beta}(x,y) \tag{3.24}$$

where K is the constant in (2.1) and $G_{\alpha,\beta}(x,y)$ is a function defined by

$$G_{\alpha,\beta}(x,y) = \frac{\pi\cosh x}{(1+\mathrm{e}^{-\pi\sinh(x)\cos y})^{\alpha}(1+\mathrm{e}^{\pi\sinh(x)\cos y})^{\beta}\cos^{\alpha+\beta}(\frac{\pi}{2}\sin y)}. \tag{3.25}$$

In the case $\tilde{f} \in \mathbf{L}_{\alpha,\beta}(\psi_{\mathrm{DE}}(\mathscr{D}_d^1))$ (needed in Section 5), let $\tilde{F}(\zeta) = \tilde{f}(\psi_{\mathrm{DE}}(\zeta))$. Then it holds that

$$|\tilde{F}(x+\mathrm{i}\,y)| \leq \frac{K(b-a)^{\alpha+\beta}}{\pi}G_{\alpha,\beta}(x,y). \tag{3.26}$$

In what follows we prove Lemma 3.4 and Lemma 3.5. We begin with the proof of Lemma 3.5, which is easier.

Proof of Lemma 3.5. It clearly follows from (3.24) of Lemma 3.7 that

$$\begin{aligned}
&\int_{-\infty}^{-(\lceil\frac{M}{2}\rceil+\frac{1}{4})h}|F(x)|\,\mathrm{d}x + \int_{(\lceil\frac{N}{2}\rceil+\frac{1}{4})h}^{\infty}|F(x)|\,\mathrm{d}x\\
&\leq K(b-a)^{\alpha+\beta-1}\left\{\int_{-\infty}^{-Mh/2}G_{\alpha,\beta}(x,0)\,\mathrm{d}x + \int_{Nh/2}^{\infty}G_{\alpha,\beta}(x,0)\,\mathrm{d}x\right\},
\end{aligned} \tag{3.27}$$

and it also follows that

$$\begin{aligned}
&h\sum_{j=-M}^{-\lceil\frac{M}{2}\rceil-1}|F(jh)| + h\sum_{j=\lceil\frac{N}{2}\rceil+1}^{N}|F(jh)|\\
&\leq K(b-a)^{\alpha+\beta-1}\left\{h\sum_{j=-\infty}^{-\lceil\frac{M}{2}\rceil-1}G_{\alpha,\beta}(jh,0) + h\sum_{j=\lceil\frac{N}{2}\rceil+1}^{\infty}G_{\alpha,\beta}(jh,0)\right\}\\
&\leq K(b-a)^{\alpha+\beta-1}C_{\alpha,\beta,c}\left\{\int_{-\infty}^{-Mh/2}G_{\alpha,\beta}(x,0)\,\mathrm{d}x + \int_{Nh/2}^{\infty}G_{\alpha,\beta}(x,0)\,\mathrm{d}x\right\},
\end{aligned} \tag{3.28}$$

for some constant $C_{\alpha,\beta,c}$ depending only on α, β, c. Noting the relation among M, N and n, for the first term of the RHS of (3.27), we have

$$\begin{aligned} K(b-a)^{\alpha+\beta-1}\int_{-\infty}^{-Mh/2} G_{\alpha,\beta}(x,0)\,\mathrm{d}x &\leq \frac{K(b-a)^{\alpha+\beta-1}}{\alpha}\int_{-\infty}^{-Mh/2} \pi\alpha\cosh(x)\,\mathrm{e}^{\pi\alpha\sinh x}\,\mathrm{d}x \\ &= \frac{K(b-a)^{\alpha+\beta-1}}{\alpha}\,\mathrm{e}^{-\pi\alpha\sinh(Mh/2)} \\ &\leq \frac{K(b-a)^{\alpha+\beta-1}}{\mu}\,\mathrm{e}^{\frac{\pi}{2}\nu}\,\mathrm{e}^{-\frac{\pi}{2}\mu\exp(nh/2)}. \end{aligned} \tag{3.29}$$

The estimate of the second term is similar, and the same applies to the estimate for (3.28). Then substituting $h = \log(cn)/n$, we have the desired inequality. ■

Proof of Lemma 3.4. We define the following integral paths:

$$\varGamma_{\mathrm{vr}}^{+}(x,y) = \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta = x,\ 0 \leq \operatorname{Im}\zeta \leq y\}, \tag{3.30}$$
$$\varGamma_{\mathrm{vr}}^{-}(x,y) = \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta = x,\ -y \leq \operatorname{Im}\zeta \leq 0\}, \tag{3.31}$$
$$\varGamma_{\mathrm{vl}}^{+}(x,y) = \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta = -x,\ 0 \leq \operatorname{Im}\zeta \leq y\}, \tag{3.32}$$
$$\varGamma_{\mathrm{vl}}^{-}(x,y) = \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta = -x,\ -y \leq \operatorname{Im}\zeta \leq 0\}, \tag{3.33}$$
$$\varGamma_{\mathrm{hr}}^{\pm}(x,y) = \{\zeta \in \mathbb{C} : 0 \leq \operatorname{Re}\zeta \leq x,\ \operatorname{Im}\zeta = \pm y\}, \tag{3.34}$$
$$\varGamma_{\mathrm{hl}}^{\pm}(x,y) = \{\zeta \in \mathbb{C} : -x \leq \operatorname{Re}\zeta \leq 0,\ \operatorname{Im}\zeta = \pm y\}. \tag{3.35}$$

The directions of them are set to be counterclockwise with respect to the origin (see Figure 9).

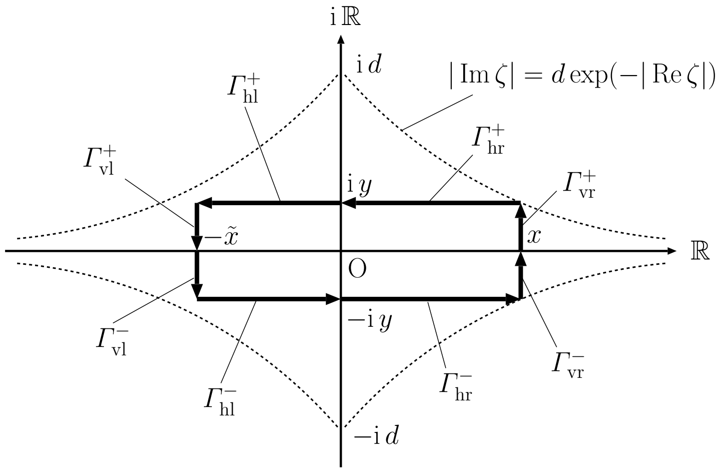

Figure 9. Integral paths $\varGamma_{\mathrm{vr}}^{\pm}(x,y)$, $\varGamma_{\mathrm{vl}}^{\pm}(\tilde{x},y)$ (vertical) and $\varGamma_{\mathrm{hr}}^{\pm}(x,y)$, $\varGamma_{\mathrm{hl}}^{\pm}(\tilde{x},y)$ (horizontal) in the case $x > \tilde{x}$.

We set $x_{M,h} = (\lceil M/2\rceil + 1/4)h$, $x_{N,h} = (\lceil N/2\rceil + 1/4)h$, $x_{n,h} = (\lceil n/2\rceil + 1/4)h$, $y_{n,h} = d(1-\epsilon)\exp(-x_{n,h})$ and consider the contour

$$\begin{aligned} &\varGamma_{\mathrm{vr}}^{+}(x_{N,h},y_{n,h}) + \varGamma_{\mathrm{hr}}^{+}(x_{N,h},y_{n,h}) + \varGamma_{\mathrm{hl}}^{+}(x_{M,h},y_{n,h}) + \varGamma_{\mathrm{vl}}^{+}(x_{M,h},y_{n,h}) \\ &+ \varGamma_{\mathrm{vl}}^{-}(x_{M,h},y_{n,h}) + \varGamma_{\mathrm{hl}}^{-}(x_{M,h},y_{n,h}) + \varGamma_{\mathrm{hr}}^{-}(x_{N,h},y_{n,h}) + \varGamma_{\mathrm{vr}}^{-}(x_{N,h},y_{n,h}), \end{aligned} \tag{3.36}$$

where $0 < \epsilon < 1$. For simplicity of the notations, we set $k = \lceil M/2\rceil$, $l = \lceil N/2\rceil$ and omit the expressions $(x_{M,h}, y_{n,h})$ and $(x_{N,h}, y_{n,h})$. Using complex contour integrals, we have

$$\int_{-(k+1/4)h}^{(l+1/4)h} F(x)\,\mathrm{d}x = \frac{1}{2}\left\{-\int_{\Gamma_{\mathrm{vr}}^{+}+\Gamma_{\mathrm{hr}}^{+}+\Gamma_{\mathrm{hl}}^{+}+\Gamma_{\mathrm{vl}}^{+}} F(\zeta)\,\mathrm{d}\zeta + \int_{\Gamma_{\mathrm{vl}}^{-}+\Gamma_{\mathrm{hl}}^{-}+\Gamma_{\mathrm{hr}}^{-}+\Gamma_{\mathrm{vr}}^{-}} F(\zeta)\,\mathrm{d}\zeta\right\}, \tag{3.37}$$

$$h\sum_{j=-k}^{l} F(jh) = \frac{1}{2\mathrm{i}}\int_{\Gamma_{\mathrm{vr}}^{+}+\Gamma_{\mathrm{hr}}^{+}+\Gamma_{\mathrm{hl}}^{+}+\Gamma_{\mathrm{vl}}^{+}+\Gamma_{\mathrm{vl}}^{-}+\Gamma_{\mathrm{hl}}^{-}+\Gamma_{\mathrm{hr}}^{-}+\Gamma_{\mathrm{vr}}^{-}} \frac{F(\zeta)}{\tan(\pi\zeta/h)}\,\mathrm{d}\zeta, \tag{3.38}$$

which follows from Cauchy's theorem and the residue theorem, respectively. Then we have the inequality

$$\begin{aligned}
&\left|\int_{-(k+1/4)h}^{(l+1/4)h} F(x)\,\mathrm{d}x - h\sum_{j=-k}^{l} F(jh)\right| \\
&\leq \left|\frac{1}{2}\left(-\int_{\Gamma_{\mathrm{hr}}^{+}+\Gamma_{\mathrm{hl}}^{+}} F(\zeta)\,\mathrm{d}\zeta + \int_{\Gamma_{\mathrm{hl}}^{-}+\Gamma_{\mathrm{hr}}^{-}} F(\zeta)\,\mathrm{d}\zeta\right) - \frac{1}{2\mathrm{i}}\int_{\Gamma_{\mathrm{hr}}^{+}+\Gamma_{\mathrm{hl}}^{+}+\Gamma_{\mathrm{hl}}^{-}+\Gamma_{\mathrm{hr}}^{-}} \frac{F(\zeta)}{\tan(\pi\zeta/h)}\,\mathrm{d}\zeta\right| \\
&\quad + \left|\frac{1}{2}\left(-\int_{\Gamma_{\mathrm{vr}}^{+}+\Gamma_{\mathrm{vl}}^{+}} F(\zeta)\,\mathrm{d}\zeta + \int_{\Gamma_{\mathrm{vl}}^{-}+\Gamma_{\mathrm{vr}}^{-}} F(\zeta)\,\mathrm{d}\zeta\right) - \frac{1}{2\mathrm{i}}\int_{\Gamma_{\mathrm{vr}}^{+}+\Gamma_{\mathrm{vl}}^{+}+\Gamma_{\mathrm{vl}}^{-}+\Gamma_{\mathrm{vr}}^{-}} \frac{F(\zeta)}{\tan(\pi\zeta/h)}\,\mathrm{d}\zeta\right|.
\end{aligned} \tag{3.39}$$

The remaining task is to estimate these two terms of the RHS, say E_{h} and E_{v}, respectively.
(i) Estimate of the second term E_{v}. Let us first estimate the second term. Using (3.24) of Lemma 3.7 and noting the relation among M, N and n, for the integral on Γ_{vr}^{+} we have

$$\begin{aligned}
\int_{\Gamma_{\mathrm{vr}}^{+}} |F(\zeta)||\mathrm{d}\zeta| &\leq K(b-a)^{\alpha+\beta-1}\int_{0}^{y_{n,h}} G_{\alpha,\beta}(x_{N,h}, y)\,\mathrm{d}y \\
&\leq \frac{K\pi(b-a)^{\alpha+\beta-1}}{\cos^{\alpha+\beta}(\frac{\pi}{2}\sin y_{n,h})}\int_{0}^{y_{n,h}} \cosh(x_{N,h})\,\mathrm{e}^{-\pi\beta\sinh(x_{N,h})\cos(y_{n,h})}\,\mathrm{d}y \\
&= \frac{K\pi(b-a)^{\alpha+\beta-1}}{\cos^{\alpha+\beta}(\frac{\pi}{2}\sin y_{n,h})} d(1-\epsilon)\frac{\cosh(x_{N,h})}{\exp(x_{n,h})}\,\mathrm{e}^{-\pi\beta\sinh(x_{N,h})\cos(y_{n,h})} \\
&\leq \frac{K\pi(b-a)^{\alpha+\beta-1}d}{\cos^{\alpha+\beta}(\frac{\pi}{2}\sin y_{n,h})}\,\mathrm{e}^{-\pi\beta\sinh((l+1/4)h)\cos(y_{n,h})} \\
&\leq \frac{K\pi(b-a)^{\alpha+\beta-1}d\,\mathrm{e}^{\frac{\pi}{2}\nu\cos d}}{\cos^{\alpha+\beta}(\frac{\pi}{2}\sin y_{n,h})}\,\mathrm{e}^{-\frac{\pi}{2}\mu\exp(nh/2)\cos(y_{n,h})},
\end{aligned} \tag{3.40}$$

$$\int_{\Gamma_{\mathrm{vr}}^{+}} \left|\frac{F(\zeta)}{\tan(\pi\zeta/h)}\right||\mathrm{d}\zeta| \leq \frac{K\pi(b-a)^{\alpha+\beta-1}d\,\mathrm{e}^{\frac{\pi}{2}\nu\cos d}}{\cos^{\alpha+\beta}(\frac{\pi}{2}\sin y_{n,h})}\,\mathrm{e}^{-\frac{\pi}{2}\mu\exp(nh/2)\cos(y_{n,h})}, \tag{3.41}$$

where $\mu = \min\{\alpha,\,\beta\}$ and $\nu = \max\{\alpha,\,\beta\}$. The estimate of the latter follows from the former one and the following equality:

$$\begin{aligned}
|1/\tan(\pi(x_{N,h}+\mathrm{i}\,y)/h)| &= \sqrt{\frac{\cosh(2\pi y/h)+\cos(2\pi x_{N,h}/h)}{\cosh(2\pi y/h)-\cos(2\pi x_{N,h}/h)}} \\
&= \sqrt{\frac{\cosh(2\pi y/h)+\cos(\pi(2l+1/2))}{\cosh(2\pi y/h)-\cos(\pi(2l+1/2))}} \\
&= \sqrt{\frac{\cosh(2\pi y/h)+0}{\cosh(2\pi y/h)-0}} \\
&= 1,
\end{aligned} \tag{3.42}$$

which holds for $x_{N,h} + \mathrm{i}\, y \in \Gamma_{\mathrm{vr}}^{+}$. We can also obtain similar estimates for the integrals on Γ_{vr}^{-} and $\Gamma_{\mathrm{vl}}^{\pm}$. Moreover, noting

$$x_{n,h} = (\lceil n/2 \rceil + 1/4)h \geq (n/2 + 1/4)h \geq \frac{nh}{2} = \frac{\log(cn)}{2} \tag{3.43}$$

and $n > 1/c$, we have

$$y_{n,h} = d(1-\epsilon)\exp(-x_{n,h}) \leq \frac{d}{\sqrt{cn}} \leq d, \tag{3.44}$$

which implies $\cos(y_{n,h}) \geq \cos d$. Then we have

$$\begin{aligned}\lim_{\epsilon\to+0} E_{\mathrm{v}} &\leq \frac{4K\pi(b-a)^{\alpha+\beta-1} d\,\mathrm{e}^{\frac{\pi}{2}\nu\cos d}}{\cos^{\alpha+\beta}(\frac{\pi}{2}\sin d)}\,\mathrm{e}^{-\frac{\pi}{2}\mu\exp(nh/2)\cos d} \\ &= \frac{4K\pi(b-a)^{\alpha+\beta-1} d\,\mathrm{e}^{\frac{\pi}{2}\nu\cos d}}{\cos^{\alpha+\beta}(\frac{\pi}{2}\sin d)}\,\mathrm{e}^{-\frac{\pi}{2}\mu\sqrt{cn}\cos d}. \end{aligned}\tag{3.45}$$

