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Damper Placement Optimization in a Shear Building Model with Discrete Design Variables: A Mixed-Integer Second-Order Cone Programming Approach

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Abstract

Supplemental damping is known as an efficient and practical means to improve seismic response of building structures. Presented in this paper is a mixed-integer programming approach to find the optimal placement of supplemental dampers in a given shear building model. The damping coefficients of dampers are treated as discrete design variables. It is shown that a minimization problem of the sum of the transfer function amplitudes of the interstory drifts can be formulated as a mixed-integer second-order cone programming problem. The global optimal solution of the optimization problem is then found with an existing algorithm. Two numerical examples in literature are solved with discrete design variables. In one of these examples, the proposed method finds a better solution than an existing steepest-decent-type method for a continuous version of the optimization problem.

Keywords

Optimal damper placement; aseismic design; transfer function; structural control; mixed-integer programming; global optimization.

1 Introduction

Supplemental damping has became a reliable and practical seismic design strategy including habilitation and retrofit of existing buildings. The placement of dampers is a key to efficiency of seismic design. This paper concerns optimization of placement of viscous dampers in a given shear building model. Particularly, an attempt is made to treat the damper damping coefficients as discrete design variables.

Several methods have been proposed for design of passive damping. Shukla and Datta [22] used a controllability index to define the optimal placement of dampers and proposed a procedure to introduce damper units sequentially. This heuristic approach was called a *sequential search algorithm* later by López García [17] and has been studied extensively [17, 22, 35]. Gluck *et al.* [7] adopted the linear-quadratic regulator in the theory of optimal control to design of damper distribution. Agranovich and Ribakov [1] proposed a heuristic method modifying this solution as a realistic one with reference to price of damper devices and energy required to activate dampers. To optimize damper placement in a shear building model, Takewaki [26] formulated a minimization problem of the sum of the transfer functions of the interstory drifts evaluated at the undamped natural frequency of the

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building model. Then the optimization problem was solved with a steepest-descent-type algorithm based on the first-order sensitivity analysis. Subsequently, this method has been extended for more complex structures [27, 28, 30]. A gradient-based optimization method was also proposed by Singh and Moreschi [23]. Lavan and Levy [15] proposed an iterative procedure consisting of analysis and redesign. This procedure was inspired by the fully stressed design of a truss, that minimizes the structural weight under the stress constraints; see also Levy and Lavan [16] for comparison of this method with an optimal control using the Riccati equation. López García [17] performed numerical experiments with recorded ground motions and found that the damper placement obtained by the method of Takewaki [26] effectively reduces the sum of interstory drifts. Whittle *et al.* [33] performed comparison of three representative methods for damper placement, i.e., a simplified sequential search algorithm [17], minimization of sum of the transfer functions of interstory drifts [26], and a fully stressed design approach [15], with the uniform and stiffness-proportional damping distributions. In this comparison these three methods showed broadly comparable performance in reduction of peak interstory drift, absolute acceleration, and residual interstory drift.

Among these methods, this paper revisits the seminal work of Takewaki [26] in optimization of passive damper placement. While Takewaki [26] performed continuous optimization, in this paper the damping coefficients of dampers, i.e., the design variables, are considered discrete variables. Specifically, the damping coefficient of the damper placed at each story is supposed to be chosen from $\{0, \bar{c}, 2\bar{c}, \dots, p\bar{c}\}$, where a scalar $\bar{c} > 0$ and an integer $p > 0$ are specified. We show that this discrete optimization problem can be reduced to a *mixed-integer programming* (MIP) problem. More precisely, this problem is of the form

$$\begin{aligned} \min \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{r}^{\mathrm{T}}\boldsymbol{y} \\ \mathrm{s.\,t.} \quad & \|A_i\boldsymbol{x} + G_i\boldsymbol{y} - \boldsymbol{b}_i\| \le \boldsymbol{d}_i^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{e}_i^{\mathrm{T}}\boldsymbol{y} - h_i, \quad i = 1, \dots, k, \\ & \boldsymbol{x} \in \{0,1\}^n, \\ & \boldsymbol{y} \in \mathbb{R}^p. \end{aligned}$$

Here, $\boldsymbol{x}$ and $\boldsymbol{y}$ are variables to be optimized, $A_i \in \mathbb{R}^{m_i \times n}$ and $G_i \in \mathbb{R}^{m_i \times p}$ $(i = 1, \dots, k)$ are constant matrices, $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{r} \in \mathbb{R}^p$, $\boldsymbol{b}_i \in \mathbb{R}^{m_i}$, $\boldsymbol{d}_i \in \mathbb{R}^n$, and $\boldsymbol{e}_i \in \mathbb{R}^p$ $(i = 1, \dots, k)$ are constant vectors, and $h_i \in \mathbb{R}$ $(i = 1, \dots, k)$ are constant scalars. This optimization problem is called a *mixed-integer second-order cone programming problem* (also called a *mixed-integer conic quadratic programming problem*). If we relax binary constraints to continuous constraints, $0 \le x_j \le 1$ $(j = 1, \dots, n)$, then this problem is reduced to a *second-order cone programming* (SOCP) problem. Since an SOCP problem is a convex optimization problem, a global solution of a mixed-integer second-order cone programming problem can be found by using, e.g., a branch-and-bound method; see, e.g., Atamtürk and Narayanan [4], Drewes and Pokutt [6] and Vielma *et al.* [32] for more account. This guaranteed global optimality is a major advantage of the proposed approach to the existing local and/or heuristic algorithms for design of damper distribution.

The formulation proposed in this paper can be viewed as a natural extension of MIP formulations of topology optimization of trusses with discrete member cross-sectional areas [11, 20] and that of continua with binary design variables [25]. There exists a difference, however, that the optimization problems in the literature cited above are mixed-integer linear programming problems, while this paper addresses a mixed-integer nonlinear programming problem.

Optimization with discrete damper coefficients that performed in the present paper might have the following importance:

- It is often in practice that a damper capacity should be chosen among available candidates due to manufacturing and commercial convenience. In such a situation discrete optimization provides us with more realistic solutions than continuous optimization.

- The proposed approach can find a global optimal solution of the discrete optimization problem. Guarantee of global optimality is a distinguished feature of this approach. A numerical example in section 4.2 will demonstrate that the proposed method finds a better solution than the existing method in [26].
- The proposed approach solves a mixed-integer second-order cone programming problem. To this purpose several well-developed software packages, e.g., Gurobi Optimizer [8], CPLEX [9], and MOSEK [19], are available. Therefore, there is no need to implement optimization algorithms. Also, algorithms specialized for damper placement optimization are not required.
- Combinatorial constraints on damper placement can be treated within the framework of the proposed approach. Typical examples of such constraints will appear in section 3.4.

A potential disadvantage of the proposed approach is that computational cost to solve the optimization problem might increase drastically as the number of design variables increases and it might be difficult to solve large-scale problems. This is because the approach is essentially based on enumeration of solutions using, e.g., a branch-and-bound method.

In view of discrete optimization, it is relevant that meta-heuristics have been applied to damper placement problems with discrete design variables [5, 13, 24, 31, 34]. Among them, Singh and Moreschi [24] used a genetic algorithm to solve an optimal placement problem of viscous and viscoelastic dampers with discrete damping coefficients. Lavan and Dargush [13] proposed a genetic algorithm for multi-objective optimization in which type of a damper is also treated as a design variable. Again, a potential advantage of the MIP approach to these meta-heuristics is guaranteed convergence to a global optimal solution.

The paper is organized as follows. In section 2 we recall the optimal damper placement problem that minimizes the sum of the transfer functions of interstory drifts. In section 3 this problem with discrete design variables is formulated as a mixed-integer second-order cone programming problem. Two numerical examples are demonstrated in section 4. We conclude in section 5.

A few words regarding our notation: All vectors are column vectors. We use $\mathbf{1} = (1, 1, \dots, 1)^{\mathrm{T}}$ to denote the all-ones vector. We use $\mathrm{diag}(\boldsymbol{a})$ to denote the $n \times n$ diagonal matrix with a vector $\boldsymbol{a} \in \mathbb{R}^n$ on its diagonal. We denote by i the imaginary unit. For a complex number $z \in \mathbb{C}$, we use $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ to denote its real and imaginary parts, respectively. We denote by $|z|$ the modulus of $z \in \mathbb{C}$, i.e., $|z| = [(\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2]^{1/2}$.

2 Definition of optimization problem

In section 2.1 the equation of motion in the frequency domain is recalled for a shear building model. Section 2.2 summarizes the optimal damper placement problem based upon minimizing the sum of the transfer functions of the interstory drifts evaluated at the undamped natural frequency.

2.1 Fundamentals of frequency-domain formulations

Consider an n-story shear building model with added supplementary viscous dampers. Figure 1 shows the case of $n = 3$. Let k_i and m_i denote the story stiffness and the mass, respectively, of story i. We use c_i to denote the damping coefficient of the viscous damper introduced to story

i. For simplicity, we assume that the inherent structural damping is negligible compared with the damping of the added dampers. Throughout the paper we suppose that $k_1, \dots, k_n$ and $m_1, \dots, m_n$ are given and that $c_1, \dots, c_n$ are design variables to be optimized.

