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Trichotomy for Integer Linear Systems Based on Their Sign Patterns*

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Abstract

In this paper, we consider solving the integer linear systems, i.e., given a matrix $A \in \mathbb{R}^{m\times n}$, a vector $b \in \mathbb{R}^m$, and a positive integer d, to compute an integer vector $x \in D^n$ such that $Ax \geq b$, where m and n denote positive integers, $\mathbb{R}$ denotes the set of reals, and $D = \{0, 1, \ldots, d-1\}$. The problem is one of the most fundamental NP-hard problems in computer science.

For the problem, we propose a complexity index η which is based only on the sign pattern of A. For a real γ, let $\mathrm{ILS}_=(\gamma)$ denote the family of the problem instances I with $\eta(I) = \gamma$. We then show the following trichotomy:

- $\mathrm{ILS}_=(\gamma)$ is linearly solvable, if $\gamma < 1$,
- $\mathrm{ILS}_=(\gamma)$ is weakly NP-hard and pseudo-polynomially solvable, if $\gamma = 1$, and
- $\mathrm{ILS}_=(\gamma)$ is strongly NP-hard, if $\gamma > 1$.

This, for example, includes the existing results that quadratic systems and Horn systems can be solved in pseudo-polynomial time.

1 Introduction

Integer linear systems

Let A denote a matrix $A \in \mathbb{R}^{m\times n}$, b denote a vector $b \in \mathbb{R}^m$, where m and n denote positive integers, and $\mathbb{R}$ denote the set of reals. For a positive integer d, let $D = \{0, 1, \ldots, d-1\}$. In this paper, we consider the problem of computing an integer vector $x \in D^n$ such that $Ax \geq b$, which we denote by ILS. The ILS problem is one of the most fundamental and important problems in computer science, and have been studied extensively from both theoretical and practical points of view [18, 27]. It is known that the ILS problem is strongly NP-hard, and can be solved in polynomial time, if m or n are bounded by some constant [23], or A is totally unimodular and

*A preliminary version appears in Proceedings of the 29th Symposium on Theoretical Aspects of Computer Science (STACS 2012) [20].

b is integral [15]. When A is quadratic (also called TVPI, i.e., each row of A contains at most two nonzero elements) or Horn (i.e., each row of A contains at most one positive element), the ILS problem is known to be weakly NP-hard, but it can be solved in time polynomial in the input length and d, and hence in pseudo-polynomial time [21, 14, 30]. The best known bounds for quadratic and Horn systems require $\mathrm{O}(md)$ time [2] and $\mathrm{O}(n^2md)$ time, respectively. For unit linear systems, i.e., $A \in \{0,-1,+1\}^{m\times n}$, it is known that the problem is still strongly NP-hard, but it can be solved in $\mathrm{O}(nm)$ [22] and $\mathrm{O}(n\log n+m)$ time [28] if A is in addition quadratic, and can be solved in $\mathrm{O}(n^2m)$ time [9, 29] if A is in addition Horn. Finally, for the difference constraint systems, i.e., $A \in \{0,-1,+1\}^{m\times n}$ and each row of A contains one positive element and one negative element, it is known that the problem is equivalent to the negative cycle detection in network theory and can be solved in $\mathrm{O}(nm)$ [3, 11, 25] and $\mathrm{O}(\sqrt{n}m\log C)$ [12], where C denotes the maximum absolute value of the negative elements in b.

A complexity index for integer linear systems

In this paper, we introduce a complexity index η for the ILS problem, which sharply distinguishes between the classes of *easy*, *semi-hard* and *hard* integer linear systems. The complexity index is based only on the sign pattern of A.