<u>(ii) Estimate of the first term E_{h}.</u> Next, we bound the first term of the RHS of (3.39). Noting

$$-\int_{\Gamma_{\mathrm{hr}}^{+}+\Gamma_{\mathrm{hl}}^{+}} F(\zeta)\,\mathrm{d}\zeta + \int_{\Gamma_{\mathrm{hr}}^{-}+\Gamma_{\mathrm{hl}}^{-}} F(\zeta)\,\mathrm{d}\zeta = \int_{-x_{M,h}}^{x_{N,h}} \{F(x+\mathrm{i}\,y_{n,h}) + F(x-\mathrm{i}\,y_{n,h})\}\,\mathrm{d}x \tag{3.46}$$

and

$$\int_{\Gamma_{\mathrm{hr}}^{+}+\Gamma_{\mathrm{hl}}^{+}+\Gamma_{\mathrm{hl}}^{-}+\Gamma_{\mathrm{hr}}^{-}} \frac{F(\zeta)}{\tan(\pi\zeta/h)}\,\mathrm{d}\zeta = \int_{-x_{M,h}}^{x_{N,h}} \left\{-\frac{F(x+\mathrm{i}\,y_{n,h})}{\tan(\pi(x+\mathrm{i}\,y_{n,h})/h)} + \frac{F(x-\mathrm{i}\,y_{n,h})}{\tan(\pi(x-\mathrm{i}\,y_{n,h})/h)}\right\}\mathrm{d}x, \tag{3.47}$$

we have

$$\begin{aligned} E_{\mathrm{h}} &= \left| \int_{-x_{M,h}}^{x_{N,h}} \left\{ F(x-\mathrm{i}\,y_{n,h}) \frac{\mathrm{e}^{-\mathrm{i}\,2\pi(x-\mathrm{i}\,y_{n,h})/h}}{1-\mathrm{e}^{-\mathrm{i}\,2\pi(x-\mathrm{i}\,y_{n,h})/h}} - F(x+\mathrm{i}\,y_{n,h}) \frac{\mathrm{e}^{\mathrm{i}\,2\pi(x+\mathrm{i}\,y_{n,h})/h}}{1-\mathrm{e}^{\mathrm{i}\,2\pi(x+\mathrm{i}\,y_{n,h})/h}} \right\} \mathrm{d}x \right| \\ &\leq \frac{\mathrm{e}^{-2\pi y_{n,h}/h}}{1-\mathrm{e}^{-2\pi y_{n,h}/h}} \int_{-x_{M,h}}^{x_{N,h}} \{|F(x-\mathrm{i}\,y_{n,h})| + |F(x+\mathrm{i}\,y_{n,h})|\}\,\mathrm{d}x. \end{aligned}\tag{3.48}$$

As for the integral of the RHS of (3.48), using (3.24) of Lemma 3.7 we have

$$\begin{aligned} &\int_{-x_{M,h}}^{x_{N,h}} \{|F(x-\mathrm{i}\,y_{n,h})| + |F(x+\mathrm{i}\,y_{n,h})|\}\,\mathrm{d}x \\ &\leq 2K(b-a)^{\alpha+\beta-1} \int_{-x_{M,h}}^{x_{N,h}} G_{\alpha,\beta}(x, y_{n,h})\,\mathrm{d}x \\ &\leq \frac{4K\pi(b-a)^{\alpha+\beta-1}}{\cos^{\alpha+\beta}(\frac{\pi}{2}\sin d)} \int_0^{\infty} \cosh(x)\,\mathrm{e}^{-\pi\mu\sinh(x)\cos d}\,\mathrm{d}x \\ &= \frac{4K(b-a)^{\alpha+\beta-1}}{\mu\cos(d)\cos^{\alpha+\beta}(\frac{\pi}{2}\sin d)}. \end{aligned}\tag{3.49}$$

With this estimate and

$$y_{n,h} = d(1-\epsilon)\exp(-(\lceil n/2 \rceil + 1/4)h) \geq d(1-\epsilon)\exp(-nh/2)\exp(-3h/4), \tag{3.50}$$

we have

$$\lim_{\epsilon\to+0} E_{\mathrm{h}} \leq \frac{4K(b-a)^{\alpha+\beta-1}}{\mu\cos(d)\cos^{\alpha+\beta}(\frac{\pi}{2}\sin d)} \left\{ \frac{\mathrm{e}^{-2\pi d\exp(-nh/2)\exp(-3h/4)/h}}{1-\mathrm{e}^{-2\pi d\exp(-nh/2)\exp(-3h/4)/h}} \right\}. \tag{3.51}$$

Moreover, noting

$$\exp\left(-\frac{3h}{4}\right) = \exp\left(-\frac{3c}{4}\frac{\log(cn)}{(cn)}\right) \geq \exp\left(-\frac{3c}{4\,\mathrm{e}}\right), \tag{3.52}$$

$$\frac{\exp(-nh/2)}{h} = \frac{1}{c}\frac{\sqrt{cn}}{\log(cn)} = \frac{1}{2c}\frac{\sqrt{cn}}{\log\sqrt{cn}} \geq \frac{\mathrm{e}}{2c}, \tag{3.53}$$

we can bound the denominator of $\{\,\cdot\,\}$ part in (3.51) as

$$\frac{1}{(1-\,\mathrm{e}^{-2\pi d\exp(-nh/2)\exp(-3h/4)/h})} \leq \frac{1}{(1-\,\mathrm{e}^{-2\pi d/c'})}, \tag{3.54}$$

where $c' = 2c/\exp(1-3c/(4\,\mathrm{e}))$. To bound the numerator of $\{\,\cdot\,\}$ part in (3.51), we use the fact that

$$\lim_{n\to\infty}\frac{\sqrt{c_1 n}}{\log(c_2 n)}\left(\frac{1}{(c_2 n)^{c_3/n}} - 1\right) = 0, \tag{3.55}$$

which is shown by l'Hôpital's theorem. Then we have

$$\begin{aligned}\exp\left(-\frac{2\pi d\exp(-nh/2)\exp(-3h/4)}{h}\right) &= \exp\left(-\frac{2\pi d}{\sqrt{c}}\frac{\sqrt{n}}{\log(cn)}\frac{1}{\{(cn)^{3/4}\}^{1/n}}\right)\\ &\leq C_{c,d}\exp\left(-\frac{2\pi d}{\sqrt{c}}\frac{\sqrt{n}}{\log(cn)}\right),\end{aligned} \tag{3.56}$$

for a certain constant $C_{c,d}$ depending only on c and d. Combining the estimates of the denominator and the numerator above, we have

$$\lim_{\epsilon\to+0} E_{\mathrm{h}} \leq \frac{4K(b-a)^{\alpha+\beta-1}C_{c,d}}{\mu\cos(d)\cos^{\alpha+\beta}(\frac{\pi}{2}\sin d)(1-\,\mathrm{e}^{-2\pi d/c'})}\exp\left(-\frac{2\pi d}{\sqrt{c}}\frac{\sqrt{n}}{\log(cn)}\right). \tag{3.57}$$

From this estimate and (3.45), we obtain the conclusion. ■

4 Existing/new convergence theorems for the Sinc approximation and the Sinc indefinite integration

The Sinc approximation and the Sinc indefinite integration are approximation formulas frequently combined with the SE transformation or the DE transformation, like the trapezoidal rule (described later in detail). If the Sinc approximation is combined with the SE transformation, we call it the "SE-Sinc approximation," and the same applies to the other combination.

As for these approximation formulas, there also exist functions for which the formulas with the SE transformation works good but the formulas with DE does not. In this section, for the DE-Sinc approximation and the DE-Sinc indefinite integration, we give similar results to Theorem 2.5 under the SE's assumptions. In a similar manner to Section 2, after reviewing the standard theorems for the SE/DE-Sinc approximation, we present a new theorem for DE's error under the SE's assumption. After that, we do the same for the SE/DE-Sinc indefinite integration. We prove these theorems in Sections 5 and 6, respectively.

4.1 Existing convergence theorems for the SE/DE-Sinc approximation under the standard assumptions

The Sinc approximation is an approximation formula for a function F defined on $\mathbb{R}$ as:

$$F(x) \approx \sum_{j=-M}^{N} F(jh)S(j,h)(x), \quad x\in\mathbb{R}, \tag{4.1}$$

where $S(j,h)$ is the so-called Sinc function defined by

$$S(j,h)(x) = \frac{\sin \pi[(x/h) - j]}{\pi[(x/h) - j]}. \tag{4.2}$$

Even if a function f is defined on the finite interval $(a,\, b)$, we can apply the Sinc approximation combining with the SE/DE transformation as follows:

$$f(t) = f(\psi_{\mathrm{SE}}(\psi_{\mathrm{SE}}^{-1}(t))) \approx \sum_{j=-M}^{N} f(\psi_{\mathrm{SE}}(jh))S(j,h)(\psi_{\mathrm{SE}}^{-1}(t)), \tag{4.3}$$

$$f(t) = f(\psi_{\mathrm{DE}}(\psi_{\mathrm{DE}}^{-1}(t))) \approx \sum_{j=-M}^{N} f(\psi_{\mathrm{DE}}(jh))S(j,h)(\psi_{\mathrm{DE}}^{-1}(t)), \tag{4.4}$$

for $t \in (a,\, b)$. The following convergence theorems have been known for these approximations.

Theorem 4.1 (Stenger [11, Theorem 4.2.5])**.** Let $f \in \mathbf{L}_{\alpha,\beta}(\psi_{\mathrm{SE}}(\mathscr{D}_d))$ for d with $0 < d < \pi$. Let $\mu = \min\{\alpha,\, \beta\}$, n be a positive integer, and h be selected by the formula

$$h = \sqrt{\frac{\pi d}{\mu n}}. \tag{4.5}$$

Furthermore, let M and N be positive integers defined by (2.6). Then there exists a constant C independent of n such that

$$\sup_{a<t<b} \left| f(t) - \sum_{j=-M}^{N} f(\psi_{\mathrm{SE}}(jh))S(j,h)(\psi_{\mathrm{SE}}^{-1}(t)) \right| \leq C\sqrt{n}\,\mathrm{e}^{-\sqrt{\pi d \mu n}}. \tag{4.6}$$

Theorem 4.2 (Okayama et al. [6, Theorem 2.11])**.** Let $f \in \mathbf{L}_{\alpha,\beta}(\psi_{\mathrm{DE}}(\mathscr{D}_d))$ for d with $0 < d < \pi/2$. Let $\mu = \min\{\alpha,\, \beta\}$, $\nu = \max\{\alpha,\, \beta\}$, n be a positive integer with $n > \nu/(2d)$, and h be selected by the formula

$$h = \frac{\log(2dn/\mu)}{n}. \tag{4.7}$$

Furthermore, let M and N be positive integers defined by (2.9). Then there exists a constant C independent of n such that

$$\sup_{a<t<b} \left| f(t) - \sum_{j=-M}^{N} f(\psi_{\mathrm{DE}}(jh))S(j,h)(\psi_{\mathrm{DE}}^{-1}(t)) \right| \leq C\,\mathrm{e}^{-\pi d n/\log(2dn/\mu)}. \tag{4.8}$$

4.2 New convergence theorem for the DE-Sinc approximation under the SE-Sinc assumption

The condition on f in Theorem 4.2 is different from the one in Theorem 4.1. Under the same assumption as Theorem 4.1, we have the next theorem.

Theorem 4.3. Let $f \in \mathbf{L}_{\alpha,\beta}(\psi_{\mathrm{SE}}(\mathscr{D}_d))$ for d with $0 < d < \pi$, and put $d' = \arcsin(d/\pi)$. Let $\mu = \min\{\alpha, \beta\}$, $\nu = \max\{\alpha, \beta\}$, and let c be a positive number. For a positive integer n with $n > \nu/(c\mu)$, define h as (2.19), and define M and N as (2.9). Then there exists a positive constant $C_{\alpha,\beta,c,d'}$ depending only on α, β, c, d' such that

$$\sup_{a<t<b} \left| f(t) - \sum_{j=-M}^{N} f(\psi_{\mathrm{DE}}(jh))S(j,h)({\psi_{\mathrm{DE}}}^{-1}(t)) \right| \leq K(b-a)^{\alpha+\beta} C_{\alpha,\beta,c,d'}\sqrt{n}\exp\left(-\frac{\pi d'}{\sqrt{c}}\frac{\sqrt{n}}{\log(cn)}\right), \tag{4.9}$$

where K is the constant in (2.1).

4.3 Existing convergence theorems for the SE/DE-Sinc indefinite integration under the standard assumptions

The Sinc indefinite integration is an approximation formula for the indefinite integral, derived by integrating the both sides of (4.1) as follows:

$$\int_{-\infty}^{x} F(\xi)\,\mathrm{d}\xi \approx \int_{-\infty}^{x}\left\{\sum_{j=-M}^{N} F(jh)S(j,h)(\xi)\right\}\mathrm{d}\xi = \sum_{j=-M}^{N} F(jh)J(j,h)(x), \tag{4.10}$$

where $J(j,h)(x) = \int_{-\infty}^{x} S(j,h)(\xi)\,\mathrm{d}\xi$. This approximation can be also applied to the indefinite integral over the finite interval, by combining with the SE transformation or the DE transformation as follows:

$$\int_{a}^{t} f(s)\,\mathrm{d}s = \int_{-\infty}^{\psi_{\mathrm{SE}}^{-1}(t)} f(\psi_{\mathrm{SE}}(\xi))\psi_{\mathrm{SE}}'(\xi)\,\mathrm{d}\xi \approx \sum_{j=-M}^{N} f(\psi_{\mathrm{SE}}(jh))\psi_{\mathrm{SE}}'(jh)J(j,h)(\psi_{\mathrm{SE}}^{-1}(t)), \tag{4.11}$$

$$\int_{a}^{t} f(s)\,\mathrm{d}s = \int_{-\infty}^{\psi_{\mathrm{DE}}^{-1}(t)} f(\psi_{\mathrm{DE}}(\xi))\psi_{\mathrm{DE}}'(\xi)\,\mathrm{d}\xi \approx \sum_{j=-M}^{N} f(\psi_{\mathrm{DE}}(jh))\psi_{\mathrm{DE}}'(jh)J(j,h)(\psi_{\mathrm{DE}}^{-1}(t)). \tag{4.12}$$

For each approximation a convergence theorem has been given as below.

