

MATHEMATICAL ENGINEERING TECHNICAL REPORTS

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METR 2013-05

April 2013

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WWW page: <http://www.keisu.t.u-tokyo.ac.jp/research/techrep/index.html>

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Markov degree of the Birkhoff model

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April, 2013

Abstract

We prove the conjecture by Diaconis and Eriksson (2006) that the Markov degree of the Birkhoff model is three. In fact we prove the conjecture in a generalization of the Birkhoff model, where each voter is asked to rank a fixed number, say r , of candidates among all candidates. We also give an exhaustive characterization of Markov bases for small r .

Keywords and phrases: algebraic statistics, Markov basis, normality of semigroup, ranking model.

1 Preliminaries

Diaconis and Eriksson [6] conjectured that the Markov degree of the Birkhoff model is three, i.e., the toric ideal associated with the Birkhoff model is generated by binomials of degree at most three. In this paper we give a proof of this conjecture in a generalization of the Birkhoff model, where each voter is asked to rank a fixed number of most preferred candidates among all candidates. Our proof is based on arguments of Jacobson and Matthews [7] for Latin squares. The set of Latin squares is a particular fiber in our setting and our result is also a generalization of [7]. See [3] for terminology of algebraic statistics and toric ideals used in this paper.

Consider an election, where there are n candidates and N voters. Each voter is asked to give r ($1 \leq r \leq n$) preferred candidates and to rank them. For example, let $n = 5$, $r = 3$ and let the candidates be labeled as a, b, c, d, e . A vote (a, c, d) by a voter means that he/she ranks a first, c second and d third. For a positive integer m , denote $[m] = \{1, \dots, m\}$. When the candidates are labeled as $1, \dots, n$, the set of possible votes is

$$S_{n,r} = \{\sigma = (\sigma(1), \dots, \sigma(r)) \mid \sigma : \text{injection from } [r] \text{ to } [n]\}, \quad |S_{n,r}| = \frac{n!}{(n-r)!},$$

where $\sigma(j)$ denotes the candidate chosen in the j -th position in the vote $\sigma = (\sigma(1), \dots, \sigma(r))$. Let ψ_{jk} , $j \in [r], k \in [n]$, be positive parameters and define a probability distribution over $S_{n,r}$ by

$$p(\sigma) = \frac{1}{Z} \prod_{j=1}^r \psi_{j\sigma(j)}, \quad Z = \sum_{\sigma \in S_{n,r}} \prod_{j=1}^r \psi_{j\sigma(j)}. \quad (1)$$

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If ψ_{jk} is large, then the candidate k is likely to be ranked in the j -th position. When $r = n$, this model is called the Birkhoff model ([6], [9]). In this paper we call (1) an (n, r) -Birkhoff model. The sufficient statistic of the (n, r) -Birkhoff model consists of numbers of times the candidate k is ranked in the j -th position, $j \in [r], k \in [n]$. We denote the sufficient statistic as $(t_{jk})_{j \in [r], k \in [n]}$.

Define a 0-1 matrix $A = A_{n,r}$ of size $rn \times (n!/(n-r)!)$, called a configuration matrix for the (n, r) -Birkhoff model, whose columns are labeled by $\sigma \in S_{n,r}$ and rows are labeled by $(j, k) = (\text{position, candidate})$, such that the $((j, k), \sigma)$ -element of A is one if and only if $\sigma(j) = k$. For example, for $n = 4, r = 3$, the configuration matrix $A_{4,3}$ with labels for its rows and columns is displayed as follows.

$$\begin{array}{cccccccccccccccccccccccc}
& (123) & (124) & (132) & (134) & (142) & (143) & (213) & (214) & (231) & (234) & (241) & (243) & (312) & (314) & (321) & (324) & (341) & (342) & (412) & (413) & (421) & (423) & (431) & (432) \\
(1, 1) & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(1, 2) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(1, 3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
(1, 4) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
(2, 1) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
(2, 2) & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
(2, 3) & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
(2, 4) & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
(3, 1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
(3, 2) & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
(3, 3) & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
(3, 4) & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} \tag{2}$$

Let $x(\sigma) \in \mathbb{N} = \{0, 1, \dots\}$ be the frequency of voters choosing a vote $\sigma \in S_{n,r}$ and let $\mathbf{x} = \{x(\sigma) \mid \sigma \in S_{n,r}\}$ be the vector of frequencies. Then $\mathbf{t} = A_{n,r}\mathbf{x}$ is the vector of sufficient statistic. For a given \mathbf{t} , $\mathcal{F}_{\mathbf{t}} = \{\mathbf{x} \in \mathbb{N}^{|S_{n,r}|} \mid A\mathbf{x} = \mathbf{t}\}$ is called the \mathbf{t} -fiber.

Let K be any field and let $K[\{p(\sigma), \sigma \in S_{n,r}\}]$ be the polynomial ring in the indeterminates $p(\sigma), \sigma \in S_{n,r}$. Similarly let $K[\{\psi_{jk}, j \in [r], k \in [n]\}]$ be the polynomial ring in the indeterminates $\psi_{jk}, j \in [r], k \in [n]$. Let

$$\pi_{n,r} : K[\{p(\sigma) \mid \sigma \in S_{n,r}\}] \rightarrow K[\{\psi_{jk}, j \in [r], k \in [n]\}]$$

be a homomorphism defined by

$$\pi_{n,r} : p(\sigma) \mapsto \prod_{j=1}^r \psi_{j\sigma(j)}.$$

Then the toric ideal $I_A = I_{A_{n,r}}$ for the (n, r) -Birkhoff model is defined to be the kernel of $\pi_{n,r}$. Elements of the integer kernel $\ker_{\mathbb{Z}} A_{n,r} = \{\mathbf{z} \in \mathbb{Z}^{|S_{n,r}|} \mid A\mathbf{z} = 0\}$ of $A_{n,r}$ are called moves for $A_{n,r}$.

Note that if a voter ranks $r = n-1$ most preferred candidates, then he/she automatically ranks the last candidate. It can be easily seen that the configuration matrix $A_{n,n-1}$ for the $(n, n-1)$ -Birkhoff model and the configuration matrix $A_{n,n}$ for the Birkhoff model have the same number of columns and their integer kernels are the same: $\ker_{\mathbb{Z}} A_{n,n-1} = \ker_{\mathbb{Z}} A_{n,n}$.

2 Main result and its proof

The main result of this paper is the following theorem.

