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Abstract

In this paper we consider stochastic multiarmed bandit problems. Recently a policy, DMED, is proposed and proved to achieve the asymptotic bound for the model that each reward distribution is supported in a known bounded interval, e.g. $[0, 1]$. However, the derived regret bound is described in an asymptotic form and the performance in finite time has been unknown. We inspect this policy and derive a finite-time regret bound by refining large deviation probabilities to a simple finite form. Further, this observation reveals that the assumption on the lower-boundedness of the support is not essential and can be replaced with a weaker one, the existence of the moment generating function.

1 Introduction

In the multiarmed bandit problem a gambler pulls arms of a slot machine sequentially so that the total reward is maximized. There is a tradeoff between exploration and exploitation since he cannot know the most profitable arm unless pulling all arms infinitely many times.

There are two main formulations for this problem: stochastic and nonstochastic bandits. In the stochastic setting rewards of each arm follow an unknown distribution (Gittins, 1989; Agrawal, 1995; Vermorel and Mohri, 2005), whereas the rewards are detemined by an adversary in the nonstochastic setting (Auer et al., 2002b). In this paper we consider the stochastic bandit, where rewards of arm $i \in \{1, \ldots, K\}$ are i.i.d. sequence from unknown distribution $F_i \in \mathcal{F}$ with expectation μ_i for a model $\mathcal{F}$ known to the gambler. For the maximum expectation $\mu^* \equiv \max_i \mu_i$, we call an arm i optimal if $\mu_i = \mu^*$ and suboptimal otherwise. If the gambler knows each μ_i beforehand, it is best to choose optimal arms at every round. A *policy* is a strategy of the gambler for choosing arms based on the past result of plays. The performance of a policy is measured by the loss called expected regret or *regret*, in short, given by

$$\sum_{i:\mu_i < \mu^*} (\mu^* - \mu_i) \mathrm{E}[T_i(n)] \ ,$$

where $T_i(n)$ is the number of plays of arm i through the first n rounds. Since we regard each μ_i as a unknown constant fixed in advance, we consider how we can reduce $\mathrm{E}[T_i(n)]$ for each suboptimal arm i to achieve a small regret.

Robbins (1952) first considered this setting and Lai and Robbins (1985) gave a framework for determining an optimal policy by establishing a theoretical bound for the regret. Later this theoretical bound was extended to multiparameter or nonparametric models $\mathcal{F}$ by Burnetas and Katehakis (1996). In their paper, it was proved that any policy satisfying a mild regularity condition satisfies

$$\mathrm{E}[T_i(n)] \leq \frac{1 - \mathrm{o}(1)}{D_{\inf}(F_i, \mu^*; \mathcal{F})} \log n \ , \tag{1}$$

where $D_{\inf}(F, \mu; \mathcal{F})$ is defined in terms of Kullback-Leibler divergence $D(\cdot \| \cdot)$ by

$$D_{\inf}(F_i, \mu^*; \mathcal{F}) = \inf_{G \in \mathcal{F}:\mathrm{E}_G[X] > \mu^*} D(F_i \| G) \ .$$

The most popular model in the nonparametric setting is the family of distributions with supports contained in a known bounded interval, e.g. $[0, 1]$. For this model, which we denote by $\mathcal{A}_0$, it is known

that fine performance can be obtained by policies called Upper Confidence Bound (UCB) (Auer et al., 2002a; Audibert et al., 2009; Garivier and Cappé, 2011). However, although some bounds for regrets of UCB policies have been obtained in a non-asymptotic form, they do not necessarily achieve the asymptotic theoretical bound.

Recently Honda and Takemura (2010) proposed Deterministic Minimum Empirical Divergence (DMED) policy, which chooses arms based on an index $D_{\inf}(\hat{F}_i, \mu; \mathcal{A}_0)$, or simply written as $D_{\inf}(\hat{F}_i, \mu)$, for empirical distribution $\hat{F}_i$ of arm i. Whereas DMED achieves the theoretical bound asymptotically, the evaluation heavily depends on an asymptotic analysis and any finite-time regret bound has been unknown. Further, in the analysis of DMED, the assumption on the lower bound of the support seems to be a technical one needed for the proof. For example, the gambler does not have to know that the lower bound of the support is zero if he knows that the upper bound is one.

Our Contribution. Based on the above observation, we consider the family $\mathcal{A}$ of distributions on $(-\infty, 1]$ instead of $\mathcal{A}_0$. We first show that $D_{\inf}(F, \mu; \mathcal{A}_0) = D_{\inf}(F, \mu; \mathcal{A})$ for all $F \in \mathcal{A}_0$. Thus, although the gambler has more candidates for the true distribution of each arm in the model $\mathcal{A}$ than in $\mathcal{A}_0$, the theoretical bound (1) does not vary between $\mathcal{A}_0$ and $\mathcal{A}$.

Next we provide a finite-time regret bound of DMED for all distributions in $\mathcal{A}$ with moment generating functions existing in some neighborhood of the origin. Since nonstochastic bandits inevitably require the boundedness of the support, we can now assert that an advantage of assuming stochastic bandits is that the semi-bounded rewards can be dealt with in the nonparametric setting.

Technical Approach. In the evaluation of DMED it is essential to evaluate the probability that $D_{\inf}(\hat{F}_i, \mu)$ deviates from $D_{\inf}(F_i, \mu)$. Note that for policies based on the index $D_{\inf}(\hat{F}_i, \mu)$, finite-time regret bounds have been derived for the case that each distribution is supported in a *finite* subset of $[0, 1]$ (Maillard et al., 2011; Honda and Takemura, 2011). The advantage of assuming finiteness is that Sanov's theorem gives a non-asymptotic large deviation probability. However the regret bounds derived by this technique contain a finite but exceedingly large term

$$\sum_{t=1}^{\infty} t^{|\mathrm{supp}(F_i)|} \, \mathrm{e}^{-at} \quad,$$

where $|\mathrm{supp}(F_i)|$ denotes the size of the support of F_i and the polynomial $t^{|\mathrm{supp}(F)|}$ appears as a total number of possible empirical distributions from t samples from F_i. Similarly, whereas non-asymptotic Sanov's theorem is also known for continuous support distributions (see Dembo and Zeitouni (1998, Ex. 6.2.19)), it requires the total number of ϵ-balls to cover a set of distributions as a coefficient. Thus, although it is not impossible to derive a finite-time regret bound by a naive application of the non-asymptotic Sanov's theorem, it becomes very complicated and unrealistic.

To avoid counting or covering the possible empirical distributions, we exploit the following fact

$$D_{\inf}(\hat{F}_i, \mu) = \max_{0 \le \nu \le \frac{1}{1-\mu}} \mathrm{E}_{\hat{F}_i}[\log(1 - (X - \mu)\nu)] \quad. \tag{2}$$

Although it involves a maximization operation, it is merely an empirical mean of random variables $\log(1 - (X_t - \mu)\nu)$ where each X_t follows distribution F_i. By Cramér's theorem we can bound the large deviation probability for such a finite dimensional empirical mean by an exponential function with a simple coefficient.

Another difficulty for our setting is that $D_{\inf}(F, \mu) = D_{\inf}(F, \mu; \mathcal{A})$ is neither bounded nor continuous in $F \in \mathcal{A}$ unlike the case of $\mathcal{A}_0$, which makes the evaluation of the exponential rate for the large deviation probability of $D_{\inf}(\hat{F}_i, \mu)$ much harder. The key to this problem also lies in (2). Since it is an expectation of a logarithmic function on X, the effect of the tail weight is weaker than the polynomial function $X^1 = X$. Thus the large deviation probability of the joint distribution of $(D_{\inf}(\hat{F}_i, \mu), \mathrm{E}_{\hat{F}_i}[X])$ can be evaluated on the same regularity condition as that for the empirical mean $\mathrm{E}_{\hat{F}_i}[X]$ alone, namely, the existence of the moment generating function of F_i in some neighborhood of the origin.

Paper Outline. In Sect. 2 we give definitions used throughout this paper and introduce DMED policy proposed for distributions on $[0, 1]$. In Sect. 3, we give the main results of this paper on the finite-time regret bound of DMED for distributions on $(-\infty, 1]$. The remaining sections are devoted to the proof of the main results. We extend some results for the support $[0, 1]$ to $(-\infty, 1]$ in Sect. 4. We derive a large deviation probability for $D_{\inf}(F, \mu)$ in a non-asymptotic form in Sect. 5. We conclude this paper in Sect. 6. We give some results on large deviation principle in Appendix A. A proof of the main theorem is given in Appendix B.

Algorithm 1 DMED Policy

Parameter: $r \in (0, 1)$.
Initialization: $L_C, L_R := \{1, \cdots, K\}$, $L_N := \emptyset$, $n := K$. Pull each arm once.
Loop:
   1. For $i \in L_C$ in ascending order,
      1.1. $n := n + 1$ and pull arm i. $L_R := L_R \setminus \{i\}$.
      1.2. $L_N := L_N \cup \{j\}$ for all $j \notin L_R$ such that the following $J'_n(j)$ occurs:

$$J'_n(j) \equiv \{(1 - r)T_i(n)D_{\inf}(\hat{F}_i(n), \hat{\mu}^*(n); \mathcal{A}_a) \le \log n\}. \tag{4}$$

   2. $L_C, L_R := L_N$ and $L_N := \emptyset$.