Theorem 4.4 (Okayama et al. [6, Theorem 2.7])**.** Let $fQ \in \mathbf{L}_{\alpha,\beta}(\psi_{\mathrm{SE}}(\mathscr{D}_d))$ for d with $0 < d < \pi$. Let $\mu = \min\{\alpha,\ \beta\}$, n be a positive integer, and h be selected by the formula (4.5). Furthermore, let M and N be positive integers defined by (2.6). Then there exists a constant C independent of n such that

$$\sup_{a<t<b}\left|\int_{a}^{t} f(s)\,\mathrm{d}s - \sum_{j=-M}^{N} f(\psi_{\mathrm{SE}}(jh))\psi_{\mathrm{SE}}'(jh)J(j,h)(\psi_{\mathrm{SE}}^{-1}(t))\right| \leq C\,\mathrm{e}^{-\sqrt{\pi d\mu n}}. \tag{4.13}$$

Theorem 4.5 (Okayama et al. [6, Theorem 2.13])**.** Let $fQ \in \mathbf{L}_{\alpha,\beta}(\psi_{\mathrm{DE}}(\mathscr{D}_d))$ for d with $0 < d < \pi/2$. Let $\mu = \min\{\alpha,\ \beta\}$, $\nu = \max\{\alpha,\ \beta\}$, n be a positive integer with $n > \nu/(2d)$, and h be selected by the formula (4.7). Furthermore, let M and N be positive integers defined by (2.9). Then there exists a constant C independent of n such that

$$\sup_{a<t<b}\left|\int_{a}^{t} f(s)\,\mathrm{d}s - \sum_{j=-M}^{N} f(\psi_{\mathrm{DE}}(jh))\psi_{\mathrm{DE}}'(jh)J(j,h)(\psi_{\mathrm{DE}}^{-1}(t))\right| \leq C\frac{\log(2dn/\mu)}{n}\,\mathrm{e}^{-\pi dn/\log(2dn/\mu)}. \tag{4.14}$$

4.4 New convergence theorem for the DE-Sinc indefinite integration under the SE-Sinc assumption

Under the same assumption as Theorem 4.4, we have the next theorem.

Theorem 4.6. Let $fQ \in \mathbf{L}_{\alpha,\beta}(\psi_{\mathrm{SE}}(\mathscr{D}_d))$ for d with $0 < d < \pi$, and put $d' = \arcsin(d/\pi)$. Let $\mu = \min\{\alpha,\beta\}$, $\nu = \max\{\alpha,\ \beta\}$, and let c be a positive number. For a positive integer n with $n > \nu/(c\mu)$, define h as (2.19) and define M and N as (2.9). Then there exists a positive constant $C_{\alpha,\beta,c,d'}$ depending only on α, β, c, d' such that

$$\begin{aligned}&\sup_{a<t<b}\left|\int_{a}^{t} f(s)\,\mathrm{d}s - \sum_{j=-M}^{N} f(\psi_{\mathrm{DE}}(jh))\psi_{\mathrm{DE}}'(jh)J(j,h)(\psi_{\mathrm{DE}}^{-1}(t))\right|\\ &\leq K(b-a)^{\alpha+\beta-1}C_{\alpha,\beta,c,d'}\frac{\log(cn)}{\sqrt{n}}\exp\left(-\frac{\pi d'}{\sqrt{c}}\frac{\sqrt{n}}{\log(cn)}\right),\end{aligned} \tag{4.15}$$

where K is the constant in (2.1).

5 Proof of Theorem 4.3 (for the DE-Sinc approximation)

5.1 Useful inequalities

To prove Theorem 4.3 (in this section) and Theorem 4.6 (in the next section), we use similar estimates to (3.29), (3.40) and (3.49). Here we prepare a lemma giving such estimates.

Lemma 5.1. Let α, β and c be positive constants, and put $\mu = \min\{\alpha, \beta\}$ and $\nu = \max\{\alpha, \beta\}$. For a positive integer n with $n > \nu/(c\mu)$, let M, N be defined by (2.9), and h be defined by (2.19). Then we have

$$\int_{-\infty}^{-Mh/2} G_{\alpha,\beta}(x, 0)\,\mathrm{d}x \leq \frac{1}{\mu}\,\mathrm{e}^{\frac{\pi}{2}\nu}\,\mathrm{e}^{-\frac{\pi}{2}\mu\sqrt{cn}}, \tag{5.1}$$

$$\int_{Nh/2}^{\infty} G_{\alpha,\beta}(x, 0)\,\mathrm{d}x \leq \frac{1}{\mu}\,\mathrm{e}^{\frac{\pi}{2}\nu}\,\mathrm{e}^{-\frac{\pi}{2}\mu\sqrt{cn}}, \tag{5.2}$$

where $G_{\alpha,\beta}(x, y)$ is the function defined in (3.25). Furthermore, for a positive real constant $\tilde{d}$ with $\tilde{d} < \pi/2$ and a real constant δ, set $\tilde{x}_{M,h} = (\lceil M/2\rceil + \delta)h$, $\tilde{x}_{N,h} = (\lceil N/2\rceil + \delta)h$, $\tilde{x}_{n,h} = (\lceil n/2\rceil + \delta)h$, and $\tilde{y}_{n,h} = \tilde{d}\exp(-\tilde{x}_{n,h})$. Then we have the following inequalities:

$$\int_{0}^{\tilde{y}_{n,h}} G_{\alpha,\beta}(-\tilde{x}_{M,h}, y)\,\mathrm{d}y \leq \frac{\tilde{C}_{\alpha,\beta,c,\delta}\pi\tilde{d}\,\mathrm{e}^{\frac{\pi}{2}\nu\cos\tilde{d}}}{\cos^{\alpha+\beta}(\frac{\pi}{2}\sin\tilde{d})}\,\mathrm{e}^{-\frac{\pi}{2}\mu\sqrt{cn}\cos\tilde{d}}, \tag{5.3}$$

$$\int_{0}^{\tilde{y}_{n,h}} G_{\alpha,\beta}(\tilde{x}_{N,h}, y)\,\mathrm{d}y \leq \frac{\tilde{C}_{\alpha,\beta,c,\delta}\pi\tilde{d}\,\mathrm{e}^{\frac{\pi}{2}\nu\cos\tilde{d}}}{\cos^{\alpha+\beta}(\frac{\pi}{2}\sin\tilde{d})}\,\mathrm{e}^{-\frac{\pi}{2}\mu\sqrt{cn}\cos\tilde{d}}, \tag{5.4}$$

$$\int_{-\tilde{x}_{M,h}}^{\tilde{x}_{N,h}} G_{\alpha,\beta}(x, \tilde{y}_{n,h})\,\mathrm{d}x \leq \frac{2}{\mu\cos(\tilde{d})\cos^{\alpha+\beta}(\frac{\pi}{2}\sin\tilde{d})}, \tag{5.5}$$

where $\tilde{C}_{\alpha,\beta,c,\delta}$ is a positive constant depending only on α, β, c and δ. Moreover, if $\delta > 0$, $\tilde{C}_{\alpha,\beta,c,\delta}$ can be taken as $\tilde{C}_{\alpha,\beta,c,\delta} = 1$.

Proof. It suffices to prove (5.1), (5.4) and (5.5). First, (5.1) is the same inequality as (3.29) except for the constant $K(b-a)^{\alpha+\beta-1}$. Next, (5.5) is the same inequality as (3.49) except for the constant $2K(b-a)^{\alpha+\beta-1}$. Finally we prove (5.4). In a similar manner to (3.40), we have

$$\int_{0}^{\tilde{y}_{n,h}} G(\tilde{x}_{N,h}, y)\,\mathrm{d}y \leq \frac{\pi\tilde{d}\,\mathrm{e}^{-\pi\beta\sinh(\tilde{x}_{N,h})\cos(\tilde{y}_{n,h})}}{\cos^{\alpha+\beta}(\frac{\pi}{2}\sin\tilde{y}_{n,h})} \leq \frac{\pi\tilde{d}\,\mathrm{e}^{-\pi\beta\sinh(\tilde{x}_{N,h})\cos(\tilde{d})}}{\cos^{\alpha+\beta}(\frac{\pi}{2}\sin\tilde{d})}. \tag{5.6}$$

As for $\beta\sinh(\tilde{x}_{N,h})$ in (5.6), we can obtain its lower bound as follows:

$$\begin{aligned}
\beta\sinh(\tilde{x}_{N,h}) &\geq \beta\frac{\mathrm{e}^{\tilde{x}_{N,h}} - 1}{2} \\
&\geq \frac{\beta}{2}\exp\{(\lceil N/2\rceil + \delta)h\} - \frac{\nu}{2} \\
&\geq \frac{\beta}{2}\exp(Nh/2)\exp(\delta h) - \frac{\nu}{2} \\
&\geq \begin{cases} (\beta/2)\exp(nh/2)\exp(\delta h) - \nu/2 & (\alpha \geq \beta) \\ (\beta/2)\exp(nh/2)\exp(-\log(\beta/\alpha)/2)\exp(\delta h) - \nu/2 & (\alpha < \beta) \end{cases} \\
&\geq \frac{\mu}{2}\exp(nh/2)\exp(\delta h) - \frac{\nu}{2} \\
&= \frac{\mu}{2}\sqrt{cn}\cdot(cn)^{\delta/n} - \frac{\nu}{2}.
\end{aligned} \tag{5.7}$$

To see the effect of $(cn)^{\delta/n}$ in (5.7), we note that

$$\lim_{n\to\infty} \frac{\exp(-A\sqrt{cn}\cdot(cn)^{\delta/n})}{\exp(-A\sqrt{cn})} = \lim_{n\to\infty} \exp\{-A\sqrt{cn}((cn)^{\delta/n}-1)\} = 1 \tag{5.8}$$

for an arbitrary positive constant A. Combining (5.6), (5.7) and (5.8), we obtain the desired inequality. Finally, if $\delta > 0$, it follows from (5.7) that

$$\beta \sinh(\tilde{x}_{N,h}) \geq \frac{\mu}{2}\sqrt{cn} - \frac{\nu}{2}. \tag{5.9}$$

Therefore we can take $\tilde{C}_{\alpha,\beta,c,\delta}$ as $\tilde{C}_{\alpha,\beta,c,\delta} = 1$. ■

In a similar manner to the proof of Theorem 2.5, we write d' as d and assume that $f \in \mathbf{L}_{\alpha,\beta}(\psi_{\mathrm{DE}}(\mathit{\Delta}_d^1))$ throughout the following proofs in this section.

5.2 Sketch of the proof

The idea of the proof is similar to that of Theorem 2.5: we split the error into several terms and estimate each of them. Setting $F(x) = f(\psi_{\mathrm{DE}}(x))$, we have

$$\begin{aligned}
(\text{The LHS of (4.9)}) &= \sup_{-\infty<x<\infty} \left| F(x) - \sum_{j=-M}^{N} F(jh)S(j,h)(x) \right| \\
&\leq \sup_{-\infty<x<\infty} \left| F(x) - \sum_{j=-\lceil \frac{M}{2} \rceil}^{\lceil \frac{N}{2} \rceil} F(jh)S(j,h)(x) \right| \\
&\quad + \sup_{-\infty<x<\infty} \left\{ \left| \sum_{j=-M}^{-\lceil \frac{M}{2} \rceil -1} F(jh)S(j,h)(x) \right| + \left| \sum_{j=\lceil \frac{N}{2} \rceil +1}^{N} F(jh)S(j,h)(x) \right| \right\}.
\end{aligned} \tag{5.10}$$

The first term of the RHS, say $E_0(F,n)$, can be further estimated as

$$\begin{aligned}
E_0(F,n) &\leq \max \left\{ \sup_{-\lceil \frac{M}{2} \rceil h \leq x \leq \lceil \frac{N}{2} \rceil h} \left| F(x) - \sum_{j=-\lceil \frac{M}{2} \rceil}^{\lceil \frac{N}{2} \rceil} F(jh)S(j,h)(x) \right|, \right. \\
&\qquad \left. \sup_{x \leq -\lceil \frac{M}{2} \rceil h,\ \lceil \frac{N}{2} \rceil h \leq x} \left| F(x) - \sum_{j=-\lceil \frac{M}{2} \rceil}^{\lceil \frac{N}{2} \rceil} F(jh)S(j,h)(x) \right| \right\} \\
&\leq \max \left\{ E_1(F,n), E_2(F,n) + E_3(F,n) \right\},
\end{aligned} \tag{5.11}$$

where $E_1(F,n)$, $E_2(F,n)$, $E_3(F,n)$ are defined by

$$E_1(F,n) = \sup_{-\lceil \frac{M}{2} \rceil h \leq x \leq \lceil \frac{N}{2} \rceil h} \left| F(x) - \sum_{j=-\lceil \frac{M}{2} \rceil}^{\lceil \frac{N}{2} \rceil} F(jh)S(j,h)(x) \right|, \tag{5.12}$$

$$E_2(F,n) = \sup_{x \leq -\lceil \frac{M}{2} \rceil h,\ \lceil \frac{N}{2} \rceil h \leq x} \left| \sum_{j=-\lceil \frac{M}{2} \rceil}^{\lceil \frac{N}{2} \rceil} F(jh)S(j,h)(x) \right|, \tag{5.13}$$

$$E_3(F,n) = \sup_{x \leq -\lceil \frac{M}{2} \rceil h,\ \lceil \frac{N}{2} \rceil h \leq x} |F(x)|, \tag{5.14}$$

respectively. And we define $E_4(F,n)$ as

$$E_4(F,n) = \sup_{-\infty<x<\infty}\left|\sum_{j=-M}^{-\lceil\frac{M}{2}\rceil-1} F(jh)S(j,h)(x)\right| + \sup_{-\infty<x<\infty}\left|\sum_{j=\lceil\frac{N}{2}\rceil+1}^{N} F(jh)S(j,h)(x)\right|. \tag{5.15}$$

Their estimates are given by the following lemmas.

Lemma 5.2. Under the same assumptions of Theorem 4.3, for $F(x) = f(\psi_{\mathrm{DE}}(x))$, there exists a positive constant $C_{\alpha,\beta,c,d}^{(1)}$ depending only on α, β, c, d such that

$$E_1(F,n) \leq K(b-a)^{\alpha+\beta}C_{\alpha,\beta,c,d}^{(1)}\sqrt{n}\exp\left(-\frac{\pi d}{\sqrt{c}}\frac{\sqrt{n}}{\log(cn)}\right). \tag{5.16}$$

Lemma 5.3. Under the same assumptions of Theorem 4.3, for $F(x) = f(\psi_{\mathrm{DE}}(x))$, there exists a positive constant $C_{\alpha,\beta,c,d}^{(2)}$ depending only on α, β, c, d such that

$$E_2(F,n) \leq K(b-a)^{\alpha+\beta}C_{\alpha,\beta,c,d}^{(2)}\sqrt{n}\exp\left(-\frac{\pi d}{\sqrt{c}}\frac{\sqrt{n}}{\log(cn)}\right). \tag{5.17}$$

Lemma 5.4. Under the same assumptions of Theorem 4.3, for $F(x) = f(\psi_{\mathrm{DE}}(x))$, it holds that

$$E_3(F,n) \leq K(b-a)^{\alpha+\beta}\,\mathrm{e}^{\frac{\pi}{2}\nu}\,\mathrm{e}^{-\frac{\pi}{2}\mu\sqrt{cn}}. \tag{5.18}$$

Lemma 5.5. Under the same assumptions of Theorem 4.3, for $F(x) = f(\psi_{\mathrm{DE}}(x))$, there exists a positive constant $\tilde{C}_{\alpha,\beta,c}$ depending only on α, β, c such that

$$E_4(F,n) \leq K(b-a)^{\alpha+\beta}\tilde{C}_{\alpha,\beta,c}\frac{n}{\log(cn)}\,\mathrm{e}^{-\frac{\pi}{2}\mu\sqrt{cn}}. \tag{5.19}$$

By showing each lemma above, we obtain Theorem 4.3.

5.3 Proofs of Lemmas 5.2–5.5

First, we begin with the proof of Lemma 5.4: the estimate of E_3.

