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Abstract

In circuit simulation, differential-algebraic equations (DAEs) arising from the modified nodal analysis (MNA) have been analyzed from the viewpoint of index, which is a measure of the numerical difficulty of DAEs. While MNA is the most popular method, the hybrid analysis has been shown to have inherent advantage over MNA in terms of index.

For nonlinear time-varying circuits with dependent sources, we give a necessary condition for DAEs arising from the hybrid analysis to have index at most two. Moreover, we show that this condition is also sufficient for linear time-invariant circuits if dependent sources satisfy the genericity assumption. This result remains valid for nonlinear time-varying circuits unless unlucky numerical cancellations occur. The obtained necessary and sufficient condition is simple and reasonable, and is satisfied by commonly-used circuits. Thus, the hybrid analysis results in a DAE with index at most two in most cases, while MNA is known to lead to DAEs with index greater than two in some cases.

1 Introduction

Mathematical modeling and numerical computation are of great importance in circuit simulation. Circuits are described by *differential-algebraic equations* (*DAEs*), which consist of algebraic equations and differential operations. The numerical difficulty of DAEs is measured by the *index*. In general, the higher the index is, the more difficult it is to solve the DAE.

Numerical computation step for DAEs has been actively studied. For example, Gear [5] proposed the backward difference formulae, which were implemented in the DASSL code by Petzold (cf. [2]). Hairer and Wanner [7] implemented an implicit Runge-Kutta method in their RADAU5 code. These methods are applicable to DAEs with low index, and more general methods for high index DAEs have been developed recently (see [11, Chapter 8]).

While many DAE solvers have been implemented, modeling step is very critical to accuracy of numerical solutions, because the difficulty of a DAE increases with its index. In circuit simulation, the most popular method in modeling step is to apply the *modified nodal analysis* (*MNA*). It is shown in [4] that the *tractability index* of a DAE obtained by applying MNA

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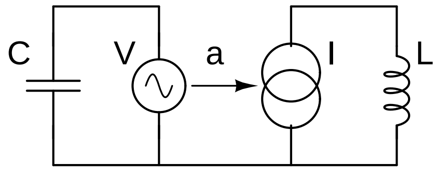

Figure 1: A circuit with a dependent current source.

to nonlinear time-varying circuits that may contain a large class of dependent sources does not exceed two. However, for a circuit with dependent sources which are not included in this class, MNA sometimes leads to a DAE with index greater than two. Figure 1 depicts an example of such a circuit with index three [6], which contains a dependent current source I.

The wide use of MNA is attributed to an automatic setup of model equations. To put it the other way around, however, MNA has no flexibility in modeling step. In contrast, the *hybrid analysis* has an advantage that we can choose a model description which reduces the numerical difficulties. In the hybrid analysis, we select a *partition* of elements and a *reference tree* in the network. This selection determines DAEs, called the *hybrid equations*, to be solved numerically.

Recently, the index of the hybrid equations has been analyzed to make a comparison with MNA theoretically. An algorithm for finding an optimal pair of a partition and a reference tree which minimizes the index of the hybrid equations is proposed in [9] for linear time-invariant circuits that may contain dependent sources. For linear time-invariant RLC circuits, it is proved in [14] that the index of the hybrid equations is at most one. A structural characterization of circuits with index zero is also given in [14]. It is also shown that the index of the hybrid equations never exceeds the index of DAEs arising from MNA. The structural characterizations given in [14] are extended to nonlinear time-varying circuits with dependent sources in [10].

In this paper, for nonlinear time-varying circuits, we give a necessary condition for the hybrid equations to have tractability index at most two. Moreover, we show that this condition is also sufficient for linear time-invariant circuits if dependent sources satisfy the genericity assumption. By combining these results with [10], we obtain criteria for tractability index zero/one/two.

The genericity assumption on dependent sources is motivated by the fact that dependent sources are inherently different from the other elements such as resistors, capacitors, and inductors. In contrast to the latter elements, dependent sources are used to describe an equivalent circuit model of an active device such as a transistor. The constitutive equations of dependent sources are determined by a voltage-amplification factor, a current gain, and so on. We assume that these values are independent parameters in the proof of a sufficient condition. We remark that this genericity assumption makes sense only for linear time-invariant circuits, but our results remain valid for nonlinear time-varying circuits unless unlucky numerical cancellations occur.

The organization of this paper is as follows. In Section 2, we describe nonlinear time-varying circuits and present the hybrid equations. Section 3 is devoted to the definition of the tractability index of DAEs. We analyze the hybrid equations in Section 4. Section 5 gives structural characterizations of the hybrid equations with index at most two. Finally, Section 6 concludes this paper.

2 Hybrid Analysis of Nonlinear Time-Varying Circuits

In this section, we describe nonlinear time-varying circuits composed of resistors, inductors, capacitors, and independent/dependent voltage/current sources.

We denote the vector of currents through all branches of the circuit by $\boldsymbol{i}$, and the vector of voltages across all branches by $\boldsymbol{u}$. Let V, J, C, and L denote the sets of independent voltage sources, independent current sources, capacitors, and inductors, respectively. Dependent voltage/current sources are denoted by S_U and S_I. We define the set of resistors later.

The vectors of currents through V, J, C, L, S_U, and S_I are denoted by $\boldsymbol{i}_V$, $\boldsymbol{i}_J$, $\boldsymbol{i}_C$, $\boldsymbol{i}_L$, $\boldsymbol{i}_U$, and $\boldsymbol{i}_I$. Similarly, the vectors of voltages are denoted by $\boldsymbol{u}_V$, $\boldsymbol{u}_J$, $\boldsymbol{u}_C$, $\boldsymbol{u}_L$, $\boldsymbol{u}_U$, and $\boldsymbol{u}_I$. Independent voltage and current sources simply read as

$$\boldsymbol{u}_V = \boldsymbol{v}_s(t) \quad \text{and} \quad \boldsymbol{i}_J = \boldsymbol{j}_s(t). \tag{1}$$

We assume that the constitutive equations of capacitors and inductors are described by

$$\boldsymbol{i}_C = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q}(\boldsymbol{u}_C, t) \quad \text{and} \quad \boldsymbol{u}_L = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}(\boldsymbol{i}_L, t). \tag{2}$$

Dependent current sources and dependent voltage sources are modeled by

$$\boldsymbol{i}_I = \boldsymbol{j}_I(\boldsymbol{u}_C, \boldsymbol{u}_V, \boldsymbol{i}_L, \boldsymbol{i}_J, t) \quad \text{and} \quad \boldsymbol{u}_U = \boldsymbol{v}_U(\boldsymbol{u}_C, \boldsymbol{u}_V, \boldsymbol{i}_L, \boldsymbol{i}_J, t). \tag{3}$$

Such dependent current/voltage sources appear in many circuits [13, §6.2.6.3].

In order to provide constitutive equations of resistors, we describe the definition of an admissible partition. Let $\Gamma = (W, E)$ be the network graph with vertex set W and edge set E. An edge in Γ corresponds to a branch that contains one element in the circuit. For a consistent model description, Γ contains no cycles consisting only of independent voltage sources and no cutsets consisting only of independent current sources, where a *cutset* is a set of edges whose deletion increases the number of connected components in Γ. We denote the set of edges corresponding to independent voltage sources and independent current sources by E_v and E_j, respectively. We split $E_* := E \setminus (E_v \cup E_j)$ into E_y and E_z, i.e., $E_y \cup E_z = E_*$ and $E_y \cap E_z = \emptyset$. A partition (E_y, E_z) is called an *admissible partition*, if E_y includes all the capacitors and all the dependent current sources, and E_z includes all the inductors and all the dependent voltage sources.

