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Abstract

In this paper, we formulate a mathematical model of autonomous distributed control for smart grids based on price mechanism and network structure. Then, we propose a controller design of price mechanism for attaining optimal supply and demand balancing of the whole of smart grids. This control system is a generalized version of the well-known Uzawa's primal-dual algorithm. Furthermore, we extend the results for a case that subsystems contain their inherent dynamics and it is positive real. Finally, we show effective improvement of control performance of price mechanism by using our proposing controller design method through numerical simulations.

1 Introduction

In recent years, environmental and energy issue has been discussed worldwide and one of possible solutions is to employ renewable energy such as solar or wind energy in power grids. However, introducing such renewable energy may cause instability of systems and it is not easy to balance demands and supplies between agents. In order to solve such difficulties, recently, the idea of smart grids has been actively discussed and developed in order to stabilize the whole systems by using information and communication technology. One of important ideas of smart grids is to control consumers, which is called "demand response". The demand response by using price mechanism has been actively investigated in the research field of micro economics (see, e.g., [5]).

Based on the above background, in this paper, we extend the model of [5] to a case that it has a multi-regional network structure, and propose a control strategy of the whole systems as follows.

Our model contains several agents of consumers, suppliers, transmissioners of electricity between regions, and ISOs (independent system operators). The ISO of a region observes a gap between the total demand and supply between agents in the region, decides its price of electricity, and broadcasts it in order to maximize a social welfare function. On the other hand, other agents (consumers, suppliers, transmissioners) acts egoistically to optimize their own evaluation functions based on the prices only.

The scale of our considering power systems is large and centralized controls are difficult to be implemented due to limitation on information capacity and computation complexity, therefore, the control strategy of the whole systems is a distributed form. Moreover, ordinary numerical optimization algorithms do not suppose physical disturbance, because they are computed only on computers. However, the optimization algorithms of demand response for smart grids is supposed to be executed on physical systems and disturbances or time delays are unavoidable in sensing physical quantity or communicating each other.

Along this consideration, at first, we examine the mechanism of the ordinary distributed optimization algorithm by control theory, in particular, a notion of passivity. We clarify that the convergence of the algorithm can be proved through the passivity. Then, we propose another control strategy for pricing by ISOs which attains the global stability on the optimal solution. Moreover, we extend this result for cases that the agents have inherent dynamics. We show that the stability can be also guaranteed if the inherent dynamics is in a class of positive real functions. Finally we show effective improvement of control performance is attained by our proposing controller with numerical simulations.

Notations: The following are notations used in this paper: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{R}_+$, $\mathbb{R}_{++}$ denote the set of nonnegative real numbers and the set of positive real numbers, respectively, $\mathbb{R}^N$ and $\mathbb{R}^{n\times m}$ denote n dimensional Euclid vector space and the set of $n\times m$ real matrices, bold letter $\boldsymbol{x}$ means a vector, and $A^\top$ denotes transpose of a matrix. $\mathrm{diag}\{a_1,\dots,a_n\}$ is a diagonal matrix.

2 Model of smart grids

We assume that there exist independent N electricity local markets at N regions (see Fig. 1). In ith region, there exist a consumer and a supplier, and they determine electricity consumption quantity d_i and electricity supply quantity s_i depending on the local electricity price p_i, respectively. The regions are connected by electricity transmission lines and there exists a transmissioner at each transmission line, who determines electricity transmission quantity q_l between the connected two regions depending on their electricity price difference. Finally, at each region, there exists an ISO (Independent System Operators) and it coordinates the local electricity market from a neutral standpoint. For simplicity of notations, in this paper, we assume that an agent in each region represents a set of the consumers or the suppliers and the following results can be easily extended for a case of multiple consumers or suppliers.

Table1 represents all variables in this problem setting.