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2 Preliminaries

Let $\mathcal{A}_a$, $a \in (-\infty, 1)$, be the family of probability distributions on $[a, 1]$. We denote the family of distributions on $(-\infty, 1]$ by $\mathcal{A}_{-\infty}$ or simply $\mathcal{A}$. For $x \in \mathbb{R}$ and $F \in \mathcal{A}$, the cumulative distribution is denoted by $F(x) = F((-\infty, x])$. For the metric of $\mathcal{A}_a$ we use Lévy distance

$$d_{\mathrm{L}}(F, G) \equiv \inf\{h > 0 : F(x - h) - h \le G(x) \le F(x + h) + h\} \ .$$

$\mathrm{E}_F[\cdot]$ denotes the expectation under $F \in \mathcal{A}$. When we write e.g. $\mathrm{E}_F[u(X)]$ for a function $u : \mathbb{R} \to \mathbb{R}$, X denotes a random variable with distribution F. The expectation of F is denoted by $\mathrm{E}(F) \equiv \mathrm{E}_F[X]$. We always assume that the moment generating function $\mathrm{E}_F[\mathrm{e}^{\lambda X}]$ is finite in some neighborhood of the origin $\lambda = 0$.

Let $T_i(n)$ be the number of times that arm i has been pulled through the first n rounds. $\hat{F}_{i,t}$ and $\hat{\mu}_{i,t}$ denote the empirical distribution and the mean of arm i when arm i is pulled t times. $\hat{F}_i(n) \equiv \hat{F}_{i,T_i(n)}$ and $\hat{\mu}_i(n) \equiv \hat{\mu}_{i,T_i(n)}$ denote the empirical distribution and the mean of arm i at the n-th round. The largest empirical mean after the first n rounds is denoted by $\hat{\mu}^*(n) \equiv \max_i \hat{\mu}_i(n)$.

In this paper we analyze DMED policy proposed by Honda and Takemura (2010). It is described as Algorithm 1, where

$$D_{\inf}(F, \mu; \mathcal{A}_a) \equiv \inf_{G \in \mathcal{A}_a : \mathrm{E}(G) > \mu} D(F\|G) \ . \tag{3}$$

Note that this policy is parametrized by $r \in (0, 1)$ in this paper, which was fixed to $r = 0$ in the original proposal. This parameter arises because some properties on $D_{\inf}(F, \mu; \mathcal{A}_a)$, such as boundedness and continuity, do not hold for $a = -\infty$. For $r > 0$ we conservatively (i.e. more often) choose seemingly suboptimal arms. As a result, the coefficient of the logarithmic term becomes $1/(1 - r)$ times the theoretical bound.

Another minor change is that $\log n$ in (4) was $\log n - \log T_i(n)$ in the original proposal. It is described in Honda and Takemura (2010) that the term $\log T_i(n)$ is only for improvement of simulation results and has no importance for the asymptotic analysis. In this paper we avoid this term since it makes the constant term in the finite-time analysis much more complicated.

For the setting of $a = 0$, the regret of DMED is evaluated as follows.

Proposition 1 (Honda and Takemura (2010, Theorem 4)) *Let $\epsilon > 0$ be arbitrary. Under DMED policy with $r = 0$, it holds for all $(F_1, \ldots, F_K) \in \mathcal{A}_0^K$ and suboptimal arms i that*

$$\mathrm{E}[T_i(n)] \le \frac{1 + \epsilon}{D_{\inf}(F_i, \mu^*; \mathcal{A}_0)} \log n + \mathrm{O}(1).$$

This bound is asymptotically optimal in view of the theoretical bound (1).

Now define

$$\begin{aligned} L(\nu; F, \mu) &\equiv \mathrm{E}_F[\log(1 - (X - \mu)\nu)] \ , \\ L_{\max}(F, \mu) &\equiv \max_{0 \le \nu \le \frac{1}{1-\mu}} L(\nu; F, \mu) \ . \end{aligned} \tag{5}$$

Functions L and $L_{\max}$ correspond to the Lagrangian function and the dual problem of $D_{\inf}(F, \mu; \mathcal{A}_a)$, respectively.

Proposition 2 (Honda and Takemura (2010, Theorem 5)) *For all $F \in \mathcal{A}_0$ and $\mu < 1$ it holds that $D_{\inf}(F, \mu; \mathcal{A}_0) = L_{\max}(F, \mu)$.*

3 Main Results

We now state the main result of this paper in Theorems 3 and 4. We show that the theoretical bound does not depend on knowledge of the lower bound of the support in Theorem 3 and that the theoretical bound is actually achievable by DMED in Theorem 4.

Theorem 3 *Let $a \in [-\infty, 1)$ and $F \in \mathcal{A}_a$ be arbitrary. (i) $D_{\inf}(F, \mu; \mathcal{A}_a) = D_{\inf}(F, \mu; \mathcal{A})$. (ii) If $\mu < 1$ then $D_{\inf}(F, \mu; \mathcal{A}) = L_{\max}(F, \mu)$.*

We prove this theorem in the next section. The part (i) of this theorem means that the theoretical bound does not depend on whether we know that the support of distributions is bounded from below by a or we have to consider the possibility that the support of distributions may not be lower-bounded. Furthermore, from (ii), we can express the theoretical bound in the same expression as $\mathcal{A}_0$ for any distribution in $\mathcal{A}$. In view of this theorem we sometimes write $D_{\inf}(F, \mu)$ instead of more precise $D_{\inf}(F, \mu; \mathcal{A}_a)$ or $D_{\inf}(F, \mu; \mathcal{A})$.

Let $\mathcal{I}_{\text{opt}} \equiv \{i : \mu_i = \mu^*\} \subset \{1, \cdots, K\}$ be the set of optimal arms and $\mu' \equiv \max_{i \notin \mathcal{I}_{\text{opt}}} \mu_i$ be the second optimal expected value. Define Fenchel-Legendre transform of the moment generating function of F_k as

$$\Lambda_k^*(x) \equiv \sup_{\lambda \in \mathbb{R}} \{\lambda x - \log \mathrm{E}_{F_k}[e^{\lambda X}]\} \ . \tag{6}$$

Then $\mathrm{E}[T_i(n)]$ is bounded for $\xi_{i,\epsilon,\delta} \equiv \epsilon D_{\inf}(F_i, \mu^*) - \delta/(1 - \mu^*)$ as follows.

Theorem 4 *Assume that $\mu^* < 1$. Let $\epsilon > 0$ and $i \notin \mathcal{I}_{\text{opt}}$ be arbitrary and fix any $\delta \in (0, \mu^* - \mu')$ such that $\xi_{i,\epsilon,\delta} > 0$. Then for all $n > 0$*

$$\mathrm{E}[T_i(n)] \leq \frac{\log n}{(1 - \epsilon)(1 - r)D_{\inf}(F_i, \mu^*)} + C \ ,$$

where, for $\tilde{\Lambda}^*(\cdot, \cdot, \cdot)$ defined in (13), the constant term is given by

$$C = \frac{1}{1 - e^{-\tilde{\Lambda}^*(\xi_{i,\epsilon,\delta}, \mu_i, \mu^*)}} + \sum_{k \in \mathcal{I}_{\text{opt}}} \frac{K}{1 - e^{-\Lambda_k^*(\mu^* - \delta)}} + \sum_{k \notin \mathcal{I}_{\text{opt}}} \frac{K}{1 - e^{-\Lambda_k^*(\mu' + \delta)}}$$

$$+ \min_{k \in \mathcal{I}_{\text{opt}}} \left\{ \frac{2(1 + K)}{1 - e^{-\Lambda_k^*(\mu' + \delta)}} + \frac{2e}{r(1 - e^{-r\Lambda_k^*(\mu' + \delta)})^2} \right\} \ .$$

We prove this theorem in Appendix B. The proof is largely the same as that of Honda and Takemura (2010, Theorem 4), with difference that asymptotic large deviation probabilities are replaced with non-asymptotic forms in Theorems 11 and 12.

As described in Prop. 14 (iii) of Appendix A, $\Lambda_k^*(\cdot)$ corresponds to the exponential rate of the probability on the sample size that the empirical mean of arm k deviates from its expectation. We can bound this rate in an explicit form for some cases. For example, it can be bounded by the variance for the case that the support of F_k is bounded from below (Hoeffding, 1963, Theorem 1). However, it seems to be impossible to bound the rate by its finite-degree moments for an optimal arms $k \in \mathcal{I}_{\text{opt}}$ in general case, although it is possible for suboptimal arms $k \notin \mathcal{I}_{\text{opt}}$ (Hoeffding, 1963, Theorem 3).

Remark 5 *The derived bound is somewhat weaker than that for the bounded support model in Prop. 1 since the bound in this theorem contains the coefficient $1/(1 - r)$ in the logarithmic term. We can remove the effect of the parameter r from the logarithmic term by letting r depend on $T_i(n)$, e.g., $r = 1/\sqrt{T_i(n)}$. However, it makes the analysis longer and we omit the evaluation of this version for lack of space.*

4 Properties of $D_{\inf}$ in the Semi-bounded Support Model

In the analysis of DMED it is essential to investigate the function $D_{\inf}(F, \mu; \mathcal{A})$. In this section we extend some results on $D_{\inf}(F, \mu; \mathcal{A}_0)$ in Honda and Takemura (2010) for our model $\mathcal{A} = \mathcal{A}_{-\infty}$ and prove Theorem 3.

First we consider the function $L(\nu; F, \mu) = \mathrm{E}_F[\log(1 - (X - \mu))\nu]$. The integrand $l(x, \nu) \equiv \log(1 - (x - \mu)\nu)$ is differentiable in $\nu \in (0, (1 - \mu)^{-1})$ for all $x \in (-\infty, 1]$ with

$$\frac{\partial l(x, \nu)}{\partial \nu} = -\frac{x - \mu}{1 - (x - \mu)\nu} \ , \qquad \frac{\partial^2 l(x, \nu)}{\partial \nu^2} = -\frac{(x - \mu)^2}{(1 - (x - \mu)\nu)^2} \ .$$

Since they are bounded in $x \in (-\infty, 1]$, the integral $L(\nu; F, \mu)$ is differentiable in ν with

$$L'(\nu; F, \mu) \equiv \frac{\partial L(\nu; F, \mu)}{\partial \nu} = -\mathrm{E}_F \left[\frac{X - \mu}{1 - (X - \mu)\nu} \right] ,$$

$$L''(\nu; F, \mu) \equiv \frac{\partial^2 L(\nu; F, \mu)}{\partial \nu^2} = -\mathrm{E}_F \left[\frac{(X - \mu)^2}{(1 - (X - \mu)\nu)^2} \right] .$$

From these derivatives the optimal solution $\nu^*(F, \mu) \equiv \mathrm{argmax}_{0 \le \nu \le (1-\mu)^{-1}} L(\nu; F, \mu)$ of (5) exists uniquely and satisfies the following lemma.