We split $\boldsymbol{i}$ and $\boldsymbol{u}$ into

$$\boldsymbol{i} = (\boldsymbol{i}_V, \boldsymbol{i}_C, \boldsymbol{i}_I, \boldsymbol{i}_Y, \boldsymbol{i}_Z, \boldsymbol{i}_U, \boldsymbol{i}_L, \boldsymbol{i}_J)^\top \quad \text{and} \quad \boldsymbol{u} = (\boldsymbol{u}_V, \boldsymbol{u}_C, \boldsymbol{u}_I, \boldsymbol{u}_Y, \boldsymbol{u}_Z, \boldsymbol{u}_U, \boldsymbol{u}_L, \boldsymbol{u}_J)^\top,$$

where the subscripts Y and Z correspond to the resistors in E_y and E_z. Resistors are modeled by constitutive equations in the form of

$$\boldsymbol{i}_Y = \boldsymbol{g}(\boldsymbol{i}_Z, \boldsymbol{u}_Y, t) \quad \text{and} \quad \boldsymbol{u}_Z = \boldsymbol{h}(\boldsymbol{i}_Z, \boldsymbol{u}_Y, t). \tag{4}$$

A vector $(\boldsymbol{i}, \boldsymbol{u})$ satisfying (1), (3), (4), and Kirchhoff's current/voltage laws at a given time t is called an *operating point* at t [13]. For a matrix A, we denote the (i, j) entry of A by $(A)_{ij}$. For a vector valued function $\boldsymbol{f}$, we denote the ith component of $\boldsymbol{f}$ by $(\boldsymbol{f})_i$. The capacitance matrix C and the inductance matrix L are given by

$$(C)_{ij} = \frac{\partial(\boldsymbol{q})_i}{\partial(\boldsymbol{u}_C)_j} \quad \text{and} \quad (L)_{ij} = \frac{\partial(\boldsymbol{\phi})_i}{\partial(\boldsymbol{i}_L)_j}.$$

The matrices Z, H, G, Y are defined by

$$(Z)_{ij} = \frac{\partial(\boldsymbol{h})_i}{\partial(\boldsymbol{i}_Z)_j}, \quad (H)_{ij} = \frac{\partial(\boldsymbol{h})_i}{\partial(\boldsymbol{u}_Y)_j}, \quad (G)_{ij} = \frac{\partial(\boldsymbol{g})_i}{\partial(\boldsymbol{i}_Z)_j}, \quad (Y)_{ij} = \frac{\partial(\boldsymbol{g})_i}{\partial(\boldsymbol{u}_Y)_j}.$$

A square matrix A is called *positive definite* if $\boldsymbol{x}^\top A \boldsymbol{x} > 0$ for all $\boldsymbol{x} \neq \boldsymbol{0}$. We assume the following conditions throughout this paper.

Assumption 2.1. The capacitance matrix C and the inductance matrix L are positive definite at all operating points.

Assumption 2.2. The hybrid immittance matrix $\begin{pmatrix} Z & H \\ G & Y \end{pmatrix}$ is positive definite at all operating points and $H = -G^\top$.

The positive definiteness in Assumptions 2.1 and 2.2 means that capacitors, inductors, and resistors are strictly locally passive elements at all operating points. For dependent sources, we do not assume the positive definiteness, but impose the genericity assumption in the proof of a sufficient condition for the hybrid equations with index at most two. This assumption is given in Section 5.4.

A *spanning tree* in a connected graph is a maximal set of edges which contains no cycles. We call a spanning tree T of the network graph Γ a *reference tree* if T contains all edges in E_v, no edges in E_j, and as many edges in E_y as possible. Note that a reference tree T may contain some edges in E_z. A reference tree is called *normal* if it contains as many edges as possible in the order corresponding to V, C, S_I, Y, Z, S_U, and L. The cotree of T is denoted by $\overline{T} = E \setminus T$. Normal trees have already been used in [3] for state approaches for linear RLC networks. The results have been extended in [12] for linear circuits containing ideal transformers, nullors, independent/dependent sources, resistors, inductors, capacitors, and, under a topological restriction, gyrators.

The hybrid equations are determined by an admissible partition (E_y, E_z) and a reference tree T, which is not necessarily normal. In this paper, we adopt a normal reference tree. With respect to a normal reference tree T, we further split $\boldsymbol{i}$ and $\boldsymbol{u}$ into

$$\boldsymbol{i} = (\boldsymbol{i}_V, \boldsymbol{i}_C^\tau, \boldsymbol{i}_I^\tau, \boldsymbol{i}_Y^\tau, \boldsymbol{i}_Z^\tau, \boldsymbol{i}_U^\tau, \boldsymbol{i}_L^\tau, \boldsymbol{i}_C^\lambda, \boldsymbol{i}_I^\lambda, \boldsymbol{i}_Y^\lambda, \boldsymbol{i}_Z^\lambda, \boldsymbol{i}_U^\lambda, \boldsymbol{i}_L^\lambda, \boldsymbol{i}_J)^\top$$

and

$$\boldsymbol{u} = (\boldsymbol{u}_V, \boldsymbol{u}_C^\tau, \boldsymbol{u}_I^\tau, \boldsymbol{u}_Y^\tau, \boldsymbol{u}_Z^\tau, \boldsymbol{u}_U^\tau, \boldsymbol{u}_L^\tau, \boldsymbol{u}_C^\lambda, \boldsymbol{u}_I^\lambda, \boldsymbol{u}_Y^\lambda, \boldsymbol{u}_Z^\lambda, \boldsymbol{u}_U^\lambda, \boldsymbol{u}_L^\lambda, \boldsymbol{u}_J)^\top,$$

where the superscripts τ and λ designate the tree T and the cotree $\overline{T}$. With respect to a normal reference tree T, the vector valued function $\boldsymbol{g}$ is also split into $\boldsymbol{g}^\tau$ and $\boldsymbol{g}^\lambda$. This means

$\boldsymbol{i}_Y^\tau = \boldsymbol{g}^\tau(\boldsymbol{i}_Z, \boldsymbol{u}_Y, t)$ and $\boldsymbol{i}_Y^\lambda = \boldsymbol{g}^\lambda(\boldsymbol{i}_Z, \boldsymbol{u}_Y, t)$. Similarly, we split $\boldsymbol{h}$, $\boldsymbol{q}$, $\boldsymbol{\phi}$, $\boldsymbol{j}_I$, and $\boldsymbol{v}_U$. The matrix Y is written in the form of $\begin{pmatrix} Y_\tau^\tau & Y_\lambda^\tau \\ Y_\tau^\lambda & Y_\lambda^\lambda \end{pmatrix}$, where

$$(Y_\tau^\tau)_{ij} = \frac{\partial(\boldsymbol{g}^\tau)_i}{\partial(\boldsymbol{u}_Y^\tau)_j}, \quad (Y_\lambda^\tau)_{ij} = \frac{\partial(\boldsymbol{g}^\tau)_i}{\partial(\boldsymbol{u}_Y^\lambda)_j}, \quad (Y_\tau^\lambda)_{ij} = \frac{\partial(\boldsymbol{g}^\lambda)_i}{\partial(\boldsymbol{u}_Y^\tau)_j}, \quad (Y_\lambda^\lambda)_{ij} = \frac{\partial(\boldsymbol{g}^\lambda)_i}{\partial(\boldsymbol{u}_Y^\lambda)_j}.$$

The matrices C, L, Z, H, G are written in a similar way.

By the definition of a normal reference tree, the *fundamental cutset matrix* F is given by

$$F = \begin{array}{c} \begin{array}{cccccccccccccc} \boldsymbol{i}_V & \boldsymbol{i}_C^\tau & \boldsymbol{i}_I^\tau & \boldsymbol{i}_Y^\tau & \boldsymbol{i}_Z^\tau & \boldsymbol{i}_U^\tau & \boldsymbol{i}_L^\tau & \boldsymbol{i}_C^\lambda & \boldsymbol{i}_I^\lambda & \boldsymbol{i}_Y^\lambda & \boldsymbol{i}_Z^\lambda & \boldsymbol{i}_U^\lambda & \boldsymbol{i}_L^\lambda & \boldsymbol{i}_J \end{array} \\ \begin{pmatrix} I & O & O & O & O & O & O & A_{VC} & A_{VI} & A_{VY} & A_{VZ} & A_{VU} & A_{VL} & A_{VJ} \\ O & I & O & O & O & O & O & A_{CC} & A_{CI} & A_{CY} & A_{CZ} & A_{CU} & A_{CL} & A_{CJ} \\ O & O & I & O & O & O & O & O & A_{II} & A_{IY} & A_{IZ} & A_{IU} & A_{IL} & A_{IJ} \\ O & O & O & I & O & O & O & O & O & A_{YY} & A_{YZ} & A_{YU} & A_{YL} & A_{YJ} \\ O & O & O & O & I & O & O & O & O & O & A_{ZZ} & A_{ZU} & A_{ZL} & A_{ZJ} \\ O & O & O & O & O & I & O & O & O & O & O & A_{UU} & A_{UL} & A_{UJ} \\ O & O & O & O & O & O & I & O & O & O & O & O & A_{LL} & A_{LJ} \end{pmatrix} \end{array}.$$