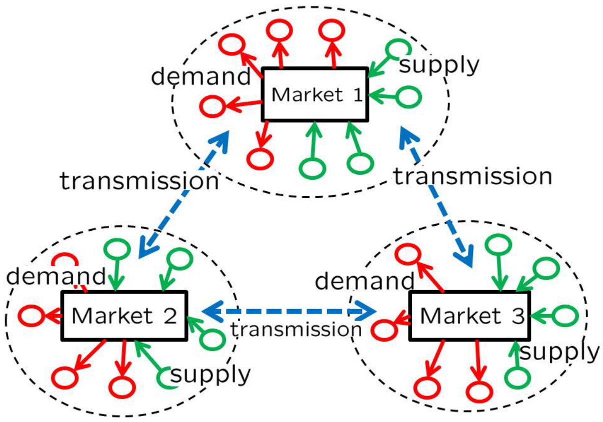

Figure 1: Multi-regional electricity markets

$N \in \mathbb{N}$	number of regions
$p_i \in \mathbb{R}_{++}$	electricity price of region i
$d_i \in \mathbb{R}_{++}$	electricity demand of region i
$s_i \in \mathbb{R}_{++}$	electricity supply of region i
$u_i(\cdot) \in C^2$	utility function of region i's consumer
$c_i(\cdot) \in C^2$	cost function of region i's supplier
$E \in \mathbb{N}$	number of transmission lines
$q_l \in \mathbb{R}$	power transmission of line l's agent
$t_l(\cdot) \in C^2$	transmission cost function of line l's agent

Table 1: Definition of variables

In this paper, we assume each agent obtains profit through dealing of electricity as follows:

Consumer Consumer of region i obtains consumer surplus $u_i(d_i) - p_i d_i$ by buying electricity d_i quantity with the price p_i per unit.

Supplier Supplier of region i obtains consumer surplus $p_i s_i - c_i(s_i)$ by selling electricity s_i quantity with the price p_i per unit.

Transmissioner Transmissioner of line l obtains profit $(D^\top \boldsymbol{p})_l q_l - t_l(q_l)$ by transmitting electricity q_l quantity between regions.

We assume that consumers, suppliers and transmissioners are all price-takers, and ISOs do not pursue the profit and take a neutral standpoint such as government or quasi-government organizations. Moreover, we also assume that utility function $u_i(\cdot) \in C^2$ of each consumer i is monotonically increasing and strictly concave, cost function $c_i(\cdot) \in C^2$ of each supplier i is monotonically increasing and strictly convex, and cost function $t_l(\cdot) \in C^2$ of each transmissoner l is strictly convex.

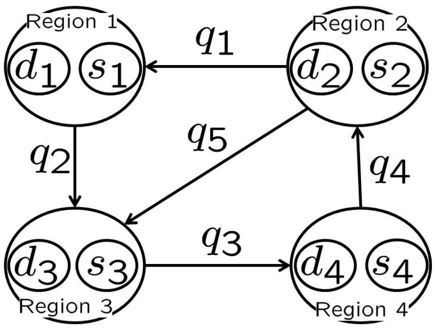

Figure 2: Example of network structure($N = 4, E = 5$)

Finally, we assume graph structures are connected. In Fig. 2, each node represents a region, each edge represents transmission line, and q_l denotes electricity which flows along the direction of an edge l. Then, the network constraint equation should satisfy

$$s_i - d_i = -\sum_{i\leftarrow} q_l + \sum_{i\rightarrow} q_l.$$

By using the incident matrix $D \in \mathbb{R}^{N\times E}$, this constraint is also represented by

$$\boldsymbol{s} - \boldsymbol{d} = -D\boldsymbol{q}.$$

3 Social welfare problem

In this section, we introduce a resource allocation problem in order to maximize the sum of profits of all agents under network constraints. The sum of profits is given by

$$
\begin{aligned}
SW &= \sum_i (u_i(d_i) - p_i d_i) + \sum_i (p_i s_i - c_i(s_i)) \\
&\quad + \sum_l \{ \left(D^\top \boldsymbol{p} \right) q_l - t_l(q_l) \} \\
&= \sum_i (u_i(d_i)) - \sum_i (c_i(s_i)) - \sum_l (t_l(q_l)) . \qquad (1)
\end{aligned}
$$

Hereafter, we call $SW(\boldsymbol{d}, \boldsymbol{s}, \boldsymbol{q})$ as the *social welfare function* (see, e.g., [6]). Note that terms including prices $\boldsymbol{p}$ are cancelled, in other words, social welfare function does not explicitly depend on electricity prices, and is determined by assignment to each agent of electricity only.