Let $\boldsymbol{u} \in \mathbb{R}^n$ denote the displacement vector, where u_i is the displacement of mass m_i. We use $K \in \mathbb{R}^{n \times n}$ and $M \in \mathbb{R}^{n \times n}$ to denote the system stiffness and mass matrices, respectively, which are constant matrices. The damping matrix, denoted $C \in \mathbb{R}^{n \times n}$, depends on design variables $\boldsymbol{c} = (c_1, \dots, c_n)^{\mathrm{T}} \in \mathbb{R}^n$. This dependency is sometimes written explicitly as $C(\boldsymbol{c})$. Suppose that the structure undergoes base acceleration $\ddot{u}_{\mathrm{g}}$. Then the equation of motion is written as

$$K\boldsymbol{u} + C\dot{\boldsymbol{u}} + M\ddot{\boldsymbol{u}} = -M\ddot{u}_{\mathrm{g}}\mathbf{1}, \tag{1}$$

where $\mathbf{1} = (1, \dots, 1)^{\mathrm{T}} \in \mathbb{R}^n$.

Let d_i denote the interstory drift of the ith story and let $\boldsymbol{d} = (d_1, \dots, d_n)^{\mathrm{T}} \in \mathbb{R}^n$. The relation between $\boldsymbol{d}$ and $\boldsymbol{u}$ can be written as

$$\boldsymbol{d} = H^{\mathrm{T}}\boldsymbol{u}, \tag{2}$$

where $H \in \mathbb{R}^{n \times n}$ is a constant matrix with the form

$$H = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{3}$$

Let $\boldsymbol{v}(\omega)$ and $\ddot{v}_{\mathrm{g}}(\omega)$ denote the Fourier transforms of $\boldsymbol{u}$ and $\ddot{u}_{\mathrm{g}}$, respectively, where ω is the circular frequency of excitation. Fourier transformation of (1) reads

$$(K + \mathrm{i}\omega C - \omega^2 M)\boldsymbol{v}(\omega) = -M\ddot{v}_{\mathrm{g}}(\omega)\mathbf{1}, \tag{4}$$

which is the equation of motion in the frequency domain. Let $\delta_i(\omega)$ denote the Fourier transform of the interstory drift, d_i. It follows from (2) that δ_i's are related to $\boldsymbol{v}$ by

$$\boldsymbol{\delta}(\omega) = H^{\mathrm{T}}\boldsymbol{v}(\omega). \tag{5}$$

2.2 Minimization of transfer functions of interstory drifts

The optimal damper placement problem proposed by Takewaki [26] is recalled in this section. The problem attempts to minimize the sum of the transfer functions of the interstory drifts.

According to [26] (see also [28, Chap. 2]), attention is focused on the steady-state response at the undamped fundamental frequency, which usually plays a crucial role in dynamic behavior of a structure. Let $\bar{\omega}$ denote the fundamental natural circular frequency of the undamped structure, i.e., $\bar{\omega}$ is the minimum value of ω solving eigenvalue problem $K\boldsymbol{\phi} = \omega^2 M\boldsymbol{\phi}$. Define $\hat{\boldsymbol{v}} \in \mathbb{C}^n$ by

$$\hat{\boldsymbol{v}} = \frac{\boldsymbol{v}(\bar{\omega})}{\ddot{v}_{\mathrm{g}}(\bar{\omega})}. \tag{6}$$

Note that $\hat{v}_i$ is the transfer function of the floor displacement, u_i, evaluated at the undamped fundamental frequency of the structure. In other words, $\hat{v}_i$ is the resonant amplitude at the undamped fundamental frequency. It follows from (4) and (6) that $\hat{\boldsymbol{v}}$ is a solution of the following equation:

$$(K + \mathrm{i}\bar{\omega}C - \bar{\omega}^2 M)\hat{\boldsymbol{v}} = -M\mathbf{1}. \tag{7}$$

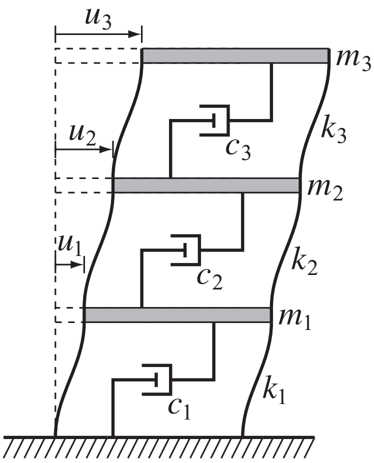

Figure 1: A three-story shear building model with supplementary viscous dampers.

Thus, $\hat{\boldsymbol{v}}$ is independent of the base acceleration, $\ddot{u}_{\rm g}$. Let $\hat{\boldsymbol{\delta}}$ denote a vector of the transfer functions of the interstory drifts evaluated at $\bar{\omega}$, i.e.,

$$\hat{\boldsymbol{\delta}} = \frac{\boldsymbol{\delta}(\bar{\omega})}{\ddot{v}_{\rm g}(\bar{\omega})}.$$

By using (5) and (6), $\hat{\boldsymbol{\delta}}$ can be written in terms of $\hat{\boldsymbol{v}}$ as

$$\hat{\boldsymbol{\delta}} = H^{\rm T}\hat{\boldsymbol{v}}. \tag{8}$$

Following Takewaki [26], we adopt the moduli of transfer functions of interstory drifts, $|\hat{\delta}_i(\boldsymbol{c})|$ ($i = 1, \dots, n$), as measures of structural response that to be minimized. Specifically, the minimization problem of the sum of $|\hat{\delta}_i(\boldsymbol{c})|$'s is formulated as

$$\min_{\boldsymbol{c}} \quad \sum_{i=1}^{n} |\hat{\delta}_i(\boldsymbol{c})| \tag{9a}$$

$$\text{s. t.} \quad \sum_{i=1}^{n} c_i \le c_{\rm sum}^{\rm max}, \tag{9b}$$

$$c_i \ge 0, \quad i = 1, \dots, n. \tag{9c}$$

Here, $c_{\rm sum}^{\rm max} > 0$ is the specified upper bound for the sum of the damper damping coefficients. Takewaki [26] derived the first-order optimality condition of problem (9) and proposed a steepest-descent-type algorithm.

3 Mixed-integer programming formulation

In section 3.1 the optimal damper placement problem, (9), is restated with discrete design variables. This optimization problem is reduced to a mixed-integer programming problem with second-order cone constraints in section 3.2. As a variant, the minimization problem of the maximum interstory drift is studied in section 3.3. Section 3.4 collects some combinatorial constraints on damper placement that can be treated with the proposed approach.

3.1 Discrete damping coefficients

In practical applications it is often that the damper capacity is chosen among available candidates for manufacturing and commercial reasons. This motivates us to treat damper damping coefficients as discrete design variables. Specifically, suppose that c_i is chosen from the finitely many given candidate values as

$$c_i \in \{0, \bar{c}, 2\bar{c}, \dots, p\bar{c}\}, \tag{10}$$

where a scalar $\bar{c} > 0$ and an integer $p > 0$ are constants. With reference to (9b), integer p is chosen as

$$p = \left\lfloor \frac{c_{\mathrm{sum}}^{\max}}{\bar{c}} \right\rfloor,$$

i.e., as the largest integer not greater than $c_{\mathrm{sum}}^{\max}/\bar{c}$. Alternatively, besides the constraints of problem (9), we might consider the constraints

$$c_i \le c^{\max}, \quad i = 1, \dots, n,$$

where $c^{\max}$ is the specified upper bound for c_i. In this case, p is given by

$$p = \left\lfloor \frac{c^{\max}}{\bar{c}} \right\rfloor.$$

When p is small and $\bar{c}$ is relatively large, $\bar{c}$ is considered the damping coefficient of a unit damper and p is the maximum number of unit dampers which can be placed at each story. Alternatively, when p is large and $\bar{c}$ is sufficiently small, (10) is considered an approximation of the continuous model, $0 \le c_i \le p\bar{c}$.