Lemma 6 *Assume that* $\mathrm{E}(F) \le \mu < 1$ *holds. If* $\mathrm{E}_F[(1 - \mu)/(1 - X)] \le 1$ *then* $\nu^*(F, \mu) = (1 - \mu)^{-1}$ *and therefore* $\mathrm{E}_F[1/(1 - (X - \mu)\nu^*)] \le 1$. *Otherwise,* $L'(\nu^*; F, \mu) = 0$ *and* $\mathrm{E}_F[1/(1 - (X - \mu)\nu^*)] = 1$.

The differentiability of $L_{\max}(F, \mu)$ in μ also holds as in the case of bounded support.

Lemma 7 *For* $\mu > \mathrm{E}(F)$, $D_{\inf}(F, \mu)$ *is differentiable with*

$$\frac{\mathrm{d}D_{\inf}(F, \mu)}{\mathrm{d}\mu} = \nu^*(F, \mu) \le \frac{1}{1 - \mu} .$$

We omit the proofs of Lemmas 6 and 7 since they are the same as Theorems 3 and 5 of Honda and Takemura (2011) where the assumption on the support is not exploited.

Define $F_{(a)} \in \mathcal{A}_a$ as the distribution obtained by transferring the probability of $(-\infty, a)$ under F to $x = a$, that is,

$$F_{(a)}(x) \equiv \begin{cases} 0 & x < a , \\ F(x) & x \ge a . \end{cases}$$

Now we give the key to extension for the semi-bounded support in the following lemma, which shows that the effect of the tail weight is bounded uniformly if the expectation is bounded from below.

Lemma 8 *Fix arbitrary* $\mu, \tilde{\mu} < 1$ *and* $\epsilon > 0$. *Then there exists* $a(\epsilon)$ *such that* $|L_{\max}(F_{(a)}, \mu) - L_{\max}(F, \mu)| \le \epsilon$ *for all* $a \le a(\epsilon)$ *and* $F \in \mathcal{A}$ *such that* $\mathrm{E}(F) \ge \tilde{\mu}$.

Proof: Take sufficiently small $a < \min\{0, \mu\}$ and define $A = (-\infty, a), B = [a, 1]$. Note that $F(A) + F(B) = 1$. First we have

$$F(A) \le \frac{1 - \tilde{\mu}}{1 - a} \tag{7}$$

$$\int_A x \mathrm{d}F(x) \ge \tilde{\mu} - 1 + F(A) \tag{8}$$

from

$$\mathrm{E}(F) \le aF(A) + 1 \cdot F(B) = 1 - (1 - a)F(A) , \qquad \mathrm{E}(F) \le \int_A x \mathrm{d}F(x) + 1 \cdot F(B) ,$$

respectively. Next, $L_{\max}(F, \mu)$ can be written as

$$L_{\max}(F, \mu) = \max_{0 \le \nu \le \frac{1}{1-\mu}} \mathrm{E}_F[\log(1 - (X - \mu)\nu)]$$

$$= \max_{0 \le \nu \le \frac{1}{1-\mu}} \left\{ \int_A \log \frac{1 - (x - \mu)\nu}{1 - (a - \mu)\nu} \mathrm{d}F(x) + \int_B \log(1 - (x - \mu)\nu) \mathrm{d}F_{(a)}(x) \right\}. \tag{9}$$

Since $(1 - (x - \mu)\nu)/(1 - (a - \mu)\nu)$ is increasing in ν for $x \le a$, substituting 0 and $(1 - \mu)^{-1}$ into ν, we can bound the first term as

$$0 \le \int_A \log \frac{1 - (x - \mu)\nu}{1 - (a - \mu)\nu} \mathrm{d}F(x) \le \int_A \log \frac{1 - x}{1 - a} \mathrm{d}F(x)$$

$$\le F(A) \int_A \log(1 - x) \frac{\mathrm{d}F(x)}{F(A)} \qquad \text{(by } a \le 0\text{)}$$

$$\le F(A) \log \left(\int_A (1 - x) \frac{\mathrm{d}F(x)}{F(A)} \right) \qquad \text{(Jensen's inequality)}$$

$$\le F(A) \log \frac{1 - \tilde{\mu}}{F(A)} . \qquad \text{(by (8))}$$

From $\lim_{x\to 0} x \log x = 0$ and (7), the first term of (9) converges to 0 as $a \to -\infty$. The second term of (9) equals $L_{\max}(F_{(a)}, \mu)$ and the proof is completed. ∎

Now we show Theorem 3 based on the preceding lemmas.

Proof of Theorem 3: (i) The proof is straightforward since $D(F\|G) \geq D(F\|G_{(a)})$ always holds for $F \in \mathcal{A}_a$.

(ii) First we consider the case that F has a bounded support, i.e. $F \in \mathcal{A}_a$ for some $a \in (-\infty, 1)$. It is easily checked that $L_{\max}(F, \mu)$ defined in (5) is invariant under the scale transformation $[0, 1] \to [a, 1] : x \mapsto a + (1 - a)x$. Further, $D_{\inf}(F, \mu; \mathcal{A}_a)$ defined in (3) is also invariant with respect to scale from the invariance of the divergence. Since $D_{\inf}(F, \mu; \mathcal{A}_a) = L_{\max}(F, \mu)$ holds for $a = 0$ from Prop. 2, it also holds for all finite $a < 1$.

Next we consider the case that the support F is not bounded from below. We show $D_{\inf}(F, \mu; \mathcal{A}) \leq L_{\max}(F, \mu)$ and $D_{\inf}(F, \mu; \mathcal{A}) \geq L_{\max}(F, \mu)$ separately. We omit the proof for the former part for lack of space, but it can be proved in a similar procedure as the proof of Honda and Takemura (2010, Theorem 8).

Now we consider the latter inequality. Take arbitrary $\epsilon > 0$ and let $a < \mu$ be sufficiently small. Partitioning $(-\infty, 1]$ into $A = (-\infty, a)$ and $B = [a, 1]$ we can bound $D_{\inf}(F, \mu; \mathcal{A})$ as

$$
\begin{aligned}
\inf_{G \in \mathcal{A}: \mathrm{E}(G) > \mu} D(F\|G) &\geq \inf_{G \in \mathcal{A}: \mathrm{E}(G) > \mu} D(F_{(a)}\|G_{(a)}) \\
&\geq \inf_{G_{(a)} \in \mathcal{A}_a: \mathrm{E}(G_{(a)}) > \mu} D(F_{(a)}\|G_{(a)}) \quad (\text{by } \mathrm{E}(G) \leq \mathrm{E}(G_{(a)})) \\
&= L_{\max}(F_{(a)}, \mu) \\
&\geq L_{\max}(F, \mu) - \epsilon \quad (\text{by Lemma 8})
\end{aligned}
$$

and we complete the proof by letting $\epsilon \downarrow 0$. ∎

Finally we consider the continuity of $D_{\inf}(F, \mu; \mathcal{A})$ in F.

Lemma 9 *If $a < 1$ is finite then $D_{\inf}(F, \mu; \mathcal{A}_a)$ is continuous in $F \in \mathcal{A}_a$.*

This lemma is proved for the case $a = 0$ in Honda and Takemura (2010, Theorem 7). The extension for general bounded supports is straightforward from the scale transformation.

For the case of semi-bounded support distributions, the continuity does not hold any more. However, we can show the continuity over distributions with expectations bounded from below. Here recall that in view of Theorem 3 we write $D_{\inf}(F, \mu)$ instead of $D_{\inf}(F, \mu; \mathcal{A}_{-\infty}) = L_{\max}(F, \mu)$ when no confusion arises.

Lemma 10 *Let $\epsilon > 0$ and $\mu, \tilde{\mu} < 1$ be arbitrary. There exists $\delta > 0$ such that*

$$|D_{\inf}(G, \mu) - D_{\inf}(F, \mu)| \leq \epsilon \tag{10}$$

for all $G \in \mathcal{A}$ such that $\mathrm{E}(G) \geq \tilde{\mu}$ and $d_{\mathrm{L}}(F, G) \leq \delta$.

Proof: Applying Lemma 8 twice to F and G, there exists $a(\epsilon)$ such that

$$|D_{\inf}(G, \mu) - D_{\inf}(F, \mu)| \leq |D_{\inf}(G_{(a)}, \mu) - D_{\inf}(F_{(a)}, \mu)| + \epsilon/2 \tag{11}$$

for all $a \leq a(\epsilon)$ and G such that $\mathrm{E}(G) \geq \tilde{\mu}$. From the continuity of $D_{\inf}(\cdot, \mu)$ for bounded distribution in Lemma 9, there exists $\delta(\epsilon, F_{(a)})$ such that

$$|D_{\inf}(G_{(a)}, \mu) - D_{\inf}(F_{(a)}, \mu)| \leq \epsilon/2 \tag{12}$$

for all $G_{(a)}$ such that $d_{\mathrm{L}}(G_{(a)}, F_{(a)}) \leq \delta(\epsilon, F_{(a)})$. Note that $d_{\mathrm{L}}(G_{(a)}, F_{(a)}) \leq d_{\mathrm{L}}(G, F)$ obviously holds from the definition of Lévy distance. Therefore, from (11) and (12), we obtain (10) for all $G \in \mathcal{A}$ such that $\mathrm{E}(G) \geq \tilde{\mu}$ and $d_{\mathrm{L}}(F, G) \leq \delta(\epsilon, F_{(a(\epsilon))})$. ∎

5 Large Deviation Probabilities for $D_{\inf}$

In this section we consider the behavior of $D_{\inf}(\hat{F}_t, \mu)$ where $\hat{F}_t$ is the empirical distribution of t samples from distribution F, which approaches $D_{\inf}(F, \mu)$ as t increases. For our case of semi-bounded support, it is sometimes convenient to consider the joint distribution of empirical mean $\hat{\mu}_t = \mathrm{E}(\hat{F}_t)$ and distribution $\hat{F}_t$, since the convergence of the empirical distribution does not mean that of the empirical mean.