For a real a, its sign is defined as

$$\mathrm{sgn}(a)=\begin{cases} + & (a>0)\\ 0 & (a=0)\\ - & (a<0),\end{cases} \tag{1}$$

and the sign of a real matrix $A\in\mathbb{R}^{m\times n}$ is the matrix $\mathrm{sgn}(A)\in\{0,-,+\}^{m\times n}$ which is obtained from A by replacing each element by its sign. For example, for a matrix

$$A=\begin{pmatrix}1 & -3 & 0\\ 4 & 2 & -5\end{pmatrix}, \tag{2}$$

we have

$$\mathrm{sgn}(A)=\begin{pmatrix}+ & - & 0\\ + & + & -\end{pmatrix}. \tag{3}$$

Given an instance $I=(A,b,d)$ of the ILS problem, the index $\eta(I)$ is the optimal value of the following linear programming problem.

$$\begin{array}{lll}\text{min.} & Z & \\ \text{s.t.} & \sum_{j:\mathrm{sgn}(a_{ij})=+}\alpha_j+\sum_{j:\mathrm{sgn}(a_{ij})=-}(1-\alpha_j)\le Z & (i=1,\dots,m)\\ & 0\le\alpha_j\le 1 & (j=1,\dots,n).\end{array} \tag{4}$$

We note that neither numerical information of A, b nor d is used for our index $\eta(I)$, and it depends *only* on $\mathrm{sgn}(A)$, i.e., two problem instances I and I' have $\eta(I)=\eta(I')$ if the corresponding matrices have the same sign pattern.

The idea of this index originates from the works by Boros et al. [5], which introduced a complexity index for the Boolean satisfiability problem (SAT): Given a CNF $\varphi=\bigwedge_{i=1}^{m}\Big(\bigvee_{j\in P_i}x_j\vee\bigvee_{j\in N_i}\overline{x}_j\Big)$ of n variables, where $P_i,N_i\subseteq\{1,2,\dots,n\}$ with $P_i\cap N_i=\emptyset$, determine whether or not φ is satisfiable, i.e., whether or not there is $x\in\{0,1\}^n$ such that $\varphi(x)=1$. Their index

distinguishes between the classes of easy and hard SAT instances. We can see that our index is a generalization of theirs to integer linear systems, since the Boolean satisfiability problem can be represented as integer linear systems with unit matrices $A \in \{0, -1, +1\}^{m \times n}$.

The results obtained in this paper

For a real γ, let $\mathrm{ILS}_{=}(\gamma)$ denote the family of the problem instances I with $\eta(I) = \gamma$. We then have the following main result.

Theorem 1. (1) $\mathrm{ILS}_{=}(\gamma)$ *is linearly solvable, if* $\gamma < 1$,

(2) $\mathrm{ILS}_{=}(\gamma)$ *is weakly NP-hard and pseudo-polynomially solvable, if* $\gamma = 1$, *and*

(3) $\mathrm{ILS}_{=}(\gamma)$ *is strongly NP-hard, if* $\gamma > 1$.

Here we assume that $\mathrm{ILS}_{=}(\gamma) \neq \emptyset$ holds.

We also show that $\eta(I) < 1$, $= 1$, and > 1 can be checked in linear time. This theorem implies the existing results [2, 14, 30] that quadratic (i.e., TVPI) systems and Horn systems can be solved in pseudo-polynomial time, since quadratic systems and Horn systems are included in $\mathrm{ILS}_{=}(\gamma)$ with $\gamma \leq 1$, which will be discussed later.

If we restrict integer linear systems to Boolean satisfiability problem, then Boros et al. [5] showed that $\mathrm{ILS}_{=}(\gamma)$ is linearly solvable if $\gamma \leq 1$, and $\mathrm{ILS}_{=}(\gamma)$ is strongly NP-hard if $\gamma > 1$. Instead of their result, which partitions the SAT problem into *two* classes of *easy* and *hard* SAT instances, we partition integer linear systems into *three* classes of *easy*, *semi-hard* and *hard* systems.

For unit linear systems, i.e., $A \in \{0, -1, +1\}^{m \times n}$, we have the following result.

Theorem 2. *Let* A *be a unit matrix, i.e.,* $A \in \{0, -1, +1\}^{m \times n}$. *Then we have*

(1) $\mathrm{ILS}_{=}(\gamma)$ *is polynomially solvable if* $\gamma \leq 1$.

(2) $\mathrm{ILS}_{=}(\gamma)$ *is strongly NP-hard if* $\gamma > 1$.

We note that Theorem 2 includes polynomial solvability for Horn and quadratic unit systems [1, 8, 9, 17, 22, 28, 29], and tractability of SAT problem (i.e., the satisfiability problem for 2-, Horn, renamable Horn, and q-Horn CNFs can be solved in polynomial time) [10, 13, 24, 4].