Theorem 2.1. *For $r \geq 2$ and $n \geq 3$, the toric ideal I_A for the (n, r) -Birkhoff model is generated by binomials of degree two and three.*

For $r = 1$ or $r = n = 2$, the toric ideal I_A is trivial. For $r \geq 2$ and $n \geq 3$, any set of generators for I_A contains a binomial of degree three (see Section 3.2). In the terminology of algebraic statistics, Theorem 2.1 states that the Markov degree of the (n, r) -Birkhoff model is three for $r \geq 2$ and $n \geq 3$.

The rest of this section is devoted to a proof of this theorem.

We define some notation and terminology for our proof, mainly following [7]. We denote candidates either by alphabets a, b, c, \dots or by numbers $1, \dots, n$. The set of n candidates is denoted by $[n]$, using numbers. As in Section 1, each vote, such as (a, c, d) , is denoted by a row vector. A dataset of N votes is denoted by an $N \times r$ matrix P whose entries are candidates. A row is a vote and a column is a multiset of candidates at the same position or rank. An example of a dataset for $n = 5, r = 3$ and $N = 4$, is written as

$$P = (p_{ij}) = \begin{bmatrix} a & c & d \\ e & a & b \\ c & d & a \\ b & d & c \end{bmatrix},$$

where the candidates are labeled as a, b, c, d, e . Although the order of the rows of P are arbitrary, this matrix notation is convenient for our proof. A Latin square with $N = r = n$ is a special case in our problem.

When the candidates are denoted by $1, \dots, n$, a dataset P can be regarded as a three-dimensional 0-1 cuboid $C = C_P$ with $N \times r \times n$ cells

$$C_P = (c_{ijk}), \quad i \in [N], \quad j \in [r], \quad k \in [n],$$

where $c_{ijk} = 1$ if and only if $p_{ij} = k$. Following the terminology of Latin squares, we call C_P the orthogonal array representation of P . A 0-1 cuboid C of size $N \times r \times n$ is an orthogonal array representation of an $N \times r$ dataset P if and only if the row sums $(\sum_j c_{ijk})$ are either zero or one, the vertical sums $(\sum_k c_{ijk})$ are one. The column sums

$$t_{jk} = \sum_i c_{ijk}, \quad j \in [r], \quad k \in [n]$$

are the elements of the sufficient statistic \mathbf{t} for the (n, r) -Birkhoff model. In the special case of Latin squares all the line sums are one: $1 = \sum_i c_{ijk} = \sum_j c_{ijk} = \sum_k c_{ijk}, \forall i, j, k$. In addition to “valid” votes, we consider two kinds of invalid votes.

Improper vote. The first type of invalid vote is an *improper* vote, which contains an element of the form $b + c - a$, where a, b, c are different. An example of an improper vote for $n = 5$ and $r = 4$ is

$$(a, a, b + c - a, d),$$

which contains an improper element at the third position. We can interpret this improper vote in the orthogonal array representation. Suppose that this vote is the i -th row of P . The $(i, *, *)$ -slice of C_P is written as

$$\begin{array}{l} e \\ d \\ c \\ b \\ a \end{array} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix}. \quad (3)$$

Note that the row sums of this $(i, *, *)$ -slice are zero or one and the vertical sums of this $(i, *, *)$ -slice are one, as in the case of a valid vote. We require this property for an improper vote. This requirement implies the following restriction. If a vote contains an improper element $b + c - a$ in the j -th position, then neither b nor c appears in other positions and a appears once or twice in other positions. In our proof we consider a dataset P , which contains at most one improper vote, with the row slice of C_P given as in (3). As in a dataset containing only valid votes, we further require that the column sums $t_{jk} = \sum_i c_{ijk}$ are nonnegative integers. This implies that if $b + c - a$ appears in (i, j) -element of P , then a has to appear in the j -th position in some other row of P .

In summary, we consider P such that its orthogonal array representation C_P contains one -1 and its line sums are the same as datasets consisting of valid votes only. We call such a P an *improper* dataset. We call a dataset P *proper* if it consists of valid votes only. In the proof below, we denote an improper dataset by I . The elements of I are denoted by t_{ij} , $i \in [N]$, $j \in [r]$.

The following example gives an improper dataset I for $N = 5$, $n = 6$, $r = 4$ with the candidates labeled as a, b, c, d, e, f :

$$\begin{bmatrix} f & b & c & d \\ a & d & b & c \\ a & e + f - a & a & b \\ d & a & e & c \\ c & a & d & f \end{bmatrix}.$$

Vote with collision. The second type of an invalid vote is a vote containing a candidate twice, such as (a, b, b) . We say that this vote contains a *collision* or the candidate b *collides* in this vote.

We also need to consider collisions for a vote containing an improper element. Let a vote contain $b + c - a$ in the j -th position. We say that b (resp. c) *collides* in this vote if b (resp. c) appears in some position other than j . We say that a *collides* in this vote if a appears three times in positions other than j . Note that in the definition of an improper vote above, we did not consider collisions. In the following, when we just refer to an improper vote, the vote should not contain a collision. If a vote contains both an improper element and a collision, we call it an improper vote with collisions.

We now consider a basic operation, which we call a *swap*, for transforming a dataset to another. This operation does not alter the sufficient statistic of the dataset. By this operation we interchange elements in the same position in two rows of P . For illustration, consider the upper-left 2×1 submatrix of P and let $p_{11} = a, p_{21} = b, a \neq b$. Then by adding

$$\begin{bmatrix} b - a \\ a - b \end{bmatrix}$$

to the submatrix, we swap a and b :

$$\begin{bmatrix} a & * & \cdots & * \\ b & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \cdots & * \end{bmatrix} + \begin{bmatrix} b - a & 0 & \cdots & 0 \\ a - b & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} b & * & \cdots & * \\ a & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \cdots & * \end{bmatrix}, \quad (4)$$

where elements denoted by $*$ are not changed. Note that a and b may collide after the swap. Note also that the addition and the equality in (4) should be understood in the three-dimensional orthogonal array representation. However, the two-dimensional display in (4) is more convenient.

We denote the simple swap in (4) by $a \leftrightarrow b$. When we want to specify the position (column) j of a and b , we denote $a \overset{j}{\leftrightarrow} b$. When we further want to specify the rows i, i' , we denote $\{i, i'\} : a \overset{j}{\leftrightarrow} b$.