Then Kirchhoff's current law is written as $F\boldsymbol{i} = \boldsymbol{0}$. Performing the hybrid analysis described in [9], we obtain the *hybrid equations* (or *hybrid equation system*)

$$\begin{aligned} -A_{VL}^\top \boldsymbol{v}_s(t) - A_{CL}^\top \boldsymbol{u}_C^\tau - A_{IL}^\top \boldsymbol{u}_I^\tau - A_{YL}^\top \boldsymbol{u}_Y^\tau - A_{ZL}^\top \boldsymbol{h}^\tau - A_{UL}^\top \boldsymbol{v}_U^\tau - A_{LL}^\top \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}^\tau + \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}^\lambda &= \boldsymbol{0}, \\ -A_{VU}^\top \boldsymbol{v}_s(t) - A_{CU}^\top \boldsymbol{u}_C^\tau - A_{IU}^\top \boldsymbol{u}_I^\tau - A_{YU}^\top \boldsymbol{u}_Y^\tau - A_{ZU}^\top \boldsymbol{h}^\tau - A_{UU}^\top \boldsymbol{v}_U^\tau + \boldsymbol{v}_U^\lambda &= \boldsymbol{0}, \\ -A_{VZ}^\top \boldsymbol{v}_s(t) - A_{CZ}^\top \boldsymbol{u}_C^\tau - A_{IZ}^\top \boldsymbol{u}_I^\tau - A_{YZ}^\top \boldsymbol{u}_Y^\tau - A_{ZZ}^\top \boldsymbol{h}^\tau + \boldsymbol{h}^\lambda &= \boldsymbol{0}, \\ \boldsymbol{g}^\tau + A_{YY}\boldsymbol{g}^\lambda + A_{YZ}\boldsymbol{i}_Z^\lambda + A_{YU}\boldsymbol{i}_U^\lambda + A_{YL}\boldsymbol{i}_L^\lambda + A_{YJ}\boldsymbol{j}_s(t) &= \boldsymbol{0}, \\ \boldsymbol{j}_I^\tau + A_{II}\boldsymbol{j}_I^\lambda + A_{IY}\boldsymbol{g}^\lambda + A_{IZ}\boldsymbol{i}_Z^\lambda + A_{IU}\boldsymbol{i}_U^\lambda + A_{IL}\boldsymbol{i}_L^\lambda + A_{IJ}\boldsymbol{j}_s(t) &= \boldsymbol{0}, \\ \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q}^\tau + A_{CC}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q}^\lambda + A_{CI}\boldsymbol{j}_I^\lambda + A_{CY}\boldsymbol{g}^\lambda + A_{CZ}\boldsymbol{i}_Z^\lambda + A_{CU}\boldsymbol{i}_U^\lambda + A_{CL}\boldsymbol{i}_L^\lambda + A_{CJ}\boldsymbol{j}_s(t) &= \boldsymbol{0}. \end{aligned}$$

Here, $\boldsymbol{q}^\tau$, $\boldsymbol{q}^\lambda$, $\boldsymbol{g}^\tau$, $\boldsymbol{g}^\lambda$, $\boldsymbol{h}^\tau$, $\boldsymbol{h}^\lambda$, $\boldsymbol{\phi}^\tau$, $\boldsymbol{\phi}^\lambda$, $\boldsymbol{j}_I^\tau$, $\boldsymbol{j}_I^\lambda$, $\boldsymbol{v}_U^\tau$, $\boldsymbol{v}_U^\lambda$ are given by

$$\begin{aligned} \boldsymbol{q}^* &= \boldsymbol{q}^*(\boldsymbol{u}_C^\tau, A_{VC}^\top \boldsymbol{v}_s(t) + A_{CC}^\top \boldsymbol{u}_C^\tau, t), \\ \boldsymbol{g}^* &= \boldsymbol{g}^*(\boldsymbol{\alpha}, \boldsymbol{i}_Z^\lambda, \boldsymbol{u}_Y^\tau, \boldsymbol{\beta}, t), \\ \boldsymbol{h}^* &= \boldsymbol{h}^*(\boldsymbol{\alpha}, \boldsymbol{i}_Z^\lambda, \boldsymbol{u}_Y^\tau, \boldsymbol{\beta}, t), \\ \boldsymbol{\phi}^* &= \boldsymbol{\phi}^*(-A_{LL}\boldsymbol{i}_L^\lambda - A_{LJ}\boldsymbol{j}_s(t), \boldsymbol{i}_L^\lambda, t), \\ \boldsymbol{j}_I^* &= \boldsymbol{j}_I^*(\boldsymbol{u}_C^\tau, A_{VC}^\top \boldsymbol{v}_s(t) + A_{CC}^\top \boldsymbol{u}_C^\tau, \boldsymbol{u}_V, -A_{LL}\boldsymbol{i}_L^\lambda - A_{LJ}\boldsymbol{j}_s(t), \boldsymbol{i}_L^\lambda, \boldsymbol{i}_J, t), \\ \boldsymbol{v}_U^* &= \boldsymbol{v}_U^*(\boldsymbol{u}_C^\tau, A_{VC}^\top \boldsymbol{v}_s(t) + A_{CC}^\top \boldsymbol{u}_C^\tau, \boldsymbol{u}_V, -A_{LL}\boldsymbol{i}_L^\lambda - A_{LJ}\boldsymbol{j}_s(t), \boldsymbol{i}_L^\lambda, \boldsymbol{i}_J, t), \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{\alpha} &= -A_{ZZ}\boldsymbol{i}_Z^\lambda - A_{ZU}\boldsymbol{i}_U^\lambda - A_{ZL}\boldsymbol{i}_L^\lambda - A_{ZJ}\boldsymbol{j}_s(t), \\ \boldsymbol{\beta} &= A_{VY}^\top \boldsymbol{v}_s(t) + A_{CY}^\top \boldsymbol{u}_C^\tau + A_{IY}^\top \boldsymbol{u}_I^\tau + A_{YY}^\top \boldsymbol{u}_Y^\tau, \end{aligned}$$

and $*$ represents τ or λ. The idea of its derivation is to use all constitutive equations so that Kirchhoff's current/voltage laws provide a system that depends only on $\boldsymbol{u}_C^\tau$, $\boldsymbol{u}_I^\tau$, $\boldsymbol{u}_Y^\tau$ and $\boldsymbol{i}_Z^\lambda$, $\boldsymbol{i}_U^\lambda$, $\boldsymbol{i}_L^\lambda$.

3 DAEs with Properly Stated Leading Term

In this section, we briefly explain DAEs with properly stated leading term and the tractability index. Consider a DAE in the form of

$$A(\boldsymbol{x}(t), t)\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{d}(\boldsymbol{x}(t), t) + \boldsymbol{b}(\boldsymbol{x}(t), t) = \mathbf{0} \tag{5}$$

for $\boldsymbol{x} \in \mathcal{D} \subseteq \mathbb{R}^m$ and $t \in \mathcal{I} \subseteq \mathbb{R}$. Let $A(\boldsymbol{x}(t), t)$ be an $m \times n$ matrix. We define

$$D(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{d}(\boldsymbol{x}, t)}{\partial \boldsymbol{x}}, \quad B(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{b}(\boldsymbol{x}, t)}{\partial \boldsymbol{x}}, \quad \text{and} \quad M_0(\boldsymbol{x}, t) = A(\boldsymbol{x}, t)D(\boldsymbol{x}, t).$$

A matrix $Q(\boldsymbol{x}, t)$ satisfying $Q(\boldsymbol{x}, t)^2 = Q(\boldsymbol{x}, t)$ is called a *projector*. A projector $Q(\boldsymbol{x}, t)$ is called a *projector onto* a subspace Π if $\operatorname{Im} Q(\boldsymbol{x}, t) = \Pi$.

Definition 3.1 ([8, Definition 26, Lemma A.1])**.** The equation (5) is said to be a DAE with properly stated leading term if the size of $D(\boldsymbol{x}, t)$ is $n \times m$, the three conditions

$$\operatorname{Im} M_0(\boldsymbol{x}, t) = \operatorname{Im} A(\boldsymbol{x}, t),\ \operatorname{Ker} M_0(\boldsymbol{x}, t) = \operatorname{Ker} D(\boldsymbol{x}, t),\ \operatorname{Ker} A(\boldsymbol{x}, t) \cap \operatorname{Im} D(\boldsymbol{x}, t) = \{\mathbf{0}\}$$

hold for all $\boldsymbol{x} \in \mathcal{D}$ and $t \in \mathcal{I}$, and there is an $n \times n$ projector function $P(t)$ continuously differentiable with respect to t such that $\operatorname{Ker} P(t) = \operatorname{Ker} A(\boldsymbol{x}, t)$, $\operatorname{Im} P(t) = \operatorname{Im} D(\boldsymbol{x}, t)$, and $\boldsymbol{d}(\boldsymbol{x}, t) = P(t)\boldsymbol{d}(\boldsymbol{x}, t)$ for all $\boldsymbol{x} \in \mathcal{D}$ and $t \in \mathcal{I}$.

We focus on a further special form of a DAE:

$$A\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{d}(\boldsymbol{x}(t), t) + \boldsymbol{b}(\boldsymbol{x}(t), t) = \mathbf{0}, \tag{6}$$

where A is invariant of $\boldsymbol{x}(t)$ and t. Most DAEs which appear in circuit simulation are in the form of (6). DAEs with index at most two are characterized as follows.

