MATHEMATICAL ENGINEERING TECHNICAL REPORTS

On Half-integrality of Network Synthesis Problem

Than Nguyen HAU, Hiroshi HIRAI, and Nobuyuki TSUCHIMURA

METR 2012-15 September 2012

DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

WWW page: http://www.keisu.t.u-tokyo.ac.jp/research/techrep/index.html

On Half-integrality of Network Synthesis Problem

Than Nguyen HAU*, Hiroshi HIRAI†, and Nobuyuki TSUCHIMURA‡

September, 2012

Abstract

Network synthesis problem is the problem of constructing a minimum cost network satisfying a given flow-requirement. A classical result of Gomory and Hu is that if the cost is uniform and the flow requirement is integer-valued, then there exists a half-integral optimal solution. They also gave a simple algorithm to find a half-integral optimal solution.

In this note, we show that this half-integrality and Gomory-Hu algorithm can be extended to a class of fractional cut-covering problems.

1 Introduction

Let K_V be a complete (undirected) graph on node set V. We are given a nonnegative integer-valued flow-requirement $r_{ij} \in \mathbf{Z}_+$ for each (unordered) pair ij of nodes. A nonnegative edge-capacity $x : E(K_V) \to \mathbf{R}_+$ is said to be *feasible* if, for every node-pair ij, the maximum value of an (i,j)-flow under the capacity x is at least r_{ij}. We are also given a nonnegative edge-cost $a : E(K_V) \to \mathbf{R}_+$. The *network synthesis problem* (NSP) is the problem of finding a feasible edge-capacity of the minimum cost, where the cost of edge-capacity x is defined as $\sum_{e \in E(K_V)} a(e)x(e)$. By max-flow min-cut theorem [2], NSP is formulated as:

$$\begin{aligned} \text{Min.} \quad & \sum_{e \in E(K_V)} a(e)x(e) \\ \text{s.t.} \quad & \sum_{e \in \delta X} x(e) \geq r_{ij} \quad (i, j \in V, X \subseteq V : i \in X \not\ni j), \\ & x : E(K_V) \to \mathbf{R}_+. \end{aligned}$$

Here δ_X denotes the set of edges joining X and $V \setminus X$.

A classical result by Gomory and Hu [6] is that NSP admits a half-integral optimal solution provided the edge cost is uniform.

Theorem 1.1 ([6]). *Suppose $a(e) = 1$ for $e \in E(K_V)$. Then we have the following:*

(1) *The optimal value of NSP is equal to $\sum_{i \in V} \max\{r_{ij} \mid j \in V \setminus \{i\}\}/2$.*

*Department of Mathematical Informatics, Graduate School of Information Science and Technology, The University of Tokyo, Tokyo, 113-8656, Japan. E-mail: than_nguyen_hau@mist.i.u-tokyo.ac.jp

†Department of Mathematical Informatics, Graduate School of Information Science and Technology, The University of Tokyo, Tokyo, 113-8656, Japan. E-mail: hirai@mist.i.u-tokyo.ac.jp

‡Department of Informatics, School of Science and Technology, Kwansei Gakuin University, Hyogo, 669-1337, Japan. E-mail: tutimura@kwansei.ac.jp

(2) *There exists a half-integral optimal solution in NSP.*

See [2, Chapter 4], [4, Section 7.2.3] and [8, Section 62.3]. Gomory and Hu [6] presented a simple algorithm to find a half-integral optimal solution, sketched as follows. Define an edge-weight r on K_V by $r(ij) := r_{ij}$. Compute a maximum weight spanning tree T of K_V with respect to r. Restrict r to $E(T)$. Then $r : E(T) \to \mathbf{Z}_+$ is uniquely decomposed as $r = \sum_{F \in \mathcal{T}} \sigma(F) 1_{E(F)}$ for a nested family $\mathcal{T}$ of subtrees in T and a positive integral weight σ on $\mathcal{T}$. For each subtree $F \in \mathcal{T}$, take a cycle C_F (in K_V) of vertices $V(F)$. Then $x = (\sum_{F \in \mathcal{T}} \sigma(F) 1_{E(C_F)})/2$ is an optimal solution of NSP with unit edge-cost. Here 1_Y denotes the incidence vector of a set Y, and a family $\mathcal{T}$ of subtrees of a tree T is said to be *nested* if, for $F, F' \in \mathcal{T}$, one of $V(F) \subseteq V(F')$, $V(F') \subseteq V(F)$, and $V(F) \cap V(F') = \emptyset$ holds.