In (4) we swapped a and b which were already in P . The result was $\begin{bmatrix} b \\ a \end{bmatrix}$, where both elements were proper. However we also consider swapping a and b in $\begin{bmatrix} a \\ c \end{bmatrix}$, where a, b, c are different. This operation is written as

$$\begin{bmatrix} a \\ c \end{bmatrix} + \begin{bmatrix} b - a \\ a - b \end{bmatrix} = \begin{bmatrix} b \\ a + c - b \end{bmatrix}. \quad (5)$$

We call the swap in (5) *improper* and the swap in (4) *proper*. In an improper swap $a \leftrightarrow b$ for the i -th and i' -th rows at the j -th column of P , we require that one of a, b is the (i, j) -element P or the (i', j) -element of P . Then the notation $\{i, i'\} : a \overset{j}{\leftrightarrow} b$ is well defined.

We now consider a sequence of swaps. Consider swapping a and b in two different positions j, j' in the same i -th and i' -th rows. In our proof below, we often perform these two swaps sequentially, i.e., we swap a and b in the j -th column first and then in the j' -th column. We denote this operation as

$$a \overset{j}{\leftrightarrow} b \overset{j'}{\leftrightarrow} a \quad \text{or} \quad \{i, i'\} : a \overset{j}{\leftrightarrow} b \overset{j'}{\leftrightarrow} a$$

and call this a *double swap*. The double swap corresponds to the basic move for no three-factor interaction model (cf. [4]) in the orthogonal array representations of datasets. As an example, a double swap $a \overset{1}{\leftrightarrow} b \overset{2}{\leftrightarrow} a$, where the second swap is improper, is written as

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} + \begin{bmatrix} b - a & a - b \\ a - b & b - a \end{bmatrix} = \begin{bmatrix} b & a \\ a & b + c - a \end{bmatrix}. \quad (6)$$

More generally, we consider a sequence of m swaps in columns j_1, \dots, j_m , such that two consecutive swaps involve a common candidate, and denote it as

$$a_1 \xleftrightarrow{j_1} a_2 \xleftrightarrow{j_2} \dots \xleftrightarrow{j_{m-1}} a_m \xleftrightarrow{j_m} a_{m+1} \quad (7)$$

or indicating the rows as

$$\{i, i'\} : a_1 \xleftrightarrow{j_1} a_2 \xleftrightarrow{j_2} \dots \xleftrightarrow{j_{m-1}} a_m \xleftrightarrow{j_m} a_{m+1}. \quad (8)$$

We call (7) (or (8)) a *chain swap* of length m (even when $a_1 = a_{m+1}$, i.e., we do not make a distinction between a chain and a loop). A chain swap of length one is just a swap.

Suppose that we perform several chain swaps for the same two rows and ignore the order of swaps. Note that an even number of swaps at the same column results in no swap and an odd number of swaps results in a single swap. Hence the end result of several chain swaps is a set of simultaneous swaps of a subset of columns among the two rows. We call this a swap operation for a subset of columns among two rows, or simply a *swap operation among two rows*. When we apply a swap operation to P for a subset J of columns among two rows $R = \{i, i'\}$ and the result is P' , we denote the operation by a long double sided arrow:

$$P \xleftrightarrow{R} P',$$

where we omit J , because it is often cumbersome to specify J . We use the same notation when the datasets are improper, although we also need the condition of compatibility defined in Definition 2.6.

We now give a proof of Theorem 2.1 in a series of lemmas. Let P and P' be two datasets with the same sufficient statistic. Our strategy for a proof is to perform operations to P , involving at most three rows of P at each step, to increase the number of the common elements in P and P' . In each operation, elements at the same position of the three rows of P are permuted. This corresponds to a move of degree at most three. In fact, each operation will be further decomposed into a series of swap operations among two rows, which involve intermediate improper datasets. For the i -th row of P and the i' -th row of P'

$$(p_{i1}, \dots, p_{ir}), \quad (p'_{i'1}, \dots, p'_{i'r}),$$

let

$$V = V_{i,i'} = |\{j \mid p_{ij} = p'_{i'j}\}| \quad (9)$$

be the number of the same candidates in the same positions in these two rows. We call V the number of *concurrences*. If $V = r$, then we can remove these two votes from P and P' and consider other $N - 1$ votes. On the other hand, we will show that, if $V < r$ then we can always increase V by a series of operations involving at most three rows of P . The i -th row of P will eventually coincide with the i' -th row of P' . Then Theorem 2.1 is proved by induction on N .

Our first lemma concerns resolving collisions.

Lemma 2.2. *Suppose that two rows of P contain collisions and they do not contain an improper element. If each candidate appears at most twice in these two rows, then we can resolve the collisions by a swap operation among these two rows.*

Remark 2.3. We can prove this lemma based on the normality of the semigroup generated by the configuration matrix $A_{n,r}$ such as $A_{4,3}$ in (2). The normality follows from results in [8], [10] and [5]. We will discuss this point again in Section 4.2. However we give our own proof of Lemma 2.2, because we will use similar arguments for improper datasets. Arguments based on the normality cannot be applied to improper datasets.

Proof. We first consider the case that there is only one collision in one of the votes. Let a denote the colliding candidate. Relabeling the rows and the positions, without loss of generality, the two rows are displayed as

$$\begin{bmatrix} a & a & d & * & \cdots & * \\ b & c & * & * & \cdots & * \end{bmatrix},$$

where $b \neq c$. We choose one of the two a 's arbitrarily, say in the second position, and make a swap $a \stackrel{2}{\leftrightarrow} c$ with the following result:

$$\begin{bmatrix} a & c & d & * & \cdots & * \\ b & a & * & * & \cdots & * \end{bmatrix}. \quad (10)$$

Since a appears at most twice in these two rows, a does not collide in the second row. However c might again collide in the first row, e.g.,

$$\begin{bmatrix} a & c & d & c & * & \cdots & * \end{bmatrix}.$$

We then make a swap for c , which was in the first row from the beginning (in this example $c \stackrel{4}{\leftrightarrow} *$). If we continue this process, we always have collisions in the first row. If this process ends in finite number of steps, then by a chain swap we resolve the collisions of a and subsequent collisions due to swaps. We claim that this process indeed ends in finite number of steps. Actually we show a stronger result that no candidate appears twice in this process of resolving collisions.

Suppose otherwise. Then there is a candidate, say α , which is swapped twice for the first time. We consider two cases $\alpha = a$ and $\alpha \neq a$.