Proposition 3.2 ([15, Definition A.14, Remark A.18])**.** The DAE (6) has *tractability index at most two* if and only if there exist continuous projectors $Q_i(\boldsymbol{x}, t)$ onto $\operatorname{Ker} M_i(\boldsymbol{x}, t)$ for $i = 0, 1$ such that

$$M_2(\boldsymbol{x}, t) := M_1(\boldsymbol{x}, t) + B(\boldsymbol{x}, t)(I - Q_0(\boldsymbol{x}, t))Q_1(\boldsymbol{x}, t) \tag{7}$$

is nonsingular for all $\boldsymbol{x} \in \mathcal{D}$ and $t \in \mathcal{I}$, where

$$M_1(\boldsymbol{x}, t) = M_0(\boldsymbol{x}, t) + B(\boldsymbol{x}, t)Q_0(\boldsymbol{x}, t).$$

This characterization makes it easier to analyze a DAE with index at most two. The reader is referred to [15] for the precise definition of the tractability index.

4 Hybrid Equations with Properly Stated Leading Term

In this section, we rewrite the hybrid equation system as a DAE with properly stated leading term. A reflexive generalized inverse [1] of a matrix A is a matrix A^- which satisfies $AA^-A = A$ and $A^-AA^- = A^-$. We now define

$$A = \begin{pmatrix} O & -A_{LL}^\top & I & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & I & A_{CC} & O \end{pmatrix}, \quad \boldsymbol{x}(t) = \begin{pmatrix} \boldsymbol{i}_L^\lambda \\ \boldsymbol{i}_U^\lambda \\ \boldsymbol{i}_Z^\lambda \\ \boldsymbol{u}_Y^\tau \\ \boldsymbol{u}_I^\tau \\ \boldsymbol{u}_C^\tau \end{pmatrix}, \quad \boldsymbol{d}(\boldsymbol{x}, t) = A^-A \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{\phi}^\tau \\ \boldsymbol{\phi}^\lambda \\ \boldsymbol{q}^\tau \\ \boldsymbol{q}^\lambda \\ \boldsymbol{0} \end{pmatrix},$$

and

$$\boldsymbol{b}(\boldsymbol{x}, t) = \begin{pmatrix} -A_{VL}^\top \boldsymbol{v}_s(t) - A_{CL}^\top \boldsymbol{u}_C^\tau - A_{IL}^\top \boldsymbol{u}_I^\tau - A_{YL}^\top \boldsymbol{u}_Y^\tau - A_{ZL}^\top \boldsymbol{h}^\tau - A_{UL}^\top \boldsymbol{v}_U^\tau \\ -A_{VU}^\top \boldsymbol{v}_s(t) - A_{CU}^\top \boldsymbol{u}_C^\tau - A_{IU}^\top \boldsymbol{u}_I^\tau - A_{YU}^\top \boldsymbol{u}_Y^\tau - A_{ZU}^\top \boldsymbol{h}^\tau - A_{UU}^\top \boldsymbol{v}_U^\tau + \boldsymbol{v}_U^\lambda \\ -A_{VZ}^\top \boldsymbol{v}_s(t) - A_{CZ}^\top \boldsymbol{u}_C^\tau - A_{IZ}^\top \boldsymbol{u}_I^\tau - A_{YZ}^\top \boldsymbol{u}_Y^\tau - A_{ZZ}^\top \boldsymbol{h}^\tau + \boldsymbol{h}^\lambda \\ \boldsymbol{g}^\tau + A_{YY} \boldsymbol{g}^\lambda + A_{YZ} \boldsymbol{i}_Z^\lambda + A_{YU} \boldsymbol{i}_U^\lambda + A_{YL} \boldsymbol{i}_L^\lambda + A_{YJ} \boldsymbol{j}_s(t) \\ \boldsymbol{j}_I^\tau + A_{II} \boldsymbol{j}_I^\lambda + A_{IY} \boldsymbol{g}^\lambda + A_{IZ} \boldsymbol{i}_Z^\lambda + A_{IU} \boldsymbol{i}_U^\lambda + A_{IL} \boldsymbol{i}_L^\lambda + A_{IJ} \boldsymbol{j}_s(t) \\ A_{CI} \boldsymbol{j}_I^\lambda + A_{CY} \boldsymbol{g}^\lambda + A_{CZ} \boldsymbol{i}_Z^\lambda + A_{CU} \boldsymbol{i}_U^\lambda + A_{CL} \boldsymbol{i}_L^\lambda + A_{CJ} \boldsymbol{j}_s(t) \end{pmatrix}.$$

By $A = AA^-A$, this gives the hybrid equation system in the form of (6), where $\mathcal{D}$ denotes the set of $\boldsymbol{x}(t)$ such that $(\boldsymbol{i}, \boldsymbol{u})$ is an operating point at t.

The matrix $M_0(\boldsymbol{x}, t)$ is given by

$$M_0(\boldsymbol{x}, t) = AD(\boldsymbol{x}, t) = \begin{pmatrix} M_L(\boldsymbol{x}, t) & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & M_C(\boldsymbol{x}, t) \end{pmatrix},$$

where

$$M_L(\boldsymbol{x}, t) = \begin{pmatrix} -A_{LL}^\top & I \end{pmatrix} \begin{pmatrix} L_\tau^\tau & L_\lambda^\tau \\ L_\tau^\lambda & L_\lambda^\lambda \end{pmatrix} \begin{pmatrix} -A_{LL} \\ I \end{pmatrix},$$

$$M_C(\boldsymbol{x}, t) = \begin{pmatrix} I & A_{CC} \end{pmatrix} \begin{pmatrix} C_\tau^\tau & C_\lambda^\tau \\ C_\tau^\lambda & C_\lambda^\lambda \end{pmatrix} \begin{pmatrix} I \\ A_{CC}^\top \end{pmatrix}.$$

Lemma 4.1 ([10, Lemma 4.2])**.** Under Assumption 2.1, $M_L(\boldsymbol{x}, t)$ and $M_C(\boldsymbol{x}, t)$ are positive definite.

We now have the following proposition.

Proposition 4.2 ([10, Proposition 4.6])**.** Under Assumption 2.1, the hybrid equation system in the form of (6) is a DAE with properly stated leading term.

We conclude this section by summarizing necessary and sufficient conditions for the hybrid equations with index zero and at most one, which are given in [10]. Let us describe the *Resistor-Acyclic condition* below.

[Resistor-Acyclic condition]

- Each resistor in Y and each dependent current source in S_I belong to a cycle consisting of independent voltage sources, capacitors, and itself.
- Each resistor in Z and each dependent voltage source in S_U belong to a cutset consisting of inductors, independent current sources, and itself.

A necessary and sufficient condition for index zero is as follows.

Theorem 4.3 ([10, Theorem 5.2])**.** Under Assumption 2.1, the tractability index of the hybrid equations is zero if and only if the admissible partition (E_y, E_z) satisfies the Resistor-Acyclic condition.

We define a *CVU-loop* as a cycle consisting of capacitors, independent voltage sources, and/or dependent voltage sources, and an *LJI-cutset* as a cutset consisting of inductors, independent current sources, and/or dependent current sources. In order to describe a necessary and sufficient condition for index at most one, we further assume the following condition for resistors.

Assumption 4.4. The principal submatrices Z and Y of the hybrid immittance matrix $\begin{pmatrix} Z & H \\ G & Y \end{pmatrix}$ are symmetric.

Theorem 4.5 ([10, Corollary 5.7])**.** Under Assumptions 2.1, 2.2, and 4.4, the tractability index of the hybrid equations is at most one if and only if the network graph Γ contains neither CVU-loops with at least one dependent voltage source nor LJI-cutsets with at least one dependent current source.

5 Criteria for index at most two

This section gives a necessary and sufficient condition for the hybrid equations to have index at most two. In Section 5.1, we give projectors $Q_0(\boldsymbol{x}, t), Q_1(\boldsymbol{x}, t)$ and compute $M_2(\boldsymbol{x}, t)$. After giving some lemmas in Section 5.2, we present a necessary condition for nonlinear time-varying circuits in Section 5.3. In Section 5.4, we prove that the necessary condition is also sufficient for linear time-invariant circuits if dependent sources satisfy the genericity assumption.