In problem (9), $\hat{\delta}_i$ is defined by (7) and (8) and its modulus is written as

$$|\hat{\delta}_i| = \sqrt{(\mathrm{Re}\,\hat{\delta}_i)^2 + (\mathrm{Im}\,\hat{\delta}_i)^2}.$$

Moreover, by introducing a new variable y_i satisfying $y_i \ge |\hat{\delta}_i|$ for each $i = 1, \dots, n$, minimization of $\sum_{i=1}^{n} |\hat{\delta}_i|$ is converted to minimization of $\sum_{i=1}^{n} y_i$. By using these relations and incorporating constraint (10), problem (9) is reduced to

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n} y_i && \text{(11a)}\\
\mathrm{s.\,t.} \quad & y_i \ge \left\| \begin{bmatrix} \mathrm{Re}\,\hat{\delta}_i \\ \mathrm{Im}\,\hat{\delta}_i \end{bmatrix} \right\|, \quad i = 1, \dots, n, && \text{(11b)}\\
& \hat{\boldsymbol{\delta}} = H^{\mathrm{T}} \hat{\boldsymbol{v}}, && \text{(11c)}\\
& (K + \mathrm{i}\bar{\omega} C(\boldsymbol{c}) - \bar{\omega}^2 M)\hat{\boldsymbol{v}} = -M\mathbf{1}, && \text{(11d)}\\
& \sum_{i=1}^{n} c_i \le c_{\mathrm{sum}}^{\max}, && \text{(11e)}\\
& c_i \in \{0, \bar{c}, 2\bar{c}, \dots, p\bar{c}\}, \quad i = 1, \dots, n. && \text{(11f)}
\end{aligned}$$

Here, $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{y} \in \mathbb{R}^n$, $\hat{\boldsymbol{\delta}} \in \mathbb{C}^n$, and $\hat{\boldsymbol{v}} \in \mathbb{C}^n$ are variables to be optimized. Note that the constraints in (11b) are called *second-order cone constraints*. A continuous optimization problem with a linear objective function and some second-order cone constraints are called a *second-order cone programming* (SOCP) problem. See, e.g., Alizadeh and Goldfarb [2] and Anjos and Lasserre [3] for fundamentals of SOCP; applications of SOCP in applied mechanics and structural engineering are found in [10, 12, 18, 36].

3.2 Reformulation to mixed-integer second-order cone programming

In this section we reformulate problem (11) to a form that can be solved by using an algorithm with guaranteed convergence to a global optimal solution. A key idea for this reformulation is making use of 0–1 variables to express (11f), i.e., the discreteness constraint on c_i. Specifically, for each story $i = 1, \dots, n$, we introduce variables $x_{ij} \in \{0, 1\}$ $(j = 1, \dots, p)$ satisfying

$$x_{i1} \ge x_{i2} \ge \cdots \ge x_{ip}. \tag{12}$$

Then (11f) can be rewritten as

$$c_i = \bar{c} \sum_{j=1}^{p} x_{ij} \tag{13}$$

for each $i = 1, \dots, n$.

Example 3.1. Suppose that $x_{i3} = 1$ and $x_{i4} = 0$. Since $x_{ij} \in \{0, 1\}$, (12) implies $x_{i1} = x_{i2} = 1$ and $x_{i5} = \cdots = x_{ip} = 0$. Therefore, the right-hand side of (13) is reduced to

$$\bar{c} \sum_{j=1}^{p} x_{ij} = 3\bar{c}.$$

This corresponds to $c_i = 3\bar{c}$. Also, $x_{i1} = 0$ corresponds to $c_i = 0$ and $x_{ip} = 1$ corresponds to $c_i = p\bar{c}$. ∎

Among the constrains of problem (11), constraint (11d) is nonconvex in terms of $\boldsymbol{c}$ and $\hat{\boldsymbol{v}}$. We next show that this constraint can be reduced to a tractable form by using integer variables x_{ij} $(i = 1, \dots, n;\ j = 1, \dots, p)$. Specifically, attention is focused on the bilinear term, $C(\boldsymbol{c})\hat{\boldsymbol{v}}$. Since we consider a shear building model, $C(\boldsymbol{c})$ in (11d) can be written as

$$C(\boldsymbol{c}) = H \operatorname{diag}(\boldsymbol{c}) H^{\mathrm{T}}, \tag{14}$$

where H is defined by (3). Let $\boldsymbol{q} \in \mathbb{C}^n$ and $\tilde{\boldsymbol{w}} \in \mathbb{C}^n$ be additional variables defined by

$$q_i = c_i \tilde{w}_i, \quad i = 1, \dots, n, \tag{15}$$

$$\tilde{w}_i = \boldsymbol{h}_i^{\mathrm{T}} \hat{\boldsymbol{v}}, \quad i = 1, \dots, n, \tag{16}$$

where $\boldsymbol{h}_i \in \mathbb{R}^n$ is the ith column vector of H, i.e.,

$$H = \left[\boldsymbol{h}_1 \mid \boldsymbol{h}_2 \mid \dots \mid \boldsymbol{h}_n \right].$$

From (14), we obtain $C(\boldsymbol{c})\hat{\boldsymbol{v}} = H\boldsymbol{q}$ when $\boldsymbol{q}$ and $\tilde{\boldsymbol{w}}$ satisfy (15) and (16). With this observation, we can see that (11d) can be written as

$$(K - \bar{\omega}^2 M)\hat{\boldsymbol{v}} + \mathrm{i}\bar{\omega} H \boldsymbol{q} = -M\boldsymbol{1} \tag{17}$$

under constraints (15) and (16). Note that (16) and (17) are linear equality constraints. Constraint (15) requires further reformulation, because both c_i and $\tilde{w}_i$ are variables. Define w_{ij} $(i = 1, \dots, n;$ $j = 1, \dots, p)$ by

$$w_{ij} = x_{ij} \tilde{w}_i \tag{18}$$

$$= \begin{cases} 0 & \text{if } x_{ij} = 0, \\ \boldsymbol{h}_i^{\mathrm{T}} \hat{\boldsymbol{v}} & \text{if } x_{ij} = 1. \end{cases} \tag{19}$$

By substituting (13) and (18) into (15), we obtain

$$\begin{aligned} q_i &= \bar{c}\sum_{j=1}^{p} x_{ij}\tilde{w}_i \\ &= \bar{c}\sum_{j=1}^{p} w_{ij}. \end{aligned} \tag{20}$$

On the other hand, (19) is equivalent to

$$|w_{ij}| \le \mu x_{ij}, \tag{21}$$

$$|w_{ij} - \boldsymbol{h}_i^{\mathrm{T}}\hat{\boldsymbol{v}}| \le \mu(1 - x_{ij}), \tag{22}$$

where $\mu \gg 0$ is a sufficiently large constant. Thus, (15) and (16) can be rewritten as (20), (21), and (22).

Example 3.2. In continuation of Example 3.1, suppose that

$$x_{i1} = x_{i2} = x_{i3} = 1, \quad x_{i4} = \cdots = x_{ip} = 0, \tag{23}$$

i.e., $c_i = 3\bar{c}$. If $x_{ij} = 1$, (22) reads

$$|w_{ij} - \boldsymbol{h}_i^{\mathrm{T}}\hat{\boldsymbol{v}}| \le 0,$$

while (21) reads a redundant constraint $|w_{ij}| \le \mu$ because μ is large. In contrast, if $x_{ij} = 1$, (21) reads

$$|w_{ij}| \le 0,$$

while (22) reads a redundant constraint $|\boldsymbol{h}_i^{\mathrm{T}}\hat{\boldsymbol{v}}| \le \mu$. With this observation, we see that (21) and (22) with (23) imply

$$w_{ij} = \boldsymbol{h}_i^{\mathrm{T}}\hat{\boldsymbol{v}}, \quad j = 1, 2, 3, \tag{24a}$$

$$w_{ij} = 0, \qquad j = 4, \dots, p. \tag{24b}$$

Substitution of (24) into (20) yields

$$q_i = 3\bar{c}\boldsymbol{h}_i^{\mathrm{T}}\hat{\boldsymbol{v}}. \tag{25}$$

This corresponds to the relation between q_i and $\hat{\boldsymbol{v}}$ in (15) and (16); indeed, by eliminating $\tilde{w}_i$ from (15) and (16) and using $c_i = 3\bar{c}$, we obtain (25). ■

The upshot of the discussion above is that constraint (11d) can be rewritten as (17), (20), (21), and (22). In conjunction with (12) and (13), we see that problem (11) is reduced to

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n} y_i && && \text{(26a)}\\
\text{s.t.} \quad & y_i \ge \left\| \begin{bmatrix} \operatorname{Re} \hat{\delta}_i \\ \operatorname{Im} \hat{\delta}_i \end{bmatrix} \right\|, && \forall i, && \text{(26b)}\\
& \hat{\boldsymbol{\delta}} = H^{\mathrm{T}} \hat{\boldsymbol{v}}, && && \text{(26c)}\\
& (K - \bar{\omega}^2 M)\hat{\boldsymbol{v}} + \mathrm{i}\bar{\omega} H \boldsymbol{q} = -M\mathbf{1}, && && \text{(26d)}\\
& q_i = \bar{c} \sum_{j=1}^{p} w_{ij}, && \forall i, && \text{(26e)}\\
& |w_{ij}| \le \mu x_{ij}, && \forall i;\ \forall j, && \text{(26f)}\\
& |w_{ij} - \boldsymbol{h}_i^{\mathrm{T}} \hat{\boldsymbol{v}}| \le \mu (1 - x_{ij}), && \forall i;\ \forall j, && \text{(26g)}\\
& \bar{c} \sum_{i=1}^{n} \sum_{j=1}^{p} x_{ij} \le c_{\mathrm{sum}}^{\max}, && && \text{(26h)}\\
& x_{i1} \ge x_{i2} \ge \dots \ge x_{ip}, && \forall i, && \text{(26i)}\\
& x_{ij} \in \{0, 1\}, && \forall i;\ \forall j. && \text{(26j)}
\end{aligned}
$$