In this note, we show that Theorem 1.1 together with Gomory-Hu algorithm can be extended to a class of fractional cut-covering problems. Let $f : 2^V \to \mathbf{Z}_+$ be a symmetric nonnegative integer-valued set function on V satisfying $f(\emptyset) = f(V) = 0$. Here a set function is called *symmetric* if it satisfies

$$f(X) = f(V \setminus X) \quad (X \subseteq V). \tag{1.1}$$

As above, we are given an edge-cost $a : E(K_V) \to \mathbf{R}_+$. Consider the following fractional cut-covering problem:

$$\begin{array}{lll} \text{NSP}[f]: & \text{Min.} & \displaystyle\sum_{e \in E(K_V)} a(e)x(e) \\ & \text{s.t.} & \displaystyle\sum_{e \in \delta X} x(e) \geq f(X) \quad (X \subseteq V), \\ & & x : E(K_V) \to \mathbf{R}_+. \end{array}$$

NSP is a special case of NSP[f]. Indeed, for flow-requirement r_{ij}, define R by

$$R(X) := \max\{r_{ij} \mid i \in X \not\ni j\} \quad (\emptyset \neq X \subset V), \tag{1.2}$$

and $R(\emptyset) = R(V) = 0$. Then NSP[R] coincides with NSP.

Our result is about a half-integrality property of NSP[f] for a special set function f and a special edge-cost a. Let us introduce a few notions to mention our result. A symmetric set function f is *skew-supermodular* if it satisfies

$$f(X) + f(Y) \leq \max\{f(X \cap Y) + f(X \cup Y), f(X \setminus Y) + f(Y \setminus X)\} \quad (X, Y \subseteq V). \tag{1.3}$$

We say that a skew-supermodular function f is *normal* if it satisfies

$$f(X) + f(Y) - f(X \cup Y) \geq 0 \quad (X, Y \subseteq V : X \cap Y = \emptyset), \tag{1.4}$$

and is *evenly-normal* if it satisfies

$$f(X) + f(Y) - f(X \cup Y) \in 2\mathbf{Z}_+ \quad (X, Y \subseteq V : X \cap Y = \emptyset). \tag{1.5}$$

We say that a pair of $X, Y \subseteq V$ is *crossing* if all $X \cap Y$, $V \setminus (X \cup Y)$, $X \setminus Y$, and $Y \setminus X$ are nonempty. A family $\mathcal{F} \subseteq 2^V$ is said to be *cross-free* if $\mathcal{F}$ has no crossing pair. The main result of this note is the following.

Theorem 1.2. *Suppose that f is evenly-normal skew-supermodular and there exist a cross-free family $\mathcal{F}$ and a nonnegative weight $l : \mathcal{F} \to \mathbf{R}_+$ with $a = \sum_{X \in \mathcal{F}} l(X) 1_{\delta X}$. Then we have the following:*

(1) *The optimal value of NSP[f] is equal to* $\sum_{X\in\mathcal{F}} l(X)f(X)$.

(2) *There exists an integral optimal solution in NSP[f].*

This theorem includes the half-integrality for NSP[f] for a normal skew-supermodular function f. One can see this fact from: (1) if f is normal skew-supermodular, then $2f$ is evenly-normal skew-supermodular, and (2) if x is optimal to NSP[$2f$], then $x/2$ is optimal to NSP[f]. Also Theorem 1.2 includes Theorem 1.1. Indeed, it is easy to see that R is normal skew-supermodular (the skew-supermodularity of R is well-known [4, Lemma 8.1.9]). Since the unit cost is represented as $\sum_{i\in V}(1/2)1_{\delta\{i\}}$, we can take $\{\{i\} \mid i \in V\}$ as $\mathcal{F}$, with $l(\{i\}) := 1/2$ $(i \in V)$. Applying Theorem 1.2 to NSP[$2R$], we obtain Theorem 1.1.

The proof of Theorem 1.2 is given in the next section. Our proof is algorithmic, and gives a simple greedy-type algorithm extending Gomory-Hu algorithm.

2 Proof

We need two lemmas. The first lemma is a general property of a symmetric skew-supermodular function. We denote $\sum_{e\in F} x(e)$ by $x(F)$ for $F \subseteq E(K_V)$.

Lemma 2.1. *Let* $f : 2^V \to \mathbf{Z}_+$ *be a symmetric skew-supermodular function and* $\mathcal{F}$ *a cross-free family on* V. *If* $x : E(K_V) \to \mathbf{R}_+$ *satisfies* $x(\delta X) = f(X)$ *for all* $X \in \mathcal{F}$, *then one of the following holds:*

(1) x *satisfies* $x(\delta X) \geq f(X)$ *for all* $X \subseteq V$.