5.1 Computation of $M_2(\boldsymbol{x}, t)$

In this section, we define projectors Q_0 and $Q_1(\boldsymbol{x}, t)$ and compute $M_2(\boldsymbol{x}, t)$. The matrix Q_0 given by

$$Q_0 = \begin{pmatrix} O & O & O & O & O & O \\ O & I & O & O & O & O \\ O & O & I & O & O & O \\ O & O & O & I & O & O \\ O & O & O & O & I & O \\ O & O & O & O & O & O \end{pmatrix}$$

is a projector onto $\operatorname{Ker} M_0(\boldsymbol{x}, t)$. Let us define

$$A_Z = \begin{pmatrix} -A_{ZU}^\top & O \\ -A_{ZZ}^\top & I \end{pmatrix}, \quad A_Y = \begin{pmatrix} I & A_{YY} \\ O & A_{IY} \end{pmatrix}, \quad N = \begin{pmatrix} A_{YU} & A_{YZ} \\ A_{IU} & A_{IZ} \end{pmatrix}.$$

The computation of $M_1(\boldsymbol{x}, t)$ gives

$$M_1(\boldsymbol{x}, t) = \begin{pmatrix} O & O & O & O & O & O \\ O & O & O & -N^\top & O \\ O & N & O & O & O \\ O & O & O & O & O & O \end{pmatrix} + \begin{pmatrix} O & O & O & O & O & O \\ O & A_Z Z A_Z^\top & A_Z H A_Y^\top & O \\ O & A_Y G A_Z^\top & A_Y Y A_Y^\top & O \\ O & O & O & O & O & O \end{pmatrix}$$
$$+ \begin{pmatrix} M_L(\boldsymbol{x}, t) & A_{ZL}^\top Z_\tau^\tau A_{ZU} & * & * & -A_{IL}^\top - A_{ZL}^\top H_\lambda^\tau A_{IY}^\top & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & -A_{CY} G_\tau^\lambda A_{ZU} + A_{CU} & * & * & A_{CY} Y_\lambda^\lambda A_{IY}^\top & M_C(\boldsymbol{x}, t) \end{pmatrix}.$$

Let Q_U and Q_I be projectors onto $\operatorname{Ker} \begin{pmatrix} A_{IU} \\ A_{YU} \\ A_{ZU} \end{pmatrix}$ and $\operatorname{Ker} \begin{pmatrix} A_{IY}^\top \\ A_{IZ}^\top \\ A_{IU}^\top \end{pmatrix}$, respectively. We define $Q_1(\boldsymbol{x}, t)$ by

$$Q_1(\boldsymbol{x}, t) = \begin{pmatrix} O & O & O & O & R_{LI}(\boldsymbol{x}, t) & O \\ O & Q_U & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & Q_I & O \\ O & R_{CU}(\boldsymbol{x}, t) & O & O & O & O \end{pmatrix},$$

where

$$R_{CU}(\boldsymbol{x}, t) = -M_C(\boldsymbol{x}, t)^{-1} A_{CU} Q_U \quad \text{and} \quad R_{LI}(\boldsymbol{x}, t) = M_L(\boldsymbol{x}, t)^{-1} A_{IL}^\top Q_I.$$

Then $Q_1(\boldsymbol{x}, t)$ is a projector onto $\operatorname{Ker} M_1(\boldsymbol{x}, t)$ under Assumptions 2.1 and 2.2, which is shown in Appendix A.

We now compute $M_2(\boldsymbol{x}, t)$ in the following. Let us define

$$A_U = \begin{pmatrix} -A_{UU} \\ I \end{pmatrix}, \quad A_I = \begin{pmatrix} I \\ A_{II}^\top \end{pmatrix}, \quad A_L = \begin{pmatrix} -A_{LL} \\ I \end{pmatrix}, \quad A_C = \begin{pmatrix} I \\ A_{CC}^\top \end{pmatrix}.$$

We also define the Jacobian matrices V, $\bar{V}$, J, $\bar{J}$ for dependent sources by

$$(V)_{ij} = \frac{\partial(\boldsymbol{v}_U)_i}{\partial(\boldsymbol{u}_C)_j}, \quad (\bar{V})_{ij} = \frac{\partial(\boldsymbol{v}_U)_i}{\partial(\boldsymbol{i}_L)_j}, \quad (J)_{ij} = \frac{\partial(\boldsymbol{j}_I)_i}{\partial(\boldsymbol{i}_L)_j}, \quad (\bar{J})_{ij} = \frac{\partial(\boldsymbol{j}_I)_i}{\partial(\boldsymbol{u}_C)_j}.$$

Then $M_2(\boldsymbol{x}, t)$ is given by

$$\begin{pmatrix} M_L(\boldsymbol{x},t) & * & O \\ O & \tilde{M}_2(\boldsymbol{x},t) & O \\ O & * & M_C(\boldsymbol{x},t) \end{pmatrix},$$

where

$$\begin{aligned}
\tilde{M}_2(\boldsymbol{x},t) = & \begin{pmatrix} O & -N^\top \\ N & O \end{pmatrix} + \begin{pmatrix} A_Z & O \\ O & A_Y \end{pmatrix} \begin{pmatrix} Z & H \\ G & Y \end{pmatrix} \begin{pmatrix} A_Z^\top & O \\ O & A_Y^\top \end{pmatrix} \\
& + \begin{pmatrix} A_Z & O \\ O & A_Y \end{pmatrix} \begin{pmatrix} Z & H \\ G & Y \end{pmatrix} \begin{pmatrix} O & O & O & -A_{ZL}R_{LI}(\boldsymbol{x},t) \\ O & O & O & O \\ O & O & O & O \\ A_{CY}^\top R_{CU}(\boldsymbol{x},t) & O & O & O \end{pmatrix} \\
& + \begin{pmatrix} -A_{CU}^\top R_{CU}(\boldsymbol{x},t) & O & O & O \\ -A_{CZ}^\top R_{CU}(\boldsymbol{x},t) & O & O & O \\ O & O & O & A_{YL}R_{LI}(\boldsymbol{x},t) \\ O & O & O & A_{IL}R_{LI}(\boldsymbol{x},t) \end{pmatrix} \\
& + \begin{pmatrix} A_U^\top & O \\ O & O \\ O & O \\ O & A_I^\top \end{pmatrix} \begin{pmatrix} V & \bar{V} \\ \bar{J} & J \end{pmatrix} \begin{pmatrix} A_C R_{CU}(\boldsymbol{x},t) & O & O & O \\ O & O & O & A_L R_{LI}(\boldsymbol{x},t) \end{pmatrix}.
\end{aligned}$$

By Lemma 4.1, the nonsingularity of $M_2(\boldsymbol{x}, t)$ is equivalent to that of $\tilde{M}_2(\boldsymbol{x}, t)$.

5.2 Preliminaries

We hereafter denote $\begin{pmatrix} \boldsymbol{w}_U \\ \boldsymbol{w}_Z \\ \boldsymbol{w}_Y \\ \boldsymbol{w}_I \end{pmatrix}$ by $\boldsymbol{w}$. Let us define

$$\Theta = \{\boldsymbol{w} \mid R_{CU}(\boldsymbol{x},t)\boldsymbol{w}_U = \boldsymbol{0}, R_{LI}(\boldsymbol{x},t)\boldsymbol{w}_I = \boldsymbol{0}, \boldsymbol{w}_U \in \operatorname{Im} Q_U, \boldsymbol{w}_I \in \operatorname{Im} Q_I, \boldsymbol{w}_Z = \boldsymbol{0}, \boldsymbol{w}_Y = \boldsymbol{0}\}.$$

We use

$$F_U = \begin{pmatrix} A_{CU} \\ A_{IU} \\ A_{YU} \\ A_{ZU} \end{pmatrix} \quad \text{and} \quad F_I = \begin{pmatrix} A_{IY}^\top \\ A_{IZ}^\top \\ A_{IU}^\top \\ A_{IL}^\top \end{pmatrix}$$

for convenience. Then Θ is rewritten as follows.

Lemma 5.1. Under Assumption 2.1, $\Theta = \{\boldsymbol{w} \mid \boldsymbol{w}_U \in \operatorname{Ker} F_U, \boldsymbol{w}_I \in \operatorname{Ker} F_I, \boldsymbol{w}_Z = \boldsymbol{0}, \boldsymbol{w}_Y = \boldsymbol{0}\}$ holds.

Proof. Let $\boldsymbol{w}_U$ satisfy $R_{CU}(\boldsymbol{x},t)\boldsymbol{w}_U = \mathbf{0}$ and $\boldsymbol{w}_U \in \operatorname{Im} Q_U$. By $\boldsymbol{w}_U \in \operatorname{Im} Q_U$, there exist a vector $\boldsymbol{x}$ such that $\boldsymbol{w}_U = Q_U\boldsymbol{x}$. Hence we have $R_{CU}(\boldsymbol{x},t)Q_U\boldsymbol{x} = \mathbf{0}$, which is equivalent to $A_{CU}Q_U^2\boldsymbol{x} = \mathbf{0}$ by Lemma 4.1. Since Q_U is a projector, $A_{CU}\boldsymbol{w}_U = \mathbf{0}$ holds. Hence we have $\boldsymbol{w}_U \in \operatorname{Ker} A_{CU}$, which implies $\boldsymbol{w}_U \in \operatorname{Ker} F_U$ because $\boldsymbol{w}_U \in \operatorname{Im} Q_U$.

Next, let $\boldsymbol{w}_U \in \operatorname{Ker} F_U$. Then we have $\boldsymbol{w}_U \in \operatorname{Im} Q_U$ and $R_{CU}(\boldsymbol{x},t)\boldsymbol{w}_U = \mathbf{0}$. Thus,

$$\boldsymbol{w}_U \in \operatorname{Ker} F_U \quad \text{is equivalent to} \quad R_{CU}(\boldsymbol{x},t)\boldsymbol{w}_U = \mathbf{0}, \boldsymbol{w}_U \in \operatorname{Im} Q_U.$$

We can prove a similar statement for $\boldsymbol{w}_I$, which completes the proof. □

Let us define a *VU-loop* as a cycle consisting of independent voltage sources and/or dependent voltage sources, and a *JI-cutset* as a cutset consisting of independent current sources, and/or dependent current sources. An equivalent condition for $\Theta = \{\mathbf{0}\}$ is given by the following lemma.