We generalize the results above by considering nonconstant γ. More precisely, we regard γ as a function of the number of variables n and d, and for such γ, let $\mathrm{ILS}_{\leq}(\gamma)$ denotes the family of the problem instances I with $\eta(I) \leq \gamma$. We have the following results.

Theorem 3. (1) $\mathrm{ILS}_{\leq}(\gamma)$ *is linearly solvable, if* $\gamma < 1$.

(2) $\mathrm{ILS}_{\leq}(\gamma)$ *is weakly NP-hard and pseudo-polynomially solvable, if* $1 \leq \gamma \leq 1 + \frac{c \log_d n}{n}$ *for some constant* $c > 0$.

(3) $\mathrm{ILS}_{\leq}(\gamma)$ *is strongly NP-hard, if* $\gamma \geq 1 + \frac{1}{n^{\delta}}$ *for some constant* $\delta < 1$.

Theorem 4. *Let* A *be a unit matrix, i.e.,* $A \in \{0, -1, +1\}^{m \times n}$. *Then we have*

(1) $\mathrm{ILS}_{\leq}(\gamma)$ *is polynomially solvable, if* $\gamma \leq 1 + \frac{c \log_d n}{n}$.

(2) $\mathrm{ILS}_{\leq}(\gamma)$ *is strongly NP-hard, if* $\gamma \geq 1 + \frac{1}{n^{\delta}}$ *for some constant* $\delta < 1$.

Finally, we mention that there exists a line of research for *sign solvability* for linear systems [7, 26], linear programming problem [16], and linear complementarity problem [19]. They mainly study sign patterns of the input data, that always determine sign patterns of solutions. Their works are motivated by the fact that the input data are uncertain but the structural properties are preserved in most practical situations. While both their and our works concern the sign patterns of the input, ours differs from theirs in that our work studies the *integer* solutions and does not concern sign patterns of the solutions.

2 Integer linear systems with index smaller than 1

For a given problem instance $I = (A, b, d)$, we denote by $(Z, \alpha_1, \ldots, \alpha_n)$ an optimal solution of (4). Let $V = \{1, \ldots, n\}$. In this paper, we assume without loss of generality that each variable is not redundant, i.e., A contains no column whose elements are all 0, since otherwise we can fix all redundant variables to 0, for example.

In this section, we consider the case in which $\eta(I) < 1$, i.e., $Z < 1$, and prove (1) in Theorem 3, which implies Theorems 1, 2, 4 when $\eta(I) < 1$.

2.1 The case of $\eta(I) < 1/2$

Let us first consider the case in which $Z = \eta(I) < 1/2$. Then, there exists no $j \in V$ with $\alpha_j = 1/2$, since otherwise we have $Z \geq 1/2$, a contradiction. If $\alpha_j > 1/2$ for some $j \in V$, then by $Z < 1/2$, the j-th column of A is nonpositive. Similarly, $\alpha_j < 1/2$ implies that the j-th column of A is nonnegative. These imply that $Z = 0$, $\alpha_j > 1/2 \Rightarrow \alpha_j = 1$, and $\alpha_j < 1/2 \Rightarrow \alpha_j = 0$. Therefore, we have the following lemma.

Lemma 1. *If Problem* (4) *has the optimal value* $Z < 1/2$*, then we have* $Z = 0$*, and there exists a unique* 0-1 *optimal solution for* (4)*.*

Moreover, $\eta(I) < 1/2$ (and hence $\eta(I) = 0$) holds if and only if each column of A is either nonnegative or nonpositive. Let y be a n-dimensional vector such that $y_j = d - 1$ if j-th column of A is nonnegative, and 0, otherwise (i.e., if j-th column of A is nonpositive). Then it is not difficult to see that there exists a vector $x \in D^n$ with $Ax \geq b$ if and only if y satisfies $Ay \geq b$. These lead to the following lemma.

Lemma 2. *Let* $I = (A, b, d)$ *be a problem instance. Then we can check whether* $\eta(I) < 1/2$ *in linear time, and if so, the problem can be solved in linear time.*

2.2 The case of $\eta(I) = 1/2$

We next consider the case in which $Z = \eta(I) = 1/2$.