Suppose $\alpha = a$. The process of swaps is displayed as follows:

$$a \leftrightarrow c \leftrightarrow s_1 \leftrightarrow \cdots \leftrightarrow s_{l-1} \leftrightarrow a \leftrightarrow \cdots .$$

Since the collision always occurs in the first row, the candidate a was moved from the second row to the first row in the swap $s_{l-1} \leftrightarrow a$. By (10) we have $c = s_{l-1}$, which contradicts the assumption that $\alpha = a$ is the first candidate colliding twice.

Consider the case $\alpha \neq a$. The process of swaps is displayed as follows:

$$a \leftrightarrow c \leftrightarrow s_1 \leftrightarrow \cdots \leftrightarrow s_{l-1} \leftrightarrow \alpha \leftrightarrow s_{l+1} \leftrightarrow \cdots \leftrightarrow s_{m-1} \leftrightarrow \alpha \leftrightarrow \cdots . \quad (11)$$

Considering the subprocess of (11) which starts from the first α , we can apply the discussion for the $\alpha = a$ and confirm that there exists a contradiction. We have shown the lemma for the case that there is only one collision.

Now suppose that there are m colliding candidates a_1, a_2, \dots, a_m . Each of these candidates appear in one of the rows twice. Temporarily, we assign different labels, say $a'_l, a''_l, l = 2, \dots, m$, to candidates except for a_1 , namely, we ignore collisions of a_2, \dots, a_m . Then by the above procedure we resolve the collision of a_1 and subsequent collisions. When this procedure is finished, we restore the labels $a'_l, a''_l \rightarrow a_l, l = 2, \dots, m$. Then some collisions of a_2, \dots, a_m may have been already resolved, but we do not have any new collisions. Hence by the above procedure we decrease the number of collisions. As long as there is a remaining collision, we can repeat this procedure and resolve all the collisions. \square

So far we discussed resolving collisions. We now consider resolving an improper element by a swap operation among two rows. Let $b + c - a$ be an improper element in the j -th column in an improper dataset I . Since the elements of the sufficient statistic of I are assumed to be nonnegative, there is a row of I containing a in the same position as $b + c - a$. We first consider a swap between these two elements. If we make a swap $a \leftrightarrow b$, then $b + c - a$ becomes c and a becomes b :

$$\begin{bmatrix} b + c - a \\ a \end{bmatrix} + \begin{bmatrix} a - b \\ b - a \end{bmatrix} = \begin{bmatrix} c \\ b \end{bmatrix}. \quad (12)$$

Similarly $a \leftrightarrow c$ results in $\begin{bmatrix} b \\ c \end{bmatrix}$. Note that $\begin{bmatrix} c \\ b \end{bmatrix}$ and $\begin{bmatrix} b \\ c \end{bmatrix}$ are swaps of each other. Hence the result of several swaps can be regarded as a single swap $a \leftrightarrow b$ or $a \leftrightarrow c$. Although there is an ambiguity between $a \leftrightarrow b$ or $a \leftrightarrow c$, the result of a swap between these rows at the j -th column is either $\begin{bmatrix} b \\ c \end{bmatrix}$ or $\begin{bmatrix} c \\ b \end{bmatrix}$. By allowing this ambiguity a swap operation among two rows (for a subset of columns) is defined for an improper dataset I . We now have the following lemma.

Lemma 2.4. *Let I be an improper dataset, containing an element $b + c - a$. By a swap operation among two rows, I can be transformed to a proper dataset.*

Proof. Without loss of generality, assume that the first row contains $b + c - a$ and the second row contains a . We can then make a swap $\{1, 2\} : a \leftrightarrow b$, as in (12). Here a may collide in the first row and b may collide in the second row. However both a and b appear at most twice in these two rows. Hence we can now resolve these possible collisions by Lemma 2.2 by a swap operation among these two rows. \square

The operation of this lemma is denoted by

$$I \xleftrightarrow{R} P, \quad (13)$$

where R is a set of two rows of I .

At this point we make the following two definitions.

Definition 2.5. We call two rows in Lemma 2.4 of the form

$$\begin{matrix} i_{\text{im}} \\ i_{\text{pr}} \end{matrix} \begin{bmatrix} * & \cdots & * & b + c - a & * & \cdots & * \\ * & \cdots & * & a & * & \cdots & * \end{bmatrix}$$

a *resolvable pair*. Here i_{im} is an improper row and i_{pr} is a proper row. A resolvable pair is denoted as $[i_{\text{im}}, i_{\text{pr}}]$.

Note that any improper dataset I contains a resolvable pair $[i_{\text{im}}, i_{\text{pr}}]$ and R in (13) is the rows of a resolvable pair.

Furthermore we consider a swap between two elements in $\begin{bmatrix} b+c-a \\ d \end{bmatrix}$, $d \neq a, b, c$. We allow $b \leftrightarrow d$ or $c \leftrightarrow d$ between these two elements. After $b \leftrightarrow d$ we have

$$\begin{bmatrix} b+c-a \\ d \end{bmatrix} \rightarrow \begin{bmatrix} d+c-a \\ b \end{bmatrix}$$

and to $\begin{bmatrix} d+c-a \\ b \end{bmatrix}$ we can make further swaps $c \leftrightarrow b$ or $d \leftrightarrow b$. The end result of several swaps is one of the following three cases

$$\begin{bmatrix} c+d-a \\ b \end{bmatrix}, \begin{bmatrix} b+d-a \\ c \end{bmatrix} \text{ or } \begin{bmatrix} b+c-a \\ d \end{bmatrix}.$$

These three cases correspond to single swaps $b \leftrightarrow d$, $c \leftrightarrow d$ and to no swap to $\begin{bmatrix} b+c-a \\ d \end{bmatrix}$. Hence if we allow this ambiguity, a swap operation between an improper row and a proper row (for a subset of columns) of an improper dataset I , resulting in another improper dataset I' , is denoted as

$$I \xleftrightarrow{R} I'. \quad (14)$$

By the notation (14) we also consider the case that a swap operation is applied to two proper rows of I and I' . We now make the following definition concerning (14).

Definition 2.6. A swap operation among two rows $R = \{i, i'\}$ in (14) is *compatible* with improper datasets I and I' if there exists a common resolvable pair $[i_{\text{im}}, i_{\text{pr}}]$ of I and I' such that $R \cap \{i_{\text{im}}, i_{\text{pr}}\} \neq \emptyset$, or equivalently $|R \cup \{i_{\text{im}}, i_{\text{pr}}\}| \leq 3$.