In this problem, $\boldsymbol{x} \in \mathbb{R}^{n \times p}$, $\boldsymbol{y} \in \mathbb{R}^n$, $\hat{\boldsymbol{\delta}} \in \mathbb{C}^n$, $\boldsymbol{q} \in \mathbb{C}^n$, $\hat{\boldsymbol{v}} \in \mathbb{C}^n$, and $\boldsymbol{w} \in \mathbb{C}^{n \times p}$ are variables to be optimized. The constraints consist of linear equality constraints in (26c), (26d), and (26e), linear inequality constraints in (26f), (26g), (26h), and (26i), second-order cone constraints in (26b), and integrality constraints in (26j). Thus all the constraints other than the integrality constraints are convex constraints. Moreover, the objective function is a linear function. This means that a convex relaxation problem can be obtained by replacing the integrality constraints, (26j), with the linear inequalities,

$$
0 \le x_{ij} \le 1, \quad i = 1, \dots, n;\ j = 1, \dots, p. \tag{27}
$$

Therefore, problem (26) can be solved globally by using, e.g., a branch-and-bound algorithm. Specifically, the relaxation problem is a second-order cone programming problem, and thence problem (26) is called a *mixed-integer second-order cone programming problem*. Several software packages, e.g., Gurobi Optimizer [8] and CPLEX [9], are available for computing a global optimal solution of this problem.

Problem (26) includes complex variables. In practice, we solve problem (26) by converting these complex variables to real variables. Specifically, introducing new variables $\hat{\delta}_i^{\mathrm{R}} \in \mathbb{R}$ and $\hat{\delta}_i^{\mathrm{I}} \in \mathbb{R}$ by

$$
\hat{\delta}_i^{\mathrm{R}} = \operatorname{Re} \hat{\delta}_i, \quad \hat{\delta}_i^{\mathrm{I}} = \operatorname{Im} \hat{\delta}_i,
$$

we can rewrite (26b) as

$$
y_i \ge \left\| \begin{bmatrix} \hat{\delta}_i^{\mathrm{R}} \\ \hat{\delta}_i^{\mathrm{I}} \end{bmatrix} \right\|, \quad \forall i.
$$

Similarly, let $\boldsymbol{q} = \boldsymbol{q}^{\mathrm{R}} + \mathrm{i}\boldsymbol{q}^{\mathrm{I}}$ and $\hat{\boldsymbol{v}} = \hat{\boldsymbol{v}}^{\mathrm{R}} + \mathrm{i}\hat{\boldsymbol{v}}^{\mathrm{I}}$. Then (26c), (26d), and (26e) are rewritten in real variables as

$$\begin{bmatrix} \hat{\boldsymbol{\delta}}^{\rm R} & \hat{\boldsymbol{\delta}}^{\rm I} \end{bmatrix} = H^{\rm T} \begin{bmatrix} \hat{\boldsymbol{v}}^{\rm R} & \hat{\boldsymbol{v}}^{\rm I} \end{bmatrix},$$

$$\begin{bmatrix} O & -\bar{\omega} H & K - \bar{\omega}^2 M & O \\ \bar{\omega} H & O & O & K - \bar{\omega}^2 M \end{bmatrix} \begin{bmatrix} \boldsymbol{q}^{\rm R} \\ \boldsymbol{q}^{\rm I} \\ \hat{\boldsymbol{v}}^{\rm R} \\ \hat{\boldsymbol{v}}^{\rm I} \end{bmatrix} = \begin{bmatrix} -M\boldsymbol{1} \\ \boldsymbol{0} \end{bmatrix},$$

$$\begin{bmatrix} q_i^{\rm R} \\ q_i^{\rm I} \end{bmatrix} = \begin{bmatrix} \bar{c} \displaystyle\sum_{j=1}^{p} w_{ij}^{\rm R} \\ \bar{c} \displaystyle\sum_{j=1}^{p} w_{ij}^{\rm I} \end{bmatrix}, \quad \forall i.$$

Also, let $w_{ij} = w_{ij}^{\rm R} + {\rm i} w_{ij}^{\rm I}$. Since μ is supposed to be sufficiently large, (26f) and (26g) can be replaced by

$$\begin{aligned} &|w_{ij}^{\rm R}| \le \mu x_{ij}, & &|w_{ij}^{\rm I}| \le \mu x_{ij}, & &\forall i;\ \forall j, \\ &|w_{ij}^{\rm R} - \boldsymbol{h}_i^{\rm T} \hat{\boldsymbol{v}}^{\rm R}| \le \mu(1 - x_{ij}), & &|w_{ij}^{\rm I} - \boldsymbol{h}_i^{\rm T} \hat{\boldsymbol{v}}^{\rm I}| \le \mu(1 - x_{ij}), & &\forall i;\ \forall j. \end{aligned}$$

The remaining constraints, (26h), (26i), and (26j), as well as the objective function includes only real variables. Thus, problem (26) can be converted to an optimization problem with real variables.

3.3 Minimization of maximum interstory drift

In section 3.2 we have shown that the minimization problem of the sum of the transfer functions of interstory drifts can be reduced to a mixed-integer second-order cone programming problem. Other objective functions can also be handled within the framework of mixed-integer second-order cone programming.

For instance, some authors adopt the maximum, instead of the sum, of the interstory drifts as a measure of response of the structure; see, e.g., Lavan and Levy [15], Levy and Lavan [16], and Takewaki *et al.* [29, section 7]. Specifically, consider a minimization problem of the maximum value of interstory drift transfer functions, evaluated at the undamped natural frequency. This problem is formally stated as

$$\min_{\boldsymbol{c}} \quad \max\{|\hat{\delta}_1(\boldsymbol{c})|, \dots, |\hat{\delta}_n(\boldsymbol{c})|\} \tag{28a}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} c_i \le c_{\rm sum}^{\max}, \tag{28b}$$

$$c_i \in \{0, \bar{c}, 2\bar{c}, \dots, p\bar{c}\}, \quad i = 1, \dots, n. \tag{28c}$$

In a manner similar to problem (26), problem (28) is reduced to a mixed-integer second-order programming problem as follows. In problem (26), replace constraint (26b) by

$$y \ge \left\| \begin{bmatrix} {\rm Re}\, \hat{\delta}_i \\ {\rm Im}\, \hat{\delta}_i \end{bmatrix} \right\|, \quad \forall i.$$

Here, y serves as an upper bound for $\max\{|\hat{\delta}_1(\boldsymbol{c})|, \dots, |\hat{\delta}_n(\boldsymbol{c})|\}$. Consider a minimization problem of y, instead of the objective function in (26a). This optimization problem is equivalent to problem (28) and is a mixed-integer second-order cone programming problem.

3.4 More constraints on damper placement

As stated in section 3.2, one of advantages of the presented MIP approach is that algorithms with guaranteed convergence to a global optimal solution are applicable. Besides this, various combinatorial constraints on damper placement, which are usually difficult to be dealt with in continuous optimization, can be treated within the framework of the MIP approach. In this section we explore some of them.

The first one is an upper bound constraint on the number of stories at which dampers are placed. For instance, Ribakov and Agranovich [21] performed parametric study aimed for finding a small number of damped stories that is necessary to achieve sufficient effect in aseismic control. Suppose that dampers can be introduced to at most γ stories, where γ is a specified value. This constraint can be written as

$$\sum_{i=1}^{n} x_{i1} \leq \gamma, \tag{29}$$

because $x_{i1} = 1$ implies that a damper is placed at story i and vice versa.

Another example is a lower bound constraint on the damper damping coefficients. Even if the damping coefficient is small, a damper occupies space of the building of the corresponding story. Therefore, introducing dampers with small damping coefficients might be undesirable in practical applications from a viewpoint of floor obstruction. The lower bound constraint on c_i avoids such small damping coefficients. Specifically, let $c^{\min}$ denote the minimum value of the damping coefficient of the existing damper. In other words, the damping coefficient, c_i, should satisfy the constraint

$$(c_i = 0) \vee (c_i \geq c^{\min}), \tag{30}$$

where $\vee$ denotes the logical "or." By using (13), constraint (30) can be rewritten as

$$\Bigl(\bar{c}\sum_{j=1}^{p} x_{ij} = 0\Bigr) \vee \Bigl(\bar{c}\sum_{j=1}^{p} x_{ij} \geq c^{\min}\Bigr),$$

and hence there exists $\check{p}$ such that

$$\Bigl(\sum_{j=1}^{p} x_{ij} = 0\Bigr) \vee \Bigl(\sum_{j=1}^{p} x_{ij} \geq \check{p}\Bigr). \tag{31}$$