If $\alpha_j > 1/2$ (resp., $\alpha_j < 1/2$) for some $j \in V$, then $Z = 1/2$ implies that the j-th column of A is nonpositive (resp., nonnegative). Define a vector $\alpha^* \in \mathbb{R}^n$ by $\alpha^*_j = 1$ if the j-th column of A is nonpositive, 0 if the j-th column of A is nonnegative, and $1/2$, otherwise. Then we can see that this α^* is also an optimal solution of (4).

Lemma 3. *If Problem* (4) *has the optimal value* $Z = 1/2$*, then it has a half-integral optimal solution.*

Moreover, $\alpha^*_j = 1/2$ if and only if the j-th column of A contains both positive and negative elements, and if $a_{ij} \neq 0$ for such j, then the i-th row of A contains no nonzero element a_{ik} with $k \neq j$ and $\alpha^*_k = 1/2$. Let us fix $x_j = 0$ for all $j \in V$ with $\alpha^*_j = 1$, and $x_j = d - 1$ for all $j \in V$ with $\alpha^*_j = 0$. Then each inequality of the resulting integer linear system contains at most one variable, and hence it can be easily solved.

Lemma 4. *Let $I = (A, b, d)$ be a problem instance. Then we can check whether $\eta(I) = 1/2$ in linear time, and if so, the problem can be solved in linear time.*

2.3 The case of $1/2 < \eta(I) < 1$

In this section, we consider the case in which $1/2 < Z = \eta(I) < 1$. Note that in this case Problem (4) might have no (half-)integral optimal solution. For example, let A be a $(n+1) \times n$ matrix such that $a_{ij} = -1$ if $i = j$, 1 if $i = n + 1$, and 0 otherwise. Then the problem has a unique optimal solution $Z = \frac{n}{n+1}$ and $\alpha_j = \frac{1}{n+1}$ for all j.

For a subset $S \subset \mathbb{R}$, let $V_S = \{j \in V \mid \alpha_j \in S\}$. For two reals a and b with $a < b$, $[a, b) = \{z \in \mathbb{R} \mid a \leq z < b\}$, $(a, b] = \{z \in \mathbb{R} \mid a < z \leq b\}$ and $[a, b] = \{z \in \mathbb{R} \mid a \leq z \leq b\}$. Let ε be a positive number that satisfies $Z \leq 1 - \varepsilon$ and $2k\varepsilon = 1$ for some integer k, where we note that ε might depend on m and n. We then partition $[0, 1]$ into $2k + 1$ sets

$$[0, 1] = \bigcup_{\ell=1}^{k} [(\ell - 1)\varepsilon, \ell\varepsilon) \cup \{1/2\} \cup \bigcup_{\ell=1}^{k} (1 - \ell\varepsilon, 1 - (\ell - 1)\varepsilon]. \tag{5}$$

For $i = 1, 2, \ldots, m$, let $P_i = \{j \in V \mid a_{ij} > 0\}$ and $N_i = \{j \in V \mid a_{ij} < 0\}$. Then we have the following properties.

Lemma 5. *Let $I = (A, b, d)$ be a problem instance with $1/2 < \eta(I) < 1$, and let ε be defined as above. Then*

(i) *$V_{(1-\varepsilon,1]} \cap P_i = \emptyset$ and $V_{[0,\varepsilon)} \cap N_i = \emptyset$ hold for all $i = 1, 2, \ldots, m$.*

(ii) *If $j \in V_{(1-(\ell+1)\varepsilon,1-\ell\varepsilon]} \cap P_i$ for some $\ell = 1, 2, \ldots, k$ and $i = 1, 2, \ldots, m$, then we have $P_i - \{j\} \subseteq V_{[0,\ell\varepsilon)}$ and $N_i \subseteq V_{(1-\ell\varepsilon,1]}$.*

(iii) *If $j \in V_{[\ell\varepsilon,(\ell+1)\varepsilon)} \cap N_i$ for some $\ell = 1, 2, \ldots, k$ and $i = 1, 2, \ldots, m$, then we have $P_i \subseteq V_{[0,\ell\varepsilon)}$ and $N_i - \{j\} \subseteq V_{(1-\ell\varepsilon,1]}$.*