Lemma 2.7. *Let P, P' be two proper datasets with the same sufficient statistic. Choose any i -th row from P and any i' -th row from P' , which are different, and let $V, V < r$, in (9), be the number of concurrences in these two rows. If we allow improper datasets, then V can be increased by at most three steps of swap operations among two rows applied to P , such that 1) if the resulting data set is improper then its improper row and the i -th row form a resolvable pair; and 2) each intermediate swap operation between two consecutive improper datasets is compatible with them.*

Proof. Without loss of generality we consider the first rows of P and P' . We consider two disjoint cases.

Case 1 There is a candidate in different positions in the two rows.

Let b be the candidate appearing in different positions in two rows. Relabeling the positions, without loss of generality, let $p_{11} = a$, $p'_{11} = b$, $a \neq b$, and $p_{12} = b$. Since P' contains b in the first column and the sufficient statistic for P and P' is common, P has to

contain b in the first column, say $p_{21} = b$. We now perform a double swap $a \xleftrightarrow{1} b \xleftrightarrow{2} a$ to P :

$$\begin{bmatrix} a & b & * & \cdots & * \\ b & c & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix} \rightarrow \begin{bmatrix} b & a & * & \cdots & * \\ a & b+c-a & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix},$$

where $*$'s are not changed. By this double swap V is increased. If $c = a$, the swap results in a proper dataset. Otherwise, the swap results in an improper dataset, where $[2, 1]$ forms a resolvable pair. Therefore, V is increased by a process of the form $P \xleftrightarrow{\{1,2\}} P$ or $P \xleftrightarrow{\{1,2\}} I$.

Case 2 Every candidate appearing twice in the two rows appears in the same position.

Again let $p_{11} = a$, $p'_{11} = b$, $a \neq b$. b does not appear in the first row of P and a does not appear in the first row of P' . As in Case 1, we can assume $p_{21} = b$. Since the first row of P' does not contain a , the total frequency of the candidate a in P' is less than N . Since the sufficient statistic is common, it follows that there is a row of P which does not contain a .

If the second row does not contain a , we can make a swap $a \xleftrightarrow{1} b$ among the first two rows and increase V without causing collision. This process is of the form $P \xleftrightarrow{\{1,2\}} P$.

If the second row contains a , without loss of generality, we let $p_{22} = a$ and also assume that the third row of P does not contain a . Let $p_{32} = c \neq a$. Since a is chosen in the second row and not chosen in the third row and both rows have the same number r of candidates, there is a candidate d , who is chosen in the third row but is not chosen in the second row. If d is in the position $j > 2$, then by relabeling of positions we assume that $p_{33} = d$. Then P looks like

$$\begin{bmatrix} a & * & * & \cdots & * \\ b & a & * & \cdots & * \\ d & c & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix} \text{ or } \begin{bmatrix} a & * & * & \cdots & * \\ b & a & * & \cdots & * \\ * & d(=c) & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix} \text{ or } \begin{bmatrix} a & * & * & \cdots & * \\ b & a & * & \cdots & * \\ * & c & d & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix}.$$

We perform a swap $\{2, 3\} : a \xleftrightarrow{2} d$ to the second column of the second and third rows:

$$\begin{bmatrix} a & * & * & \cdots & * \\ b & d & * & \cdots & * \\ d & a+c-d & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix} \text{ or } \begin{bmatrix} a & * & * & \cdots & * \\ b & d & * & \cdots & * \\ * & a & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix} \text{ or } \begin{bmatrix} a & * & * & \cdots & * \\ b & d & * & \cdots & * \\ * & a+c-d & d & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix}.$$

After the swap the second row does not contain a . The result is proper if $c = d$ (the middle case) and improper if $c \neq d$.

Now we apply a swap $\{1, 2\} : a \xleftrightarrow{1} b$ for the first column of the first and the second rows and increase V . In the case $c \neq d$, the last swap was performed on an improper dataset, but it is compatible with the datasets. Furthermore we can resolve the improper element $a + c - d$ by Lemma 2.4, since $[3, 2]$ is a resolvable pair. The process in this case is summarized as $P \xleftrightarrow{\{2,3\}} P \xleftrightarrow{\{1,2\}} P$ or $P \xleftrightarrow{\{2,3\}} I \xleftrightarrow{\{1,2\}} I \xleftrightarrow{\{2,3\}} P$.

This proves the lemma. □

Lemma 2.8. *Let I be an improper dataset and P' be a proper dataset with the same sufficient statistic. Choose any i' -th row from P' and choose any resolvable pair $[i_{\text{im}}, i_{\text{pr}}]$ of I . Then by at most three swap operations among two rows to I , we can 1) increase the number of concurrences $V_{i_{\text{pr}}, i'}$, or 2) make I proper without changing the i_{pr} -th row of I . Furthermore, 1) if the resulting data set is improper then its improper row and the i_{pr} -th row form a resolvable pair, and 2) each intermediate swap operation between two consecutive improper datasets is compatible with them.*

To prove Lemma 2.8, we need the following two lemmas.

Lemma 2.9. *Let I be an improper dataset with $\iota_{i_{\text{im}}j} = b + c - a$. Suppose that $\iota_{ij'} = a$ where $i \neq i_{\text{im}}, j' \neq j$ and the pair of i_{im} and i is not a resolvable pair. Then, letting $\iota_{ij} = d \neq a$, I can be transformed by a swap operation among two rows $R = \{i_{\text{im}}, i\}$ to another improper dataset I' containing the improper i_{im} -th row where $\iota'_{i_{\text{im}}j'} = a$ and $\iota'_{i_{\text{im}}j}$ is either of $b + c - a, b + d - a$ or $c + d - a$.*

Proof. Without loss of generality, assume that $i_{\text{im}} = j = 1$ and $i = j' = 2$. Let $\iota_{12} = e \neq b, c$. We first make a swap of $\{1, 2\} : a \xleftrightarrow{2} e$ to I :

$$\begin{bmatrix} b+c-a & e & * & \cdots & * \\ d & a & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix} \rightarrow \begin{bmatrix} b+c-a & a & * & \cdots & * \\ d & e & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix}.$$

After the swap, a may collide in the first row and e may collide in the second row. If there is no collision, the claim of this lemma is proved. Otherwise, we can resolve these possible collisions in the following way.

We try to resolve the collision of e in the second row as in Lemma 2.2 considering a swap process:

$$\{1, 2\} : e \leftrightarrow s_1 \leftrightarrow s_2 \leftrightarrow \cdots . \quad (15)$$

In this process the collisions always occur in the second row.