Note that, in this section and Appendix A, we sometimes consider moment generating functions and their Fenchel-Legendre transforms of random variables on domains other than $\mathbb{R}$. Since the underlying distribution is obvious from the context, we write e.g. $\Lambda^*_{\mathbb{R}^2}$ to clarify the domain, whereas the subscript was used to indicate the arm such as Λ^*_k in previous sections.

Theorem 11 *If $\mu < \mathrm{E}(F)$ and $u > D_{\inf}(F, \mu)$ then*

$$P_F[D_{\inf}(\hat{F}_t, \mu) \geq u \cap \hat{\mu}_t \leq \mu] \quad \leq \quad \begin{cases} 2\mathrm{e}^{-t\Lambda_{\mathbb{R}}^*(\mu)} & u \leq \Lambda_{\mathbb{R}}^*(\mu), \\ 2\mathrm{e}(1+t)\mathrm{e}^{-tu} & \text{otherwise,} \end{cases}$$

where $\Lambda_{\mathbb{R}}^*(x) \equiv \sup_{\lambda \in \mathbb{R}} \{\lambda x - \log \mathrm{E}_F[\mathrm{e}^{\lambda X}]\}$.

Theorem 12 *Fix arbitrary $\mu > \mathrm{E}(F)$ and $v > 0$. Then it holds for $c_0 \geq 2.163$ that*

$$P_F[D_{\inf}(\hat{F}_t, \mu) \leq D_{\inf}(F, \mu) - v] \leq \mathrm{e}^{-t\tilde{\Lambda}^*(v, \mathrm{E}(F), \mu)}$$

where

$$\tilde{\Lambda}^*(v, \mathrm{E}(F), \mu) \equiv \begin{cases} \frac{v^2}{2(c_0 + \frac{1-\mathrm{E}(F)}{1-\mu})} & v \leq \frac{1}{2}(c_0 + \frac{1-\mathrm{E}(F)}{1-\mu}) \;, \\ \frac{v}{2} - \frac{1}{8}(c_0 + \frac{1-\mathrm{E}(F)}{1-\mu}) \;, & \text{otherwise.} \end{cases} \tag{13}$$

We prove these theorems using Prop. 14, Theorem 15 and Prop. 16 in Appendix A. Before proving Theorem 11, we show its asymptotic version in the following.

Lemma 13 *If $\mu < \mathrm{E}(F)$ and $u > D_{\inf}(F, \mu)$ then*

$$\limsup_{t \to \infty} \frac{1}{t} \log P_F[D_{\inf}(\hat{F}_t, \mu) \geq u \cap \hat{\mu}_t \leq \mu] \leq -\max\{u, \Lambda_{\mathbb{R}}^*(\mu)\} \;.$$

Proof: Define $C \equiv \{(G, \mathrm{E}(G)) : G \in \mathcal{A}, D_{\inf}(G, \mu) \geq u \cap \mathrm{E}(G) \leq \mu\} \subset \mathcal{A} \times \mathbb{R}$ and let $\bar{C}$ be its closure. First we show that $D_{\inf}(G, \mu) \geq u$ and $v \leq \mu$ for all $(G, v) \in \bar{C}$.

From the definition of closure, there exists a sequence $\{(G_l, \mathrm{E}(G_l)) \in C\}_l$ such that $(G_l, \mathrm{E}(G_l)) \to (G, v)$, i.e., $G_l \to G$ and $\mathrm{E}(G_l) \to v$. Thus $\mathrm{E}(G_l) \geq v - \epsilon$ holds for all sufficiently large l where $\epsilon > 0$ is arbitrary. Therefore, from Lemma 10 we obtain

$$D_{\inf}(G, \mu) = \lim_{l \to \infty} D_{\inf}(G_l, \mu) \geq \liminf_{l \to \infty} u = u \;.$$

The inequality $v \leq \mu$ is obvious from $\mathrm{E}(G_l) \to v$ and $\mathrm{E}(G_l) \leq \mu$.

Now we obtain from Theorem 15 that

$$\limsup_{t \to \infty} \frac{1}{t} \log P_F[D_{\inf}(\hat{F}_t, \mu) \geq u \cap \hat{\mu}_t \leq \mu]$$

$$\leq \quad \limsup_{t \to \infty} \frac{1}{t} \log P_F[(\hat{F}_t, \hat{\mu}_t) \in \bar{C}]$$

$$\leq \quad - \inf_{(G,v)\,:\,D_{\inf}(G,\mu) \geq u \cap v \leq \mu} \sup_{(\phi, \lambda) \in C_{\mathrm{b}}(\mathbb{R}) \times \mathbb{R}} \left\{ \int \phi(x)\mathrm{d}G(x) + \lambda v - \log \int \mathrm{e}^{\phi(x) + \lambda x}\mathrm{d}F(x) \right\}$$

$$\leq \quad - \inf_{(G,v)\,:\,D_{\inf}(G,\mu) \geq u \cap v \leq \mu} \max\{\Lambda_{\mathbb{R}}^*(v),\, D(G\|F)\} \tag{14}$$

$$\leq \quad - \inf_{(G,v)\,:\,D_{\inf}(G,\mu) \geq u \cap v \leq \mu} \max\{\Lambda_{\mathbb{R}}^*(v),\, D_{\inf}(G, \mu)\} \qquad (\text{by } \mu < \mathrm{E}(F))$$

$$\leq \quad - \max\{\Lambda_{\mathbb{R}}^*(\mu), u\} \;, \qquad (\Lambda_{\mathbb{R}}^*(v) \text{ is decreasing in } v \leq \mu < \mathrm{E}(F))$$

where (14) follows from $(\{0\} \times \mathbb{R}) \cup (C_{\mathrm{b}}(\mathbb{R}) \times \{0\}) \subset C_{\mathrm{b}}(\mathbb{R}) \times \mathbb{R}$ and Prop. 16. $\blacksquare$

Proof of Theorem 11: Let $\delta > 0$ be arbitrary and define $\nu_i \equiv 1/(2(1-\mu)) + i\delta$ for $i = -M_\delta, -M_\delta + 1, \ldots, M_\delta - 1, M_\delta$, where $M_\delta \equiv \lfloor 1/(2(1-\mu)\delta) \rfloor$. Further define $\nu_{-M_\delta - 1} \equiv 0$ and $\nu_{M_\delta + 1} \equiv 1/(1-\mu)$. Then $\{[\nu_i, \nu_{i+1}]\}$ partitions $[0, (1-\mu)^{-1}]$ into intervals with length not larger than δ. Therefore the event $\{D_{\inf}(F, \mu) \geq u\}$ can be expressed as

$$\{D_{\inf}(\hat{F}_t, \mu) \geq u\} = \left\{ \exists \nu \in \left[0, \frac{1}{1-\mu}\right], L(\nu; \hat{F}_t, \mu) \geq u \right\}$$

$$= \bigcup_{i=-M_\delta - 1}^{-1} \left\{ \exists \nu \in [\nu_i, \nu_{i+1}], L(\nu; \hat{F}_t, \mu) \geq u \right\} \cup \bigcup_{i=1}^{M_\delta + 1} \left\{ \exists \nu \in [\nu_{i-1}, \nu_i], L(\nu; \hat{F}_t, \mu) \geq u \right\}. \tag{15}$$

Since $\nu_{i+1} - \nu_i \leq \delta$ and $L(\nu; \hat{F}_t, \mu)$ is concave in ν, it holds for $i \leq -1$ that

$$\left\{ \exists \nu \in [\nu_i, \nu_{i+1}], L(\nu; \hat{F}_t, \mu) \geq u \right\} \subset \left\{ L(\nu_{i+1}; \hat{F}_t, \nu) - \delta \min\{0, L'(\nu_{i+1}; \hat{F}_t, \nu)\} \geq u \right\}$$

$$\subset \left\{ L(\nu_{i+1}; \hat{F}_t, \nu) - \delta \min\{0, L'(\nu_0; \hat{F}_t, \nu)\} \geq u \right\}. \tag{16}$$

Similarly it holds for $i \geq 1$ that

$$\left\{ \exists \nu \in [\nu_{i-1}, \nu_i],\, L(\nu; \hat{F}_t, \mu) \geq u \right\} \subset \left\{ L(\nu_{i-1}; \hat{F}_t, \mu) + \delta \max\{0, L'(\nu_0; \hat{F}_t, \mu)\} \geq u \right\}. \quad (17)$$

Here the derivative is written as

$$L'(\nu; \hat{F}_t, \mu) = -\mathrm{E}_{\hat{F}_t}\left[\frac{X - \mu}{1 - (X - \mu)\nu}\right] = \frac{1}{\nu} - \frac{1}{\nu}\mathrm{E}_{\hat{F}_t}\left[\frac{1}{1 - (X - \mu)\nu}\right] .$$

Since $1/(1 - (x - \mu)\nu)$ is positive and increasing in $x \leq 1$, it is bounded as

$$\frac{1}{\nu} \geq L'(\nu; \hat{F}_t, \mu) \geq \frac{1}{\nu} - \frac{1}{\nu}\frac{1}{1 - (1 - \mu)\nu} = -\frac{1 - \mu}{1 - (1 - \mu)\nu} .$$