Proof. (i), (ii), and (iii) follow from $Z \leq 1 - \varepsilon$. □

By (i) in Lemma 5, if $j \in V_{(1-\varepsilon,1]}$, then the j-th column of A is nonpositive, and hence we can fix $x_j = 0$. Similarly, if $j \in V_{[0,\varepsilon)}$, then the j-th column of A is nonnegative, and hence we can fix $x_j = d - 1$. After fixing variables in $V_{(1-\varepsilon,1]} \cup V_{[0,\varepsilon)}$, if $a_{ij} > 0$ for some $j \in V_{(1-2\varepsilon,1-\varepsilon]}$, then (ii) in Lemma 5 implies that the i-th inequality of the resulting system contains only one variable x_j. By solving such inequalities, we have a lower bound $x_j \geq p_j\, (\in D)$. Since all the other inequalities have $a_{ij} \leq 0$, we can fix $x_j = p_j$. Similarly, if $a_{ij} < 0$ for some $j \in V_{[\varepsilon,2\varepsilon)}$, then (iii) in Lemma 5 implies that the i-th inequality of the resulting system contains only one variable x_j. By solving such inequalities, we have an upper bound $x_j \leq p_j\, (\in D)$. Since all the other inequalities have $a_{ij} \geq 0$, we can fix $x_j = p_j$. By repeatedly applying this argument to variables in $V_{(1-(\ell+1)\varepsilon,1-\ell\varepsilon]}$ and $V_{[\ell\varepsilon,(\ell+1)\varepsilon)}$ for $\ell = 2, 3, \ldots k$, we can fix all the variables in

$V \setminus V_{\{1/2\}}$. Note that by (ii) and (iii) in Lemma 5, each inequality of the resulting system consists of at most one variable. Hence we can solve it in linear time.

Formally, we describe the algorithm in Algorithm 1. We note that the algorithm uses no information of $(Z, \alpha_1, \ldots, \alpha_n)$ of (4).

We remark that if the algorithm above solved the integer linear system, then we have $\eta(I) < 1$. Since we can check whether $\eta(I) \leq 1/2$ in linear time by Lemmas 2 and 4, we have the following result.

Lemma 6. *Let $I = (A, b, d)$ be a problem instance. Then we can check whether $1/2 < \eta(I) < 1$ in linear time, and if so, the problem can be solved in linear time.*

By combining Lemmas 2, 4, and 6, we have (1) in Theorem 3.

Algorithm 1.

Step 1.
  for $1 \leq j \leq n$ **do**
    if *j-th column of A is nonpositive* **then** $x_j := 0$
    else if *j-th column of A is nonnegative* **then** $x_j := d - 1$
    end if
  end for
  if *the resulting system has an inconsistent inequality (with no variable)* **then** *output "infeasible" and halt*
  else *remove inequalities with no variable from the system*
  end if

Step 2.
  while *the resulting system has $j \in V$ such that $a_{ij'} = 0$ for all i and j' with $a_{ij} > 0$ and $j' \neq j$* **do**
    compute a lower bound $x_j \geq p$ by solving inequalities in $\{i \mid a_{ij} > 0\}$
    if $p \leq d$ **then** $x_j := \max\{\lceil p \rceil, 0\}$
    else *output "infeasible" and halt*
    end if
    if *the resulting system has an inconsistent inequality (with no variable)* **then** *output "infeasible" and halt*
    else *remove inequalities with no variable from the system*
    end if
  end while

Step 3.
  while *the resulting system has $j \in V$ such that $a_{ij'} = 0$ for all i and j' with $a_{ij} < 0$ and $j' \neq j$* **do**
    compute an upper bound $x_j \leq p$ by solving inequalities in $\{i \mid a_{ij} < 0\}$
    if $p \geq 0$ **then** $x_j := \min\{\lfloor p \rfloor, d - 1\}$
    else *output "infeasible" and halt*
    end if
    if *the resulting system has an inconsistent inequality (with no variable)* **then** *output "infeasible" and halt*
    else *remove inequalities with no variable from the system*
    end if
  end while

Step 4. */* Note that each inequalities of the resulting system has exactly one variable.*/*
  Solve the resulting system.