With reference to (12), we see that this condition means that $x_{i\check{p}} = 1$ if $x_{i1} = 1$ and $x_{i1} = 0$ if $x_{i\check{p}} = 0$. Therefore, (31) can be rewritten as

$$x_{i1} \leq x_{i\check{p}}.$$

The final example is a constraint avoiding simultaneously introducing dampers to two adjacent stories. For instance, if there is a damper at the first story, then this constraint means that we cannot place a damper at the second story. Also, if there is a damper at the third story, then we cannot place a damper at the second and fourth stories. The constraint excluding adjacent damped stories might be useful in practical applications when, for instance, some facilities should be placed at least every two stories and occupancy of floors by dampers is not acceptable. Since $x_{i1} = 1$ implies that a damper is introduced into the ith story, x_{i1} and $x_{i+1,1}$ cannot be equal to one simultaneously. Thus the constraint excluding adjacent damped stories can be written as

$$x_{i1} + x_{i+1,1} \leq 1, \quad i = 1, \dots, n-1. \tag{32}$$

Table 1: Optimal solutions of example (I) for minimizing $\sum_{i=1}^{6} |\hat{\delta}_i|$.

	p	$\bar{c}$ (Ns/m)	Opt. sol. ($\times 10^5$ Ns/m)						$\sum_{i=1}^{n} \lvert\hat{\delta}_i\rvert$
			c_1	c_2	c_3	c_4	c_5	c_6	
Case A	15	5×10^5	50	40	0	0	0	0	0.135236 m
Case A	30	2×10^5	48	42	0	0	0	0	0.135132 m
Case A	60	1×10^5	48	42	0	0	0	0	0.135132 m
Case B	15	5×10^5	55	0	35	0	0	0	0.149515 m
Case B	30	2×10^5	54	0	36	0	0	0	0.149494 m
Case B	60	1×10^5	54	0	36	0	0	0	0.149494 m

Similarly, suppose that at most one damped story is accepted among the adjacent three stories. For instance, if there is a damper at the first story, then this constraint means that no damper can be placed at the second and third stories. Also, if there is a damper at the third story, then no damper can be placed at the first, second, fourth, and fifth stories. This constraint can be written as

$$x_{i1} + x_{i+1,1} + x_{i+2,1} \leq 1, \quad i = 1, \dots, n-2.$$

All the constraints considered in this section are written as linear inequality constraints on x_{ij}'s. Therefore, they can be treated within the framework of mixed-integer second-order cone programming.

4 Numerical experiments

The two numerical examples solved in [26] with continuous design variables are solved in this section with discrete design variables. Computation was carried out on two 2.66 GHz 6-Core Intel Xeon Westmere processors with 64 GB RAM. The mixed-integer second-order cone programming problem formulated in section 3 was solved with commercial solvers. CPLEX Ver. 12.2 [9] and Gurobi Optimizer Ver. 5.0 [8] were used for comparison, where the data of the problem was prepared with MATLAB Ver. 7.13 in the CPLEX LP file format. The tolerance of integrality feasibility of each solver was set as 10^{-8}. The other parameters of the solvers were the default values.

4.1 Example (I): Model with uniform distribution of story stiffnesses

Consider a six-story shear building model, i.e., $n = 6$. All stories have the same stiffness, $k_i =$ 40,000 kN/m, and the same mass, $m_i =$ 80,000 kg ($i = 1, \dots, 6$). The undamped fundamental frequency is $\bar{\omega} = 5.39$ rad/s. Design variables are damper damping coefficients, $c_1, \dots, c_6$. The upper bound for the sum of c_i's is $c_{\mathrm{sum}}^{\mathrm{max}} =$ 9,000 kNs/m. This problem was solved in [26, section 4.1] for continuous design variables; see also [28, section 2.7.1].

As for the set of candidate values of c_i, we consider three cases:

- $p = 15$ and $\bar{c} = 500$ kNs/m, i.e., $c_i \in \{0, 500, 1000, \dots, 7500\}$ in kNs/m.
- $p = 30$ and $\bar{c} = 200$ kNs/m, i.e., $c_i \in \{0, 200, 400, \dots, 6000\}$ in kNs/m.
- $p = 60$ and $\bar{c} = 100$ kNs/m, i.e., $c_i \in \{0, 100, 200, \dots, 6000\}$ in kNs/m.

Table 2: Optimal solutions of example (I) for minimizing $\max\{|\hat{\delta}_1|, \ldots, |\hat{\delta}_6|\}$.

	p	$\bar{c}$ (Ns/m)	Opt. sol. ($\times 10^5$ Ns/m)						$\max_i |\hat{\delta}_i|$
			c_1	c_2	c_3	c_4	c_5	c_6	
Case C	15	5×10^5	50	40	0	0	0	0	0.0293061 m
Case C	30	2×10^5	50	40	0	0	0	0	0.0293061 m
Case C	60	1×10^5	51	39	0	0	0	0	0.0291444 m
Case D	15	5×10^5	55	0	35	0	0	0	0.0364951 m
Case D	30	2×10^5	54	0	36	0	0	0	0.0364832 m
Case D	60	1×10^5	54	0	36	0	0	0	0.0364832 m

Table 3: Comparison of computational costs of example (I).

	p	CPLEX		Gurobi	
		Time (s)	No. of nodes	Time (s)	No. of nodes
Case A	15	3.1	25,233	7.7	20,536
Case A	30	172.6	837,374	135.2	465,107
Case A	60	2,103.5	6,164,308	1,210.9	1,954,957
Case B	15	0.8	2,317	0.8	2,380
Case B	30	5.2	9,837	6.3	7,234
Case B	60	13.7	17,118	23.1	16,143
Case C	15	3.6	35,037	6.5	26,188
Case C	30	68.1	399,745	95.8	348,183
Case C	60	887.3	2,106,923	513.0	671,475
Case D	15	0.7	2,609	1.9	5,658
Case D	30	3.1	8,663	6.9	8,432
Case D	60	13.6	18,318	24.2	18,961

Then we solve the minimization problem of the sum of transfer functions of interstory drifts. This case is called case A. Besides, we consider three variations of optimization problems. Namely, we examine the minimization of the maximum value of the interstory drifts, instead of the sum of them, as discussed in section 3.3. Moreover, optimization problems together with the additional constraints in section 3.4, (29) and (32), are also solved. By summing up, we consider the following four different optimization problems:

- Case A: minimization of the sum of $|\hat{\delta}_1|, \ldots, |\hat{\delta}_6|$, i.e., the discrete version of the problem solved in [26].
- Case B: minimization of the sum of $|\hat{\delta}_1|, \ldots, |\hat{\delta}_6|$, where the upper bound for the number of damped stories is $\gamma = 3$ and adjacent damped stories are forbidden.
- Case C: minimization of the maximum of $\{|\hat{\delta}_1|, \ldots, |\hat{\delta}_6|\}$.
- Case D: minimization of the maximum of $\{|\hat{\delta}_1|, \ldots, |\hat{\delta}_6|\}$, where the upper bound for the number of damped stories is $\gamma = 3$ and adjacent damped stories are forbidden.