The purpose of this paper is to derive algorithms for the agents to attain the maximization of $SW(\boldsymbol{d}, \boldsymbol{s}, \boldsymbol{q})$ such as

$$
\begin{aligned}
&\max_{\boldsymbol{d},\boldsymbol{s},\boldsymbol{q}} \sum_i (u_i(d_i)) - \sum_i (c_i(s_i)) - \sum_l (t_l(q_l)) \\
&\text{s.t.} \quad \boldsymbol{s} - \boldsymbol{d} = -D\boldsymbol{q}.
\end{aligned}
$$

In this paper, we call this optimization problem as the *social welfare problem*. The difficulty of this problem is that the maximization of $SW(\boldsymbol{d}, \boldsymbol{s}, \boldsymbol{q})$ is a natural objective for ISOs, on the other hand, it may not be necessarily the objective for the other agents such as consumers, suppliers, or transmissioners. Therefore, the optimization algorithms should allow egoistic behavior for those agents. In order to solve this difficulty, we consider to employ Uzawa's primal-dual algorithm as explained below.

It is known that there exists the unique optimal solution $(\boldsymbol{d}^*, \boldsymbol{s}^*, \boldsymbol{q}^*)$ under the assumptions given above. In order to find the optimal solution, we define the Lagrangian of the social welfare problem by

$$
\begin{aligned}
\mathcal{L}(\boldsymbol{d}, \boldsymbol{s}, \boldsymbol{q}, \boldsymbol{p}) := &\sum_i (u_i(d_i)) - \sum_i (c_i(s_i)) - \sum_l (t_l(q_l)) \\
&+ \boldsymbol{p}^\top (\boldsymbol{s} - \boldsymbol{d} + D\boldsymbol{q}) . \qquad (2)
\end{aligned}
$$

Uzawa's primal-dual algorithm is well known as a continuous algorithm to solve a class of constraint optimization [2]. We consider to apply it to (2) and obtain the following:

$$
\begin{aligned}
\dot{\boldsymbol{d}} &= \frac{\partial \mathcal{L}(\boldsymbol{d}, \boldsymbol{s}, \boldsymbol{q}, \boldsymbol{p})}{\partial \boldsymbol{d}}, \\
\dot{\boldsymbol{s}} &= \frac{\partial \mathcal{L}(\boldsymbol{d}, \boldsymbol{s}, \boldsymbol{q}, \boldsymbol{p})}{\partial \boldsymbol{s}}, \\
\dot{\boldsymbol{q}} &= \frac{\partial \mathcal{L}(\boldsymbol{d}, \boldsymbol{s}, \boldsymbol{q}, \boldsymbol{p})}{\partial \boldsymbol{q}}, \\
\dot{\boldsymbol{p}} &= -\frac{\partial \mathcal{L}(\boldsymbol{d}, \boldsymbol{s}, \boldsymbol{q}, \boldsymbol{p})}{\partial \boldsymbol{p}}.
\end{aligned}
$$

The above equations are also written at each agent as follows:

$$\dot{d}_i = u_i^{'}(d_i) - p_i, \quad i = 1, 2, \ldots, N \tag{3}$$

$$\dot{s}_i = p_i - c_i^{'}(s_i), \quad i = 1, 2, \ldots, N \tag{4}$$

$$\dot{q}_l = -t_l^{'}(q_l) + \left(D^\top \boldsymbol{p}\right)_l, \quad l = 1, 2, \ldots, E \tag{5}$$

$$\dot{p}_i = d_i - s_i - (D\boldsymbol{q})_i, \quad i = 1, 2, \ldots, N \tag{6}$$

Note that the equilibrium point of the system satisfies the KKT condition of the social welfare problem and it is shown that optimal solution is attained by employing the result of [2].

On the algorithms (3)–(6), we know that (i) each ISO determines price p_i from a neutral standpoint, (ii) each ISO does not need to know u_i, c_i, t_l, (iii) on the other hand, the other agents (consumers, suppliers, transmissioners) act egoistically according to price signal, (iv) the algorithms are distributed form.

Hereafter we examine the stability of the dynamics of the price mechanism from a viewpoint of a notion, *passivity*, in control theory. First of all, the following proposition holds:

Proposition 3.1 In the price mechanism (3)–(6), consumers, suppliers and transmissioners have incremental passivity [3].