(2) *There exists* $W \subseteq V$ *such that* $x(\delta W) < f(W)$ *and* $\mathcal{F} \cup \{W\}$ *is cross-free.*

In particular, if $\mathcal{F}$ *is a maximal cross-free family, then* (1) *holds.*

Proof. By symmetry, we may assume $Y \in \mathcal{F} \Leftrightarrow V \setminus Y \in \mathcal{F}$. We use the induction on $|V|$. Suppose that (1) does not hold. Then there is $Z \subseteq V$ with $x(\delta Z) < f(Z)$. If $\mathcal{F} \cup \{Z\}$ is cross-free, then (2) holds, as required. Suppose that $\mathcal{F} \cup \{Z\}$ is not cross-free. Then there is $Y \in \mathcal{F}$ such that (Y, Z) is crossing. By the skew-supermodularity of f, we have

$$f(Y) + f(Z) \leq f(Y \cap Z) + f(Y \cup Z) \text{ or } f(Y) + f(Z) \leq f(Y \setminus Z) + f(Z \setminus Y).$$

By symmetry, we may assume the first case; otherwise replace Y by $V \setminus Y$. By $x(\delta Y) = f(Y)$ and $x(\delta Z) < f(Z)$, we have

$$x(\delta Y) + x(\delta Z) < f(Y) + f(Z) \leq f(Y \cap Z) + f(Y \cup Z). \tag{2.1}$$

By $x \geq 0$ we also have

$$x(\delta(Y \cap Z)) + x(\delta(Y \cup Z)) \leq x(\delta Y) + x(\delta Z). \tag{2.2}$$

By (2.1) and (2.2) we have $x(\delta(Y \cap Z)) < f(Y \cap Z)$ or $x(\delta(Y \cup Z)) < f(Y \cup Z)$. Again, by symmetry, we may assume

$$x(\delta(Y \cap Z)) < f(Y \cap Z). \tag{2.3}$$

Otherwise replace Y by $V \setminus Y$ and replace Z by $V \setminus Z$.

In K_V, contract all edges with both ends belonging to $V \setminus Y$. Then $V \setminus Y$ is contracted into one node r. The resulting graph is a complete graph $K_{V'}$ on node set $V' := Y \cup \{r\}$.

Since (Y, Z) is crossing, both $Z \setminus Y$ and $V \setminus (Y \cup Z)$ are nonempty, and hence $|V \setminus Y| \geq 2$ and $|V'| < |V|$. Define a family $\mathcal{F}'$ on V' by

$$\mathcal{F}' := \bigcup_{X \in \mathcal{F}: X \subseteq Y} \{X\} \cup \bigcup_{X \in \mathcal{F}: X \cup Y = V} \{(X \cap Y) \cup \{r\}\}.$$

Then $\mathcal{F}'$ is a cross-free family on V'. Let f' be a set function on V' defined by

$$f'(X) := \begin{cases} f(X) & \text{if } r \notin X, \\ f((X \setminus \{r\}) \cup (V \setminus Y)) & \text{if } r \in X, \end{cases} \quad (X \subseteq V' = Y \cup \{r\}). \tag{2.4}$$

Then f' is symmetric skew-supermodular on V'. By construction, we can regard $E(K_{V'})$ as $E(K_{V'}) \subseteq E(K_V)$. Let x' denote the restriction of x to $E(K_{V'})$, and let $\delta' X$ denote the set of edges joining X and $V' \setminus X$ in $K_{V'}$. Then we have

$$x'(\delta' X) = x(\delta X) \quad (X \subseteq Y). \tag{2.5}$$

Thus we have $x'(\delta' X) = f'(X)$ $(X \in \mathcal{F}')$, and $x'(\delta'(Y \cap Z)) < f'(Y \cap Z)$. Recall $|V'| < |V|$. By induction, there exists $W \subseteq V'$ such that $x'(\delta' W) < f'(W)$ and $\mathcal{F}' \cup \{W\}$ is cross-free. By symmetry, we may assume $W \subseteq Y$, i.e., W is a subset of V. Then, by (2.4) and (2.5), we have $x(\delta W) < f(W)$. Also $\mathcal{F} \cup \{W\}$ is cross-free in V. □

The second lemma is about the path decomposition of a capacitated trivalent tree. A tree is said to be *trivalent* if each node that is not a leaf has degree three, where a *leaf* of a tree is a node of degree one.