Lemma 5.2. Under Assumption 2.1, $\Theta = \{\mathbf{0}\}$ holds if and only if the network graph Γ contains neither VU-loops nor JI-cutsets.

Proof. By Lemma 5.1, $\Theta = \{\mathbf{0}\}$ is equivalent to $\operatorname{Ker} F_U = \{\mathbf{0}\}$ and $\operatorname{Ker} F_I = \{\mathbf{0}\}$. The former condition holds if and only if F_U is of full column rank, which is equivalent to the condition that

$$\begin{pmatrix} I & O & A_{VU} \\ O & O & A_{CU} \\ O & O & A_{IU} \\ O & O & A_{YU} \\ O & O & A_{ZU} \\ O & I & A_{UU} \end{pmatrix}$$

is of full column rank. This is a submatrix of F with the column set corresponding to $V \cup S_U$, and hence it is of full column rank if and only if Γ contains no cycles that consist of independent/dependent voltage sources. Similarly, $\operatorname{Ker} F_I = \{\mathbf{0}\}$ is equivalent to Γ contains no cutsets that consist of independent/dependent current sources. □

5.3 A necessary condition

In order to give a necessary condition for the hybrid equations to have index at most two, we provide the following lemma.

Lemma 5.3. It holds that $\operatorname{Ker} \tilde{M}_2(\boldsymbol{x},t) \supseteq \Theta$.

Proof. Let $\boldsymbol{w} \in \Theta$. Since $R_{CU}(\boldsymbol{x},t)\boldsymbol{w}_U = \mathbf{0}$, $R_{LI}(\boldsymbol{x},t)\boldsymbol{w}_I = \mathbf{0}$, $\boldsymbol{w}_Z = \mathbf{0}$, and $\boldsymbol{w}_Y = \mathbf{0}$, we have

$$\begin{aligned}
\tilde{M}_2(\boldsymbol{x},t)\boldsymbol{w} &= \begin{pmatrix} O & -N^\top \\ N & O \end{pmatrix}\boldsymbol{w} + \begin{pmatrix} A_Z & O \\ O & A_Y \end{pmatrix}\begin{pmatrix} Z & H \\ G & Y \end{pmatrix}\begin{pmatrix} A_Z^\top & O \\ O & A_Y^\top \end{pmatrix}\boldsymbol{w} \\
&= \begin{pmatrix} -A_{IU}^\top\boldsymbol{w}_I \\ -A_{IZ}^\top\boldsymbol{w}_I \\ A_{YU}\boldsymbol{w}_U \\ A_{IU}\boldsymbol{w}_U \end{pmatrix} + \begin{pmatrix} A_Z & O \\ O & A_Y \end{pmatrix}\begin{pmatrix} Z & H \\ G & Y \end{pmatrix}\begin{pmatrix} -A_{ZU}\boldsymbol{w}_U \\ \mathbf{0} \\ \mathbf{0} \\ A_{IY}^\top\boldsymbol{w}_I \end{pmatrix} \\
&= \mathbf{0},
\end{aligned}$$

where the last step is due to $\boldsymbol{w}_U \in \operatorname{Im} Q_U$ and $\boldsymbol{w}_I \in \operatorname{Im} Q_I$. Thus we obtain $\boldsymbol{w} \in \operatorname{Ker} \tilde{M}_2(\boldsymbol{x}, t)$. □

Lemma 5.3 leads to the following proposition.

Proposition 5.4. Suppose that the tractability index of the hybrid equations is at most two. Under Assumptions 2.1 and 2.2, the network graph Γ contains neither VU-loops nor JI-cutsets.

Proof. Since $\tilde{M}_2(\boldsymbol{x}, t)$ is nonsingular, $\operatorname{Ker} \tilde{M}_2(\boldsymbol{x}, t) = \{\mathbf{0}\}$ holds. Hence $\Theta = \{\mathbf{0}\}$ follows from Lemma 5.3. Thus, the network graph Γ contains neither VU-loops nor JI-cutsets by Lemma 5.2. □

5.4 A sufficient condition

We now focus on linear time-invariant circuits. Then, the necessary condition given in Section 5.3 is shown to be sufficient under the genericity assumption on dependent sources. For linear time-invariant circuits, the capacitance matrix C, the inductance matrix L, and the hybrid immittance matrix $\begin{pmatrix} Z & H \\ G & Y \end{pmatrix}$ are real matrices. The Jacobian matrices V, $\bar{V}$, J, $\bar{J}$ for dependent sources are also real matrices.

Let $\mathbf{K}$ be a field and $\mathbf{F}$ be an extension field of $\mathbf{K}$. A subset $(a_1, \ldots, a_q)$ of $\mathbf{F}$ is called *algebraically independent* over $\mathbf{K}$ if there exists no nontrivial polynomial $f(p_1, \ldots, p_q)$ in q indeterminates over $\mathbf{K}$ such that $f(a_1, \ldots, a_q) = 0$, where $f(p_1, \ldots, p_q)$ is called nontrivial if some of its coefficients are distinct from zero. For a subset $\mathcal{Y}$ of $\mathbf{F}$, we denote by $\mathbf{K}(\mathcal{Y})$ the field adjunction, that is, $\mathbf{K}(\mathcal{Y})$ is the extension field of $\mathbf{K}$ generated by $\mathcal{Y}$ over $\mathbf{K}$.

Let $\mathcal{S} \subseteq \mathbb{R}$ be the set of nonzero entries which appear in C, L, and $\begin{pmatrix} Z & H \\ G & Y \end{pmatrix}$, and $\mathcal{D} \subseteq \mathbb{R}$ be that for $\begin{pmatrix} V & \bar{V} \\ \bar{J} & J \end{pmatrix}$. We assume the following condition.

Assumption 5.5. The set $\mathcal{D}$ is algebraically independent over $\mathbb{Q}(\mathcal{S})$.

Assumption 5.5 means that nonzero entries of $\begin{pmatrix} V & \bar{V} \\ \bar{J} & J \end{pmatrix}$ are regarded as independent parameters. We give a key lemma as follows.

Lemma 5.6. Let $\mathbf{K}$ be a field and $\mathbf{F}$ be an extension field of $\mathbf{K}$. In addition, let A, B, and K be an $n \times n$ matrix, an $n \times m$ matrix, and an $m \times n$ matrix over $\mathbf{K}$, respectively. We assume that W is an $m \times m$ matrix over $\mathbf{F}$ such that the set of its nonzero entries is algebraically independent over $\mathbf{K}$. If A is nonsingular, then $A + BWK$ is nonsingular.

Proof. Let us define

$$\tilde{A} = \begin{pmatrix} A & O & B \\ O & W & I \\ K & I & O \end{pmatrix}.$$

The row set and the column set of $\tilde{A}$ are denoted by $R(\tilde{A})$ and $C(\tilde{A})$, and those of A are denoted by $R(A)$ and $C(A)$. Since we have

$$\begin{pmatrix} I & -B & O \\ O & I & O \\ O & O & I \end{pmatrix} \tilde{A} \begin{pmatrix} I & O & O \\ -K & I & O \\ O & O & I \end{pmatrix} = \begin{pmatrix} A+BWK & -BW & O \\ -WK & W & I \\ O & I & O \end{pmatrix},$$

the nonsingularity of $A+BWK$ is equivalent to that of $\tilde{A}$. By the generalized Laplace expansion,

$$\det \tilde{A} = \sum_{Q \subseteq C(\tilde{A}), |Q|=|P|} \operatorname{sgn}(P,Q) \cdot \det \tilde{A}[P,Q] \cdot \det \tilde{A}[R(\tilde{A}) \setminus P, C(\tilde{A}) \setminus Q],$$

where $\tilde{A}[P,Q]$ denotes the submatrix of $\tilde{A}$ with row set P and column set Q, and $\operatorname{sgn}(P,Q) = \pm 1$. If A is nonsingular, there exists a nonzero term $\tilde{a}$ corresponding to $P = R(A)$ and $Q = C(A)$. The other nonzero terms contain at least one nonzero entry of W. By the genericity assumption on W, $\tilde{a}$ does not vanish by numerical cancellations. Hence $\tilde{A}$ is nonsingular, which implies that $A + BWK$ is also nonsingular. □

Let us define

$$\begin{aligned} \tilde{M}_2'(\boldsymbol{x},t) = & \begin{pmatrix} O & -N^\top \\ N & O \end{pmatrix} + \begin{pmatrix} A_Z & O \\ O & A_Y \end{pmatrix} \begin{pmatrix} Z & H \\ G & Y \end{pmatrix} \begin{pmatrix} A_Z^\top & O \\ O & A_Y^\top \end{pmatrix} \\ & + \begin{pmatrix} A_Z & O \\ O & A_Y \end{pmatrix} \begin{pmatrix} Z & H \\ G & Y \end{pmatrix} \begin{pmatrix} O & O & O & -A_{ZL}R_{LI}(\boldsymbol{x},t) \\ O & O & O & O \\ O & O & O & O \\ A_{CY}^\top R_{CU}(\boldsymbol{x},t) & O & O & O \end{pmatrix} \\ & + \begin{pmatrix} -A_{CU}^\top R_{CU}(\boldsymbol{x},t) & O & O & O \\ -A_{CZ}^\top R_{CU}(\boldsymbol{x},t) & O & O & O \\ O & O & O & A_{YL}R_{LI}(\boldsymbol{x},t) \\ O & O & O & A_{IL}R_{LI}(\boldsymbol{x},t) \end{pmatrix}. \end{aligned}$$

Lemma 5.6 leads to the following lemma.