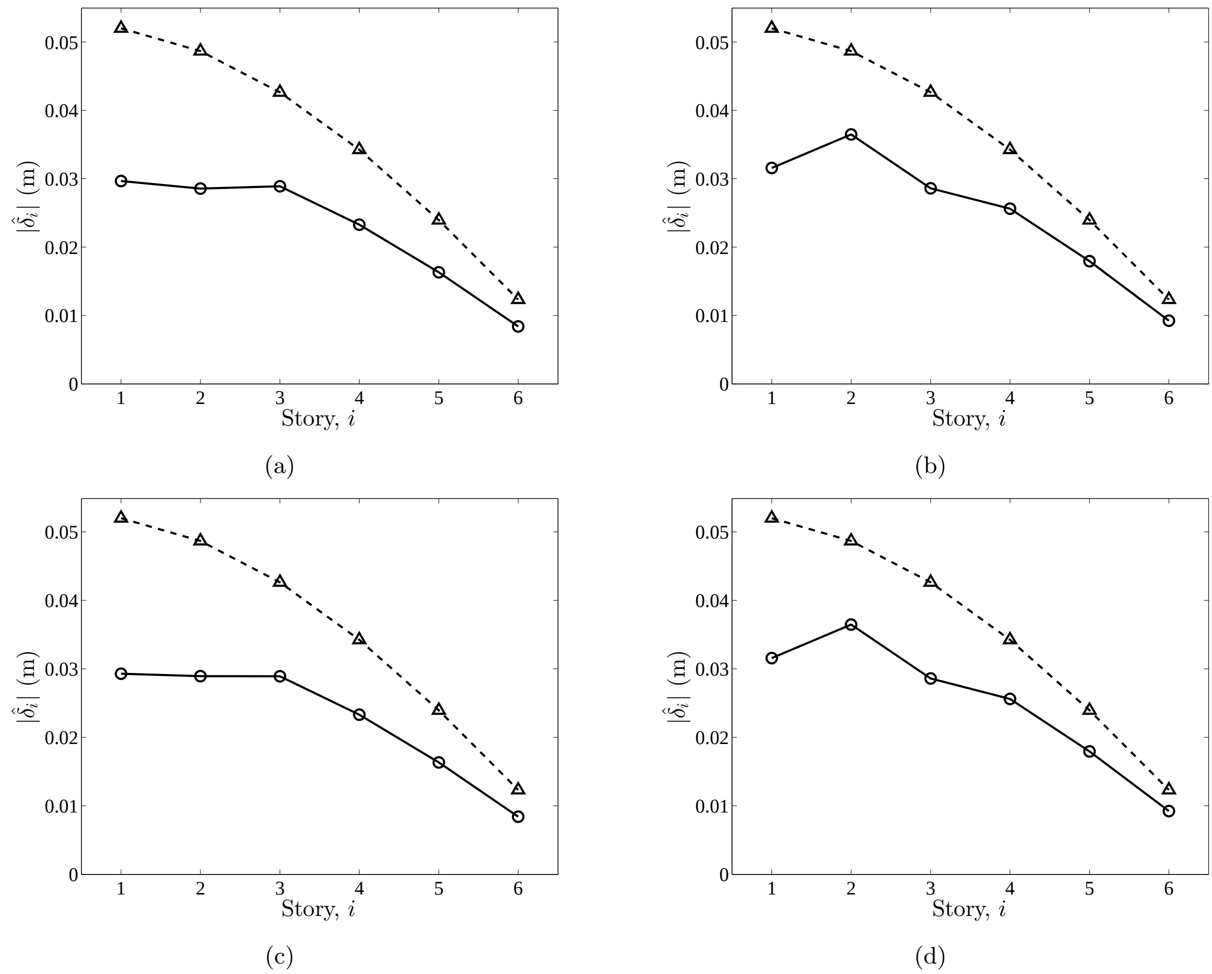

Figure 2: Transfer functions of interstory drifts evaluated at the undamped natural frequency of example (I). "——" The optimal solutions with $p = 30$ in (a) case A; (b) case B; (c) case C; and (d) case D. "– – –" The uniform damping.

Firstly, case A is investigated, where MIP problem (26) is solved. The obtained optimal solutions are listed in Table 1. Dampers are placed at the first and second stories. The optimal solution for $p = 30$ is same as that for $p = 60$. Its optimal value agrees well with the objective value reported in [26], 0.1351 m. The optimal damper distribution also agrees well with the result in [26]. The optimal value for $p = 15$ is slightly larger due to coarse discretization of the design variables. In the case of $p = 30$, the target transfer functions, $|\hat{\delta}_i| = |\delta_i(\bar{\omega})/\ddot{v}_{\rm g}(\bar{\omega})|$, at the optimal solution are shown in Figure 2(a). For comparison, the result of the uniform damping, i.e., $c_i = 1{,}500\,\mathrm{kNs/m}$ $(i = 1, \dots, 6)$, is also depicted. The objective value of the uniform damper distribution is 0.213888 m. Figure 3 shows the variation of $|\delta_i(\omega)/\ddot{v}_{\rm g}(\omega)|$ with respect to ω for the optimal design and the uniform design. It is observed that $|\hat{\delta}_i|$ is drastically decreased especially in the lower stories.

The computational costs are listed in Table 3. Here, "Time" shows the computational time and "No. of nodes" shows the number of nodes of a branch-and-bound tree generated by a solver. For $p = 30$, problem (26) has $np = 180$ binary variables, 402 continuous variables, 1615 linear inequality constraints, 36 linear equality constraints, and 6 second-order cone constraints. For $p = 60$, it has $np = 360$ binary variables, 762 continuous variables, 3235 linear inequality constraints, 36 linear equality constraints, and 6 second-order cone constraints.

We next study the problem with additional constraints on damper distribution, case B. Specifi-

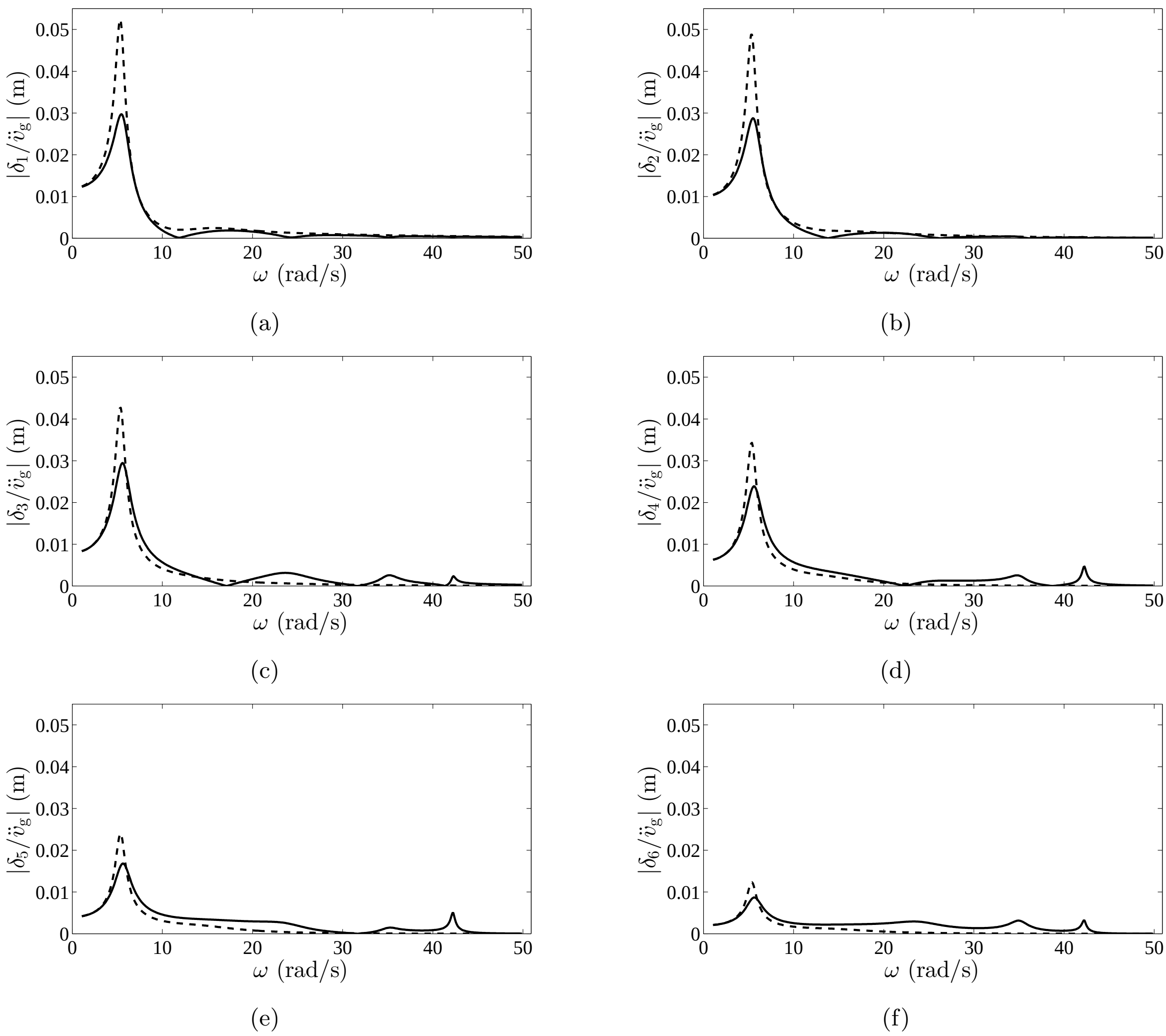

Figure 3: Transfer functions of interstory drifts, $|\delta_i(\omega)/\ddot{v}_{\rm g}(\omega)|$, of example (I) at (a) the 1st story; (b) the 2nd story; (c) the 3rd story; (d) the 4th story; (e) the 5th story; and (f) the 6th story. "——" The optimal solution in case A ($p = 30$); and "– – –" the uniform damping.

cally, the upper bound for the number of damped stories is $\gamma = 3$ and placement of dampers at two adjacent stories is not allowed. The optimal solutions are listed in Table 1. Dampers are placed at the first and third stories; only at two stories dampers are placed. The optimal values in case B are slightly larger thatn those in case A. Figure 2(b) shows $|\hat{\delta}_i|$ of the optimal solution with $p = 30$. Figure 4 depicts the transfer functions, $|\delta_i(\omega)|/|\ddot{v}_{\rm g}(\omega)|$, of the optimal solution with $p = 30$. It is observed that $|\hat{\delta}_2|$ is larger than that in case A, since in case B no damper is introduced to the second story.

In case C and case D, we solve problem (28), i.e., the minimization problem of the maximum of interstory drifts. The additional constraints considered in case D are same as those in case B. The optimal solutions are listed in Table 2. The objective value of the uniform damper distribution, i.e., $c_i = 1{,}500\,\mathrm{kNs/m}$ ($i = 1, \dots, 6$), is $0.0520132\,\mathrm{m}$. In case C dampers are placed at the first and second stories. The optimal solutions with $p = 30$ and $p = 60$ are slightly different from those in case A. The optimal solutions in case D coincide with the optimal solutions in case B. Figure 2(c) and Figure 2(d) shows $|\hat{\delta}_i|$ of the optimal solutions with $p = 30$ in case C and case D, respectively.