It is not difficult to see that the algorithm 1 above can be implemented in linear time in the input length and the number of nonzero elements of A.

3 Integer linear systems with index 1

In this section, we assume that integer linear systems have index 1, and prove Theorems 1 and 2 for this case.

Let $(Z, \alpha_1, \ldots, \alpha_n)$ be an optimal solution of (4). Then we note that $|P_i \cap V_{(1/2,1]}| \leq 1$, $|N_i \cap V_{[0,1/2)}| \leq 1$, and $|(P_i \cup N_i) \cap V_{\{1/2\}}| \leq 2$ holds for all $i = 1, 2, \ldots m$, since otherwise we have $Z > 1$, a contradiction. Moreover, $(P_i \cup N_i) \cap V_{\{1/2\}} \neq \emptyset$ implies $P_i \cap V_{(1/2,1]}, N_i \cap V_{[0,1/2)} = \emptyset$, which again follows from $Z = 1$. Define a vector $\alpha^* \in \mathbb{R}^n$ by $\alpha^*_j = 0$ if $\alpha_j < 1/2$, $\alpha^*_j = 1/2$ if $\alpha_j = 1/2$, and $\alpha^*_j = 1$, otherwise (i.e., if $\alpha_j > 1/2$). It is not difficult to see that α^* is also an optimal solution of (4).

Lemma 7 ([5]). *If Problem* (4) *has the optimal value $Z = 1$, then it has a half-integral optimal solution.*

Moreover, such a solution can be computed in linear time.

Lemma 8 ([6]). *We can decide whether Problem* (4) *has the optimal value $Z = 1$ in linear time, and if so, we can compute a half-integral optimal solution in linear time.*

Let $\alpha \in \{0, 1/2, 1\}^n$ denote an optimal solution of Problem (4). To make discussion clear, we may assume $\alpha \in \{1/2, 1\}^n$ without loss of generality. To see this, assume that $\alpha_j = 0$ holds for some j. We then replace the variable x_j to a new variable x'_j $(= d - 1 - x_j)$, i.e., we substitute $x_j := d - 1 - x'_j$ in the system. It is not difficult to see that the feasibility of the original integer linear system is equivalent to the one of the resulting system. Since the coefficient matrix of the resulting system differs A only by the sign of the j-th column of matrix A, we have a half-integral optimal solution with $\alpha_j = 1$ for the new LP problem (4). By replacing all variables j with $\alpha_j = 0$, we have the integer linear system such that problem (4) has an optimal solution $\alpha \in \{1/2, 1\}^n$. We remark that this replacement can be done in linear time.

Let $Q = V_{\{1/2\}}$ and $H = V_{\{1\}}$. By $\alpha \in \{1/2, 1\}^n$, V can be partitioned into Q and H:

$$V = Q \cup H. \tag{6}$$

Then by the discussion at the beginning of this section, we have the following properties.

Lemma 9 (QH-partition [5]). *A partition $V = Q \cup H$ satisfies the following three conditions:*

(a) *Each row i of A contains at most two nonzero elements a_{ij} with $j \in Q$. Or equivalently, $|(P_i \cup N_i) \cap Q| \leq 2$ holds for all $i = 1, 2, \ldots, m$.*

(b) *Each row i of A contains at most one positive element a_{ij} with $j \in H$. Or equivalently, $|P_i \cap H| \leq 1$ holds for all $i = 1, 2, \ldots, m$.*

(c) *If row i of A contains a positive element a_{ij} with $j \in H$, then the elements a_{ik} with $k \in Q$ are all zeros. Or equivalently, $P_i \cap H \neq \emptyset \Rightarrow (P_i \cup N_i) \cap Q = \emptyset$ for all $i = 1, 2, \ldots, m$.*