Consider the case that d is equal to b or c , say $d = b$. Since the first and the second row do not contain b other than in the first column, b does not collide in (15), which implies that c does not collide in (15). Since no a is in the second row at the beginning of (15), a does not collide

in (15). Therefore, there is no swap involving the first column in (15), which implies that the collision of e can be resolved as in Lemma 2.2.

Consider the case $d \neq b, c$. The difference of this case from Lemma 2.2 is that the process (15) may hit the first column. This happens when d appears in (15) for the first time as $s_l \xleftrightarrow{j} d$, $j \neq 1$, and $d = \iota_{1j}$ in the first row is swapped down to the second row in the j -th column. Then we need to choose b or c and make the swap $d \leftrightarrow b$ or $d \leftrightarrow c$ in the first column. By symmetry, without loss of generality, we perform $d \xleftrightarrow{1} c$:

$$\begin{bmatrix} b + c - a \\ d \end{bmatrix} \rightarrow \begin{bmatrix} b + d - a \\ c \end{bmatrix}.$$

This amounts to ignoring b and $-a$ and we look at the improper element $b + c - a$ just as a proper element c in resolving the collision of e . We leave $b - a$ intact in the $(1, 1)$ -element of I during the sequence in (15). Then just as in Lemma 2.2 it follows that no candidate appears twice in (15). Note that b and $-a$ which were left in the $(1, 1)$ -element cause no trouble, because collision occurs always in the second row. Indeed, b causes no trouble because it does not leave the first row. a causes no trouble because the second row does not initially contain a and when a is swapped from the first row to the second row, then the process in (15) ends at that point.

After the collision of e is resolved, a may still collide in the first row. Let j_1 and j_2 be the labels of rows containing a in the first row other than the first column. To resolve this collision we consider the following two swap processes:

$$\{1, 2\} : a \xleftrightarrow{j_1} s_1 \leftrightarrow s_2 \leftrightarrow \cdots, \quad (16)$$

$$\{1, 2\} : a \xleftrightarrow{j_2} s_1 \leftrightarrow s_2 \leftrightarrow \cdots, \quad (17)$$

where no swap in the j_2 -th column is involved in (16) and no swap in the j_1 -th column is involved in (17). Since every candidate in the first and second rows except a appears in at most two columns, the common candidate involved both in (16) and in (17) is a only. Then one of (16) and (17), say (16), involves neither b nor c , or involves b and no c . Therefore, ignoring c , $-a$ and a in the j_2 -th column, we see that the swap process (16) ends in finite number of steps as in Lemma 2.2. \square

Lemma 2.10. *Let I be an improper dataset with an improper element $\iota_{i_{\text{im}}j} = b + c - a$. Let $\iota_{ij} = d, i \neq i_{\text{im}}$, and suppose that $d \neq a, b, c$. Then I can be transformed to another improper dataset I' by a swap operation among two rows $R = \{i_{\text{im}}, i\}$ such that either $\iota'_{i_{\text{im}}j} = b + d - a, \iota'_{ij} = c$ or $\iota'_{i_{\text{im}}j} = c + d - a, \iota'_{ij} = b$.*

Proof. Without loss of generality assume $i_{\text{im}} = j = 1$ and $i = 2$. Then the upper-left 2×1 submatrix of I is $\begin{bmatrix} b + d - a \\ d \end{bmatrix}$. Note that $[1, 2]$ is not a resolvable pair because $d \neq a$.

We begin by considering two swaps of $\{1, 2\} : d \xleftrightarrow{1} b$ and $\{1, 2\} : d \xleftrightarrow{1} c$. If $\{1, 2\} : d \xleftrightarrow{1} b$ is applied to I , b may collide in the second row and d may collide in the first row. If $\{1, 2\} : d \xleftrightarrow{1} c$ is applied to I , c may collide in the second row and d may collide in the first row. Considering

the resolution of possible collisions in the second row for each swap, the following two swap processes are obtained:

$$\{1, 2\} : d \overset{1}{\leftrightarrow} b \leftrightarrow s_1 \leftrightarrow s_2 \leftrightarrow \cdots, \quad (18)$$

$$\{1, 2\} : d \overset{1}{\leftrightarrow} c \leftrightarrow s'_1 \leftrightarrow s'_2 \leftrightarrow \cdots. \quad (19)$$

Since the number of columns which contains a in the first or second row is at most three, one of (18) and (19), say (18), contains a at most once. Note that each candidates other than a appears in the first and second row at most twice. If (18) does not contain a , we see that (18) ends in finite number of steps as in Lemma 2.2. If (18) contains one a , the finiteness of (18) is proved by applying the similar discussion of Lemma 2.2 for the subprocess of (18) which starts from a .

After resolving the collision of b in the second row, d may still collide in the first row. At this point the second row contains at most one a . Consider a swap process

$$\{1, 2\} : d \leftrightarrow s''_1 \leftrightarrow s''_2 \leftrightarrow \cdots. \quad (20)$$

Since b has already been involved in (18), no s''_i is equal to b . If some s''_i is c , the chain swap

$$\{1, 2\} : d \leftrightarrow s''_1 \leftrightarrow s''_2 \leftrightarrow \cdots \leftrightarrow c$$

resolves the collisions in the first row. Since a appears in the second row at most once, the process (20) contains a at most once. If a does not appear in (20), the process does not hit the first column and we see that (18) ends in finite number of steps as in Lemma 2.2. If a appears in (20), the finiteness of (18) is proved by applying the similar discussion of Lemma 2.2 for the subprocess of (18) which starts from a . \square

Proof of Lemma 2.8. Without loss of generality, let $i' = 1$, $[i_{im}, i_{pr}] = [2, 1]$, $\iota_{11} = a$, and $\iota_{21} = b + c - a$. Then I looks like

$$\begin{bmatrix} a & * & \cdots & * \\ b + c - a & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \cdots & * \end{bmatrix}.$$

In the cases below, where a resulting dataset is improper, $[2, 1]$ will be a resolvable pair.

Case 1 $p'_{11} = a$.

In this case in P' and hence in I , the candidate a appears at least once in the first column. Therefore a is in the first position in some row $i > 2$ in I . Let $i = 3$. Then the rows $[2, 3]$ of I form a revolvable pair and I can be transformed to a proper dataset by Lemma 2.4.

This corresponds to 2) of the lemma and is summarized as $I \overset{\{2,3\}}{\longleftrightarrow} P$.

Case 2 $p'_{11} \neq a$, but a appears in the first row of P' .

Without loss of generality let $p'_{12} = a$. Let $d = \iota_{12}$.