Proof. In this proof, we show the incremental passivity of suppliers only (that of consumers and transmissioner are similarly shown). On ith supplier, we regard $p_i - \tilde{p}_i$ as the input, and $s_i - \tilde{s}_i$ as the output, and define a storage function candidate as $\frac{1}{2}(s_i - \tilde{s}_i)^2$. Then it holds that:

$$\begin{aligned}
\dot{S}_{s_i} &= (s_i - \tilde{s}_i)(\dot{s}_i - \dot{\tilde{s}}_i) \\
&= (s_i - \tilde{s}_i)\{(p_i - c_i^{'}(s_i)) - (\tilde{p}_i - c_i^{'}(\tilde{s}_i))\} \\
&= (s_i - \tilde{s}_i)(p_i - \tilde{p}_i) - (s_i - \tilde{s}_i)(c_i^{'}(s_i) - c_i^{'}(\tilde{s}_i)).
\end{aligned}$$

By employing the mean value theorem, there always exists $\exists \xi_{s_i} \in [s_i, \tilde{s}_i]$ satisfying $c_i^{'}(s_i) - c_i^{'}(\tilde{s}_i) = c_i^{''}(\xi_{s_i})(s_i - \tilde{s}_i)$. Then, we get

$$\dot{S}_{s_i} = (s_i - \tilde{s}_i)(p_i - \tilde{p}_i) - c_i^{''}(\xi_{s_i})(s_i - \tilde{s}_i)^2.$$

From the assumption that $c_i(\cdot)$ is strictly convex, $c_i^{''}(\xi_{s_i})$ is always greater than 0. Therefore the system is incremental passive. □

We are now ready to show the stability of the price mechanism with the storage functions of each agent.

Theorem 3.1 The price mechanism (3)–(6) converges to the optimal solution $(\boldsymbol{d}^*, \boldsymbol{s}^*, \boldsymbol{q}^*, \boldsymbol{p}^*)$ of the social welfare problem asymptotically.

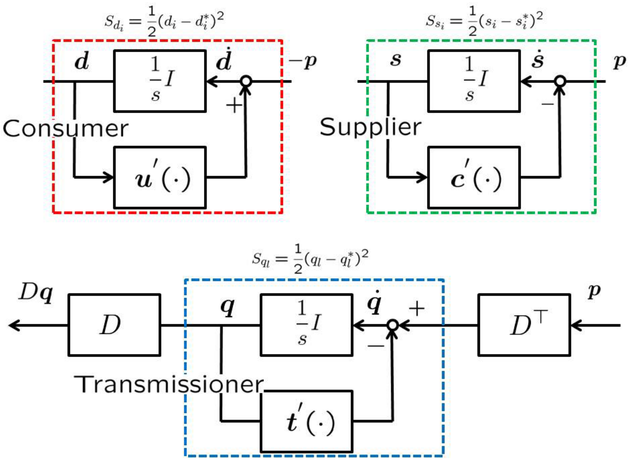

Figure 3: Block diagrams of each agent

Proof. Define a storage function $S_{d_i} = \frac{1}{2}(d_i - d_i^*)^2$ for consumer i ($i = 1, 2, \ldots, N$). Also define a storage function S_{s_i} for each supplier i ($i = 1, 2, \ldots, N$) and a storage function S_{q_l} for each transmissioner l ($l = 1, 2, \ldots, E$) similarly. Also define a Lyapunov function candidate by

$$V := \sum_i S_{d_i} + \sum_i S_{s_i} + \sum_l S_{q_l} + \sum_i \frac{1}{2}(p_i - p_i^*)^2.$$

Then we can calculate the time derivative of V as follows:

$$\begin{aligned}
\dot{V} &= \sum_i \dot{S}_{d_i} + \sum_i \dot{S}_{s_i} + \sum_l \dot{S}_{q_l} + \sum_i (p_i - p_i^*)\dot{p}_i \\
&= \sum_i \{-(d_i - d_i^*)(p_i - p_i^*) + u_i^{''}(\xi_{d_i})(d_i - d_i^*)^2\} \\
&\quad + \sum_i \{(s_i - s_i^*)(p_i - p_i^*) - c_i^{''}(\xi_{s_i})(s_i - s_i^*)^2\} \\
&\quad + \sum_l \{(q_l - q_l^*)(p_i - p_i^*) - t_l^{''}(\xi_{q_l})(q_l - q_l^*)^2\} \\
&\quad + \sum_i (p_i - p_i^*)(d_i - s_i - (D\boldsymbol{q})_i)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_i \{u_i''(\xi_{d_i})(d_i - d_i^*)^2\} + \sum_i \{-c_i''(\xi_{s_i})(s_i - s_i^*)^2\} \\
&\quad + \sum_l \{-t_l''(\xi_{q_l})(q_l - q_l^*)^2\} \\
&\quad + \sum_i \{-(d_i - d_i^*)(p_i - p_i^*)\} + \sum_i \{(s_i - s_i^*)(p_i - p_i^*)\} \\
&\quad + \sum_l \{(q_l - q_l^*) \left(D^\top (\boldsymbol{p} - \boldsymbol{p}^*)\right)_l\} \\
&\quad + \sum_i (p_i - p_i^*)(d_i - d_i^* - s_i + s_i^* - (D\boldsymbol{q})_i + (D\boldsymbol{q}^*)_i) \\
&= \sum_i \{u_i''(\xi_{d_i})(d_i - d_i^*)^2\} + \sum_i \{-c_i''(\xi_{s_i})(s_i - s_i^*)^2\} \\
&\quad + \sum_l \{-t_l''(\xi_{q_l})(q_l - q_l^*)^2\} \\
&\leq 0
\end{aligned}
$$