Lemma 5.7. Under Assumption 5.5, if $\tilde{M}_2'(\boldsymbol{x},t)$ is nonsingular, then $\tilde{M}_2(\boldsymbol{x},t)$ is nonsingular.

Proof. In Lemma 5.6, we set $\mathbf{K} = \mathbb{Q}(\mathcal{S})$ and $\mathbf{F} = \mathbb{R}$. By Assumption 5.5, we can apply Lemma 5.6 to $\tilde{M}_2(\boldsymbol{x},t)$, where $A = \tilde{M}_2'(\boldsymbol{x},t)$,

$$B = \begin{pmatrix} A_U^\top & O \\ O & O \\ O & O \\ O & A_I^\top \end{pmatrix}, \quad W = \begin{pmatrix} V & \bar{V} \\ \bar{J} & J \end{pmatrix}, \quad K = \begin{pmatrix} A_C R_{CU}(\boldsymbol{x},t) & O & O & O \\ O & O & O & A_L R_{LI}(\boldsymbol{x},t) \end{pmatrix}.$$

This completes the proof. □

The following lemma gives a sufficient condition for the nonsingularity of $\tilde{M}_2'(\boldsymbol{x},t)$.

Lemma 5.8. Under Assumptions 2.1 and 2.2, if $\Theta = \{\mathbf{0}\}$ holds, the matrix $\tilde{M}_2'(\boldsymbol{x}, t)$ is nonsingular.

Proof. We prove $\operatorname{Ker} \tilde{M}_2'(\boldsymbol{x}, t) \subseteq \Theta$. Let $\boldsymbol{w} := \begin{pmatrix} \boldsymbol{w}_U \\ \boldsymbol{w}_Z \\ \boldsymbol{w}_Y \\ \boldsymbol{w}_I \end{pmatrix} \in \operatorname{Ker} \tilde{M}_2'(\boldsymbol{x}, t)$. Then $\tilde{M}_2'(\boldsymbol{x}, t)\boldsymbol{w} = \mathbf{0}$ holds. Now we have

$$\begin{pmatrix} Q_U^\top & O & O & O \\ O & O & O & O \\ O & O & O & O \\ O & O & O & Q_I^\top \end{pmatrix} \tilde{M}_2'(\boldsymbol{x}, t)\boldsymbol{w} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix},$$

which is equivalent to $Q_U^\top A_{CU}^\top R_{CU}(\boldsymbol{x}, t)\boldsymbol{w}_U = \mathbf{0}$ and $Q_I^\top A_{IL} R_{LI}(\boldsymbol{x}, t)\boldsymbol{w}_I = \mathbf{0}$. By the definition of $R_{CU}(\boldsymbol{x}, t)$ and $R_{LI}(\boldsymbol{x}, t)$, it holds that

$$\boldsymbol{w}_U^\top Q_U^\top A_{CU}^\top M_C(\boldsymbol{x}, t)^{-1} A_{CU} Q_U \boldsymbol{w}_U = \mathbf{0} \quad \text{and} \quad \boldsymbol{w}_I^\top Q_I^\top A_{IL} M_L(\boldsymbol{x}, t)^{-1} A_{IL}^\top Q_I \boldsymbol{w}_I = \mathbf{0}.$$

Since $M_C(\boldsymbol{x}, t)$ and $M_L(\boldsymbol{x}, t)$ are positive definite by Lemma 4.1, we obtain $A_{CU} Q_U \boldsymbol{w}_U = \mathbf{0}$ and $A_{IL}^\top Q_I \boldsymbol{w}_I = \mathbf{0}$, which implies

$$R_{CU}(\boldsymbol{x}, t)\boldsymbol{w}_U = \mathbf{0} \quad \text{and} \quad R_{LI}(\boldsymbol{x}, t)\boldsymbol{w}_I = \mathbf{0}. \tag{8}$$

By substituting (8) to $\boldsymbol{w}^\top \tilde{M}_2'(\boldsymbol{x}, t)\boldsymbol{w} = 0$, we obtain

$$\boldsymbol{w}^\top \begin{pmatrix} O & -N^\top \\ N & O \end{pmatrix} \boldsymbol{w} + \boldsymbol{w}^\top \begin{pmatrix} A_Z & O \\ O & A_Y \end{pmatrix} \begin{pmatrix} Z & H \\ G & Y \end{pmatrix} \begin{pmatrix} A_Z^\top & O \\ O & A_Y^\top \end{pmatrix} \boldsymbol{w} = 0.$$

Since the first term is equal to zero and $\begin{pmatrix} Z & H \\ G & Y \end{pmatrix}$ is positive definite by Assumption 2.2, $\begin{pmatrix} A_Z^\top & O \\ O & A_Y^\top \end{pmatrix} \boldsymbol{w} = \mathbf{0}$ holds. Hence we have

$$A_{ZU}\boldsymbol{w}_U = \mathbf{0}, \quad \boldsymbol{w}_Z = \mathbf{0}, \quad \boldsymbol{w}_Y = \mathbf{0}, \quad A_{IY}^\top \boldsymbol{w}_I = \mathbf{0}. \tag{9}$$

By substituting (8) and (9) to $\tilde{M}_2'(\boldsymbol{x}, t)\boldsymbol{w} = \mathbf{0}$, we obtain $\begin{pmatrix} O & -N^\top \\ N & O \end{pmatrix} \boldsymbol{w} = \mathbf{0}$, which implies

$$A_{IU}\boldsymbol{w}_U = \mathbf{0}, \quad A_{YU}\boldsymbol{w}_U = \mathbf{0}, \quad A_{IZ}^\top \boldsymbol{w}_I = \mathbf{0}, \quad A_{IU}^\top \boldsymbol{w}_I = \mathbf{0}. \tag{10}$$

Thus, $\boldsymbol{w} \in \Theta$ follows from (8)–(10). Hence we obtain $\operatorname{Ker} \tilde{M}_2'(\boldsymbol{x}, t) \subseteq \Theta$. It follows from $\Theta = \{\mathbf{0}\}$ that $\operatorname{Ker} \tilde{M}_2'(\boldsymbol{x}, t) = \{\mathbf{0}\}$, which implies that $\tilde{M}_2'(\boldsymbol{x}, t)$ is nonsingular. □

We now obtain a sufficient condition for the hybrid equations with index at most two.

Theorem 5.9. Under Assumptions 2.1, 2.2, and 5.5, if the network graph Γ contains neither VU-loops nor JI-cutsets, the tractability index of the hybrid equations is at most two.

Proof. Lemmas 5.2 and 5.8 indicate that if the network graph Γ contains neither VU-loops nor JI-cutsets, then $\tilde{M}_2'(\boldsymbol{x}, t)$ is nonsingular. By Lemma 5.7, if $\tilde{M}_2'(\boldsymbol{x}, t)$ is nonsingular, $\tilde{M}_2(\boldsymbol{x}, t)$ is nonsingular and hence the tractability index of the hybrid equations is at most two. □

Since commonly-used circuits contain neither VU-loops nor JI-cutsets, the hybrid analysis results in a DAE with index at most two in most cases. Proposition 5.4 and Theorem 5.9 lead to the following theorem.

Theorem 5.10. Under Assumptions 2.1, 2.2, and 5.5, the tractability index of the hybrid equations is at most two if and only if the network graph Γ contains neither VU-loops nor JI-cutsets.

Remark 5.11. Although we focus on linear time-invariant circuits satisfying Assumption 5.5 in the proof of Theorem 5.9, Theorem 5.9 is applicable to nonlinear time-varying circuits if Lemma 5.7 holds. Note that, without Assumption 5.5, Lemma 5.7 possibly does not hold because of unlucky numerical cancellations. In practical situations, however, such numerical cancellations seldom occur.