Proof of Lemma 5.4. Noting

$$|F(x)| \leq \frac{K(b-a)^{\alpha+\beta}}{(1+\mathrm{e}^{-\pi\sinh x})^{\alpha}(1+\mathrm{e}^{\pi\sinh x})^{\beta}} \leq \begin{cases} K(b-a)^{\alpha+\beta}\,\mathrm{e}^{\frac{\pi}{2}\alpha}\,\mathrm{e}^{-\frac{\pi}{2}\alpha\exp(-x)} & (x<0) \\ K(b-a)^{\alpha+\beta}\,\mathrm{e}^{\frac{\pi}{2}\beta}\,\mathrm{e}^{-\frac{\pi}{2}\beta\exp(x)} & (x>0) \end{cases} \tag{5.20}$$

and the relation among M, N and n, we have

$$E_3(F,n) \leq K(b-a)^{\alpha+\beta}\,\mathrm{e}^{\frac{\pi}{2}\nu}\,\mathrm{e}^{-\frac{\pi}{2}\mu\exp(nh/2)} = K(b-a)^{\alpha+\beta}\,\mathrm{e}^{\frac{\pi}{2}\nu}\,\mathrm{e}^{-\frac{\pi}{2}\mu\sqrt{cn}}. \tag{5.21}$$

■

Next, we prove Lemma 5.5: the estimate of E_4.

Proof of Lemma 5.5. As for the first sum of E_4, using (3.26) of Lemma 3.7, we have

$$\left|\sum_{j=-M}^{-\lceil\frac{M}{2}\rceil-1} F(jh)S(j,h)(x)\right| \leq \sum_{j=-M}^{-\lceil\frac{M}{2}\rceil-1}|F(jh)| \leq \frac{K(b-a)^{\alpha+\beta}}{\pi}\sum_{j=-\infty}^{-\lceil\frac{M}{2}\rceil-1}G_{\alpha,\beta}(jh,0), \tag{5.22}$$

and using a similar manner to (3.28) and applying (5.1) of Lemma 5.1, we have

$$\frac{K(b-a)^{\alpha+\beta}}{\pi h} h \sum_{j=-\infty}^{-\lceil \frac{M}{2} \rceil - 1} G_{\alpha,\beta}(jh, 0) \le \frac{K(b-a)^{\alpha+\beta}}{\pi h} C_{\alpha,\beta,c} \int_{-\infty}^{-Mh/2} G_{\alpha,\beta}(x,0)\,\mathrm{d}x$$
$$\le C_{\alpha,\beta,c} \frac{K(b-a)^{\alpha+\beta}\,\mathrm{e}^{\frac{\pi}{2}\nu}}{\pi\mu} \frac{n}{\log(cn)}\,\mathrm{e}^{-\frac{\pi}{2}\mu\sqrt{cn}}, \tag{5.23}$$

for some constant $C_{\alpha,\beta,c}$ depending only on α, β, c. Here (2.19) is used. As for the second sum of E_4, we can obtain the same estimate as above. Therefore we have the conclusion. ■

Next, we prove Lemma 5.2: the estimate of E_1.

Proof of Lemma 5.2. We use the paths defined in the proof of Lemma 3.4. We set $x'_{M,h} = (\lceil M/2 \rceil + 1/2)h$, $x'_{N,h} = (\lceil N/2 \rceil + 1/2)h$, $x'_{n,h} = (\lceil n/2 \rceil + 1/2)h$, $y'_{n,h} = d(1-\epsilon)\exp(-x'_{n,h})$ and consider the contour

$$\Gamma^+_{\mathrm{vr}}(x'_{N,h}, y'_{n,h}) + \Gamma^+_{\mathrm{hr}}(x'_{N,h}, y'_{n,h}) + \Gamma^+_{\mathrm{hl}}(x'_{M,h}, y'_{n,h}) + \Gamma^+_{\mathrm{vl}}(x'_{M,h}, y'_{n,h})$$
$$+ \Gamma^-_{\mathrm{vl}}(x'_{M,h}, y'_{n,h}) + \Gamma^-_{\mathrm{hl}}(x'_{M,h}, y'_{n,h}) + \Gamma^-_{\mathrm{hr}}(x'_{N,h}, y'_{n,h}) + \Gamma^-_{\mathrm{vr}}(x'_{N,h}, y'_{n,h}), \tag{5.24}$$

where $0 < \epsilon < 1$. For simplicity of the notations, we set $k = \lceil M/2 \rceil$, $l = \lceil N/2 \rceil$ and omit the expressions $(x'_{M,h}, y'_{n,h})$ and $(x'_{N,h}, y'_{n,h})$. For x with $-kh \le x \le lh$, we have

$$F(x) - \sum_{j=-k}^{l} F(jh) S(j,h)(x) = \frac{\sin(\pi x/h)}{2\pi \mathrm{i}} \int_{\Gamma^+_{\mathrm{vr}} + \Gamma^+_{\mathrm{hr}} + \Gamma^+_{\mathrm{hl}} + \Gamma^+_{\mathrm{vl}} + \Gamma^-_{\mathrm{vl}} + \Gamma^-_{\mathrm{hl}} + \Gamma^-_{\mathrm{hr}} + \Gamma^-_{\mathrm{vr}}} \frac{F(\zeta)\,\mathrm{d}\zeta}{(\zeta - x)\sin(\pi\zeta/h)}. \tag{5.25}$$

As for the integral on Γ^+_{vr}, noting

$$|\zeta - x| \ge |(l + 1/2)h - x| \ge h/2, \tag{5.26}$$
$$|\sin(\pi\zeta/h)| = |(-1)^l \cosh\{\pi(\operatorname{Im}\zeta)/h\}| \ge 1 \tag{5.27}$$

for $\zeta \in \Gamma^+_{\mathrm{vr}}$, and using (3.26) of Lemma 3.7 and (5.4) of Lemma 5.1 with $\delta = 1/2$, we have

$$\left| \frac{\sin(\pi x/h)}{2\pi \mathrm{i}} \int_{\Gamma^+_{\mathrm{vr}}} \frac{F(\zeta)}{(\zeta - x)\sin(\pi\zeta/h)}\,\mathrm{d}\zeta \right| \le \frac{1}{\pi h} \int_{\Gamma^+_{\mathrm{vr}}} |F(\zeta)|\,|\mathrm{d}\zeta|$$
$$\le \frac{K(b-a)^{\alpha+\beta}}{\pi^2 h} \int_0^{y'_{n,h}} G_{\alpha,\beta}(x'_{N,h}, y)\,\mathrm{d}y$$
$$\le \frac{K(b-a)^{\alpha+\beta} d\,\mathrm{e}^{\frac{\pi}{2}\nu\cos d}}{\pi \cos^{\alpha+\beta}(\frac{\pi}{2}\sin d)} \frac{n}{\log(cn)}\,\mathrm{e}^{-\frac{\pi}{2}\mu\sqrt{cn}\cos d}. \tag{5.28}$$

Since we can obtain the same estimates for the integrals on Γ^-_{vr} and $\Gamma^{\pm}_{\mathrm{vl}}$, we have

$$\lim_{\epsilon \to 0} \left| \frac{\sin(\pi x/h)}{2\pi \mathrm{i}} \int_{\Gamma^+_{\mathrm{vr}} + \Gamma^-_{\mathrm{vr}} + \Gamma^-_{\mathrm{vl}} + \Gamma^+_{\mathrm{vl}}} \frac{F(\zeta)}{(\zeta - x)\sin(\pi\zeta/h)}\,\mathrm{d}\zeta \right|$$
$$\le \frac{4K(b-a)^{\alpha+\beta} d\,\mathrm{e}^{\frac{\pi}{2}\nu\cos d}}{\pi \cos^{\alpha+\beta}(\frac{\pi}{2}\sin d)} \frac{n}{\log(cn)}\,\mathrm{e}^{-\frac{\pi}{2}\mu\sqrt{cn}\cos d}. \tag{5.29}$$

As for the integral on $\Gamma^+_{\mathrm{hr}} + \Gamma^+_{\mathrm{hl}} + \Gamma^-_{\mathrm{hl}} + \Gamma^-_{\mathrm{hr}}$, noting

$$|\zeta - x| \ge y'_{n,h}, \tag{5.30}$$
$$|\sin(\pi\zeta/h)| = \sqrt{\cosh^2(\pi y'_{n,h}/h) - \cos^2\{\pi(\operatorname{Re}\zeta)/h\}} \ge \sinh(\pi y'_{n,h}/h) \tag{5.31}$$

for $\zeta \in \Gamma^+_{\mathrm{hr}} + \Gamma^+_{\mathrm{hl}} + \Gamma^-_{\mathrm{hl}} + \Gamma^-_{\mathrm{hr}}$, we have

$$\begin{aligned} &\left| \frac{\sin(\pi x/h)}{2\pi \mathrm{i}} \int_{\Gamma^+_{\mathrm{hr}}+\Gamma^+_{\mathrm{hl}}+\Gamma^-_{\mathrm{hl}}+\Gamma^-_{\mathrm{hr}}} \frac{F(\zeta)}{(\zeta - x)\sin(\pi\zeta/h)} \,\mathrm{d}\zeta \right| \\ &\leq \frac{1}{2\pi y'_{n,h} \sinh(\pi y'_{n,h}/h)} \int_{-x'_{M,h}}^{x'_{N,h}} \{|F(x - \mathrm{i}\, y'_{n,h})| + |F(x + \mathrm{i}\, y'_{n,h})|\} \,\mathrm{d}x. \end{aligned} \tag{5.32}$$

As for the integral of this RHS, it follows from (3.26) of Lemma 3.7 and (5.5) of Lemma 5.1 that

$$\begin{aligned} \int_{-x'_{M,h}}^{x'_{N,h}} \{|F(x - \mathrm{i}\, y'_{n,h})| + |F(x + \mathrm{i}\, y'_{n,h})|\} \,\mathrm{d}x &\leq \frac{2K(b-a)^{\alpha+\beta}}{\pi} \int_{-x'_{M,h}}^{x'_{N,h}} G_{\alpha,\beta}(x, y'_{n,h}) \,\mathrm{d}x \\ &\leq \frac{4K(b-a)^{\alpha+\beta}}{\pi\mu \cos(d) \cos^{\alpha+\beta}(\frac{\pi}{2}\sin d)}. \end{aligned} \tag{5.33}$$

With this estimate and

$$y'_{n,h} = d(1-\epsilon)\exp(-(\lceil n/2 \rceil + 1/2)h) \geq d(1-\epsilon)\exp(-nh/2)\exp(-h), \tag{5.34}$$

we have

$$\begin{aligned} &\lim_{\epsilon\to 0} \left| \frac{\sin(\pi x/h)}{2\pi \mathrm{i}} \int_{\Gamma^+_{\mathrm{hr}}+\Gamma^+_{\mathrm{hl}}+\Gamma^-_{\mathrm{hl}}+\Gamma^-_{\mathrm{hr}}} \frac{F(\zeta)}{(\zeta - x)\sin(\pi\zeta/h)} \,\mathrm{d}\zeta \right| \\ &\leq \frac{4K(b-a)^{\alpha+\beta}}{\pi^2 d\mu \cos(d)\cos^{\alpha+\beta}(\frac{\pi}{2}\sin d)} \frac{1}{\exp(-nh/2)\exp(-h)} \frac{\mathrm{e}^{-\pi d \exp(-nh/2)\exp(-h)/h}}{1 - \mathrm{e}^{-2\pi d \exp(-nh/2)\exp(-h)/h}}. \end{aligned} \tag{5.35}$$

Furthermore, similarly to (3.52)–(3.56), we have

$$\begin{aligned} &\lim_{\epsilon\to 0} \left| \frac{\sin(\pi x/h)}{2\pi \mathrm{i}} \int_{\Gamma^+_{\mathrm{hr}}+\Gamma^+_{\mathrm{hl}}+\Gamma^-_{\mathrm{hl}}+\Gamma^-_{\mathrm{hr}}} \frac{F(\zeta)}{(\zeta - x)\sin(\pi\zeta/h)} \,\mathrm{d}\zeta \right| \\ &\leq \frac{4K(b-a)^{\alpha+\beta}\,\mathrm{e}^{c/\mathrm{e}}\, C'_{c,d}}{\pi^2 d\mu \cos(d)\cos^{\alpha+\beta}(\frac{\pi}{2}\sin d)(1 - \mathrm{e}^{-2\pi d/c''})} \sqrt{cn} \exp\left(-\frac{\pi d}{\sqrt{c}} \frac{\sqrt{n}}{\log(cn)} \right), \end{aligned} \tag{5.36}$$

where $c'' = 2c/\exp(1 - c/\mathrm{e})$, and $C'_{c,d}$ is a certain constant depending only on c and d. Combining (5.29) and (5.36), we obtain the estimate of E_1. ∎

Lastly, we prove Lemma 5.3: the estimate of E_2.

Proof of Lemma 5.3. We use again the paths defined in the proof of Lemma 3.4. We set $x''_{M,h} = (\lceil M/2 \rceil - 1/2)h$, $x''_{N,h} = (\lceil N/2 \rceil - 1/2)h$, $x''_{n,h} = (\lceil n/2 \rceil - 1/2)h$, $y''_{n,h} = d(1-\epsilon)\exp(-x''_{n,h})$ and consider the contour

$$\begin{aligned} &\Gamma^+_{\mathrm{vr}}(x''_{N,h}, y''_{n,h}) + \Gamma^+_{\mathrm{hr}}(x''_{N,h}, y''_{n,h}) + \Gamma^+_{\mathrm{hl}}(x''_{M,h}, y''_{n,h}) + \Gamma^+_{\mathrm{vl}}(x''_{M,h}, y''_{n,h}) \\ &+ \Gamma^-_{\mathrm{vl}}(x''_{M,h}, y''_{n,h}) + \Gamma^-_{\mathrm{hl}}(x''_{M,h}, y''_{n,h}) + \Gamma^-_{\mathrm{hr}}(x''_{N,h}, y''_{n,h}) + \Gamma^-_{\mathrm{vr}}(x''_{N,h}, y''_{n,h}), \end{aligned} \tag{5.37}$$

where $0 < \epsilon < 1$. In a similar manner to Lemma 5.2, we set $k = \lceil M/2 \rceil$, $l = \lceil N/2 \rceil$ and omit the expressions $(x''_{M,h}, y''_{n,h})$ and $(x''_{N,h}, y''_{n,h})$. For x with $x \leq -kh$ or $lh \leq x$, we have

$$-\sum_{j=-k+1}^{l-1} F(jh)S(j,h)(x) = \frac{\sin(\pi x/h)}{2\pi \mathrm{i}} \int_{\Gamma^+_{\mathrm{vr}}+\Gamma^+_{\mathrm{hr}}+\Gamma^+_{\mathrm{hl}}+\Gamma^+_{\mathrm{vl}}+\Gamma^-_{\mathrm{vl}}+\Gamma^-_{\mathrm{hl}}+\Gamma^-_{\mathrm{hr}}+\Gamma^-_{\mathrm{vr}}} \frac{F(\zeta)}{(\zeta - x)\sin(\pi\zeta/h)} \,\mathrm{d}\zeta. \tag{5.38}$$