Thus $L'(\nu_0; \hat{F}_t, \mu) = L'(1/(2(1 - \mu)); \hat{F}_t, \mu)$ is bounded as

$$2(1 - \mu) \geq L'(\nu_0; \hat{F}_t, \mu) \geq -2(1 - \mu) .$$

Combining this inequality with (15), (16) and (17) we obtain

$$P_F[D_{\inf}(\hat{F}_t, \mu) \geq u \cap \hat{\mu}_t \leq \mu]$$
$$\leq \sum_{-M_\delta - 1 \leq i \leq M_\delta + 1,\, i \neq 0} P_F\left[L(\nu_i; \hat{F}_t, \mu) \geq u - 2(1 - \mu)\delta \cap \hat{\mu}_t \leq \mu\right]. \quad (18)$$

Now regard $Y = (Y^{(1)}, Y^{(2)}) \equiv (\log(1 - (X - \mu)\nu_i), X)$ as a random variable on $\mathbb{R}^2$. Define a closed set $C \equiv [u - 2(1 - \mu)\delta, \infty) \times (-\infty, \mu] \subset \mathbb{R}^2$ and its α-blowup $C^\alpha \equiv (u - 2(1 - \mu)\delta - \alpha, \infty) \times (-\infty, \mu + \alpha)$ for $\alpha > 0$. Then the event $\{L(\nu_i; \hat{F}_t, \mu) \geq u - 2(1 - \mu)\delta \cap \hat{\mu}_t \leq \mu\}$ is equivalent to the event that the empirical mean of Y is contained in the closed convex set C. Thus we obtain from Prop. 14 (i) that

$$P_F[L(\nu_i; \hat{F}_t, \mu) \geq u - 2(1 - \mu)\delta \cap \hat{\mu}_t \leq \mu] \leq \exp\left(-t \inf_{y \in C} \Lambda^*_{\mathbb{R}^2}(y)\right) , \quad (19)$$

where $\Lambda^*_{\mathbb{R}^2}(y)$ is defined by (27). Since $C^\alpha \supset C$ is open, the exponential rate is bounded as

$$-\inf_{y \in C} \Lambda^*_{\mathbb{R}^2}(y) \leq -\inf_{y \in C^\alpha} \Lambda^*_{\mathbb{R}^2}(y)$$
$$\leq \liminf_{t \to \infty} \frac{1}{t} \log P_F[L(\nu_i; \hat{F}_t, \mu) > u - 2(1 - \mu)\delta - \alpha \cap \hat{\mu}_t < \mu + \alpha]$$
$$\text{(by Prop. 14 (ii))}$$
$$\leq \limsup_{t \to \infty} \frac{1}{t} \log P_F[D_{\inf}(\hat{F}_t, \mu) \geq u - 2(1 - \mu)\delta - \alpha \cap \hat{\mu}_t \leq \mu + \alpha]$$
$$\leq -\max\{u - 2(1 - \mu)\delta - \alpha, \Lambda^*_{\mathbb{R}}(\mu + \alpha)\} . \quad \text{(by Lemma 13)}$$

Letting $\alpha \downarrow 0$ we obtain

$$-\inf_{y \in C} \Lambda^*_{\mathbb{R}^2}(y) \leq -\max\{u - 2(1 - \mu)\delta, \Lambda^*_{\mathbb{R}}(\mu)\} . \quad (20)$$

Finally we obtain from (18), (19) and (20) that

$$P_F[D_{\inf}(\hat{F}_t, \mu) \geq x \cap \hat{\mu}_t \leq \mu] \leq 2\left(1 + \frac{1}{2(1 - \mu)\delta}\right) \exp\left(-t \max\{u - 2(1 - \mu)\delta, \Lambda^*_{\mathbb{R}}(\mu)\}\right)$$

and we complete the proof by letting $\delta \to \infty$ for $u \leq \Lambda^*_{\mathbb{R}}(\mu)$ and $\delta = 1/(2t(1 - \mu))$ for $u > \Lambda^*_{\mathbb{R}}(\mu)$. ∎

Proof of Theorem 12: Let $u \equiv D_{\inf}(F, \mu) - v$. First we obtain for $\nu^* = \nu^*(F, \mu)$ that

$$P_F[D_{\inf}(\hat{F}_t, \mu) \leq D_{\inf}(F, \mu) - v] = P_F\left[\max_{0 \leq \nu \leq (1 - \mu)^{-1}} \mathrm{E}_{\hat{F}_t}[\log(1 - (X - \mu)\nu)] \leq u\right]$$
$$\leq P_F[\mathrm{E}_{\hat{F}_t}[\log(1 - (X - \mu)\nu^*)] \leq u] .$$

Define random variables $Y \equiv 1 - (X - \mu)\nu^*$ and $Z \equiv \log Y = \log(1 - (X - \mu)\nu^*)$ where X follows the distribution F. Let $\bar{Z}_t$ be the mean of t i.i.d. copies of Z. Then, from Prop. 14 (iii), the above probability is bounded as

$$P_F[D_{\inf}(\hat{F}_t, \mu) \leq D_{\inf}(F, \mu) - v] \leq P_F[\bar{Z}_t \leq u] \leq \mathrm{e}^{-t\Lambda^*_{\mathbb{R}}(u)} , \quad (21)$$

where $\Lambda_{\mathbb{R}}^*(u) = \sup_\lambda \left\{ \lambda u - \log \mathrm{E}_F[\mathrm{e}^{\lambda Z}] \right\} = \sup_\lambda \left\{ \lambda u - \log \mathrm{E}_F[Y^\lambda] \right\}$.

Note that $\mathrm{E}_F[\mathrm{e}^{-1 \cdot Z}] = \mathrm{E}_F[(1-(X-\mu)\nu^*)^{-1}] \le 1$ from Prop. 6 and $\mathrm{E}_F[\mathrm{e}^{1 \cdot Z}] = \mathrm{E}_F[1-(X-\mu)\nu^*] = 1 - (\mathrm{E}(F) - \mu)\nu^*$. Since they are finite, the moment generating function $\mathrm{E}_F[\mathrm{e}^{\lambda Z}] = \mathrm{E}_F[Y^\lambda]$ exists for all $\lambda \in [-1, 1]$ and infinitely differentiable in $\lambda \in (-1, 1)$.

Before evaluating $\Lambda^*(u)$ we bound $\mathrm{E}_F[Y^\lambda]$ for $\lambda \in [-1, 1]$. For $\lambda \in [-1, 0]$, we obtain from $\mathrm{E}_F[Y^{-1}] \le 1$ and the convexity of y^λ in λ that

$$\mathrm{E}_F[Y^\lambda] \le \mathrm{E}_F[(-\lambda)Y^{-1} + (1+\lambda)Y^0] \le -\lambda + (1+\lambda) = 1 \ . \tag{22}$$

Similarly, we obtain for $\lambda \in (0, 1]$ that

$$
\begin{aligned}
\mathrm{E}_F[Y^\lambda] &\le \mathrm{E}_F[(1-\lambda)Y^0 + \lambda Y^1] \\
&= (1-\lambda) + \lambda(1 - (\mathrm{E}(F) - \mu)\nu^*) \\
&= 1 + \lambda(\mu - \mathrm{E}(F))\nu^* \\
&\le 1 + 1 \cdot \frac{\mu - \mathrm{E}(F)}{1-\mu} = \frac{1 - \mathrm{E}(F)}{1-\mu} \ . \qquad \left(\text{by } \mu > \mathrm{E}(F) \text{ and } \nu^* \le \tfrac{1}{1-\mu} \right)
\end{aligned} \tag{23}
$$

Define the objective function in $\Lambda_{\mathbb{R}}^*(u)$ as $R(\lambda) \equiv \lambda u - \log \mathrm{E}_F[Y^\lambda]$. Then, for $\lambda \in [-1, 0]$,

$$R'(\lambda) = u - \frac{\mathrm{E}_F[Y^\lambda \log Y]}{\mathrm{E}_F[Y^\lambda]} \le u - \mathrm{E}_F[Y^\lambda \log Y] \ . \qquad \text{(by (22))} \tag{24}$$

We bound $R(\lambda)$ from below for $\lambda \in [-1/2, 0]$ in the following. For the second term of the right-hand side of (24), it holds for $\lambda \in [-1/2, 0]$ that

$$
\begin{aligned}
\mathrm{E}_F[Y^\lambda \log Y] &\ge \mathrm{E}_F[Y^0 \log Y] - \int_\lambda^0 \max_{\lambda \in [-\frac{1}{2}, 0]} \left\{ \frac{\mathrm{d}\mathrm{E}_F[Y^\lambda \log Y]}{\mathrm{d}\lambda} \right\} \mathrm{d}\lambda \\
&= D_{\inf}(F, \mu) + \lambda \max_{\lambda \in [-\frac{1}{2}, 0]} \mathrm{E}_F[Y^\lambda (\log Y)^2] \ .
\end{aligned} \tag{25}
$$

Note that $(\log y)^2$ is smaller than $y^{-1/2}$ for $y \to +0$ and smaller than y for $y \to \infty$. Therefore there exists $c_0 > 0$ such that $(\log y)^2 \le c_0 y^{-1/2} + y$ for all $y > 0$. In fact, this inequality holds by letting $c_0 \ge 2.163$. Then we obtain from (22) and (23) that

$$\mathrm{E}_F[Y^\lambda (\log Y)^2] \le \mathrm{E}_F[Y^\lambda (c_0 Y^{-1/2} + Y)] \le c_0 + \frac{1 - \mathrm{E}(F)}{1-\mu} \ . \tag{26}$$