This concludes the statement of the theorem. □

The distributed optimization algorithm has the following properties: No agent does not know the information on the whole system or the optimal solution $(\boldsymbol{d}^*, \boldsymbol{s}^*, \boldsymbol{q}^*, \boldsymbol{p}^*)$ in advance. This is a highly-desirable property for such large scaled systems. Nevertheless, the optimal solution $(\boldsymbol{d}^*, \boldsymbol{s}^*, \boldsymbol{q}^*, \boldsymbol{p}^*)$ can be obtained in a distributed manner. This implies that the necessary information for agents is concentrated in the prices and ISOs can control the behavior of the agents by broadcasting them. This dynamics can be also regarded as a kind of *tâtonnement* process [6], which was introduced by Walras in economics.

4 The controller design to the price mechanism

In this section, we deal with a controller design problem for ISOs in order to improve the time responses of the whole system by using price signal. The function of ISO in region i is to define p_i by using $y_i := d_i - s_i - (D\boldsymbol{q})_i$ satisfying the global stability on the same optimal points d_i^*, s_i^*, q_l^* in the original system for all i and l. The whole system is nonlinear and the problem is not trivial, however we can find a control mechanism which satisfies the stability.

Our proposing mechanism is the following PI controller from y_i to p_i in each region:

$$\dot{x}_i = y_i, \quad i = 1, 2, \ldots, N, \tag{7}$$

$$p_i = K_i^I x_i + K_i^P y_i, \quad i = 1, 2, \ldots, N, \tag{8}$$

where $x_i \in \mathbb{R}$ is the internal variable in the controller. The following theorem guarantees the stability of the augmented price mechanism accompanied with the controller:

Theorem 4.1 For any $K_i^I > 0$ and $K_i^P > 0$, the augmented price mechanism accompanied with the controller (3)–(5) and (7), (8) converges to the optimal solution of the original social welfare problem asymptotically.

Proof. Define a Lyapunov function candidate by

$$V := \sum_i S_{d_i} + \sum_i S_{s_i} + \sum_l S_{q_l} + \sum_i \frac{K_i^I}{2}(\xi_i - \xi_i^*)^2.$$

Then, the time derivative of V is given as

$$\begin{aligned}
\dot{V} &= \sum_i \dot{S}_{d_i} + \sum_i \dot{S}_{s_i} + \sum_l \dot{S}_{q_l} + \sum_i (\xi_i - \xi_i^*)\dot{\xi}_i \\
&= \sum_i \{-(d_i - d_i^*)(p_i - p_i^*) + u_i^{''}(\xi_{d_i})(d_i - d_i^*)^2\} \\
&\quad + \sum_i \{(s_i - s_i^*)(p_i - p_i^*) - c_i^{''}(\xi_{s_i})(s_i - s_i^*)^2\} \\
&\quad + \sum_l \{(q_l - q_l^*)(p_i - p_i^*) - t_l^{''}(\xi_{q_l})(q_l - q_l^*)^2\} \\
&\quad + \sum_i K_i^I(\xi_i - \xi_i^*)(y_i - y_i^*) \\
&= \sum_i \{u_i^{''}(\xi_{d_i})(d_i - d_i^*)^2\} + \sum_i \{-c_i^{''}(\xi_{s_i})(s_i - s_i^*)^2\} \\
&\quad + \sum_l \{-t_l^{''}(\xi_{q_l})(q_l - q_l^*)^2\} \\
&\quad + \sum_i \{-(d_i - d_i^*)(p_i - p_i^*)\} + \sum_i \{(s_i - s_i^*)(p_i - p_i^*)\} \\
&\quad + \sum_l \{(q_l - q_l^*)\left(D^\top(\boldsymbol{p} - \boldsymbol{p}^*)\right)_l\} \\
&\quad + \sum_i K_i^I(\xi_i - \xi_i^*)(d_i - d_i^* - s_i + s_i^* - (D\boldsymbol{q})_i + (D\boldsymbol{q}^*)_i)
\end{aligned}$$