Note that the terms for $j = -k$ and $j = l$ are not included in the LHS of (5.38). As for the RHS of (5.38), using (3.26) of Lemma 3.7, Lemma 5.1 with $\delta = -1/2$, we can obtain similar estimates to (5.29) and (5.36) (in similar manners to (3.52)–(3.56)):

$$\begin{aligned}
&\lim_{\epsilon\to 0}\left|\frac{\sin(\pi x/h)}{2\pi \mathrm{i}}\int_{\Gamma_{\mathrm{vr}}^{+}+\Gamma_{\mathrm{vr}}^{-}+\Gamma_{\mathrm{vl}}^{-}+\Gamma_{\mathrm{vl}}^{+}}\frac{F(\zeta)}{(\zeta-x)\sin(\pi\zeta/h)}\,\mathrm{d}\zeta\right| \\
&\leq \frac{4K(b-a)^{\alpha+\beta}\tilde{C}_{\alpha,\beta,c,-1/2}\,d\,\mathrm{e}^{\frac{\pi}{2}\nu\cos d}}{\pi\cos^{\alpha+\beta}(\frac{\pi}{2}\sin d)}\frac{n}{\log(cn)}\,\mathrm{e}^{-\frac{\pi}{2}\mu\sqrt{cn}\cos d}, \qquad (5.39)\\
&\lim_{\epsilon\to 0}\left|\frac{\sin(\pi x/h)}{2\pi \mathrm{i}}\int_{\Gamma_{\mathrm{hr}}^{+}+\Gamma_{\mathrm{hl}}^{+}+\Gamma_{\mathrm{hl}}^{-}+\Gamma_{\mathrm{hr}}^{-}}\frac{F(\zeta)}{(\zeta-x)\sin(\pi\zeta/h)}\,\mathrm{d}\zeta\right| \\
&\leq \frac{4K(b-a)^{\alpha+\beta}}{\pi^2 d\mu\cos(d)\cos^{\alpha+\beta}(\frac{\pi}{2}\sin d)}\frac{1}{\exp(-nh/2)}\frac{\mathrm{e}^{-\pi d\exp(-nh/2)/h}}{1-\mathrm{e}^{-2\pi d\exp(-nh/2)/h}}\\
&\leq \frac{4K(b-a)^{\alpha+\beta}}{\pi^2 d\mu\cos(d)\cos^{\alpha+\beta}(\frac{\pi}{2}\sin d)(1-\mathrm{e}^{-2\pi d/c'''})}\sqrt{cn}\exp\left(-\frac{\pi d}{\sqrt{c}}\frac{\sqrt{n}}{\log(cn)}\right), \qquad (5.40)
\end{aligned}$$

where $c''' = 2c/\mathrm{e}$. To derive the above estimates, we use the following inequalities:

$$y''_{n,h} = d(1-\epsilon)\exp(-(\lceil n/2\rceil - 1/2)h)\begin{cases}\leq d,\\ \geq d(1-\epsilon)\exp(-nh/2).\end{cases} \qquad (5.41)$$

Therefore we have

$$\begin{aligned}
E_2(F,n) &\leq \left|\sum_{j=-k+1}^{l-1}F(jh)S(j,h)(x)\right| + |F(-kh)| + |F(lh)|\\
&\leq (\text{The RHS of (5.39)}) + (\text{The RHS of (5.40)}) + 2E_3(F,n). \qquad (5.42)
\end{aligned}$$

Thus we obtain the estimate of E_2. ■

6 Proof of Theorem 4.6 (for the DE-Sinc indefinite integration)

6.1 Useful inequalities

In a similar manner to the proof of Theorem 2.5, we write d' as d and assume that $fQ \in \mathbf{L}_{\alpha,\beta}(\psi_{\mathrm{DE}}(\Delta_d^1))$ throughout the following proofs in this section.

To prove Theorem 4.6, we need some auxiliary propositions as below.

Corollary 6.1. Under the same assumptions as Theorem 4.6, for $F(t) = f(\psi_{\mathrm{DE}}(t))\psi'_{\mathrm{DE}}(t)$, there exists a positive constant $\tilde{C}_{\alpha,\beta,c,d}$ depending only on α, β, c, d such that

$$\sup_{-\infty<t<\infty}\left|F(t)-\sum_{j=-M}^{N}F(jh)S(j,h)(t)\right| \leq K(b-a)^{\alpha+\beta-1}\tilde{C}_{\alpha,\beta,c,d}\sqrt{n}\exp\left(-\frac{2\pi d}{\sqrt{c}}\frac{\sqrt{n}}{\log(cn)}\right). \qquad (6.1)$$

Proof. This can be shown in almost the same manner as that of Theorem 4.3. The difference between them is the estimate of the absolute value of the transformed function F; in Lemma 3.7 use (3.24) instead of (3.26). ■

Lemma 6.2 (Stenger [11, Lemma 3.6.5])**.** Let $h > 0$, $j \in \mathbb{Z}$, and $x \in \mathbb{R}$, and define $J(j,h)$ as $J(j,h)(x) = \int_{-\infty}^{x} S(j,h)(\xi)\,\mathrm{d}\xi$. Then

$$|J(j,h)(x)| \leq 1.1h. \qquad (6.2)$$

For $h, \xi, \eta, x_u \in \mathbb{R}$ with $h > 0$ and $x_d \in \mathbb{R} \cup \{-\infty\}$ with $x_d \leq x_u$, define $w(h, \xi, \eta, x_d, x_u)$ as

$$w(h, \xi, \eta, x_d, x_u) = \frac{1}{2\pi \mathrm{i}} \int_{x_d}^{x_u} \frac{\sin(\pi t/h)}{(\xi + \eta \mathrm{i}) - t} \, \mathrm{d}t. \tag{6.3}$$

The following two lemmas are variants of Stenger's lemma [11, Lemma 3.6.3]. The first one can be shown in almost the same manner as his lemma [11, Lemma 3.6.3], and we omit its proof. Since the proof of the second one is long, we like to leave it to the end of this section.

Lemma 6.3. For $\eta \neq 0$, it follows that

$$|w(h, \xi, \eta, x_d, x_u)| \leq \frac{h}{4|\eta|}. \tag{6.4}$$

Lemma 6.4. Let $q \in \mathbb{Z}$ and $\xi_{q,h} = (q + 1/2)h$. For $x \in \mathbb{R}$ and $p \in \mathbb{Z} \cup \{-\infty\}$, we have the following estimates.
Case I. If $p \leq q$ and $ph \leq x \leq qh$, we have

$$|w(h, \xi_{q,h}, \eta, ph, x)| \leq \frac{1}{\pi} \left(2 + \frac{5|\eta|}{h} \right). \tag{6.5}$$

Case II. If $p \geq q + 1$ and $x \geq ph$, we have the same estimate as (6.5).

Remark 6.5. Lemma 6.4 is needed for small $|\eta|$, and otherwise we use Lemma 6.3.

6.2 Sketch of the proof

The proof is done similarly to the proof of Theorem 2.5. To describe the splitting of the error, we introduce some notation here. Set $F(t) = f(\psi_{\mathrm{DE}}(t))\psi'_{\mathrm{DE}}(t)$. We define intervals I_j $(j = 1, \ldots, 5)$ as

$$I_1 = (-\infty, \, -(\lceil M/2 \rceil + 1)h], \tag{6.6}$$
$$I_2 = [-(\lceil M/2 \rceil + 1)h, \, -\lceil M/2 \rceil h], \tag{6.7}$$
$$I_3 = [-\lceil M/2 \rceil h, \, \lceil N/2 \rceil h], \tag{6.8}$$
$$I_4 = [\lceil N/2 \rceil h, \, (\lceil N/2 \rceil + 1)h], \tag{6.9}$$
$$I_5 = [(\lceil N/2 \rceil + 1)h, \, +\infty), \tag{6.10}$$

and $H(F, n; t)$ as

$$H(F, n; t) = F(t) - \sum_{j=-\lceil \frac{M}{2} \rceil}^{\lceil \frac{N}{2} \rceil} F(jh) S(j, h)(t), \tag{6.11}$$

and $E_j(F, n)$ $(j = 1, \ldots, 5)$ as

$$E_j(F, n) = \sup_{x \in I_j} \left| \int_{I_j \cap (-\infty, x]} H(F, n; t) \, \mathrm{d}t \right|. \tag{6.12}$$

Then we have

$$\begin{aligned} (\text{The LHS of (4.15)}) &= \sup_{-\infty < x < \infty} \left| \int_{-\infty}^{x} F(t) \, \mathrm{d}t - \sum_{j=-M}^{N} F(jh) J(j, h)(x) \right| \\ &\leq \sup_{-\infty < x < \infty} \left| \int_{-\infty}^{x} H(F, n; t) \, \mathrm{d}t \right| \\ &\quad + \sup_{-\infty < x < \infty} \left\{ \left| \sum_{j=-M}^{-\lceil \frac{M}{2} \rceil - 1} F(jh) J(j, h)(x) \right| + \left| \sum_{j=\lceil \frac{N}{2} \rceil + 1}^{N} F(jh) J(j, h)(x) \right| \right\}. \end{aligned} \tag{6.13}$$

Moreover, for the first term of (6.13), we have

$$\begin{aligned}\sup_{-\infty<x<\infty}\left|\int_{-\infty}^{x} H(F,n;t)\,\mathrm{d}t\right| &\leq \max_{1\leq j\leq 5}\sup_{x\in I_j}\left|\int_{-\infty}^{x} H(F,n;t)\,\mathrm{d}t\right| \\ &\leq \max_{1\leq j\leq 5}\sup_{x\in I_j}\sum_{k=1}^{j}\left|\int_{I_k\cap(-\infty,x]} H(F,n;t)\,\mathrm{d}t\right| \\ &\leq \max_{1\leq j\leq 5}\sup_{x\in I_j}\sum_{k=1}^{j} E_k(F,n). \\ &\leq \sum_{k=1}^{5} E_k(F,n). \end{aligned} \tag{6.14}$$

For the second and third term of (6.13), using Lemmas 6.2 and 5.1 we have (similarly to Lemma 3.5)

$$\begin{aligned}&\sup_{-\infty<x<\infty}\left|\sum_{j=-M}^{-\lceil\frac{M}{2}\rceil-1} F(jh)J(j,h)(x)\right| + \sup_{-\infty<x<\infty}\left|\sum_{j=\lceil\frac{N}{2}\rceil+1}^{N} F(jh)J(j,h)(x)\right| \\ &\leq 1.1h\left(\sum_{j=-M}^{-\lceil\frac{M}{2}\rceil-1}|F(jh)| + \sum_{j=\lceil\frac{N}{2}\rceil+1}^{N}|F(jh)|\right) \\ &\leq 1.1C_{\alpha,\beta,c}K(b-a)^{\alpha+\beta-1}\left(\int_{-\infty}^{-Mh/2} G_{\alpha,\beta}(x,0)\,\mathrm{d}x + \int_{Nh/2}^{\infty} G_{\alpha,\beta}(x,0)\,\mathrm{d}x\right) \\ &\leq \frac{2.2C_{\alpha,\beta,c}K(b-a)^{\alpha+\beta-1}\,\mathrm{e}^{\frac{\pi}{2}\nu}}{\mu}\,\mathrm{e}^{-\frac{\pi}{2}\mu\sqrt{cn}}, \end{aligned} \tag{6.15}$$

for some constant $C_{\alpha,\beta,c}$ depending only on α, β, c. Then what remains is to estimate $E_j(F,n)$ ($j = 1, \ldots, 5$). Their estimates are given by the following lemmas.

Lemma 6.6. Under the same assumptions as Theorem 4.6, for $F(t) = f(\psi_{\mathrm{DE}}(t))\psi'_{\mathrm{DE}}(t)$, there exist positive constants $\tilde{C}^{(i)}_{\alpha,\beta,c,d}$ ($i = 1, 5$) depending only on α, β, c, d such that for $i = 1, 5$

$$E_i(F,n) \leq K(b-a)^{\alpha+\beta-1}\tilde{C}^{(i)}_{\alpha,\beta,c,d}\frac{\log(cn)}{\sqrt{n}}\exp\left(-\frac{\pi d}{\sqrt{c}}\frac{\sqrt{n}}{\log(cn)}\right). \tag{6.16}$$

Lemma 6.7. Under the same assumptions as Theorem 4.6, for $F(t) = f(\psi_{\mathrm{DE}}(t))\psi'_{\mathrm{DE}}(t)$, there exist positive constants $\tilde{C}^{(i)}_{\alpha,\beta,c,d}$ ($i = 2, 4$) depending only on α, β, c, d such that for $i = 2, 4$

$$E_i(F,n) \leq K(b-a)^{\alpha+\beta-1}\tilde{C}^{(i)}_{\alpha,\beta,c,d}\frac{\log(cn)}{\sqrt{n}}\exp\left(-\frac{2\pi d}{\sqrt{c}}\frac{\sqrt{n}}{\log(cn)}\right). \tag{6.17}$$

Lemma 6.8. Under the same assumptions as Theorem 4.6, for $F(t) = f(\psi_{\mathrm{DE}}(t))\psi'_{\mathrm{DE}}(t)$, there exists a positive constant $\tilde{C}^{(3)}_{\alpha,\beta,c,d}$ depending only on α, β, c, d such that

$$E_3(F,n) \leq K(b-a)^{\alpha+\beta-1}\tilde{C}^{(3)}_{\alpha,\beta,c,d}\frac{\log(cn)}{\sqrt{n}}\exp\left(-\frac{\pi d}{\sqrt{c}}\frac{\sqrt{n}}{\log(cn)}\right). \tag{6.18}$$

6.3 Proofs of Lemmas 6.6–6.8, and 6.4

First, we begin with the proof of Lemma 6.7: estimates of E_2 and E_4.

Proof of Lemma 6.7. Since E_4 can be estimated in almost the same manner as E_2, it suffices to consider E_2. We have

$$\begin{aligned}
E_2(F,n) &\leq \int_{-(\lceil\frac{M}{2}\rceil+1)h}^{-\lceil\frac{M}{2}\rceil h} \left| F(t) - \sum_{j=-\lceil\frac{M}{2}\rceil}^{\lceil\frac{N}{2}\rceil} F(jh)S(j,h)(t) \right| \mathrm{d}t \\
&\leq h \sup_{-\infty<t<\infty} \left| F(t) - \sum_{j=-\lceil\frac{M}{2}\rceil}^{\lceil\frac{N}{2}\rceil} F(jh)S(j,h)(t) \right| \\
&\leq h \sup_{-\infty<t<\infty} \left| F(t) - \sum_{j=-M}^{N} F(jh)S(j,h)(t) \right| \\
&\quad + h \sup_{-\infty<t<\infty} \left| \sum_{j=-M}^{-\lceil\frac{M}{2}\rceil-1} F(jh)S(j,h)(t) \right| + h \sup_{-\infty<t<\infty} \left| \sum_{j=\lceil\frac{N}{2}\rceil+1}^{N} F(jh)S(j,h)(t) \right| \\
&\leq h \sup_{-\infty<t<\infty} \left| F(t) - \sum_{j=-M}^{N} F(jh)S(j,h)(t) \right| + h \sum_{j=-M}^{-\lceil\frac{M}{2}\rceil-1} |F(jh)| + h \sum_{j=\lceil\frac{N}{2}\rceil+1}^{N} |F(jh)|.
\end{aligned} \tag{6.19}$$

Use Corollary 6.1 for the first term. For the second and third term, use Lemma 5.1 and do the same discussion in the proof of Lemma 3.5. Then we can establish this lemma. ∎

Next, we prove Lemma 6.8: the estimate of E_3.