Table 3 collects the computational costs required by the two solvers for solving the problems in

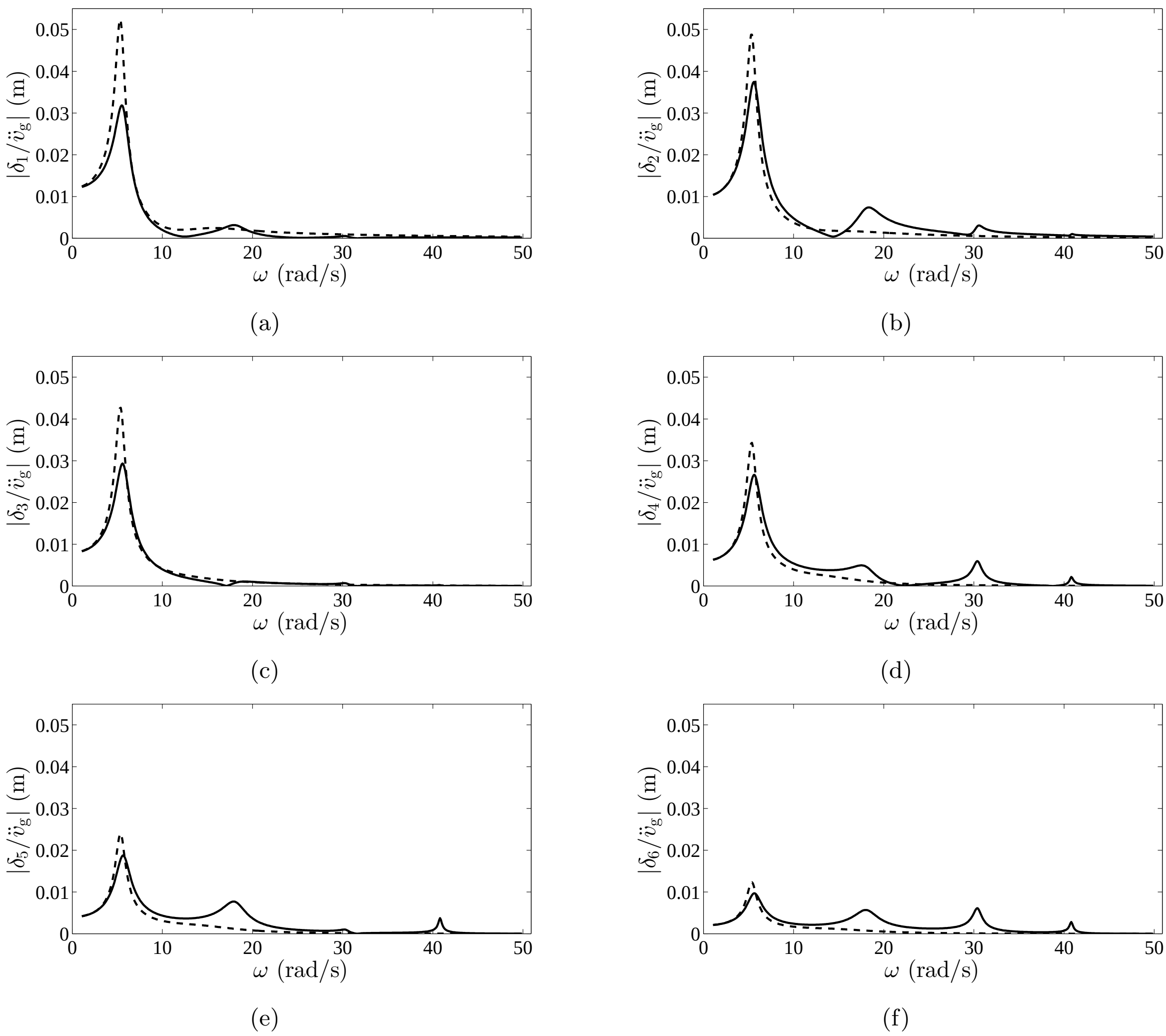

Figure 4: Transfer functions of interstory drifts, $|\delta_i(\omega)/\ddot{v}_{\rm g}(\omega)|$, of example (I) at (a) the 1st story; (b) the 2nd story; (c) the 3rd story; (d) the 4th story; (e) the 5th story; and (f) the 6th story. "——" The optimal solution in case B ($p = 30$); and "– – –" the uniform damping.

this section. It is observed that computational cost increases drastically as the number of variables increases. The computational costs in case B and case D are much smaller than those in case A and case C. This may be because the additional constraints considered in case B and case D reduce the number of feasible solutions. In many cases the computational time required by CPLEX is less than Gurobi Optimizer; exceptions are $p = 30$ and $p = 60$ in case A and $p = 60$ in case C. In contrast, in most every case the nodes of a branch-and-bound tree generated by Gurobi Optimizer are less than CPLEX; exceptions are $p = 15$ in case B and $p = 15$ and $p = 60$ in case D. In case A with $p = 60$, Gurobi Optimizer requires about 20 minutes to solve the problem. For this problem, the number of nodes generated by CPLEX is about 3.1 times larger than Gurobi Optimizer, while the computational time required by CPLEX is about 1.7 times larger than Gurobi Optimizer.

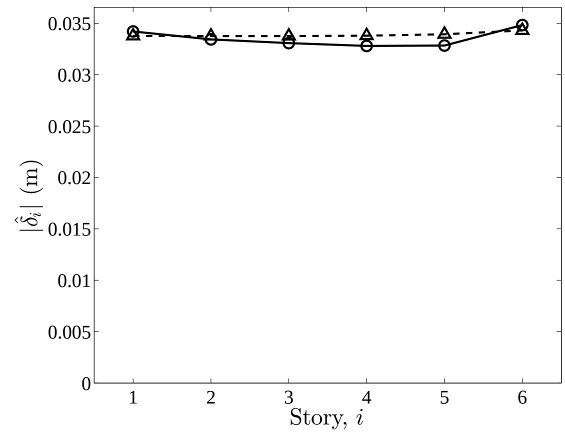

Figure 5: Transfer functions of interstory drifts evaluated at the undamped natural frequency of example (II). "——" The optimal solutions in case A with $p = 30$; and "– – –" the uniform damping.

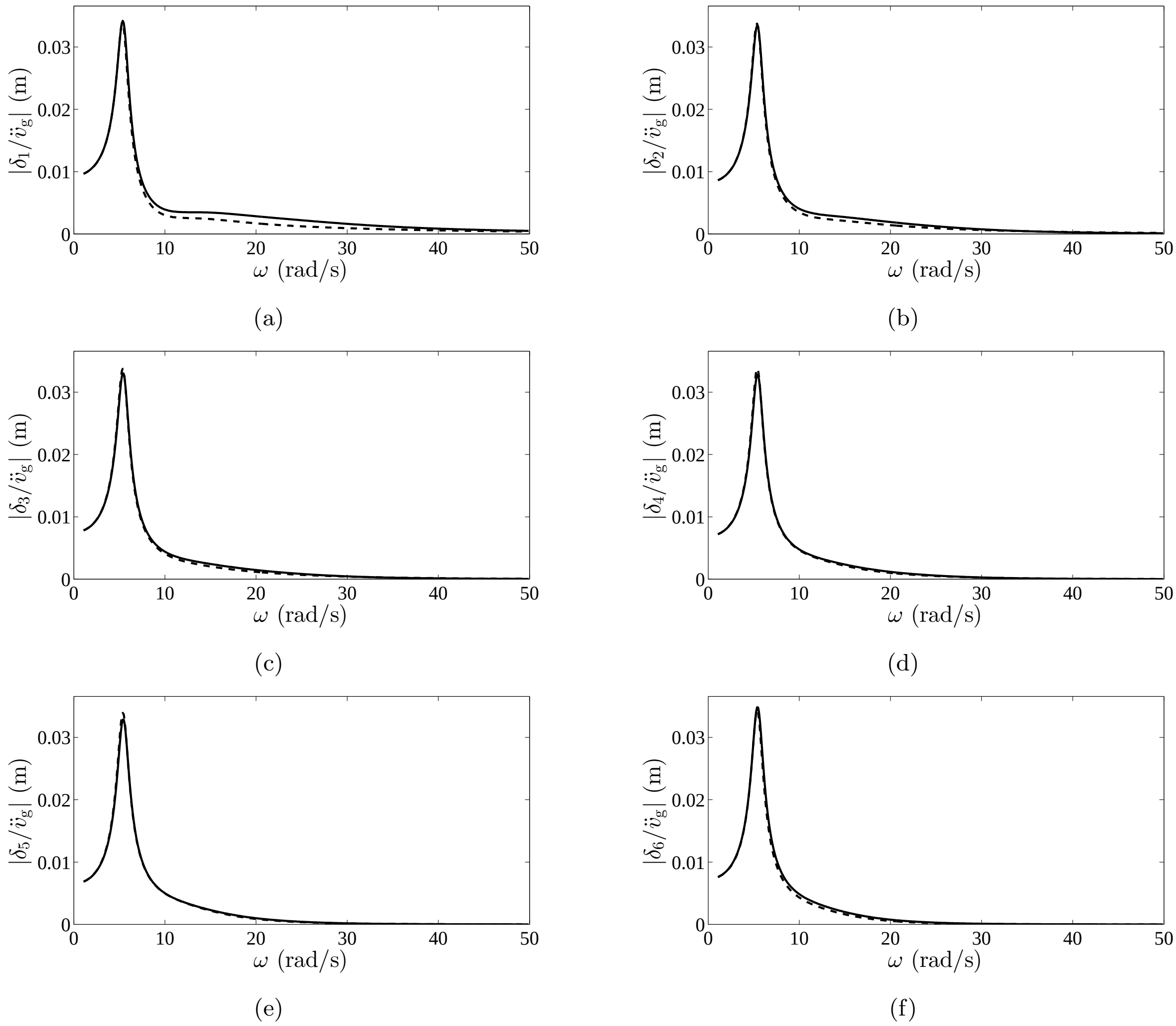

Figure 6: Transfer functions of interstory drifts, $|\delta_i(\omega)/\ddot{v}_{\mathrm{g}}(\omega)|$, of example (II) at (a) the 1st story; (b) the 2nd story; (c) the 3rd story; (d) the 4th story; (e) the 5th story; and (f) the 6th story. "——" The optimal solution in case A ($p = 30$); and "– – –" the uniform damping.