Let us discuss Assumption 5.5, which ensures Lemma 5.7. Assumption 5.5 makes sense for linear time-invariant circuits, because all the matrices C, L, $\begin{pmatrix} Z & H \\ G & Y \end{pmatrix}$, and $\begin{pmatrix} V & \bar{V} \\ \bar{J} & J \end{pmatrix}$ are constant. On the other hand, nonlinear time-varying circuits do not satisfy Assumption 5.5 in general. In fact, since nonzero entries of $\begin{pmatrix} V & \bar{V} \\ \bar{J} & J \end{pmatrix}$ vary with t, there exists a time t when some nonzero entries become rational numbers, which contradicts Assumption 5.5.

Remark 5.12. As described in Introduction, MNA often results in a DAE with tractability index greater than two. Figure 1 shows a circuit with a dependent current source I controlled by the current through V. When we apply MNA to this circuit, we obtain a DAE with index three [6]. On the other hand, the hybrid analysis results in a DAE with index two. We remark that we cannot apply Theorem 5.10 to this circuit, because we assume that the constitutive equations of dependent sources are given by (3), where the arguments are only $\boldsymbol{u}_C$ and $\boldsymbol{i}_L$ besides $\boldsymbol{u}_V$, $\boldsymbol{i}_J$, and t.

Next, suppose that the dependent current source I in Figure 1 is controlled by the voltage across C. Then this dependent current source is described in the form of (3). While MNA results in a DAE with index three, the hybrid analysis results in a DAE with index two. In fact, since the network graph Γ of this circuit contains neither VU-loops nor JI-cutsets, Theorem 5.10 implies that the index is at most two. Moreover, the index is shown to be at least two by Theorem 4.5. Thus, we can easily determine that the index of the hybrid equations is exactly two in this case.

6 Conclusion

For nonlinear time-varying circuits with dependent sources, we give a necessary condition for the hybrid equations to have tractability index at most two. Moreover, we prove that the necessary condition is also sufficient for linear time-invariant circuits if dependent sources

Table 1: Relation between our results and the previous works.

MNA	• Γ has neither CVU-loops nor LJI-cutsets and additional conditions (see [4] for details)	$\Longrightarrow$	index ≤ 2	[4]
Hybrid analysis	For circuits consisting of elements given by (1)–(4),			
	• Γ has neither CVU-loops nor LJI-cutsets (under Assumptions 2.1, 2.2, and 4.4)	$\Longleftrightarrow$	index ≤ 1	[10]
	• Γ has neither VU-loops nor JI-cutsets (under Assumptions 2.1, 2.2, and 5.5)	$\Longleftrightarrow$	index ≤ 2	this paper

satisfy the genericity assumption. Our results also remain valid for nonlinear time-varying circuits unless unlucky numerical cancellations occur.

The sufficient condition given in this paper indicates that DAEs arising from the hybrid analysis have index at most two in most cases, because commonly-used circuits contain neither VU-loops nor JI-cutsets. By combining our results with [10], we obtain criteria for tractability index zero/one/two.

Let us summarize relations between our results and the previous works [4, 10] in Table 1. In [4], the index of a DAE arising from MNA is shown to be at most two, if dependent sources satisfy certain conditions. One of these conditions is that the network graph Γ has neither CVU-loops nor LJI-cutsets. This condition appears in the characterization of the hybrid equations with index at most one (Theorem 4.5). For the hybrid analysis, we can check that the necessary and sufficient conditions for index at most two is weaker than that for index at most one.

As discussed in Remark 5.12, we restrict the constitutive equations of dependent sources to the form of (3). Extending the results in this paper to circuits containing dependent sources controlled by other variables, such as $\boldsymbol{i}_V$, $\boldsymbol{i}_Z$, and $\boldsymbol{u}_Y$, is left for future investigation.

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A Proof for a projector $Q_1(\boldsymbol{x}, t)$

Under Assumptions 2.1 and 2.2, we show that $Q_1(\boldsymbol{x}, t)$ is a projector onto $\operatorname{Ker} M_1(\boldsymbol{x}, t)$. We compute $\operatorname{Ker} M_1(\boldsymbol{x}, t)$ in the following.

Let $\boldsymbol{w} := \begin{pmatrix} \boldsymbol{w}_L \\ \boldsymbol{w}_U \\ \boldsymbol{w}_Z \\ \boldsymbol{w}_Y \\ \boldsymbol{w}_I \\ \boldsymbol{w}_C \end{pmatrix} \in \operatorname{Ker} M_1(\boldsymbol{x}, t)$. We use $\boldsymbol{w}_1 = \begin{pmatrix} \boldsymbol{w}_U \\ \boldsymbol{w}_Z \end{pmatrix}$ and $\boldsymbol{w}_2 = \begin{pmatrix} \boldsymbol{w}_Y \\ \boldsymbol{w}_I \end{pmatrix}$ for convenience.

Since $\operatorname{Ker} M_1(\boldsymbol{x}, t)\boldsymbol{w} = \boldsymbol{0}$ holds, we have

$$-N^\top \boldsymbol{w}_2 + A_Z Z A_Z^\top \boldsymbol{w}_1 + A_Z H A_Y^\top \boldsymbol{w}_2 = \boldsymbol{0}, \tag{11}$$

$$N\boldsymbol{w}_1 + A_Y G A_Z^\top \boldsymbol{w}_1 + A_Y Y A_Y^\top \boldsymbol{w}_2 = \boldsymbol{0}. \tag{12}$$

By computing $\boldsymbol{w}_1^\top \times (11) + \boldsymbol{w}_2^\top \times (12)$, we obtain

$$\boldsymbol{w}_1^\top A_Z Z A_Z^\top \boldsymbol{w}_1 + \boldsymbol{w}_1^\top A_Z H A_Y^\top \boldsymbol{w}_2 + \boldsymbol{w}_2^\top A_Y G A_Z^\top \boldsymbol{w}_1 + \boldsymbol{w}_2^\top A_Y Y A_Y^\top \boldsymbol{w}_2 = \boldsymbol{0},$$

which is equivalent to

$$\boldsymbol{w}_1^\top A_Z Z A_Z^\top \boldsymbol{w}_1 + \boldsymbol{w}_2^\top A_Y Y A_Y^\top \boldsymbol{w}_2 = \boldsymbol{0},$$

because $H = -G^\top$ by Assumption 2.2. Since Z and Y are positive definite, $A_Z^\top \boldsymbol{w}_1 = \boldsymbol{0}$ and $A_Y^\top \boldsymbol{w}_2 = \boldsymbol{0}$ hold. Hence we obtain

$$A_{ZU}\boldsymbol{w}_U = \boldsymbol{0}, \quad \boldsymbol{w}_Z = \boldsymbol{0}, \quad \boldsymbol{w}_Y = \boldsymbol{0}, \quad A_{IY}^\top \boldsymbol{w}_I = \boldsymbol{0}. \tag{13}$$

By substituting (13) into $\operatorname{Ker} M_1(\boldsymbol{x}, t)\boldsymbol{w} = \boldsymbol{0}$, we have

$$M_L(\boldsymbol{x}, t)\boldsymbol{w}_L - A_{IL}^\top \boldsymbol{w}_I = \boldsymbol{0}, \tag{14}$$

$$A_{IU}^\top \boldsymbol{w}_I = \boldsymbol{0}, \quad A_{IZ}^\top \boldsymbol{w}_I = \boldsymbol{0}, \quad A_{YU}\boldsymbol{w}_U = \boldsymbol{0}, \quad A_{IU}\boldsymbol{w}_U = \boldsymbol{0}, \tag{15}$$

$$A_{CU}^\top \boldsymbol{w}_U + M_C(\boldsymbol{x}, t)\boldsymbol{w}_C = \boldsymbol{0}. \tag{16}$$

The conditions (13)–(16) are equivalent to

$$\begin{aligned} &\boldsymbol{w}_Z = \boldsymbol{0}, \quad \boldsymbol{w}_Y = \boldsymbol{0}, \\ &\boldsymbol{w}_U \in \operatorname{Ker} \begin{pmatrix} A_{IU} \\ A_{YU} \\ A_{ZU} \end{pmatrix} = \operatorname{Im} Q_U, \quad \boldsymbol{w}_I \in \operatorname{Ker} \begin{pmatrix} A_{IY}^\top \\ A_{IZ}^\top \\ A_{IU}^\top \end{pmatrix} = \operatorname{Im} Q_I, \\ &\boldsymbol{w}_L = M_L(\boldsymbol{x}, t)^{-1} A_{IL}^\top \boldsymbol{w}_I, \quad \boldsymbol{w}_C = -M_C(\boldsymbol{x}, t)^{-1} A_{CU}^\top \boldsymbol{w}_U, \end{aligned}$$

because $M_L(\boldsymbol{x}, t)$ and $M_C(\boldsymbol{x}, t)$ are nonsingular by Lemma 4.1. Thus, we obtain $\operatorname{Ker} M_1(\boldsymbol{x}, t) = \operatorname{Im} Q_1(\boldsymbol{x}, t)$.