Lemma 2.2. *Let T be a trivalent tree, and $c : E(T) \to \mathbf{Z}_+$ an integer-valued edge-capacity. If $c(e) + c(e') - c(e'') \in 2\mathbf{Z}_+$ holds for every pairwise-incident triple (e, e', e'') of edges, then there exist a set $\mathcal{P}$ of simple paths connecting leaves and an integral weight $\lambda : \mathcal{P} \to \mathbf{Z}_+$ such that $\sum_{P \in \mathcal{P}} \lambda(P) 1_{E(P)} = c$.*

Proof. For every incident pair e, e' of edges, define $l(e, e')$ by

$$l(e, e') := (c(e) + c(e') - c(e''))/2,$$

where e'' is the third edge incident to e and to e'. Then $l(e, e')$ is a nonnegative integer, and $c(e) = l(e, e') + l(e, e'')$. $(\mathcal{P}, \lambda)$ is constructed as follows, where let $\mathcal{P} := \emptyset$ initially.

Take edge $e = uv$ with $c(e) > 0$. Suppose that u is not a leaf. Then there is an edge e' incident to u with $l(e, e') > 0$. Necessarily $c(e') > 0$ (otherwise $c(e') = 0$ and $l(e, e') = 0$). Hence we can extend e to a simple path $P = (e_0, e_1, \ldots, e_k)$ connecting leaves. Add P to $\mathcal{P}$. Define $\lambda(P) := \min_{i=1,\ldots,k} l(e_{i-1}, e_i)$ (> 0). Let $\tilde{c} := c - \lambda(P) 1_{E(P)}$. Then $\tilde{c}$ satisfies the condition of this lemma. To see this, take an arbitrary pairwise-incident triple (e, e', e'') of edges. We show $\tilde{c}(e) + \tilde{c}(e') - \tilde{c}(e'') \in 2\mathbf{Z}_+$. Here $E(P) \cap \{e, e', e''\}$ is $\emptyset$, $\{e', e''\}$, $\{e, e''\}$, or $\{e, e'\}$. For the first three cases, we have $\tilde{c}(e) + \tilde{c}(e') - \tilde{c}(e'') = c(e) + c(e') - c(e'') \in 2\mathbf{Z}_+$. For the last case, we have $\tilde{c}(e) + \tilde{c}(e') - \tilde{c}(e'') = c(e) + c(e') - c(e'') - 2\lambda(P)$, which must be an nonnegative even integer by definition of $\lambda(P)$.

Let $c \leftarrow \tilde{c}$, and repeat this process. In each step, at least one of $l(e, e')$ is zero. After $O(|V(T)|)$ step, we have $c = 0$ and obtain a desired $(\mathcal{P}, \lambda)$. □

Proof of Theorem 1.2. Consider the LP-dual of NSP$[f]$, which is given by

$$\begin{array}{lll} \text{DualNSP}[f]: & \text{Max.} & \displaystyle\sum_{X \subseteq V} \pi(X) f(X) \\ & \text{s.t.} & \displaystyle\sum_{X \subseteq V} \pi(X) 1_{\delta X} \leq a \\ & & \pi : 2^V \to \mathbf{R}_+. \end{array}$$

Suppose that a is represented by $a = \sum_{X \in \mathcal{F}} l(X) 1_{\delta X}$ for some cross-free family $\mathcal{F}$ and some positive weight l on $\mathcal{F}$. Define $\pi : 2^V \to \mathbf{R}_+$ by

$$\pi(X) = \begin{cases} l(X) & \text{if } X \in \mathcal{F}, \\ 0 & \text{otherwise,} \end{cases} \qquad (X \subseteq V).$$

Then π is feasible to DualNSP[f] with the objective value $\sum_{X \in \mathcal{F}} l(X) f(X)$. We are going to construct a feasible integral solution x in NSP[f] satisfying

$$x(\delta X) = f(X) \quad (X \in \mathcal{F}). \tag{2.6}$$

If this is possible, then, by the complementary slackness, x is optimal to NSP[f] and π is optimal to DualNSP[f]; hence Theorem 1.2 is proved.

Take a maximal cross-free family $\mathcal{F}^*$ including $\mathcal{F}$. Here recall the tree-representation of a cross-free family; see [4, Section 1.4] and [8, Section 13.4]. By the maximality of $\mathcal{F}^*$, there exists a trivalent tree T on vertex set $V \cup I$ with the following properties:

(2.7) (1) V is the set of leaves of T, and I is the set of non-leaf nodes.