For a QH-partition of V, let S denote the set of rows i of A such that $a_{ij} = 0$ for all $j \in Q$. Let $A[S, H]$ denote the submatrix of A whose row and column sets are S and H, respectively, and let b_H and x_H respectively denote the restriction of b and x to H. Then by Lemma 9 (a), linear system $A[S, H]x_H \geq b_H$ is Horn, i.e., each row of $A[S, H]$ contains at most one positive element. It is known that any Horn system has a unique minimal solution if it is feasible. Let $x^*_H \in D^H$ be such a solution for $A[S, H]x_H \geq b_H$. Since Lemma 9 (c) implies that any element a_{ij} with $i \notin S$ and $j \in H$ is nonpositive, we can see that the original integer linear system is feasible if and only if so is the system obtained from it by substituting $x_H = x^*_H$. Thus we consider the system obtained by fixing $x_H = x^*_H$. Since the resulting system is quadratic (i.e., each row contains at most two nonzero elements), we can solve it, for example, by the algorithm proposed in [14]. We summarize this algorithm in Algorithm 2.

Algorithm 2.

Step 1.

Compute a QH-partition of V

Step 2.

if *the integer linear system $x_H \in D^H$ and $A[S, H]x_H \geq b_H$ is infeasible* **then** *output "infeasible" and halt*

else *compute a unique minimal solution $x^*_H \in D^H$ of the system and substitue $x_H := x^*_H$*

end if

Step 3.

if *the resulting system is infeasible* **then** *output "infeasible" and halt*

else *compute an integer solution $x^*_Q \in D^Q$ of the resulting system, and output the vector (x^*_H, x^*_Q) and halt*

end if

Lemma 10. *Algorithm 2 solves the integer linear system with index* 1 *in time polynomial in n, m and d.*

Proof. Since the correctness of algorithm 2 follows from the discussion before the description of the algorithm, we discuss its time complexity only.

By [6], Step 1 can be executed in linear time. Steps 2 and 3 can be done in polynomial time in n, m, and d [30, 14]. Therefore, in total, the algorithm requires polynomial time in n, m, and d. □

Lemma 11. *For unit matrix A, Algorithm 2 solves the integer linear system with index* 1 *in polynomial time.*

Proof. The lemma follows from the fact that Horn and quadratic integer linear systems are solvable in polynomial time, if A is unit [8, 17]. □

We next show the weak NP-hardness of the problem.

Lemma 12. ILS$_=$(1) *is weakly NP-hard.*

Proof. It is known [21] that solving Horn or quadratic system is weakly NP-hard. We show that Horn and quadratic systems both have index at most 1. Since the integer linear system with index less than 1 is solvable in linear time, this proves the lemma.

Let $I = (A, b, d)$ be a Horn system. Then we assign all the variables α_j to 1. Since each row of A contains at most one positive element, we have $\eta(I) \leq 1$. On the other hand if I is quadratic, then by assigning all the variables α_j to $1/2$, we have $\eta(I) \leq 1$, since each row of A contains at most two nonzero elements. □

4 Integer linear systems with index η with $1 < \eta \leq 1 + \frac{c \log_d n}{n}$

In this section, we consider the case in which $1 < \eta(I) \leq 1 + \frac{c \log_d n}{n}$, and complete the proof of (2) in Theorem 3 and (1) in Theorem 4. Our positive results can be regarded as generalizations of the ones for $\mathrm{ILS}_{=}(1)$.

A partition of V into Q, H, and Y, i.e., $V = Q \cup H \cup Y$ is called QHY-partition, if Q and H satisfy all the conditions in Lemma 9.

If we have a QHY-partition with small Y, then the integer linear system can be solved by assigning all possible assignments to variables in Y. For this purpose, we make use of the following result.

Lemma 13 ([5]). *A QHY-partition with $|Y| < 6n(\eta(I) - 1)$ can be computed in polynomial time.*

By using this lemma, if γ is a function of n with $\gamma \leq 1 + \frac{c \log_d n}{n}$, then we have a QHY-partition with $|Y| \leq 6c \log_d n$. Note that each of the $d^{|Y|}$ assignments to the variables of Y produces a problem instance I^* with $\eta(I^*) \leq 1$. Each such instance is solvable in pseudo-polynomial time by Lemma 10, and if A is unit, it is solved in polynomial time by Lemma 11. Moreover, since $d^{|Y|} \leq n^{6c}$, we have that the integer linear systems can be solved in pseudo-polynomial time if the system has index at most $1 + \frac{c \log_d n}{n}$ for some constant c, and in polynomial time if the system is in addition unit.