Table 4: Optimal solutions of example (II).

	p	$\bar{c}$ (Ns/m)	Opt. sol. ($\times 10^5$ Ns/m) c_1	c_2	c_3	c_4	c_5	c_6	$\sum_{i=1}^{n} \lvert\hat{\delta}_i\rvert$
Case A	15	5×10^5	0	0	30	25	20	15	0.201222 m
Case A	30	2×10^5	0	18	20	20	18	14	0.201162 m
Case A	60	1×10^5	0	19	20	19	18	14	0.201158 m
Case B	15	5×10^5	0	40	0	30	0	20	0.212097 m
Case B	30	2×10^5	0	40	0	32	0	18	0.211510 m
Case B	60	1×10^5	0	40	0	32	0	18	0.211510 m
Case C	15	5×10^5	0	65	0	0	0	25	0.234505 m
Case C	40	2×10^5	0	66	0	0	0	24	0.234311 m
Case C	70	1×10^5	0	67	0	0	0	23	0.234301 m
Case D	15	5×10^5	0	0	45	0	25	20	0.209588 m
Case D	30	2×10^5	0	0	44	0	28	18	0.209076 m
Case D	60	1×10^5	0	0	44	0	28	18	0.209076 m

Table 5: Comparison of computational costs of example (II).

		CPLEX		Gurobi	
	p	Time (s)	No. of nodes	Time (s)	No. of nodes
Case A	15	26.3	146,817	16.6	100,999
Case A	30	1,455.6	6,158,001	880.6	4,623,129
Case A	60	62,021.6	128,500,335	33,917.6	88,934,141
Case B	15	2.4	4,421	2.6	7,457
Case B	30	8.9	18,339	9.9	20,329
Case B	60	32.3	54,149	52.1	58,396
Case C	15	0.7	1,524	0.9	1,858
Case C	40	4.0	4,404	10.7	7,046
Case C	70	9.5	7,000	31.0	9,374
Case D	15	5.2	10,213	3.7	10,419
Case D	30	16.6	31,347	15.3	29,641
Case D	60	42.3	63,727	60.6	60,439

4.2 Example (II): Model with uniform distribution of amplitudes of transfer functions

This section deals with the example model presented in [26, section 4.2]; see also [28, section 2.7.2]. We consider another six-story shear building model, where the story stiffnesses are different from those in section 4.1. The mass of each story and the upper bound for $\sum_{i=1}^{6} c_i$ are same as in section 4.1, i.e., $m_i = 80{,}000\,\mathrm{kg}$ $(i = 1, \dots, 6)$ and $c_{\mathrm{sum}}^{\max} = 9{,}000\,\mathrm{kN/s}$. The stiffness of each story is given by $k_1 = 51{,}310\,\mathrm{kN/m}$, $k_2 = 48{,}100\,\mathrm{kN/m}$, $k_3 = 42{,}600\,\mathrm{kN/m}$, $k_4 = 34{,}760\,\mathrm{kN/m}$, $k_5 = 24{,}440\,\mathrm{kN/m}$, and $k_6 = 11{,}000\,\mathrm{kN/m}$. The undamped fundamental frequency is $\bar{\omega} = 5.39\,\mathrm{rad/s}$. With the uniform damping, i.e., $c_i = 1{,}500\,\mathrm{kNs/m}$ $(i = 1, \dots, 6)$, the distribution of $|\hat{\delta}_i|$ becomes uniform as depicted by a dotted line in Figure 5.

As for the set of candidate values of c_i, we consider the three cases in section 4.1. The minimization problem of the sum of $|\hat{\delta}_1|, \dots, |\hat{\delta}_6|$ is solved. With additional constraints on damper distribution in section 3.4, we consider four cases:

- Case A: the problem without additional constraints, i.e., the discrete version of the problem solved in [26].
- Case B: the problem with $\gamma = 3$ maximum damped stories and no adjacent damped stories.
- Case C: the problem with $\gamma = 2$ maximum damped stories and no adjacent damped stories.
- Case D: the problem with $\gamma = 3$ maximum damped stories.

The optimal solutions are listed in Table 4. The objective value of the uniform damper distribution is 0.203292 m. Computational costs are listed in Table 5. In case A, the obtained optimal solutions are different from the solution presented in [26]. That is, dampers are placed at all stories in the solution in [26], while dampers are not placed at the first story in the solutions in Table 4. In the optimal solution with $p = 15$, the second story has also no damper. The objective value reported in [26] is 0.2027 m, which is slightly larger than the optimal values in case A in Table 4. This means that the solution in [26] is not optimal. Thus the method proposed by Takewaki [26] does not necessarily converge to an optimal solution. However, this weak point should not be exaggerated, because the difference of the objective values of this example is not large. In addition, this optimization problem seems to be a difficult one, since large computational time is required by the proposed method as observed in Table 5. Figure 5 shows the transfer functions at the undamped natural frequency, $|\hat{\delta}_i|$, of the optimal solution with $p = 30$. The variations of the transfer functions, $|\delta_i(\omega)|/|\ddot{v}_{\mathrm{g}}(\omega)|$, with respect to ω are shown in Figure 6. It is observed that the structural response is not improved drastically from the uniform damping design.

At the optimal solutions in case B, dampers are placed at the second, fourth, and sixth stories. The optimal values are slightly larger than those in case A. In case C dampers are placed at the second and sixth stories. At the optimal solutions with $p = 30$ and $p = 60$, the upper bound constraint for c_2 becomes active, i.e., $c_2 = 6{,}000$ kNs/m. Therefore, $p = 40$ and $p = 70$ were examined for $\bar{c} = 200$ kNs/m and $\bar{c} = 100$ kNs/m, respectively. Then the upper bound constraint on c_2 becomes inactive as shown in Table 4. In case D, dampers are placed at the third, fifth, and sixth stories. Thus the optimal set of damped stories highly depends on the additional combinatorial constraints on damper distribution. The optimal values in cases B, C, and D are larger than the objective value of the uniform damper distribution.

It is observed in Table 5 that case A requires the largest computational costs. For all problems in case A, both the computational time and the number of nodes required by Gurobi Optimizer are smaller than those required by CPLEX. Particularly, for $p = 60$, CPLEX spent about 17 hours, which is more than 1.8 times larger than Gurobi Optimizer. The computational costs in case B, case C, and case D are much less than those in case A. For most every problems in these three cases, the computational costs required by CPLEX are less than Gurobi Optimizer.

5 Conclusions

In designing building structures, it is often that the design variables are essentially considered discrete. Nonetheless, many research articles on optimization of such structures still treat continuous optimization. Approximate optimal solutions for the original discrete optimization problem may be obtained by rounding the optimal solution of the continuous optimization problem. Outside of a few exceptions, however, it remains unobvious what kind of rounding rules can generate good feasible discrete solutions. This paper has fully addressed discrete optimization of damper placement in a shear building model.

In this paper we have supposed that design variables, i.e., damper damping coefficients, are chosen among multiples of a specified unit value. Then it has been shown that the minimization problem of the sum of the transfer functions of the interstory drifts can be formulated as a mixed-integer second-order cone programming problem. Several well-developed software packages are available for finding the global optimal solution of this optimization problem.

The proposed method can handle discrete design variables without resorting any approximation. Guaranteed convergence to a global optimal solution is a distinguished attribute. Besides this, the proposed method can deal with various practical constraints on damper distribution, e.g., the upper bound constraint on damped stories. A potential disadvantage of the method is that computational cost may possibly increase drastically as the number of variables increases. However, it is worth noting that the proposed method can provide benchmark examples for evaluating performances of the other local and/or heuristic algorithms that are applicable to large-scale damper placement optimization problems.

This paper has addressed only shear building models with viscous dampers. Extensions to the other structural models and the other damper types remain to be explored.

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