(2) $\mathcal{F}^* \setminus \{\emptyset, V\} = \bigcup_{e \in E(T)} \{A_e, B_e\}$, where $\{A_e, B_e\}$ denotes the bipartition of V such that A_e (or B_e) is the set of leaves of one of components of $T - e$.

Define edge-weight $c : E(T) \to \mathbf{Z}_+$ by

$$c(e) := f(A_e) (= f(B_e)) \quad (e \in E(T)). \tag{2.8}$$

By symmetry (1.1) and the evenly-normal property (1.5) of f, for each pairwise-incident triple (e, e', e'') of edges in T, we have

$$c(e) + c(e') - c(e'') = f(A_e) + f(A_{e'}) - f(A_{e''}) \in 2\mathbf{Z}_+,$$

where we can assume $A_e \cap A_{e'} = \emptyset$ and $A_{e''} = A_e \cup A_{e'}$. By Lemma 2.2, there exist a set $\mathcal{P}$ of simple paths connecting V and a positive integral weight λ on $\mathcal{F}$ with $\sum_{P \in \mathcal{P}} \lambda(P) 1_{E(P)} = c$. Define $x : E(K_V) \to \mathbf{Z}_+$ by

$$x(ij) := \begin{cases} \lambda(P) & \text{if } \exists P \in \mathcal{P} : P \text{ connects } i \text{ and } j, \\ 0 & \text{otherwise,} \end{cases} \qquad (ij \in E(K_V)). \tag{2.9}$$

Since each P is simple, we have

$$x(\delta A_e) = c(e) = f(A_e) \quad (e \in E(T)).$$

By (2.7) (2), this implies

$$x(\delta X) = f(X) \quad (X \in \mathcal{F}^*).$$

By Lemma 2.1, x is feasible to NSP[f]. By $\mathcal{F} \subseteq \mathcal{F}^*$, x satisfies (2.6). Therefore, x is an integral optimal solution in NSP[f], π is an optimal solution in DualNSP[f], and the optimal value is equal to $\sum_{X \in \mathcal{F}} l(X) 1_{\delta X}$. $\square$

Algorithm to find an integral optimal solution in Theorem 1.2. Our proof gives the following simpler $O(n\gamma + n^2)$ algorithm to find an integral optimal solution, where $n := |V|$, and γ denotes the time complexity of evaluating f.

step 1: Take a maximal cross-free family $\mathcal{F}^*$ including $\mathcal{F}$.

step 2: Construct a trivalent tree T with (2.7).

step 3: Define edge-weight c by (2.8).

step 4: Decompose c as $c = \sum_{P \in \mathcal{P}} \lambda(P) 1_{E(P)}$ according to the proof of Lemma 2.2.

step 5: Define x by (2.9), and then x is an integral optimal solution in NSP$[f]$.

Steps 1,2 can be done in $O(n)$ time, step 3 can be done by $O(n)$ calls of f, and steps 4,5 can be done in $O(n^2)$ time.

Gomory-Hu algorithm reconsidered. Gomory-Hu algorithm can be viewed as a special case of our algorithm. First note that, in the case of unit cost, we can take an arbitrary maximal cross-free family in step 1. Consider a maximum spanning tree T on K_V with respect to r. For $e \in E(T)$, let $\{A_e, B_e\}$ denote the bipartition of V determined by $T - e$. Then $\mathcal{F} := \bigcup_{e \in E(T)} \{A_e, B_e\}$ is cross-free. Extend $\mathcal{F}$ to a maximal cross-free family $\mathcal{F}^*$. Take a trivalent tree $\bar{T}$ corresponding to $\mathcal{F}^*$. Define $c : E(\bar{T}) \to \mathbf{Z}_+$ by (2.8) with $f := R$. Here we note that R has the following property, which is stronger than (1.4):

$$\max\{R(A), R(B)\} \geq R(A \cup B) \quad (A, B \subseteq V : A \cap B = \emptyset).$$

By symmetry, the maximum of $R(A)$, $R(B)$, and $R(A \cup B)$ is attained at least twice. This in turn implies the following property of c:

(2.10) For each pairwise-incident triple (e, e', e'') of edges, the maximum of $c(e)$, $c(e')$, and $c(e'')$ is attained at least twice.