5 Strong NP-hardness for integer linear systems

In this section, we show the strong NP-hardness for the integer linear systems, i.e., we prove (2) in Theorems 2 and 4, which implies (3) in Theorems 1 and 3.

We first show that $\mathrm{ILS}_{\leq}(\gamma)$ is NP-hard, if $\gamma \geq 1 + \frac{1}{n^\delta}$ for some constant $\delta < 1$. To do this, we reduce the Boolean satisfiability problem (SAT) to our problem.

Given a CNF $\varphi = \bigwedge_{i=1}^{m} \Big(\bigvee_{j \in P_i} x_j \vee \bigvee_{j \in N_i} \overline{x}_j \Big)$, we construct an integer linear system as follows:

$$\begin{array}{rcll} \sum_{j \in P_i} x_j + \sum_{j \in N_i} (1 - x_j) & \geq & 1 & (i = 1, \ldots, m) \\ x & \in & \{0,1\}^n. & \end{array} \tag{7}$$

Namely, A is a matrix defined by

$$a_{ij} = \begin{cases} 1 & j \in P_i \\ -1 & j \in N_i \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

b is a vector defined by

$$b_i = 1 - |N_i| \qquad (i = 1, \ldots, m), \tag{9}$$

and $d = 2$.

It is not difficult to see that φ is satisfiable if and only if there exists a $x \in D^n$ such that $Ax \geq b$. Since this reduction is polynomial, solving the integer linear system is in general NP-hard. Moreover, as mentioned in the introduction, our index η is a generalization of the complexity index of SAT defined by Boros et al. [5].

Lemma 14. *Let $Z(\varphi)$ denote the complexity index of CNF φ defined in* [5]*, and $\eta(I)$ denote the complexity index of the integer linear system defined as* (7)*. Then we have $Z(\varphi) = \eta(I)$.*

We now refer the following theorem due to Boros et al. [5], where $\mathrm{SAT}(\gamma)$ denotes the set of instances φ of SAT such that $Z(\varphi) \leq \gamma$.

Theorem 5 ([5])**.** *SAT(γ) is strongly NP-hard, if $\gamma \geq 1 + \frac{1}{n^\delta}$ for some constant $\delta < 1$.*

By combining Theorem 5 with Lemma 14, we have the following result.

Lemma 15. *Let γ be a function of n such that $\gamma \geq 1 + \frac{1}{n^\delta}$ for some constant $\delta < 1$. Then $\mathrm{ILS}_{\leq}(\gamma)$ is strongly NP-hard, even if A is unit.*

Note that Lemma 15 implies that for any constant $\gamma > 1$, $\mathrm{ILS}_{\leq}(\gamma)$ is NP-hard, even if A is unit. In order to show (2) in Theorems 2, we consider the following simple reduction.

Let A (resp., A') be a unit $m \times n$ (resp., $m' \times n'$) matrix with the optimal value γ (resp., γ') of (4). Consider the following integer linear system:

$$\begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} \geq \begin{pmatrix} 0 \\ b' \end{pmatrix},$$

where 0 denote a zero matrix (or vector) of appropriate size, and b' denote a vector in $\mathbb{R}^{m'}$. We can see that this system has a solution if and only if $A'x' \geq b'$ has a solution, since $x = 0$ clearly satisfies $Ax \geq 0$. If we choose $A'x' \geq b'$ from strongly NP-hard instances with $\gamma' \leq \gamma$, we have the following results.

Lemma 16. *Let γ be a constant with $\gamma > 1$ and $\mathrm{ILS}_{=}(\gamma) \neq \emptyset$. Then $\mathrm{ILS}_{=}(\gamma)$ is strongly NP-hard, even if A is unit.*

6 Conclusion

We have introduced a complexity index for the integer linear systems. The index is defined as the optimal value of a linear programming problem whose input is based only on the sign pattern of the coefficient matrix. We have shown that the complexity index partitions the integer linear systems into *three* classes of *easy*, *semi-hard* and *hard* systems, and if all entries of the coefficient matrix are $-1, 0$ or 1, then it partitions the integer linear systems into *two* classes of *easy* and *hard* systems. Our results contain the existing ones for quadratic systems, Horn systems and Boolean satisfiability problem.

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