Case 2-1 $\iota_{22} = a$.

We perform the double swap $a \xleftrightarrow{1} d \xleftrightarrow{2} a$ to the first two rows

$$\begin{bmatrix} a & d & * & \cdots & * \\ b+c-a & a & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix} \rightarrow \begin{bmatrix} d & a & * & \cdots & * \\ b+c-d & d & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix}.$$

This increases V_{11} . This corresponds to 1) of the lemma and is summarized as $I \xleftrightarrow{\{1,2\}} I$.

Case 2-2 $\iota_{22} \neq a$.

Since $p'_{12} = a$, a has to appear in the second column of I . Without loss of generality, let $\iota_{32} = a$. Let $e = \iota_{31}$ and $f = \iota_{22}$. Then P looks like

$$\begin{bmatrix} a & d & * & \cdots & * \\ b+c-a & f & * & \cdots & * \\ e & a & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix}.$$

From Lemma 2.9 applied to rows $\{2, 3\}$, this case is reduced to Case 2-1. This case together with the subsequent operation of Case 2-1 is summarized as $I \xleftrightarrow{\{2,3\}} I \xleftrightarrow{\{1,2\}} I$.

Case 3 a does not appear in the first row of P' .

Let $d = p'_{11}$, $d \neq a$. We will change $a = \iota_{11}$ to d and increase V .

If $d = b$ or $d = c$, we directly go to the Cases 3-1 and 3-2 below. If $d \neq b, c$, we need an extra step. Let $\iota_{31} = d$ without loss of generality. Then I looks like

$$\begin{bmatrix} a & * & \cdots & * \\ b+c-a & * & \cdots & * \\ d & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \cdots & * \end{bmatrix}.$$

By Lemma 2.10 applied to rows $\{2, 3\}$, we move d to the second row resolving the possible collisions. At this point the $(2, 1)$ -element of I may be $b+d-a$ or $c+d-a$. We consider

the former case without loss of generality. Then I looks like

$$\begin{bmatrix} a & * & \cdots & * \\ b+d-a & * & \cdots & * \\ c & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \cdots & * \end{bmatrix}. \quad (21)$$

Now we try to move d to the first row. We further distinguish two cases.

Case 3-1 d appears in the first row of I .

We apply a double swap $a \xleftrightarrow{1} d \xleftrightarrow{2} a$:

$$\begin{bmatrix} a & d & * & \cdots & * \\ b+d-a & e & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix} \rightarrow \begin{bmatrix} d & a & * & \cdots & * \\ b & d+e-a & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix}.$$

This case is summarized as $I \xleftrightarrow{\{1,2\}} I$ or $I \xleftrightarrow{\{2,3\}} I \xleftrightarrow{\{1,2\}} I$, where $I \xleftrightarrow{\{2,3\}} I$ is needed for the case $d \neq b, c$. We do not repeat this comment for the other cases below.

Case 3-2 d does not appear in the first row of I .

If a appears only once in the second row in the j -th column, $j > 1$, then we can make the swap $\{1, 2\} : a \leftrightarrow d$ to make I proper, which is summarized as $I \xleftrightarrow{\{1,2\}} P$ or $I \xleftrightarrow{\{2,3\}} I \xleftrightarrow{\{1,2\}} P$.

Hence we consider the case that a appears in two columns labeled by j^1, j^2 , $1 < j^1 < j^2$ of the second row of I . Then since P' does not contain a in the first row, I has a row not containing a .

Case 3-2-1 The third row of (21) contains a .

Without loss of generality, suppose the fourth row of I does not contain a . Denote $e = \iota_{41}$. Interpreting two a 's in the second row as a collision, we try to resolve the collision by swapping a at the j^1 -th column down to the third row. Then we have a process of swaps

$$\{2, 4\} : a \xleftrightarrow{j^1} s_1 \leftrightarrow s_2 \leftrightarrow \cdots.$$

During the process the collisions occur in the second row. If d appears in this process, we discard this process and choose a in the j^2 -th column. Then the process

$$\{2, 4\} : a \xleftrightarrow{j^2} s'_1 \leftrightarrow s'_2 \leftrightarrow \cdots \quad (22)$$

does not contain d . Then as in Lemma 2.2 no candidate appears twice in (22) and (22) is a finite chain swap resolving the collisions.

At this stage I looks like

$$\begin{bmatrix} a & * & \cdots & * \\ b+d-a & * & \cdots & * \\ c & * & \cdots & * \\ e & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \cdots & * \end{bmatrix} \text{ or } \begin{bmatrix} a & * & \cdots & * \\ e+d-a & * & \cdots & * \\ c & * & \cdots & * \\ b & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \cdots & * \end{bmatrix}.$$

In either case, the swap $\{1, 2\} : a \overset{1}{\leftrightarrow} d$ increases V and makes I proper. The whole process for this case is summarized as $I \overset{\{2,4\}}{\longleftrightarrow} I \overset{\{1,2\}}{\longleftrightarrow} P$ or $I \overset{\{2,3\}}{\longleftrightarrow} I \overset{\{2,4\}}{\longleftrightarrow} I \overset{\{1,2\}}{\longleftrightarrow} P$.

Case 3-2-2 The third row of (21) does not contain a .

We can just use the third row of (21) as the fourth row of the the previous case. Hence $I \overset{\{2,4\}}{\longleftrightarrow} I$ is replaced by $I \overset{\{2,3\}}{\longleftrightarrow} I$ and this case is summarized as $I \overset{\{2,3\}}{\longleftrightarrow} I \overset{\{1,2\}}{\longleftrightarrow} P$.

□

We now summarize what we have proved so far. We will again discuss the following result in Section 4.1.

Suppose that P and P' are two proper datasets with the same sufficient statistic. Choose any i -th row from P and any i' -th row from P' , which are different. If we allow improper datasets, then by a sequence of swap operations among two rows of P , we can make the i -th row of P identical with the i' -th row of P . Then we throw away this common row from the two datasets and repeat the procedure. It should be noted that P may have been transformed to an improper dataset I when two rows coincide, but I contains a resolvable pair $[i_{\text{im}}, i_{\text{pr}}]$ with $i_{\text{pr}} \neq i$. Hence we can continue this process until P is fully transformed to P' .

In order to finish our proof of Theorem 2.1, we have to show that each intermediate improper dataset can be temporarily transformed to a proper dataset and the consecutive proper datasets are connected by operations on three rows.