Combining (24), (25) and (26) with $R(\lambda) = 0$ we obtain

$$
\begin{aligned}
R'(\lambda) &\le u - D_{\inf}(F, \mu) - \lambda \left(c_0 + \frac{1 - \mathrm{E}(F)}{1-\mu} \right) = -v - \lambda \left(c_0 + \frac{1 - \mathrm{E}(F)}{1-\mu} \right) \ , \\
R(\lambda) &= 0 + \int_0^\lambda R'(\lambda) \mathrm{d}\lambda \ge -\lambda v - \frac{\lambda^2}{2} \left(c_0 + \frac{1 - \mathrm{E}(F)}{1-\mu} \right) \ .
\end{aligned}
$$

Finally,

$$
\Lambda_{\mathbb{R}}^*(u) = \sup_\lambda R(\lambda) \ge \sup_{\lambda \in [-\frac{1}{2}, 0]} R(\lambda) \ge
\begin{cases}
\frac{v^2}{2(c_0 + \frac{1-\mathrm{E}(F)}{1-\mu})} \ , & v \le \frac{1}{2}(c_0 + \frac{1-\mathrm{E}(F)}{1-\mu}) \ , \\
\frac{v}{2} - \frac{1}{8}(c_0 + \frac{1-\mathrm{E}(F)}{1-\mu}) \ , & \text{otherwise,}
\end{cases}
$$

and we obtain the theorem with (21). $\blacksquare$

6 Concluding Remarks

We proved that the theoretical bound only depends on the upper bound of the support in the nonparametric stochastic bandits. We refined the analysis of DMED policy to a non-asymptotic form for all distributions with moment generating functions in this model.

A   Large Deviation Principle and its Application to a Joint Distribution

In this appendix we consider large deviation principle (LDP) for the empirical mean and the distribution based on Dembo and Zeitouni (1998) (DZ, hereafter). We first summarize results on LDP for empirical means of finite dimentional random variables and then we derive LDP for joint distribution of the empirical distribution and the mean in Theorem 15.

Let $\hat{S}_t$ be the empirical mean of i.i.d. random variables $X_1, \cdots, X_t \in \mathcal{X}$ with distribution F, where $\mathcal{X}$ is a general topological vector space. For a distribution on $\mathbb{R}$, we can regard its empirical distribution as the empirical mean of delta measures $\delta_{X_i} \in \mathcal{A} \subset \mathcal{V}$, where $\mathcal{V}$ is the space of all finite measures on $(-\infty, 1]$. We write $\hat{\mu}_t$ and $\hat{F}_t$ instead of $\hat{S}_t$ for empirical means of $X_i \in \mathbb{R}$ and $\delta_{X_i} \in \mathcal{A}$, respectively.

Define the logarithmic moment generating function and its Fenchel-Legendre transform for distribution F by

$$\Lambda_{\mathcal{X}}(\lambda) = \log \int_{\mathcal{X}} e^{\langle \lambda, u \rangle} dF(u) \ ,$$

$$\Lambda_{\mathcal{X}}^*(x) = \sup_{\lambda \in \mathcal{X}^*} \left\{ \langle \lambda, x \rangle - \Lambda(\lambda) \right\} \ ,$$

where $\mathcal{X}^*$ is the space of linear continuous functions on $\mathcal{X}$. Especially, for the case $\mathcal{X} = \mathbb{R}^d$ it is expressed for $\mathcal{X}^* = \mathbb{R}^d$ as

$$\langle \lambda, x \rangle = \sum_i \lambda_i x_i \ , \qquad \lambda, x \in \mathbb{R}^d \ . \tag{27}$$

Similarly, for the case $\mathcal{X} = \mathcal{V}$, it is expressed for $\mathcal{X}^* = C_{\mathrm{b}}(\mathbb{R})$ as

$$\langle \phi, G \rangle = \int \phi(u) dG(u) \ , \qquad \phi \in C_{\mathrm{b}}(\mathbb{R}),\ G \in \mathcal{A} \ ,$$

where $C_{\mathrm{b}}(\mathbb{R})$ is the space of bounded continuous functions on $\mathbb{R}$. Note that it is shown in DZ that in the scope of our paper $\Lambda_{\mathcal{X}}^*(x)$ is always a *rate function*, that is, a lower semicontinuous function with range $[0, \infty]$, although we omit this statement in the following.

Proposition 14 (DZ, Ex. 2.2.38, Theorem 2.2.30 and Lemma 2.2.5) *Let* $\mathcal{X} = \mathbb{R}^d$ *and assume that* $\Lambda_{\mathbb{R}^d}(\lambda)$ *exists around* $\lambda = 0$. *(i) For any convex closed* $C \subset \mathbb{R}^d$

$$\frac{1}{t} \log P_F[\hat{S}_t \in C] \leq - \inf_{x \in C} \Lambda_{\mathbb{R}^d}^*(x) \ .$$

(ii) For any open $A \subset \mathbb{R}^d$

$$\liminf_{t \to \infty} \frac{1}{t} \log P_F[\hat{S}_t \in A] \geq - \inf_{x \in A} \Lambda_{\mathbb{R}^d}^*(x) \ .$$

(iii) For the case $d = 1$, $\Lambda_{\mathbb{R}}^*(x)$ *is decreasing at* $x < \mathrm{E}(F)$ *and increasing at* $x > \mathrm{E}(F)$. *Consequently,*

$$\frac{1}{t} \log P_F[\hat{\mu}_t \leq x] \leq -\Lambda_{\mathbb{R}}^*(x) \ , \qquad \text{if } x < \mathrm{E}(F) \ ,$$

$$\frac{1}{t} \log P_F[\hat{\mu}_t \geq x] \leq -\Lambda_{\mathbb{R}}^*(x) \ , \qquad \text{if } x > \mathrm{E}(F) \ .$$

In well-known Sanov's theorem, LDP for the empirical distribution is considered. On the other hand, in the proof of theorem 11, we have to consider the joint probability that the empirical distribution and the mean deviate from a subset of $\mathcal{A} \times \mathbb{R}$. Theorem 15 below is an extension of Sanov's theorem for this purpose. This theorem is derived from Cramér's theorem in the same way as the derivation of Sanov's theorem.

Recall that we assume that $\mathbb{R}$ is equipped with the standard topology and $\mathcal{A}$ is equipped with the topology induced by Lévy metric $d_{\mathrm{L}}(F, G)$ for $F, G \in \mathcal{A}$. For the space $\mathcal{A} \times \mathbb{R}$ we use the product topology of $\mathcal{A}$ and $\mathbb{R}$, which is equivalent to the topology induced by the metric $\max\{d_{\mathrm{L}}(F, G), |x-y|\}$ for $(F, x),\ (G, y) \in \mathcal{A} \times \mathbb{R}$.

Theorem 15 *Let* F *be arbitrary distribution on* $\mathbb{R}$ *such that the moment generating function exists in some neighborhood of* $\lambda = 0$. *For any closed set* $C \subset \mathcal{A} \times \mathbb{R}$, *it holds that*

$$\limsup_{t \to \infty} \frac{1}{t} \log P_F[(\hat{F}_t, \hat{\mu}_t) \in C] \leq - \inf_{(G, x) \in C} \Lambda_{\mathcal{V} \times \mathbb{R}}^*((G, x)) \ , \tag{28}$$

where

$$\Lambda^*_{\mathcal{V} \times \mathbb{R}}((G, x)) = \sup_{(\phi, \lambda) \in C_{\mathrm{b}}(\mathbb{R}) \times \mathbb{R}} \left\{ \int \phi(u) \mathrm{d}G(u) + \lambda x - \log \int \mathrm{e}^{\phi(u) + \lambda u} \mathrm{d}F(u) \right\} \ .$$

For the actual computation of $\Lambda^*_{\mathcal{V} \times \mathbb{R}}(\cdot)$ the following proposition is useful.

Proposition 16 (DZ, Lemma 6.2.13) *For all $F, G \in \mathcal{A}$,*

$$\sup_{\phi \in C_{\mathrm{b}}(\mathbb{R})} \left\{ \int \phi(u) \mathrm{d}G(u) - \log \int \mathrm{e}^{\phi(u)} \mathrm{d}F(u) \right\} = D(G \| F) \ .$$

For the rest of this section we prove Theorem 15. We start with Cramér's theorem for general Hausdorff topological vector spaces $\mathcal{X}$ and probability measures F on $\mathcal{X}$.

Proposition 17 (DZ, Theorem 6.1.3) *Assume that following (a), (b) hold. (a) $\mathcal{X}$ is locally convex and there exists a closed convex subset $\mathcal{E}$ of $\mathcal{X}$ such that $P_F(\mathcal{E}) = 1$. Further, $\mathcal{E}$ can be made into a Polish space with respect to the topology induced by $\mathcal{E}$. (b) The closed convex hull of each compact $K \subset \mathcal{E}$ is compact. Then it holds for all compact closed set that*

$$\limsup_{t \to \infty} \frac{1}{t} \log P_F[\hat{S}_t \in C] \leq - \inf_{x \in C} \Lambda^*_{\mathcal{X}}(x) \ . \tag{29}$$

The assertion of this proposition is restricted to compact sets and is called *weak* LDP. We can remove this restriction to *full* LDP if the *exponential tightness* is satisfied. The laws of $\hat{S}_t$ are exponentially tight if, for every $\alpha < \infty$, there exists a compact set $K_\alpha \subset \mathcal{X}$ such that

$$\limsup_{t \to \infty} \frac{1}{t} \log P_F[\hat{S}_t \in K_\alpha^c] < -\alpha \ ,$$

where superscript "c" denotes the complement of the set.