Proof of Lemma 6.8. We use again the paths (5.24) used in the proof of Theorem 4.3. Set $k = \lceil M/2 \rceil$, $l = \lceil N/2 \rceil$. For $x \in I_3$, using a complex contour integral, we have the same expression as (5.25):

$$F(x) - \sum_{j=-k}^{l} F(jh)S(j,h)(x) = \frac{\sin(\pi x/h)}{2\pi\mathrm{i}} \int_{\mathit{\Gamma}_n} \frac{F(\zeta)}{(\zeta - x)\sin(\pi\zeta/h)} \,\mathrm{d}\zeta, \tag{6.20}$$

where

$$\mathit{\Gamma}_n = \mathit{\Gamma}_{\mathrm{vr}}^{+} + \mathit{\Gamma}_{\mathrm{hr}}^{+} + \mathit{\Gamma}_{\mathrm{hl}}^{+} + \mathit{\Gamma}_{\mathrm{vl}}^{+} + \mathit{\Gamma}_{\mathrm{vl}}^{-} + \mathit{\Gamma}_{\mathrm{hl}}^{-} + \mathit{\Gamma}_{\mathrm{hr}}^{-} + \mathit{\Gamma}_{\mathrm{vr}}^{-}. \tag{6.21}$$

Therefore we have

$$\begin{aligned}
\int_{I_3\cap(-\infty,x]} H(F,n;t)\,\mathrm{d}t &= \int_{-kh}^{x} \left(\frac{\sin(\pi t/h)}{2\pi\mathrm{i}} \int_{\mathit{\Gamma}_n} \frac{F(\zeta)}{(\zeta - t)\sin(\pi\zeta/h)} \,\mathrm{d}\zeta \right) \mathrm{d}t \\
&= \int_{\mathit{\Gamma}_n} \frac{F(\zeta)}{\sin(\pi\zeta/h)} \left(\frac{1}{2\pi\mathrm{i}} \int_{-kh}^{x} \frac{\sin(\pi t/h)}{\zeta - t} \,\mathrm{d}t \right) \mathrm{d}\zeta \\
&= \int_{\mathit{\Gamma}_n} \frac{F(\zeta)}{\sin(\pi\zeta/h)} w(h, \operatorname{Re}\zeta, \operatorname{Im}\zeta, -kh, x) \,\mathrm{d}\zeta.
\end{aligned} \tag{6.22}$$

As for the integral on $\mathit{\Gamma}_{\mathrm{vr}}^{+}$ in (6.22), it follows from Lemma 6.4 with $p = -k, q = l$ that

$$|w(h, \operatorname{Re}\zeta, \operatorname{Im}\zeta, -kh, x)| \leq \frac{1}{\pi} \left(2 + \frac{5y'_{n,h}}{h} \right) \tag{6.23}$$

for $\zeta \in \mathit{\Gamma}_{\mathrm{vr}}^{+}$. With this estimate and

$$|\sin(\pi\zeta/h)| = |(-1)^l \cosh\{\pi(\operatorname{Im}\zeta)/h\}| \geq 1 \tag{6.24}$$

for $\zeta \in \mathit{\Gamma}_{\mathrm{vr}}^{+}$, using (3.24) of Lemma 3.7 and (5.4) of Lemma 5.1 with $\delta = 1/2$, we have

$$\begin{aligned}
&\left|\int_{\mathit{\Gamma}_{\mathrm{vr}}^{+}} \frac{F(\zeta)}{\sin(\pi\zeta/h)} w(h, \operatorname{Re}\zeta, \operatorname{Im}\zeta, -kh, x)\,\mathrm{d}\zeta\right| \\
&\leq \frac{1}{\pi}\left(2 + \frac{5y'_{n,h}}{h}\right)\int_{\mathit{\Gamma}_{\mathrm{vr}}^{+}} |F(\zeta)||\,\mathrm{d}\zeta| \\
&\leq \frac{1}{\pi}\left(2 + \frac{5y'_{n,h}}{h}\right) K(b-a)^{\alpha+\beta-1}\int_{0}^{y'_{n,h}} G_{\alpha,\beta}(x'_{N,h}, y)\,\mathrm{d}y \\
&\leq \frac{1}{\pi}\left(2 + \frac{5d\sqrt{n}}{\sqrt{c}\log(cn)}\right)\frac{K(b-a)^{\alpha+\beta-1}\pi d\,\mathrm{e}^{\frac{\pi}{2}\nu\cos d}}{\cos^{\alpha+\beta}(\frac{\pi}{2}\sin d)}\,\mathrm{e}^{-\frac{\pi}{2}\mu\sqrt{cn}\cos d}.
\end{aligned} \tag{6.25}$$

Using Lemma 6.4 with $p = -k, q = l$ for the integral on $\mathit{\Gamma}_{\mathrm{vr}}^{-}$, and Lemma 6.4 with $p = -k, q = -k-1$ for the integrals on $\mathit{\Gamma}_{\mathrm{vl}}^{\pm}$, we can obtain the same estimates as (6.25) for the integrals on these paths. Then we have

$$\begin{aligned}
&\lim_{\epsilon\to 0}\left|\int_{\mathit{\Gamma}_{\mathrm{vr}}^{+}+\mathit{\Gamma}_{\mathrm{vl}}^{+}+\mathit{\Gamma}_{\mathrm{vl}}^{-}+\mathit{\Gamma}_{\mathrm{vr}}^{-}} \frac{F(\zeta)}{\sin(\pi\zeta/h)} w(h, \operatorname{Re}\zeta, \operatorname{Im}\zeta, -kh, x)\,\mathrm{d}\zeta\right| \\
&\leq \left(2 + \frac{5d\sqrt{n}}{\sqrt{c}\log(cn)}\right)\frac{4K(b-a)^{\alpha+\beta-1} d\,\mathrm{e}^{\frac{\pi}{2}\nu\cos d}}{\cos^{\alpha+\beta}(\frac{\pi}{2}\sin d)}\,\mathrm{e}^{-\frac{\pi}{2}\mu\sqrt{cn}\cos(d)}.
\end{aligned} \tag{6.26}$$

As for the integral on $\mathit{\Gamma}_{\mathrm{hr}}^{+} + \mathit{\Gamma}_{\mathrm{hl}}^{+} + \mathit{\Gamma}_{\mathrm{hl}}^{-} + \mathit{\Gamma}_{\mathrm{hr}}^{-}$, it follows from Lemma 6.3 that

$$|w(h, \operatorname{Re}\zeta, \operatorname{Im}\zeta, -kh, x)| \leq \frac{h}{4y'_{n,h}} \tag{6.27}$$

for $\zeta \in \mathit{\Gamma}_{\mathrm{hr}}^{+} + \mathit{\Gamma}_{\mathrm{hl}}^{+} + \mathit{\Gamma}_{\mathrm{hl}}^{-} + \mathit{\Gamma}_{\mathrm{hr}}^{-}$. With this estimate and

$$|\sin(\pi\zeta/h)| = \sqrt{\cosh^2(\pi y'_{n,h}/h) - \cos^2\{\pi(\operatorname{Re}\zeta)/h\}} \geq \sinh(\pi y'_{n,h}/h) \tag{6.28}$$

for $\zeta \in \mathit{\Gamma}_{\mathrm{hr}}^{+} + \mathit{\Gamma}_{\mathrm{hl}}^{+} + \mathit{\Gamma}_{\mathrm{hl}}^{-} + \mathit{\Gamma}_{\mathrm{hr}}^{-}$, we have

$$\begin{aligned}
&\left|\int_{\mathit{\Gamma}_{\mathrm{hr}}^{+}+\mathit{\Gamma}_{\mathrm{hl}}^{+}+\mathit{\Gamma}_{\mathrm{hl}}^{-}+\mathit{\Gamma}_{\mathrm{hr}}^{-}} \frac{F(\zeta)}{\sin(\pi\zeta/h)} w(h, \operatorname{Re}\zeta, \operatorname{Im}\zeta, -kh, x)\,\mathrm{d}\zeta\right| \\
&\leq \frac{h}{4y'_{n,h}\sinh(\pi y'_{n,h}/h)}\int_{-x'_{M,h}}^{x'_{N,h}} \{|F(x - \mathrm{i}\,y'_{n,h})| + |F(x + \mathrm{i}\,y'_{n,h})|\}\,\mathrm{d}x.
\end{aligned} \tag{6.29}$$

As for the integral of this RHS, it follows from (3.24) of Lemma 3.7 and (5.5) of Lemma 5.1 that

$$\begin{aligned}
\int_{-x'_{M,h}}^{x'_{N,h}} \{|F(x - \mathrm{i}\,y'_{n,h})| + |F(x + \mathrm{i}\,y'_{n,h})|\}\,\mathrm{d}x &\leq 2K(b-a)^{\alpha+\beta-1}\int_{-x'_{M,h}}^{x'_{N,h}} G_{\alpha,\beta}(x, y'_{n,h})\,\mathrm{d}x \\
&\leq \frac{4K(b-a)^{\alpha+\beta-1}}{\mu\cos(d)\cos^{\alpha+\beta}(\frac{\pi}{2}\sin d)}.
\end{aligned} \tag{6.30}$$

Then, similarly to (3.52)–(3.56), we have

$$\lim_{\epsilon\to 0}\left|\int_{\Gamma_{\mathrm{hr}}^{+}+\Gamma_{\mathrm{hl}}^{+}+\Gamma_{\mathrm{hl}}^{-}+\Gamma_{\mathrm{hr}}^{-}}\frac{F(\zeta)}{\sin(\pi\zeta/h)}w(h,\operatorname{Re}\zeta,\operatorname{Im}\zeta,-kh,x)\,\mathrm{d}\zeta\right|$$
$$\leq \frac{2K(b-a)^{\alpha+\beta-1}}{d\mu\cos(d)\cos^{\alpha+\beta}(\frac{\pi}{2}\sin d)}\frac{h}{\exp(-nh/2)\exp(-h)}\frac{\mathrm{e}^{-\pi d\exp(-nh/2)\exp(-h)/h}}{1-\mathrm{e}^{-2\pi d\exp(-nh/2)\exp(-h)/h}}$$
$$\leq \frac{2K(b-a)^{\alpha+\beta-1}\,\mathrm{e}^{c/\mathrm{e}}\,C'_{c,d}\sqrt{c}}{d\mu\cos(d)\cos^{\alpha+\beta}(\frac{\pi}{2}\sin d)(1-\mathrm{e}^{-2\pi d/c''})}\frac{\log(cn)}{\sqrt{n}}\exp\left(-\frac{\pi d}{\sqrt{c}}\frac{\sqrt{n}}{\log(cn)}\right), \tag{6.31}$$

where $c''=2c/\exp(1-c/\mathrm{e})$, and $C'_{c,d}$ is a constant depending only on c and d. Note that the RHS of (6.31) is the same as (5.36) except for $\pi^2 h/2=\pi^2\log(cn)/(2n)$. Thus we obtain the estimate of E_3. ■

As the last error estimate, we prove Lemma 6.6: the estimate of E_1 and E_5.

Proof of Lemma 6.6. Since E_5 can be estimated in almost the same manner as E_1, it suffices to consider E_1. Noting

$$E_1(F,n)\leq \sup_{x\in I_1}\left|\int_{-\infty}^{x}F(t)\,\mathrm{d}t\right|+\sup_{x\in I_1}\left|\int_{-\infty}^{x}\sum_{j=-k}^{l}F(jh)S(j,h)(t)\,\mathrm{d}t\right|, \tag{6.32}$$

we estimate each term of (6.32).

As for the first term of the RHS of (6.32), using (3.24) of Lemma 3.7 and (5.1) of Lemma 5.1, we have

$$\left|\int_{-\infty}^{x}F(t)\,\mathrm{d}t\right|\leq\int_{-\infty}^{-(k+1)h}|F(t)|\,\mathrm{d}t\leq K(b-a)^{\alpha+\beta-1}\int_{-\infty}^{-(k+1)h}G_{\alpha,\beta}(x,0)\,\mathrm{d}x$$
$$\leq\frac{K(b-a)^{\alpha+\beta-1}\,\mathrm{e}^{\frac{\pi}{2}\nu}}{\mu}\,\mathrm{e}^{-\frac{\pi}{2}\mu\sqrt{cn}} \tag{6.33}$$

for $x\in I_1$.

As for the second term of the RHS of (6.32), for $x\leq -(k+1)h$ we have

$$\left|\int_{-\infty}^{x}\sum_{j=-k}^{l}F(jh)S(j,h)(t)\,\mathrm{d}t\right|=\left|\int_{-\infty}^{x}\left(\frac{\sin(\pi t/h)}{2\pi\mathrm{i}}\int_{\Gamma_n}\frac{F(\zeta)}{(\zeta-t)\sin(\pi\zeta/h)}\,\mathrm{d}\zeta\right)\mathrm{d}t\right|$$
$$\leq\left|\int_{\Gamma_n}\frac{F(\zeta)}{\sin(\pi\zeta/h)}w(h,\operatorname{Re}\zeta,\operatorname{Im}\zeta,-\infty,x)\,\mathrm{d}\zeta\right|, \tag{6.34}$$

where the contour Γ_n is defined by (6.21), the one used in the proof of Lemma 6.8. Then, using Lemma 6.4 with $p=-\infty, q=-k-1$ or $p=-\infty, q=l$, Lemma 6.3, and similarly to (6.26) and (6.31), we have

$$\lim_{\epsilon\to 0}\left|\int_{\Gamma_{\mathrm{vr}}^{+}+\Gamma_{\mathrm{vl}}^{+}+\Gamma_{\mathrm{vl}}^{-}+\Gamma_{\mathrm{vr}}^{-}}\frac{F(\zeta)}{\sin(\pi\zeta/h)}w(h,\operatorname{Re}\zeta,\operatorname{Im}\zeta,-\infty,x)\,\mathrm{d}\zeta\right|$$
$$\leq\left(2+\frac{5d\sqrt{n}}{\sqrt{c}\log(cn)}\right)\frac{4K(b-a)^{\alpha+\beta-1}d\,\mathrm{e}^{\frac{\pi}{2}\nu\cos d}}{\cos^{\alpha+\beta}(\frac{\pi}{2}\sin d)}\,\mathrm{e}^{-\frac{\pi}{2}\mu\sqrt{cn}\cos(d)}, \tag{6.35}$$

and

$$\lim_{\epsilon\to 0}\left|\int_{\Gamma_{\rm hr}^{+}+\Gamma_{\rm hl}^{+}+\Gamma_{\rm hl}^{-}+\Gamma_{\rm hr}^{-}}\frac{F(\zeta)}{\sin(\pi\zeta/h)}w(h,\operatorname{Re}\zeta,\operatorname{Im}\zeta,-\infty,x)\,\mathrm{d}\zeta\right|$$
$$\leq \frac{2K(b-a)^{\alpha+\beta-1}\,\mathrm{e}^{c/\mathrm{e}}\,C'_{c,d}\sqrt{c}}{d\mu\cos(d)\cos^{\alpha+\beta}(\frac{\pi}{2}\sin d)(1-\mathrm{e}^{-2\pi d/c''})}\frac{\log(cn)}{\sqrt{n}}\exp\left(-\frac{\pi d}{\sqrt{c}}\frac{\sqrt{n}}{\log(cn)}\right), \tag{6.36}$$

where $c''=2c/\exp(1-c/\mathrm{e})$, and $C'_{c,d}$ is a constant depending only on c and d. Thus we obtain the estimate of E_1. ■

Finally we finish this section by proving Lemma 6.4.

Proof of Lemma 6.4. Noting

$$\begin{aligned} w(h,\xi_{q,h},\eta,ph,x) &= \frac{1}{2\pi\mathrm{i}}\left\{\int_{ph}^{x}\frac{(\xi_{q,h}-t)\sin(\pi t/h)}{(\xi_{q,h}-t)^2+\eta^2}\,\mathrm{d}t-\mathrm{i}\int_{ph}^{x}\frac{\eta\sin(\pi t/h)}{(\xi_{q,h}-t)^2+\eta^2}\,\mathrm{d}t\right\}\\ &= -\frac{1}{2\pi}\left\{\mathrm{i}\int_{ph}^{x}\frac{(\xi_{q,h}-t)\sin(\pi t/h)}{(\xi_{q,h}-t)^2+\eta^2}\,\mathrm{d}t+\int_{ph}^{x}\frac{\eta\sin(\pi t/h)}{(\xi_{q,h}-t)^2+\eta^2}\,\mathrm{d}t\right\}, \end{aligned} \tag{6.37}$$

we estimate each of the imaginary and real parts. Recall that $\xi_{q,h}=(q+1/2)h$ throughout this proof.