$$
\begin{aligned}
=&\sum_i\{u_i^{''}(\xi_{d_i})(d_i-d_i^*)^2\}+\sum_i\{-c_i^{''}(\xi_{s_i})(s_i-s_i^*)^2\}\\
&+\sum_l\{-t_l^{''}(\xi_{q_l})(q_l-q_l^*)^2\}\\
&+\sum_i\{-(d_i-d_i^*)(p_i-p_i^*)\}+\sum_i\{(s_i-s_i^*)(p_i-p_i^*)\}\\
&+\sum_i(p_i-p_i^*)\{(D\boldsymbol{q})_i-(D\boldsymbol{q}^*)_i\}\\
&+\sum_i K_i^I(\xi_i-\xi_i^*)(d_i-d_i^*-s_i+s_i^*-(D\boldsymbol{q})_i+(D\boldsymbol{q}^*)_i)\\
=&\sum_i\{u_i^{''}(\xi_{d_i})(d_i-d_i^*)^2\}+\sum_i\{-c_i^{''}(\xi_{s_i})(s_i-s_i^*)^2\}\\
&+\sum_l\{-t_l^{''}(\xi_{q_l})(q_l-q_l^*)^2\}\\
&+\sum_i(p_i-p_i^*)\{-(d_i-d_i^*)+(s_i-s_i^*)+(D\boldsymbol{q})_i-(D\boldsymbol{q}^*)_i\}\\
&+\sum_i K_i^I(\xi_i-\xi_i^*)(d_i-d_i^*-s_i+s_i^*-(D\boldsymbol{q})_i+(D\boldsymbol{q}^*)_i)\\
=&\sum_i\{u_i^{''}(\xi_{d_i})(d_i-d_i^*)^2\}+\sum_i\{-c_i^{''}(\xi_{s_i})(s_i-s_i^*)^2\}\\
&+\sum_l\{-t_l^{''}(\xi_{q_l})(q_l-q_l^*)^2\}\\
&+\sum_i\{K_i^I(\xi_i-\xi_i^*)+K_i^P(y_i-y_i^*)\}\\
&\times\{-(d_i-d_i^*)+(s_i-s_i^*)+(D\boldsymbol{q})_i-(D\boldsymbol{q}^*)_i\}\\
&+\sum_i K_i^I(\xi_i-\xi_i^*)(d_i-d_i^*-s_i+s_i^*-(D\boldsymbol{q})_i+(D\boldsymbol{q}^*)_i)\\
=&\sum_i\{u_i^{''}(\xi_{d_i})(d_i-d_i^*)^2\}+\sum_i\{-c_i^{''}(\xi_{s_i})(s_i-s_i^*)^2\}\\
&+\sum_l\{-t_l^{''}(\xi_{q_l})(q_l-q_l^*)^2\}\\
&+\sum_i\{K_i^P(y_i-y_i^*)\}\{-(d_i-d_i^*)+(s_i-s_i^*)+(D\boldsymbol{q})_i-(D\boldsymbol{q}^*)_i\}\\
=&\sum_i\{u_i^{''}(\xi_{d_i})(d_i-d_i^*)^2\}+\sum_i\{-c_i^{''}(\xi_{s_i})(s_i-s_i^*)^2\}\\
&+\sum_l\{-t_l^{''}(\xi_{q_l})(q_l-q_l^*)^2\}\\
&-\sum_i K_i^P(d_i-d_i^*-s_i+s_i^*-(D\boldsymbol{q})_i+(D\boldsymbol{q}^*)_i)^2\\
\le&0.
\end{aligned}
$$

Therefore, the augmented price mechanism accompanied with the controller (3)–(5) and (7), (8) converges to the optimal solution of the original social welfare problem asymptotically. □

Thus, it is known that PI controller is one of choices for the price mechanism in order to satisfy the stability on the same optimal however unknown equilibrium of the original systems. Note that this augmented price mechanism is a kind of generalized version of continuous Uzawa's primal-dual algorithm.