Proposition 18 (DZ, Lemma 1.2.18) *If the laws of $\hat{S}_t$ are exponentially tight then (29) holds for all closed set C.*

Proposition 19 (DZ, Lemma 6.2.6 and Discussion after Eq. (2.2.33)) *(i) The laws of the empirical distributions $\hat{F}_t \in \mathcal{A}$ are exponentially tight for all $F \in \mathcal{A}$. (ii) The laws of the empirical means $\hat{\mu}_t \in \mathbb{R}$ are exponentially tight if the moment generating function $\mathrm{E}_F[\mathrm{e}^{\lambda X}]$ exists in some neighborhood of $\lambda = 0$.*

Proof of Theorem 15: First we can obtain (28) for all closed compact $C \subset \mathcal{A} \times \mathbb{R}$ as a direct application of Prop. 17 with $\mathcal{X} := \mathcal{V} \times \mathbb{R}$ with $\mathcal{E} := \mathcal{A} \times \mathbb{R}$ by the following argument.

For the case $\mathcal{X} := \mathcal{V}$ and $\mathcal{E} := \mathcal{A}$, it is shown as Sanov's theorem that the assumption of Prop. 17 is satisfied when $\mathcal{A}$ is equipped with the topology induced by Lévy metric (see DZ, Sect. 6.1). The essential point in the proof of Sanov's theorem is that the local convexity in the assumption is satisfied if a vector space $\mathcal{X}$ is equipped with a topology called *weak topology* (see, e.g., Dunford and Schwartz (1988, Chap. V) for detail of weak topologies). Since the relative topology on $\mathcal{A}$ of the weak topology of $\mathcal{V}$ is equivalent to the topology induced by the Lévy metric, Prop. 17 is applicable for the case of Sanov's theorem. Here note that the weak topology of $\mathcal{V} \times \mathbb{R}$ is equivalent to the product topology of the weak topologies of $\mathcal{V}$ and $\mathbb{R}$. Thus it is shown in a parallel way that the assumption is also satisfied in our case.

In view of Prop. 18, we complete the proof if the exponential tightness of the laws of $(\hat{F}_t, \hat{\mu}_t)$ is proved. From Prop. 19, for every $\alpha < \infty$ there exist compact $A_\alpha \subset \mathcal{A}$ and $B_\alpha \subset \mathbb{R}$ such that

$$\limsup_{t \to \infty} \frac{1}{t} \log P_F[\hat{F}_t \in A_\alpha^c] < -\alpha \ , \qquad \limsup_{t \to \infty} \frac{1}{t} \log P_F[\hat{\mu}_t \in B_\alpha^c] < -\alpha \ . \tag{30}$$

Letting $K_\alpha := A_\alpha \times B_\alpha$ we obtain

$$P_F[(\hat{F}_t, \hat{\mu}_t) \in K_\alpha^c] \leq P_F[\hat{F}_t \in A_\alpha^c] + P_F[\hat{\mu}_t \in B_\alpha^c] \ .$$

Combining this inequality with (30) we see that the laws of $(\hat{F}_t, \hat{\mu}_t)$ are exponentially tight. ∎

B Proof of Theorem 4

Define events A_n, B_n, C_n, D_n for any $\delta > 0$ as

$$A_n \equiv \{\hat{\mu}^*(n) \geq \mu^* - \delta\}$$

$$B_n \equiv \{\hat{\mu}^*(n) \leq \mu' + \delta\} = \bigcap_{k=1}^{K}\{\hat{\mu}_k(n) \leq \mu' + \delta\}$$

$$C_n = \bigcup_{k \notin \mathcal{I}_{\text{opt}}} \{\hat{\mu}^*(n) = \hat{\mu}_k(n) \geq \mu' + \delta\}$$

$$D_n = \bigcup_{k \in \mathcal{I}_{\text{opt}}} \{\hat{\mu}^*(n) = \hat{\mu}_k(n) \leq \mu^* - \delta\} \ .$$

It is easily checked that $\{A_n \cup B_n \cup C_n \cup D_n\}$ is the whole sample space. Let $J_n(i)$ denote the event that arm i is pulled at the n-th round and recall that $J'_n(i)$ is given in Algorithm 1. Then, except for the first $2K$ rounds, the event $J_n(i)$ implies that $J'_{n'}(i)$ occurred for some $K+1 \leq n' < n$. Therefore $T_i(n)$ is bounded as

$$T_i(n)$$

$$= 2 + \sum_{t=2}^{\infty} \mathbb{1}\left[\bigcup_{m=2K}^{n-1} \{T_i(m) = t \cap J_{m+1}(i)\}\right]$$

$$\leq 2 + \sum_{t=2}^{\infty} \mathbb{1}\left[\bigcup_{m=K+1}^{n-1} \{T_i(m) = t \cap J'_m(i)\}\right]$$

$$\leq 2 + \sum_{t=2}^{\infty} \mathbb{1}\left[\bigcup_{m=K+1}^{n-1} \{T_i(m) = t \cap J'_m(i) \cap A_m\}\right] + \sum_{t=2}^{\infty} \mathbb{1}\left[\bigcup_{m=K+1}^{n-1} \{T_i(m) = t \cap J'_m(i) \cap A_m^c\}\right]$$

and we obtain for the last term that

$$\sum_{t=2}^{\infty} \mathbb{1}\left[\bigcup_{m=K+1}^{n-1} \{T_i(m) = t \cap J'_m(i) \cap A_m^c\}\right] \leq \sum_{m=K+1}^{n-1} \mathbb{1}[A_m^c]$$

$$\leq \sum_{m=K+1}^{n-1} \mathbb{1}[B_m] + \sum_{m=K+1}^{n-1} \mathbb{1}[C_m] + \sum_{m=K+1}^{n-1} \mathbb{1}[D_m] \ .$$

In the following Lemmas 20–23 we bound the expectations of these summations and they prove the theorem with $2 - 2e/r < 0$. $\blacksquare$

Lemma 20 *Let $i \notin \mathcal{I}_{\text{opt}}$ be arbitrary. If $\xi_{i,\epsilon,\delta} = \epsilon D_{\inf}(F_i, \mu^*) - \delta/(1 - \mu^*) > 0$ then*

$$\mathrm{E}\left[\sum_{t=2}^{\infty} \mathbb{1}\left[\bigcup_{m=K+1}^{n-1} \{T_i(m) = t \cap J'_m(i) \cap A_m\}\right]\right]$$

$$\leq \frac{\log n}{(1 - \epsilon)(1 - r)D_{\inf}(F_i, \mu^*)} + \frac{1}{1 - e^{-\tilde{\Lambda}^*(\xi_{i,\epsilon,\delta}, \mu_i, \mu^*)}} \ .$$

Lemma 21 *If $\delta < \mu^* - \mu'$ then*

$$\mathrm{E}\left[\sum_{m=K+1}^{n-1} \mathbb{1}[B_m]\right] \leq \min_{k \in \mathcal{I}_{\text{opt}}} \left\{\frac{2(1 + K)}{1 - e^{-\Lambda_k^*(\mu' + \delta)}} + \frac{2e}{r\left(1 - e^{-r\Lambda_k^*(\mu' + \delta)}\right)^2}\right\} - \frac{2e}{r} \ .$$

Lemma 22

$$\mathrm{E}\left[\sum_{m=K+1}^{n-1} \mathbb{1}[C_m]\right] \leq K \sum_{k \notin \mathcal{I}_{\text{opt}}} \frac{1}{1 - e^{-\Lambda_k^*(\mu' + \delta)}} \ .$$

Lemma 23

$$\mathrm{E}\left[\sum_{m=K+1}^{n-1} \mathbb{1}[D_m]\right] \leq K \sum_{k \in \mathcal{I}_{\text{opt}}} \frac{1}{1 - e^{-\Lambda_k^*(\mu^* - \delta)}} \ .$$

Proof of Lemma 20: In the same way as Honda and Takemura (2010, Lemma 15), we obtain

$$\sum_{t=2}^{\infty}\mathbb{1}\left[\bigcup_{m=K+1}^{n-1}\{J_m'(i)\cap T_i(m)=t\cap A_m\}\right]$$

$$\leq \frac{\log n}{(1-\epsilon)(1-r)D_{\inf}(F_i,\mu^*)}$$

$$+\sum_{t=\frac{\log n}{(1-\epsilon)(1-r)D_{\inf}(F_i,\mu^*)}}^{\infty}\mathbb{1}\left[\frac{\log n}{(1-\epsilon)(1-r)D_{\inf}(F_i,\mu^*)}(1-r)D_{\inf}(\hat{F}_{i,t},\mu^*-\delta)\leq \log n\right]$$

$$\leq \frac{\log n}{(1-\epsilon)(1-r)D_{\inf}(F_i,\mu^*)}+\sum_{t=1}^{\infty}\mathbb{1}\left[D_{\inf}(\hat{F}_{i,t},\mu)\leq(1-\epsilon)D_{\inf}(F_i,\mu^*)\right].\tag{31}$$

Note that it holds from Lemma 7 and Theorem 12 that

$$P_{F_i}\left[D_{\inf}(\hat{F}_{i,t},\mu^*-\delta)\leq(1-\epsilon)D_{\inf}(F_i,\mu^*)\right]$$

$$\leq\quad P_{F_i}\left[D_{\inf}(\hat{F}_{i,t},\mu^*)-\frac{\delta}{1-\mu^*}\leq(1-\epsilon)D_{\inf}(F_i,\mu^*)\right]$$

$$\leq\quad P_{F_i}\left[D_{\inf}(\hat{F}_{i,t},\mu^*)\leq D_{\inf}(F_i,\mu^*)-\left(\epsilon D_{\inf}(F_i,\mu^*)-\frac{\delta}{1-\mu^*}\right)\right]$$

$$\leq\quad \mathrm{e}^{-t\tilde{\Lambda}^*(\xi_{i,\epsilon,\delta},\mu_i,\mu^*)}\quad.\tag{32}$$