First, we begin with Case I. As for the imaginary part of (6.37), set $a=\min\{\xi_{q,h}-h/2,\ \xi_{q,h}-|\eta|\}$ and consider splitting the integral as:

$$\begin{aligned} &\int_{ph}^{x}\frac{(\xi_{q,h}-t)\sin(\pi t/h)}{(\xi_{q,h}-t)^2+\eta^2}\,\mathrm{d}t\\ &=\begin{cases}\displaystyle\sum_{m=p}^{\lfloor\frac{x}{h}\rfloor-1}(-1)^m\gamma_m+(-1)^{\lfloor\frac{x}{h}\rfloor}\int_0^{x-\lfloor\frac{x}{h}\rfloor h}\frac{(\xi_{q,h}-s-\lfloor\frac{x}{h}\rfloor h)\sin(\pi s/h)}{(\xi_{q,h}-s-\lfloor\frac{x}{h}\rfloor h)^2+\eta^2}\,\mathrm{d}s & (x\leq\lfloor\frac{a}{h}\rfloor h),\\ \displaystyle\sum_{m=p}^{\lfloor\frac{a}{h}\rfloor-1}(-1)^m\gamma_m+(-1)^{\lfloor\frac{a}{h}\rfloor}\int_0^{x-\lfloor\frac{a}{h}\rfloor h}\frac{(\xi_{q,h}-s-\lfloor\frac{a}{h}\rfloor h)\sin(\pi s/h)}{(\xi_{q,h}-s-\lfloor\frac{a}{h}\rfloor h)^2+\eta^2}\,\mathrm{d}s & (\lfloor\frac{a}{h}\rfloor h<x),\end{cases} \end{aligned} \tag{6.38}$$

where

$$\gamma_m=\int_0^h\frac{(\xi_{q,h}-s-mh)\sin(\pi s/h)}{(\xi_{q,h}-s-mh)^2+\eta^2}\,\mathrm{d}s. \tag{6.39}$$

Since

$$\frac{\xi_{q,h}-t}{(\xi_{q,h}-t)^2+\eta^2} \tag{6.40}$$

is strictly monotonically increasing on $(-\infty,\ \xi_{q,h}-|\eta|]$ with respect to t, we have

$$\gamma_{\lfloor\frac{a}{h}\rfloor-1}>\gamma_{\lfloor\frac{a}{h}\rfloor-2}>\cdots, \tag{6.41}$$

and therefore, for $m_0\leq\lfloor\frac{a}{h}\rfloor-1$,

$$(-1)^{m_0}\sum_{m=p}^{m_0}(-1)^m\gamma_m=\begin{cases}(\gamma_{m_0}-\gamma_{m_0-1})+(\gamma_{m_0-2}-\gamma_{m_0-3})+\cdots>0,\\ \gamma_{m_0}-(\gamma_{m_0-1}-\gamma_{m_0-2})-(\gamma_{m_0-3}-\gamma_{m_0-4})-\cdots<\gamma_{m_0}.\end{cases} \tag{6.42}$$

Then, noting $a \leq \xi_{q,h} - h/2$, for $m_0 \leq \lfloor \frac{a}{h} \rfloor - 1$, we have

$$\left| (-1)^{m_0} \sum_{m=p}^{m_0} (-1)^m \gamma_m \right| \leq |\gamma_{m_0}| \leq \int_0^h \left| \frac{(\xi_{q,h} - s - m_0 h)\sin(\pi s/h)}{(\xi_{q,h} - s - m_0 h)^2 + \eta^2} \right| \mathrm{d}s$$
$$\leq \int_0^h \frac{1}{|\xi_{q,h} - s - m_0 h|} \mathrm{d}s \leq \frac{2}{h} \cdot h = 2. \tag{6.43}$$

Moreover, in a similar manner to the above argument, if $x \leq \lfloor \frac{a}{h} \rfloor h$, we have

$$\left| \int_0^{x - \lfloor \frac{x}{h} \rfloor h} \frac{(\xi_{q,h} - s - \lfloor \frac{x}{h} \rfloor h)\sin(\pi s/h)}{(\xi_{q,h} - s - \lfloor \frac{x}{h} \rfloor h)^2 + \eta^2} \mathrm{d}s \right| \leq 2 \tag{6.44}$$

and if $\lfloor \frac{a}{h} \rfloor h < x$,

$$\left| \int_0^{x - \lfloor \frac{a}{h} \rfloor h} \frac{(\xi_{q,h} - s - \lfloor \frac{a}{h} \rfloor h)\sin(\pi s/h)}{(\xi_{q,h} - s - \lfloor \frac{a}{h} \rfloor h)^2 + \eta^2} \mathrm{d}s \right| \leq \frac{2}{h} \cdot \left(x - \left\lfloor \frac{a}{h} \right\rfloor h \right) \leq \frac{2}{h} \cdot (x - a + h)$$
$$\leq \frac{2}{h} \max\left\{ |\eta| - \frac{h}{2}, 0 \right\} \leq \frac{2|\eta|}{h}. \tag{6.45}$$

Combining (6.43), (6.44), and (6.45), we have

$$\left| \int_{ph}^{x} \frac{(\xi_{q,h} - t)\sin(\pi t/h)}{(\xi_{q,h} - t)^2 + \eta^2} \mathrm{d}t \right| \leq 2 + \max\left\{ 2, \frac{2|\eta|}{h} \right\}. \tag{6.46}$$

As for the real part of (6.37), we set $b = \xi_{q,h} - h/2$ and consider

$$\int_{ph}^{x} \frac{\eta \sin(\pi t/h)}{(\xi_{q,h} - t)^2 + \eta^2} \mathrm{d}t$$
$$= \begin{cases} \displaystyle\sum_{m=p}^{\lfloor \frac{x}{h} \rfloor - 1} (-1)^m \gamma'_m + (-1)^{\lfloor \frac{x}{h} \rfloor} \int_0^{x - \lfloor \frac{x}{h} \rfloor h} \frac{\eta \sin(\pi s/h)}{(\xi_{q,h} - s - \lfloor \frac{x}{h} \rfloor h)^2 + \eta^2} \mathrm{d}s & \left(x \leq \lfloor \frac{b}{h} \rfloor h \right), \\ \displaystyle\sum_{m=p}^{\lfloor \frac{b}{h} \rfloor - 1} (-1)^m \gamma'_m + (-1)^{\lfloor \frac{b}{h} \rfloor} \int_0^{x - \lfloor \frac{b}{h} \rfloor h} \frac{\eta \sin(\pi s/h)}{(\xi_{q,h} - s - \lfloor \frac{b}{h} \rfloor h)^2 + \eta^2} \mathrm{d}s & \left(\lfloor \frac{b}{h} \rfloor h < x \right), \end{cases} \tag{6.47}$$

where

$$\gamma'_m = \int_0^h \frac{\eta \sin(\pi s/h)}{(\xi_{q,h} - s - mh)^2 + \eta^2} \mathrm{d}s. \tag{6.48}$$

Since

$$\frac{\eta}{(\xi_{q,h} - t)^2 + \eta^2} \tag{6.49}$$

is strictly monotonically increasing on $(-\infty, \ \xi_{q,h}]$ with respect to t, in a similar manner to (6.43), (6.44), and (6.45), for $m_0 \leq \lfloor \frac{b}{h} \rfloor - 1$, we have

$$\left| \sum_{m=p}^{m_0} (-1)^m \gamma'_m \right| \leq |\gamma'_{m_0}| \leq \int_0^h \frac{|\eta|}{|\xi_{q,h} - s - m_0 h|^2} \mathrm{d}s \leq \frac{4|\eta|}{h^2} \cdot h = \frac{4|\eta|}{h}, \tag{6.50}$$

and moreover, if $x \leq \lfloor \frac{b}{h} \rfloor h$

$$\left| \int_0^{x - \lfloor \frac{x}{h} \rfloor h} \frac{\eta \sin(\pi s/h)}{(\xi_{q,h} - s - \lfloor \frac{x}{h} \rfloor h)^2 + \eta^2} \mathrm{d}s \right| \leq \frac{4|\eta|}{h}, \tag{6.51}$$

and if $\lfloor \frac{b}{h} \rfloor h < x$

$$\left| \int_0^{x - \lfloor \frac{b}{h} \rfloor h} \frac{\eta \sin(\pi s/h)}{(\xi_{q,h} - s - \lfloor \frac{b}{h} \rfloor h)^2 + \eta^2} \,\mathrm{d}s \right| \leq \frac{4|\eta|}{h^2} \cdot (x - b + h) \leq \frac{4|\eta|}{h^2} \cdot h = \frac{4|\eta|}{h}. \tag{6.52}$$

Combining (6.50), (6.51), and (6.52) we have

$$\left| \int_{ph}^{x} \frac{\eta \sin(\pi t/h)}{(\xi_{q,h} - t)^2 + \eta^2} \,\mathrm{d}t \right| \leq \frac{8|\eta|}{h}. \tag{6.53}$$

Finally, combining (6.46) and (6.53), we have

$$|w(h, \xi_{q,h}, \eta, ph, x)| \leq \frac{1}{2\pi} \left(2 + \max\left\{ 2, \frac{2|\eta|}{h} \right\} + \frac{8|\eta|}{h} \right) \leq \frac{1}{2\pi} \left(4 + \frac{2|\eta|}{h} + \frac{8|\eta|}{h} \right), \tag{6.54}$$

which is the desired estimate.

Next, we treat Case II. As for the imaginary part of (6.37), set $\tilde{a} = \max\{\xi_{q,h} + h/2,\ \xi_{q,h} + |\eta|\}$ and consider

$$\begin{aligned} &\int_{ph}^{x} \frac{(\xi_{q,h} - t) \sin(\pi t/h)}{(\xi_{q,h} - t)^2 + \eta^2} \,\mathrm{d}t \\ &= \begin{cases} \displaystyle\int_{ph}^{x} \frac{(\xi_{q,h} - t) \sin(\pi t/h)}{(\xi_{q,h} - t)^2 + \eta^2} \,\mathrm{d}t & \left(x \leq \left\lceil \frac{\tilde{a}}{h} \right\rceil h\right), \\ \displaystyle\int_{ph}^{\lceil \frac{\tilde{a}}{h} \rceil h} \frac{(\xi_{q,h} - t) \sin(\pi t/h)}{(\xi_{q,h} - t)^2 + \eta^2} \,\mathrm{d}t + \sum_{m = \lceil \frac{\tilde{a}}{h} \rceil}^{\lfloor \frac{x}{h} \rfloor - 1} (-1)^m \gamma_m + (-1)^{\lfloor \frac{x}{h} \rfloor} \gamma_{\lfloor \frac{x}{h} \rfloor}(x) & \left(\left\lceil \frac{\tilde{a}}{h} \right\rceil h < x\right), \end{cases} \end{aligned} \tag{6.55}$$

where γ_m is defined by (6.39) and

$$\gamma_{\lfloor \frac{x}{h} \rfloor}(x) = \int_0^{x - \lfloor \frac{x}{h} \rfloor h} \frac{(\xi_{q,h} - s - \lfloor \frac{x}{h} \rfloor h) \sin(\pi s/h)}{(\xi_{q,h} - s - \lfloor \frac{x}{h} \rfloor h)^2 + \eta^2} \,\mathrm{d}s. \tag{6.56}$$

Since

$$-\frac{\xi_{q,h} - t}{(\xi_{q,h} - t)^2 + \eta^2} \tag{6.57}$$

is strictly monotonically decreasing on $[\xi_{q,h} + |\eta|, +\infty)$ with respect to t, using similar techniques to (6.43), (6.44), and (6.45), we have

$$\left| \sum_{m = \lceil \frac{\tilde{a}}{h} \rceil}^{\lfloor \frac{x}{h} \rfloor - 1} (-1)^m \gamma_m + (-1)^{\lfloor \frac{x}{h} \rfloor} \gamma_{\lfloor \frac{x}{h} \rfloor}(x) \right| \leq \left| \gamma_{\lceil \frac{\tilde{a}}{h} \rceil} \right| \leq \int_0^h \frac{1}{|\xi_{q,h} - s - \lceil \frac{\tilde{a}}{h} \rceil h|} \,\mathrm{d}s \leq \frac{2}{h} \cdot h = 2 \tag{6.58}$$

and

$$\sup_{ph \leq x \leq \lceil \frac{\tilde{a}}{h} \rceil h} \left| \int_{ph}^{x} \frac{(\xi_{q,h} - t) \sin(\pi t/h)}{(\xi_{q,h} - t)^2 + \eta^2} \,\mathrm{d}t \right| \leq \frac{2}{h} \max\left\{ |\eta| - \frac{h}{2}, 0 \right\} \leq \frac{2|\eta|}{h}. \tag{6.59}$$

Combining (6.58) and (6.59), we have the same estimate as (6.46).

As for the real part of (6.37), set $\tilde{b} = \xi_{q,h} + h/2$ and consider

$$\int_{ph}^{x} \frac{\eta \sin(\pi t/h)}{(\xi_{q,h} - t)^2 + \eta^2}\, \mathrm{d}t$$
$$= \begin{cases} \displaystyle\int_{ph}^{x} \frac{\eta \sin(\pi t/h)}{(\xi_{q,h} - t)^2 + \eta^2}\, \mathrm{d}t & (x \leq \left\lceil \frac{\tilde{b}}{h} \right\rceil h), \\ \displaystyle\int_{ph}^{\lceil \frac{\tilde{b}}{h} \rceil h} \frac{\eta \sin(\pi t/h)}{(\xi_{q,h} - t)^2 + \eta^2}\, \mathrm{d}t + \sum_{m=\lceil \frac{\tilde{b}}{h} \rceil}^{\lfloor \frac{x}{h} \rfloor - 1} (-1)^m \gamma'_m + (-1)^{\lfloor \frac{x}{h} \rfloor} \gamma'_{\lfloor \frac{x}{h} \rfloor}(x) & (\left\lceil \frac{\tilde{b}}{h} \right\rceil h < x), \end{cases} \tag{6.60}$$

where γ'_m is defined by (6.48) and

$$\gamma'_{\lfloor \frac{x}{h} \rfloor}(x) = \int_{0}^{x - \lfloor \frac{x}{h} \rfloor h} \frac{\eta \sin(\pi s/h)}{(\xi_{q,h} - s - \lfloor \frac{x}{h} \rfloor h)^2 + \eta^2}\, \mathrm{d}s. \tag{6.61}$$

Since

$$\frac{\eta}{(\xi_{q,h} - t)^2 + \eta^2} \tag{6.62}$$

is strictly monotonically decreasing on $[\xi_{q,h}, +\infty)$ with respect to t, using similar techniques to (6.50), (6.51), and (6.52), we have

$$\left| \sum_{m=\lceil \frac{\tilde{b}}{h} \rceil}^{\lfloor \frac{x}{h} \rfloor - 1} (-1)^m \gamma'_m + (-1)^{\lfloor \frac{x}{h} \rfloor} \gamma'_{\lfloor \frac{x}{h} \rfloor}(x) \right| \leq \left| \gamma'_{\lceil \frac{\tilde{b}}{h} \rceil} \right| \leq \int_{0}^{h} \frac{|\eta|}{|\xi_{q,h} - s - \lceil \frac{\tilde{b}}{h} \rceil h|^2}\, \mathrm{d}s \leq \frac{4|\eta|}{h} \tag{6.63}$$

and

$$\sup_{ph \leq x \leq \lceil \frac{\tilde{b}}{h} \rceil h} \left| \int_{ph}^{x} \frac{\eta \sin(\pi t/h)}{(\xi_{q,h} - t)^2 + \eta^2}\, \mathrm{d}t \right| \leq \frac{4|\eta|}{h}. \tag{6.64}$$

Combining (6.63) and (6.64), we have the same estimate as (6.53). Thus we obtain the same estimate as Case I. ■

7 Numerical examples

In this section, we show numerical results that confirm the existing/new theorems. In addition to the examples for the SE/DE formula in Section 2, we present the following two examples for each of the SE/DE-Sinc approximation and the SE/DE-Sinc indefinite integration: (i) an example that can be explained by the existing theorems, and (ii) an example requiring the new theorem for the DE case. In each example, we consider a function defined on $(-1, 1)$. All computation programs used in this section were written in C with double-precision floating-point arithmetic. Throughout this section, for computation, ϵ is set to $\epsilon = 0.001$, and c in the formula of h is set optimally as described in Remark 2.7, i.e., $c = 4d'/\mu$ in the DE formula, and similarly $c = 2d'/\mu$ in the DE-Sinc approximation and the DE-Sinc indefinite integration.