Represent c as $c = \sum_{F \in \bar{\mathcal{T}}} \sigma(F) 1_{E(F)}$ for a nested family of subtrees $\bar{\mathcal{T}}$ and a positive integral weight σ on $\bar{\mathcal{T}}$. By (2.10), the set of leaves of each subtree $F \in \bar{\mathcal{T}}$ belongs to V. Therefore we may apply the path decomposition in Lemma 2.2 to each $\sigma(F) 1_{E(F)}$ independently. From the path decomposition of $\sigma(F) 1_{E(F)}$, we obtain $x_F := (\sigma(F)/2) 1_{E(C_F)}$, where a cycle C_F of vertices $V(F)$ in K_V. Then $x := \sum_{F \in \bar{\mathcal{T}}} x_F$ is optimal.

By construction, T can be regarded as a tree obtained by contracting some of edges of $\bar{T}$. So we can regard $E(T)$ as $E(T) \subseteq E(\bar{T})$. Since T is a maximum spanning tree, we have

$$r(e) = R(A_e) (= R(B_e)) \quad (e \in E(T)).$$

This means that r coincides with the restriction of c to $E(T)$. Also one can see from definition of R that the nested family obtained from $\bar{\mathcal{T}}$ by contracting the edges coincides with the nested family $\mathcal{T}$ in Gomory-Hu algorithm (see Introduction). Therefore, the above-mentioned process coincides with Gomory-Hu algorithm.

Remark 2.3. Frank [3] proved, from a general framework of edge-splitting, that in the case of uniform cost there also exists a half-integral optimal solution in NSP$[R]$ with lower-bound constraint $x \geq g$ for $g : E(K_V) \to \mathbf{Z}_+$; see [3, Section 11.1.4]. Our framework cannot explain this half-integrality (since a symmetric function $f - g \circ \delta$ defined by $(f - g \circ \delta)(X) := f(X) - g(\delta X)$ is not normal in general even if f is normal skew-supermodular).

Remark 2.4. A *tree metric* is a metric represented by the distances between a subset of vertices in a weighted tree. The cost function treated in Theorem 1.2 is nothing but a tree metric; this fact can easily be seen from the tree-representation of a cross-free family. A tree metric is a fundamental object in *phylogenetic combinatorics*, combinatorics for phylogenetic trees in biology [1]. In the literature, there are many $O(n^2)$ algorithms to construct a weighted tree (phylogenetic tree) realizing a given distance d on an n-element set V if d is a tree metric; *neighbor-joining* [7] is a popular method. By using these algorithms, the expression $a = \sum_{X \in \mathcal{F}} \pi(X) 1_{\delta X}$ in Theorem 1.2 is obtained in $O(n^2)$ time if it exists.

Remark 2.5. Lemma 2.1 is viewed as a symmetric analogue of the following well-property of submodular functions: If f is a submodular function on V and $x : V \to \mathbf{R}$ satisfies $x(Y) = f(Y)$ $(Y \in \mathcal{F})$ for some maximal chain $\mathcal{F}$ in 2^V, then $x(X) \leq f(X)$ for all $X \subseteq V$. See [4, 5, 8]. This property guarantees the correctness of the greedy algorithm for the base polytope. Also in our algorithm, Lemma 2.1 is used for a similar purpose. So our algorithm may be a symmetric analogue of the greedy algorithm.

Acknowledgments

The author is partially supported by a Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology of Japan, and is partially supported by Aihara Project, the FIRST program from JSPS.

References

[1] A. Dress, K.T. Huber, J. Koolen, V. Moulton, and A. Spillner, *Basic Phylogenetic Combinatorics*, Cambridge University Press, Cambridge, 2012.

[2] L. R. Ford, Jr. and D. R. Fulkerson, *Flows in Networks*, Princeton University Press, Princeton, 1962.

[3] A. Frank, Augmenting graphs to meet edge-connectivity requirements. *SIAM Journal on Discrete Mathematics* **5** (1992), 25–53.

[4] A. Frank, *Connections in Combinatorial Optimization*, Oxford University Press, Oxford, 2011.

[5] S. Fujishige, *Submodular Functions and Optimization, 2nd Edition*, Elsevier, Amsterdam, 2005.

[6] R. E. Gomory and T. C. Hu, Multi-terminal network flows, *Journal of the Society for Industrial and Applied Mathematics* **9** (1961), 551–570.

[7] N. Saitou and M. Nei, The neighbor-joining method: A new method for reconstructing phylogenetic trees, *Molecular Biology and Evolution* **4** (1987), 406–425.

[8] A. Schrijver, *Combinatorial Optimization*, Springer-Verlag, Berlin, 2003.