5 Generalization for inherent dynamics of agents

In the previous sections, we consider a case that the agents (consumers, suppliers, transmissioners) do not include inherent dynamics themselves but the algorithm includes differential equations. However, for example, when suppliers generate electric power by generators composed of boilers and turbines, reference electricity supply s_i is not instantly realized but physical dynamics exists from the reference and to the actual output s_i. From this, in this section, we consider to extend the previous results to the case that the agents have inherent dynamics.

From the results in the previous sections, we can assume that the essence of the mechanism of the optimization is the incremental passivity of the algorithm for agents. This suggests that if the inherent dynamics of agents is incrementally passive, then the similar stability to the optimal solutions will be derived with appropriate decentralized control inputs. This conjecture is true from the following theorem:

Theorem 5.1 If agent i has a inherent dynamics

$$z_i = f_i v_i$$

where z_i is d_i, s_i, q_i, or p_i, and f_i is a positive real function having a pole at the origin, then, with a control input

$$v_i = \begin{cases} u_i^{'}(d_i) - p_i & \text{for consumer} \\ p_i - c_i^{'}(s_i) & \text{for supplier} \\ -t_i^{'}(q_i) + \left(D^\top \boldsymbol{p}\right)_i & \text{for transmissioner} \\ d_i - s_i - (D\boldsymbol{q})_i & \text{for ISO} \end{cases}$$

for agent i and with the price mechanism (3)–(6) for other agents, the algorithm converges to the optimal solution $(\boldsymbol{d}^*, \boldsymbol{s}^*, \boldsymbol{q}^*, \boldsymbol{p}^*)$ of the social welfare problem asymptotically.

Proof. In the following, we omit the index i for simplicity. From the assumption of positive real, a minimal state space realization (A, B, C, D) of

f has solutions X, E, F of the following matrix equations from KYP lemma [1]:

$$\begin{aligned} A^{\mathrm{T}}X + XA &= -E^{\mathrm{T}}E \\ XB &= C^{\mathrm{T}} - E^{\mathrm{T}}F \\ F^{\mathrm{T}}F &= D^{\mathrm{T}} + D \\ X &= X^{\mathrm{T}} > O \end{aligned} \tag{9}$$

Then we define a storage function for this agent by

$$S = \frac{1}{2}(x - x^*)^{\mathrm{T}}X(x - x^*)$$

where x^* is a solution giving z_i^*. Existence of such x^* is guaranteed by the assumption that f has a pole at the origin. Then, by using (9), its derivative is given as follows:

$$\begin{aligned} \frac{d}{dt}S =& \frac{1}{2}\{(\dot{x} - \dot{x}^*)^{\mathrm{T}}X(x - x^*) + (x - x^*)X(\dot{x} - \dot{x}^*)\} \\ =& \frac{1}{2}\{((Ax + Bv) - (Ax^* + Bv^*))^{\mathrm{T}}X(x - x^*) \\ &+ (x - x^*)X((Ax + Bv) - (Ax^* + Bv^*))\} \\ =& -\frac{1}{2}(Ex + Fv)^{\mathrm{T}}(Ex + Fv) - \frac{1}{2}(Ex^* + Fv^*)^{\mathrm{T}}(Ex^* + Fv^*) \\ &+ v^{\mathrm{T}}z + v^{*\mathrm{T}}z^* \\ &+ \frac{1}{2}\{-(Ax + Bv)^{\mathrm{T}}Xx^* - (Ax^* + Bv^*)^{\mathrm{T}}Xx \\ &- x^{\mathrm{T}}X(Ax^* + Bv^*) - x^{*\mathrm{T}}X(Ax + Bv)\} \\ =& -\frac{1}{2}(Ex + Fv)^{\mathrm{T}}(Ex + Fv) - \frac{1}{2}(Ex^* + Fv^*)^{\mathrm{T}}(Ex^* + Fv^*) \\ &+ \frac{1}{2}(Ex + Fv)^{\mathrm{T}}(Ex^* + Fv^*) + \frac{1}{2}(Ex^* + Fv^*)^{\mathrm{T}}(Ex + Fv) \\ &+ v^{\mathrm{T}}z + v^{*\mathrm{T}}z^* - v^{\mathrm{T}}z^* - v^{*\mathrm{T}}z \\ =& -\frac{1}{2}\{(Ex + Fv) - (Ex^* + Fv^*)\}^{\mathrm{T}}\{(Ex + Fv) - (Ex^* + Fv^*)\} \\ &+ (v - v^*)^{\mathrm{T}}(z - z^*) \end{aligned}$$