From (31) and (32), we obtain

$$\mathrm{E}\left[\sum_{t=2}^{\infty}\mathbb{1}\left[\bigcup_{m=K+1}^{n-1}\{T_i(m)=t\cap J_i'(m)\cap A_m\}\right]\right]$$

$$\leq \frac{\log n}{(1-\epsilon)(1-r)D_{\inf}(F_i,\mu^*)}+\sum_{t=1}^{\infty}\mathrm{e}^{-t\tilde{\Lambda}^*(\xi_{i,\epsilon,\delta},\mu_i,\mu^*)}$$

$$= \frac{\log n}{(1-\epsilon)(1-r)D_{\inf}(F_i,\mu^*)}+\frac{1}{1-\mathrm{e}^{-\tilde{\Lambda}^*(\xi_{i,\epsilon,\delta},\mu_i,\mu^*)}}.\qquad\blacksquare$$

Proof of Lemma 21: First we simply bound $\sum_{m=K+1}^{n-1}\mathbb{1}[B_m]$ by

$$\sum_{m=K+1}^{n-1}\mathbb{1}[B_m]\quad\leq\quad\sum_{t=1}^{\infty}\sum_{m=K+1}^{\infty}\mathbb{1}[B_m\cap T_k(m)=t]\quad,\tag{33}$$

where $k\in\mathcal{I}_{\mathrm{opt}}$ is arbitrary. By the same argument as Honda and Takemura (2010, Lemma 16), the event $\{D_{\inf}(\hat{F}_{k,t},\mu'+\delta)\leq u\cap\hat{\mu}_{k,t}\leq\mu'+\delta\}$ implies

$$\sum_{m=K+1}^{\infty}\mathbb{1}[B_m\cap T_k(m)=t]\leq\mathrm{e}^{tu(1-r)}+K\quad.$$

Let $P(u)\equiv P_{F_k}[D_{\inf}(\hat{F}_{k,t},\mu'+\delta)\geq u\cap\hat{\mu}_{k,t}\leq\mu'+\delta]$. When we simply write Λ_k^* for $\Lambda_k^*(\mu'+\delta)$ given in (6), it holds from Theorem 11 that

$$\mathrm{E}\left[\sum_{m=K+1}^{\infty}\mathbb{1}[B_m\cap T_k(m)=t]\right]$$

$$\leq\quad\int_{\infty}^{0}(\mathrm{e}^{tu(1-r)}+K)\mathrm{d}P(u)$$

$$=\quad\left[(\mathrm{e}^{tu(1-r)}+K)P(u)\right]_{\infty}^{0}+t(1-r)\int_{0}^{\infty}\mathrm{e}^{tu(1-r)}P(u)\mathrm{d}u$$

$$\leq\quad 2(1+K)\mathrm{e}^{-t\Lambda_k^*}+2t(1-r)\int_{0}^{\Lambda_k^*}\mathrm{e}^{-t(\Lambda_k^*-(1-r)u)}\mathrm{d}u+2\mathrm{e}t(1-r)(1+t)\int_{\Lambda_k^*}^{\infty}\mathrm{e}^{-tru}\mathrm{d}u$$

$$\leq\quad 2(1+K)\mathrm{e}^{-t\Lambda_k^*}+2\mathrm{e}^{-tr\Lambda_k^*}+\frac{2\mathrm{e}(1-r)}{r}(1+t)\mathrm{e}^{-tr\Lambda_k^*}$$

$$\leq\quad 2(1+K)\mathrm{e}^{-t\Lambda_k^*}+\frac{2\mathrm{e}}{r}(1+t)\mathrm{e}^{-tr\Lambda_k^*}\quad.$$

Taking the summation over t with formula

$$\sum_{t=1}^{\infty}(1+t)\rho^t \;=\; \frac{1}{(1-\rho)^2}-1 \;,$$

we obtain from (33) that

$$\mathrm{E}_F\left[\sum_{m=K+1}^{n-1}\mathbb{1}[B_m]\right] \;\leq\; \frac{2(1+K)}{1-\mathrm{e}^{-\Lambda_k^*}}+\frac{2\mathrm{e}}{r\left(1-\mathrm{e}^{-r\Lambda_k^*}\right)^2}-\frac{2\mathrm{e}}{r} \;. \tag{34}$$

We complete the proof by taking $k\in\mathcal{I}_{\mathrm{opt}}$ such that (34) is minimized. ∎

Proof of Lemmas 22 and 23: We obtain from the definition of C_n that

$$\sum_{m=K+1}^{n-1}\mathbb{1}[C_m] \;\leq\; \sum_{k\notin\mathcal{I}_{\mathrm{opt}}}\sum_{m=K+1}^{\infty}\mathbb{1}[\hat{\mu}^*(m)=\hat{\mu}_k(m)\geq\mu'+\delta]$$

$$\leq\; \sum_{k\notin\mathcal{I}_{\mathrm{opt}}}\sum_{t=1}^{\infty}\sum_{m=K+1}^{\infty}\mathbb{1}[\hat{\mu}^*(m)=\hat{\mu}_{k,t}\geq\mu'+\delta\cap T_k(m)=t] \;. \tag{35}$$

By the same argument as Honda and Takemura (2010, Lemma 17), we have

$$\sum_{m=K+1}^{\infty}\mathbb{1}[\hat{\mu}^*(m)=\hat{\mu}_{k,t}\cap T_k(m)=t]\leq K \;. \tag{36}$$

On the other hand, from Prop. 14 (iii) we have

$$P_{F_k}[\hat{\mu}_{k,t}\geq\mu'+\delta]\leq\mathrm{e}^{-t\Lambda_k^*(\mu'+\delta)} \;, \tag{37}$$

where $\Lambda_k^*(x)$ is given in (6). Finally we obtain from (35)–(37) that

$$\mathrm{E}\left[\sum_{m=K-1}^{n-1}\mathbb{1}[C_m]\right]\leq K\sum_{k\notin\mathcal{I}_{\mathrm{opt}}}\sum_{t=1}^{\infty}P_{F_k}[\hat{\mu}_{k,t}\geq\mu'+\delta]\leq K\sum_{k\notin\mathcal{I}_{\mathrm{opt}}}\frac{1}{1-\mathrm{e}^{-\Lambda_k^*(\mu'+\delta)}}$$

and Lemma 22 is proved. In the same way, we obtain Lemma 23 from

$$\mathrm{E}\left[\sum_{m=K-1}^{n-1}\mathbb{1}[D_m]\right]\leq K\sum_{k\in\mathcal{I}_{\mathrm{opt}}}\sum_{t=1}^{\infty}P_{F_i}[\hat{\mu}_{k,t}\leq\mu^*-\delta]$$

$$\leq K\sum_{k\notin\mathcal{I}_{\mathrm{opt}}}\sum_{t=1}^{\infty}\mathrm{e}^{-t\Lambda_k^*(\mu^*-\delta)}$$

$$\leq K\sum_{k\notin\mathcal{I}_{\mathrm{opt}}}\frac{1}{1-\mathrm{e}^{-\Lambda_k^*(\mu^*-\delta)}} \;. \qquad\blacksquare$$

References

Rajeev Agrawal. The continuum-armed bandit problem. *SIAM Journal on Control and Optimization*, 33(6):1926–1951, 1995.

Jean-Yves Audibert, Rémi Munos, and Csaba Szepesvári. Exploration-exploitation tradeoff using variance estimates in multi-armed bandits. *Theoretical Computer Science*, 410:1876–1902, April 2009.

Peter Auer, Nicolò Cesa-Bianchi, and Paul Fischer. Finite-time analysis of the multiarmed bandit problem. *Machine Learning*, 47:235–256, 2002a.

Peter Auer, Nicolò Cesa-Bianchi, Yoav Freund, and Robert E. Schapire. The nonstochastic multi-armed bandit problem. *SIAM Journal on Computing*, 32(1):48–77, 2002b.

Apostolos N. Burnetas and Michael N. Katehakis. Optimal adaptive policies for sequential allocation problems. *Advances in Applied Mathematics*, 17(2):122–142, 1996.

Amir Dembo and Ofer Zeitouni. *Large deviations techniques and applications*, volume 38 of *Applications of Mathematics*. Springer-Verlag, New York, second edition, 1998.

Nelson Dunford and Jacob T. Schwartz. *Linear Operators, Part I: General Theory*. Wiley Classics Library. John Wiley & Sons, 1988.

Aurelien Garivier and Olivier Cappé. The KL-UCB algorithm for bounded stochastic bandits and beyond. In *Proceedings of COLT 2011*, Budapest, Hungary, 2011.

J. C. Gittins. *Multi-armed bandit allocation indices*. Wiley-Interscience Series in Systems and Optimization. John Wiley & Sons, Chichester, 1989.

Wassily Hoeffding. Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association*, 58(301):13–30, 1963.

Junya Honda and Akimichi Takemura. An asymtotically optimal bandit algorithm for bounded support models. In *Proceedings of COLT 2010*, pages 67–79, Haifa, Israel, 2010.

Junya Honda and Akimichi Takemura. An asymptotically optimal policy for finite support models in the multiarmed bandit problem. *Machine Learning*, 85(3):361–391, 2011.

Tze L. Lai and Herbert Robbins. Asymptotically efficient adaptive allocation rules. *Advances in Applied Mathematics*, 6:4–22, 1985.

Odalric-Ambrym Maillard, Rémi Munos, and Gilles Stoltz. A finite-time analysis of multi-armed bandits problems with kullback-leibler divergences. In *Proceedings of COLT 2011*, Budapest, Hungary, 2011.

Herbert Robbins. Some aspects of the sequential design of experiments. *Bulletin of the American Mathematical Society*, 58(5):527–35, 1952.

Joannès Vermorel and Mehryar Mohri. Multi-armed bandit algorithms and empirical evaluation. In *Proceedings of ECML 2005*, pages 437–448, Porto, Portugal, 2005. Springer.