7.1 Examples for the SE/DE formula

Firstly let us consider Example 1 (presented in Section 2.2), which can be naturally approximated by both of the SE and DE formulas. The result is shown in Figure 10. According to Theorems 2.2 and 2.3, the convergence rates of the SE and DE formulas are $\mathrm{O}(\exp(-c_0\sqrt{n}))$

and $\mathrm{O}(\exp(-c_1 n/\log(c_2 n)))$, respectively. From the graph we can observe the expected rates in both formulas.

Secondly consider Example 2 (presented in Section 2.3). The integrand f_2 satisfies the assumptions in Theorem 2.2 (SE case), but does not those in Theorem 2.3 (DE case). In this case Theorem 2.5 is useful. The result is shown in Figure 11. From the graph we can observe $\mathrm{O}(\exp(-c_0\sqrt{n}))$ in the SE formula, but the DE formula does not converge at the usual rate: $\mathrm{O}(\exp(-c_1 n/\log(c_2 n)))$. However, it seems to converge at a similar rate to the SE formula, which agrees with Theorem 2.5.

7.2 Examples for the SE/DE-Sinc approximation

For the SE/DE-Sinc approximation, we consider the following two examples.

Example 3. Consider the function

$$f_3(t) = (1-t^2)^{1/\sqrt{2}}\sqrt{1+t^2}. \tag{7.1}$$

The function f_3 belongs to $\mathbf{L}_{1/\sqrt{2}}(\psi_{\mathrm{SE}}(\mathscr{D}_{\pi/2}))$ and $\mathbf{L}_{1/\sqrt{2}}(\psi_{\mathrm{DE}}(\mathscr{D}_{\pi/6}))$.

Example 4. Consider the function [7, Example 9.6]

$$f_4(t) = (1-t^2)^{1/\sqrt{2}}\sqrt{\cos(4\operatorname{arctanh} t) + \cosh(2)}. \tag{7.2}$$

The function f_4 belongs to $\mathbf{L}_{1/\sqrt{2}}(\psi_{\mathrm{SE}}(\mathscr{D}_{\pi/2}))$, but does not belong to $\mathbf{L}_{1/\sqrt{2}}(\psi_{\mathrm{DE}}(\mathscr{D}_{d}))$ for any $d > 0$.

The results are shown in Figures 12 and 13, respectively. In each figure, "maximum error" denotes the maximum of the absolute values of the approximation errors evaluated at 20,000 equally-spaced points on $(-1, 1)$. We can confirm Theorems 4.1 (SE case) in both figures. Theorem 4.2 (DE case) can be confirmed in Figure 12, but not in Figure 13. The result in Figure 13 seems to behave consistently with Theorem 4.3.

7.3 Examples for the SE/DE-Sinc indefinite integration

For the SE/DE-Sinc indefinite integration, we consider the following two examples.

Example 5. Consider the function

$$f_5(t) = -\frac{t\{(\sqrt{2}+1)t^2 + (\sqrt{2}-1)\}}{(1-t^2)^{(\sqrt{2}-1)/\sqrt{2}}\sqrt{1+t^2}} \tag{7.3}$$

and its indefinite integral on $(-1, 1)$:

$$\int_{-1}^{t} f_5(s)\,\mathrm{d}s = f_3(t). \tag{7.4}$$

The function f_5 satisfies $f_5 Q \in \mathbf{L}_{1/\sqrt{2}}(\psi_{\mathrm{SE}}(\mathscr{D}_{(\pi-\epsilon)/2}))$ and $f_5 Q \in \mathbf{L}_{1/\sqrt{2}}(\psi_{\mathrm{DE}}(\mathscr{D}_{(\pi-\epsilon)/6}))$ for any ϵ with $0 < \epsilon < 1$.

Example 6. Consider the function

$$f_6(t) = -\frac{\sqrt{2}\{t\cosh(\pi) + t\cos(4\operatorname{arctanh} t) + \sqrt{2}\sin(4\operatorname{arctanh} t)\}}{(1-t^2)^{(\sqrt{2}-1)/\sqrt{2}}\sqrt{\cos(4\operatorname{arctanh} t) + \cosh(\pi)}} \tag{7.5}$$

and its indefinite integral on $(-1,\,1)$:

$$\int_{-1}^{t} f_6(s)\,\mathrm{d}s = f_4(t). \tag{7.6}$$

The function f_6 satisfies $f_6 Q \in \mathbf{L}_{1/\sqrt{2}}(\psi_{\mathrm{SE}}(\mathscr{D}_{(\pi-\epsilon)/2}))$ for any ϵ with $0 < \epsilon < 1$, but does not satisfy $f_6 Q \in \mathbf{L}_{1/\sqrt{2}}(\psi_{\mathrm{DE}}(\mathscr{D}_d))$ for any $d > 0$.

The results are shown in Figures 14 and 15, respectively. In each figure, in the same manner as in the SE/DE-Sinc approximation, "maximum error" denotes the maximum of the absolute values of the approximation errors evaluated at 20,000 equally-spaced points on $(-1,\,1)$. We can confirm Theorems 4.4 (SE case) in both figures. Theorem 4.5 (DE case) can be confirmed in Figure 14, but not in Figure 15. The result in Figure 15 seems to behave consistently with Theorem 4.6.

8 Concluding remarks

As the first contribution of this paper, we revealed the theoretical convergence rate of the DE formula under the same assumption as the SE formula. The usual convergence rates of the SE formula and the DE formula have been known as $\mathrm{O}(\exp(-c_0\sqrt{n}))$ and $\mathrm{O}(\exp(-c_1 n/\log(c_2 n)))$, respectively, as seen in Example 1. However, as seen in Example 2, there exists a case where the SE formula attains the standard rate $\mathrm{O}(\exp(-c_0\sqrt{n}))$ but the DE formula does not attain $\mathrm{O}(\exp(-c_1 n/\log(c_2 n)))$. Our result: Theorem 2.5 can explain the case, and prove the DE formula converges with the rate: $\mathrm{O}(\exp(-c_3\sqrt{n}/\log(c_4 n)))$ in such a case. This rate is slightly worse than the SE's rate: $\mathrm{O}(\exp(-c_0\sqrt{n}))$, but the difference is not so critical.

As the second contribution, we also analyzed the convergence rates in the following two cases: (i) DE-Sinc approximation under the same assumption as the SE-Sinc approximation, and (ii) the DE-Sinc indefinite integration under the same assumption as the SE-Sinc indefinite integration. The results are the same as above: $\mathrm{O}(\exp(-c_3\sqrt{n}/\log(c_4 n)))$, as shown in Theorems 4.3 and 4.6.

The results given in this paper are also useful to analyze other situations. Firstly, in this paper only the SE and DE formulas for the the integral over a *finite* interval: $(a,\,b)$ are considered, but the SE/DE formulas have also been proposed for the integral over the *semi-infinite* interval $(0,\,\infty)$ and the *infinite* interval $(-\infty,\,\infty)$ [11, 17]. And in these cases as well, the same phenomenon can happen as Example 2, i.e., the SE formula works good while the DE formula does not (an example can be found in Bornemann et al. [1, Eq. (3.24)]). The analysis strategies given in Sections 3.2 and 3.3 are quite useful to obtain the same result as Theorem 2.5 in such a case. Secondly, with the aid of Theorem 2.5, we can rigorously prove the convergence rate of the DE-Sinc scheme for weakly Volterra integral equations of the second kind [3]. We are now working on the latter issue, and the result will be reported somewhere else soon.

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References

[1] F. Bornemann, D. Laurie, S. Wagon, and J. Waldvogel, *The SIAM 100-digit challenge*, SIAM, Philadelphia, PA, 2004.

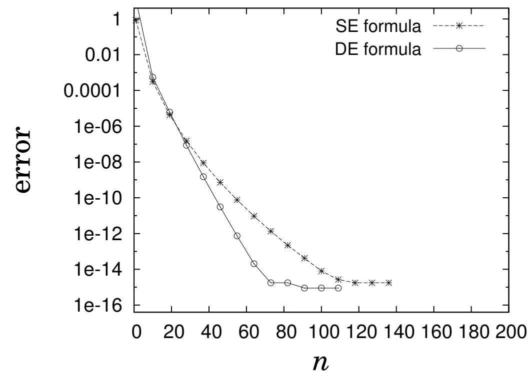

Figure 10. Error of the SE formula and the DE formula in Example 1.

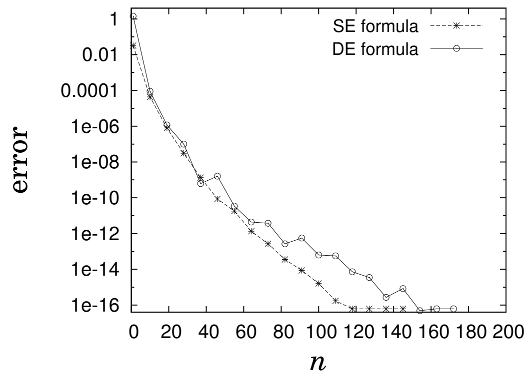

Figure 11. Error of the SE formula and the DE formula in Example 2.

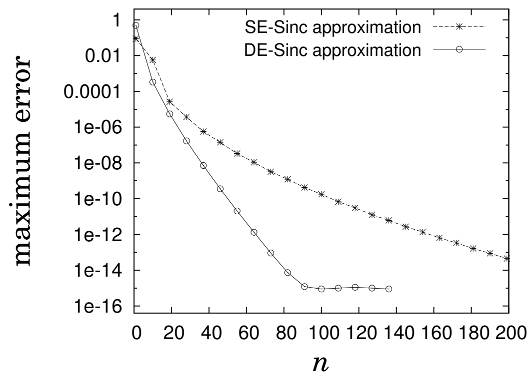

Figure 12. Error of the SE-Sinc approximation and the DE-Sinc approximation in Example 3.

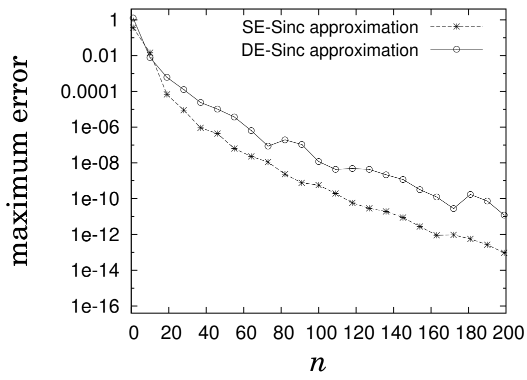

Figure 13. Error of the SE-Sinc approximation and the DE-Sinc approximation in Example 4.

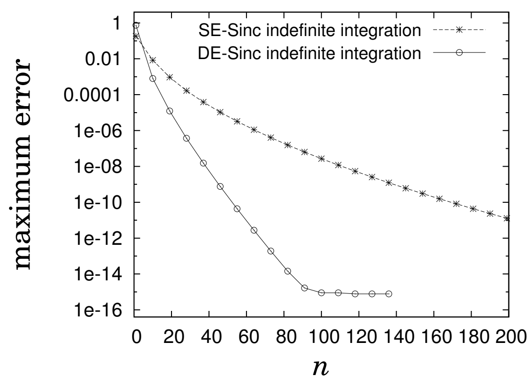

Figure 14. Error of the SE-Sinc indefinite integration and the DE-Sinc indefinite integration in Example 5.

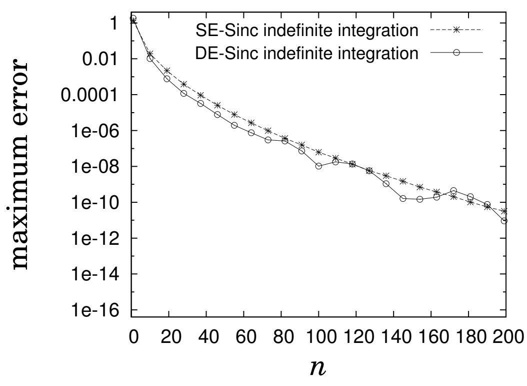

Figure 15. Error of the SE-Sinc indefinite integration and the DE-Sinc indefinite integration in Example 6.

[2] P. A. P. Moran, Numerical integration in the presence of singularities, *Acta Mathematica Scientia*, **1** (1981), 83–85.

[3] M. Mori, A. Nurmuhammad, and T. Murai, Numerical solution of Volterra integral equations with weakly singular kernel based on the DE-sinc method, *Japan Journal of Industrial and Applied Mathematics*, **25** (2008), 165–183.

[4] M. Mori and M. Sugihara, The double-exponential transformation in numerical analysis, *Journal of Computational and Applied Mathematics*, **127** (2001), 287–296.

[5] M. Muhammad and M. Mori, Double exponential formulas for numerical indefinite integration, *Journal of Computational and Applied Mathematics*, **161** (2003), 431–448.

[6] T. Okayama, T. Matsuo, and M. Sugihara, Error estimates with explicit constants for Sinc approximation, Sinc quadrature and Sinc indefinite integration, Mathematical Engineering Technical Reports 2009-01, The University of Tokyo, January 2009.

[7] ———, Improvement of a Sinc-collocation method for Fredholm integral equations of the second kind, *BIT Numerical Mathematics*, **51** (2011), 339–366.

[8] C. Schwartz, Numerical integration of analytic functions, *Journal of Computational Physics*, **4** (1969), 19–29.

[9] W. Squire, Numerical evaluation of integrals using Moran transformations, Aerospace Engineering Report No. Tr-14, 1969.

[10] F. Stenger, Optimal convergence of minimum norm approximations in H_p, *Numerische Mathematik*, **29** (1978), 345–362.

[11] ———, *Numerical Methods Based on Sinc and Analytic Functions*, Springer-Verlag, New York, 1993.

[12] ———, Summary of Sinc numerical methods, *Journal of Computational and Applied Mathematics*, **121** (2000), 379–420.

[13] M. Sugihara, Optimality of the double exponentioal formula—functional analysis approach, *Numerische Mathematik*, **75** (1997), 379–395.

[14] ———, Near optimality of the sinc approximation, *Mathematics of Computation*, **72** (2003), 767–786.

[15] M. Sugihara and T. Matsuo, Recent developments of the Sinc numerical methods, *Journal of Computational and Applied Mathematics*, **164–165** (2004), 673–689.

[16] H. Takahasi and M. Mori, Quadrature formulas obtained by variable transformation, *Numerische Mathematik*, **21** (1973–1974), 206–219.

[17] ———, Double exponential formulas for numerical integration, *Publications of the Research Institute for Mathematical Sciences,* Kyoto University, **9** (1974), 721–741.

[18] K. Tanaka, M. Sugihara, K. Murota, and M. Mori, Function classes for double exponential integration formulas, *Numerische Mathematik*, **111** (2009), 631–655.