For a case that the agent is a supplier, set a control input

$$v = p - c'(z),$$

then we get the following by the same discussion in the proof of Proposition 3.1:

$$\begin{aligned}\frac{d}{dt}S =& -\frac{1}{2}\{(Ex+Fv)-(Ex^*+Fv^*)\}^{\mathrm{T}}\{(Ex+Fv)-(Ex^*+Fv^*)\}\\&+\{(p-c'(z))-(p^*-c'(z^*))\}(z-z^*)\\=& -\frac{1}{2}\{(Ex+Fv)-(Ex^*+Fv^*)\}^{\mathrm{T}}\{(Ex+Fv)-(Ex^*+Fv^*)\}\\&+(p-p^*)(z-z^*)-(c'(z)-c'(z^*))(z-z^*)\\=& -\frac{1}{2}\{(Ex+Fv)-(Ex^*+Fv^*)\}^{\mathrm{T}}\{(Ex+Fv)-(Ex^*+Fv^*)\}\\&+(p-p^*)(z-z^*)-c''(\xi)(z-z^*)^2\end{aligned}$$

This implies agent i is still incrementally passive where we regard p is the input and z is the output. In cases of other agent types such as a consumer, the same results are derived. Finally, by employing the similar discussion in the proof of Theorem 3.1, we can conclude the statement of this theorem.

Remark 5.1 Depending on situations, the transfer function f_i can be also regarded as the closed loop dynamics composed of a inherent dynamics and a controller, or that of a controller itself, therefore, this result suggests a wide class of control mechanism including the result of Theorem 4.1 to attain the stability of the algorithm.

Remark 5.2 The similar consideration is seen in [4], however, it is not aware of the essence, that is passivity, in the primal-dual optimization, and a strictly decentralized optimization is not given.

6 Numerical example

In this section, numerical examples are shown in order to confirm effectiveness of our proposing method. Utility functions of consumers are $u_i(d) = i \log d$, $i = 1, 2, \ldots, N$, cost functions of suppliers are uniformly $c_i(s) = 0.5s^2$, $i = 1, 2, \ldots, N$, and cost functions of transmissioners are uniformly $t_l(q) = 0.1q^2$, $l = 1, 2, \ldots, E$. The network structure is given by Fig. 2. The simulation results in the cases of the price mechanism (3)–(6), and of the augmented price mechanism (3)–(5), (7), (8), where $K_i^P = 1$, $K_i^I = 1$, are shown in Fig. 4 and Fig. 5, respectively. It is known that the prices converge to different values depending on the regional utility functions from Fig. 4–5. Moreover, Fig. 5 shows that the time responses rapidly converge to the steady states and are considerably improved compared to Fig. 4. This implies an appropriate controller design for the price mechanism is very effective for control performances.

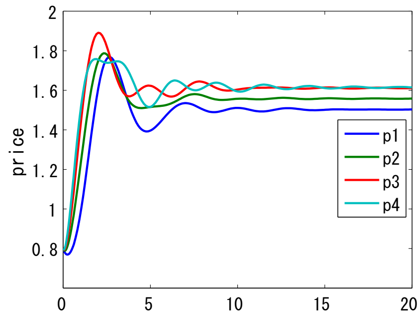

Figure 4: Time responses of prices of (3)–(6)

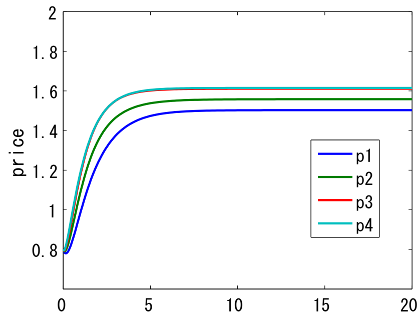

Figure 5: Time responses of prices of (3)–(5), (7), (8), where $K_i^P = 1$, $K_i^I = 1$

7 Conclusion

In this paper, we formulated a mathematical model of autonomous distributed control for smart grids based on a price mechanism considering the network structure. Then, we give a controller design in order to improve the control performance. Furthermore, we extended this result for more general cases that the agents include inherent dynamics in a class of positive real functions. Finally, we demonstrated numerical simulations to show that the control performance can be considerably improved by using our proposing controller for the price mechanism.

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