We decompose the whole process of transforming P to P' into segments, whenever there appears proper dataset. One segment is depicted as follows:

$$P_1 \longleftrightarrow I_1 \longleftrightarrow \cdots \longleftrightarrow I_i \longleftrightarrow I_{i+1} \longleftrightarrow \cdots \longleftrightarrow I_m \longleftrightarrow P_m, \quad (23)$$

where each \longleftrightarrow (omitting R) denotes a swap operation among two rows in Lemma 2.7 and Lemma 2.8. By Lemma 2.7 and Lemma 2.8, the number of concurrences in P_m is larger than in P_1 . We claim that for any consecutive improper datasets I_i, I_{i+1} , we can find proper datasets

P_i, P'_i, P'_{i+1} satisfying

$$P_i \longleftrightarrow I_i \longleftrightarrow I_{i+1} \longleftrightarrow P'_{i+1}, \quad (24)$$

$$P'_i \longleftrightarrow I_i \longleftrightarrow P_i. \quad (25)$$

The swap operation for $I_i \longleftrightarrow I_{i+1}$ is compatible with both datasets. Hence if we chose a common resolvable pair for I_i and I_{i+1} , then (24) for transforming P_i to P'_i involves three rows. On the other hand, since both $P'_i \longleftrightarrow I_i$ and $I_i \longleftrightarrow P_i$ involve an improper row, the operation of transforming P'_i to P_i involves three rows.

This completes the proof of Theorem 2.1.

3 Structure of moves of degree two and three

To analyze the structure of moves in the Markov basis, it is enough to consider the moves of degree two and three because of Theorem 2.1. It means that we only need to consider the dataset consisting of two or three votes. Then we can analyze the structure of moves by discussing the structure of fiber for the sufficient statistic. Details of computational results used in this section are available at [1].

3.1 Moves of degree two

We begin by discussing the structure of fibers for datasets which consist of two votes. Consider a sequence of multisets consisting of two elements in $[n]$ of the form:

$$M = (\{a_1, a_2\}, \dots, \{a_{2r-1}, a_{2r}\}), \quad (26)$$

where $a_j \in [n], j = 1, \dots, 2r$. Each multiset $\{a_{2j-1}, a_{2j}\}$ corresponds to the multiset of two candidates in the j -th position. This sequence is a possible observation of sufficient statistic in the (n, r) -Birkhoff model if and only if each $k \in [n]$ appears in the multiset $\{a_1, a_2, \dots, a_{2r}\}$ at most twice. For the observation $(\{a_1, a_2\}, \dots, \{a_{2r-1}, a_{2r}\})$ of the sufficient statistic, define a graph G_M on the vertex set $[r]$ as follows: for each $j, j' \in [r], j \neq j'$, an edge $\{j, j'\}$ of G_M exists if and only if $\{a_{2j-1}, a_{2j}\} \cap \{a_{2j'-1}, a_{2j'}\} \neq \emptyset$. We call the multiset $\{a_{2j-1}, a_{2j}\}$ the j -th block for $j \in [r]$.

For an isolated vertex in G_M the corresponding block has the form either $\{k, k\}$ or $\{k, k'\}$ for some $k, k' \in [n], k \neq k'$. In the former case there is no necessity to distinguish two votes by this block. In the latter case the votes might be distinguished by this block. Since every non-isolated vertex is contained by at most two edges, each connected components of G_M is a chain or a cycle if it consists of more than one vertex. Then the candidates are uniquely assigned to two votes as a subset of the votes. Let L be the number of connected components of G_M brushing aside those of the form $\{k, k\}$ for some $k \in [n]$. The number of elements of the corresponding fiber is 2^{L-1} . Especially, the move arising from the corresponding fiber is indispensable if and only if $L = 2$.

We evaluate the number of moves of degree two in a minimal Markov basis. For the case $r = 2$ the number of moves of degree two is $6\binom{n}{4}$. For the case $r = 3$, the number of moves of degree two is

$$\begin{aligned} & 1\binom{n}{6}\frac{6!}{2!2!2!}(4-1) + \binom{3}{2}\binom{n}{5}\frac{5!}{2!2!1!}(2-1) + \binom{3}{2}\binom{n}{5}\frac{5!}{2!1!1!1!}(2-1) + \binom{3}{2}\binom{n}{4}\frac{4!}{2!2!}(2-1) \\ &= 18\binom{n}{4} + 270\binom{n}{5} + 270\binom{n}{6}. \end{aligned}$$

Consider the case $r = 4$. The number of moves of degree two for the partition $(3, 1)$ of four is

$$\binom{4}{3}\binom{n}{6}\frac{6!}{1!1!1!1!2!}(2-1) + \binom{4}{3}\binom{n}{5}\frac{5!}{1!1!1!2!}(2-1) = 240\binom{n}{5} + 1440\binom{n}{6}.$$

The number of moves of degree two for the partition $(2, 2)$ of four is

$$\begin{aligned} & \frac{1}{2!}\binom{4}{2}\binom{n}{6}\frac{6!}{1!1!1!1!1!1!}(2-1) + \binom{4}{2}\binom{n}{5}\frac{5!}{1!1!1!2!}(2-1) + \frac{1}{2!}\binom{4}{2}\binom{n}{4}\frac{4!}{2!2!}(2-1) \\ &= 18\binom{n}{4} + 360\binom{n}{5} + 2160\binom{n}{6}. \end{aligned}$$

The number of moves of degree two for the partition $(2, 1, 1)$ of four is

$$\begin{aligned} & \binom{4}{2}\binom{n}{7}\frac{7!}{1!1!1!2!2!}(4-1) + \frac{4!}{2!1!1!}\binom{n}{6}\frac{6!}{1!1!1!2!1!}(2-1) + \binom{4}{2}\binom{n}{6}\frac{6!}{2!2!2!}(4-1) \\ & + \frac{4!}{2!1!1!}\binom{n}{5}\frac{5!}{2!2!1!}(2-1) \\ &= 360\binom{n}{5} + 5940\binom{n}{6} + 22680\binom{n}{7}. \end{aligned}$$

The number of moves of degree two for the partition $(1, 1, 1, 1)$ of four is

$$\begin{aligned} & n_8\frac{8!}{2!2!2!2!}(8-1) + \binom{4}{3}\binom{n}{7}\frac{7!}{2!2!2!1!1!}(4-1) + \binom{4}{2}\binom{n}{6}\frac{6!}{2!2!1!1!}(2-1) \\ &= 1080\binom{n}{6} + 7560\binom{n}{7} + 17640\binom{n}{8}. \end{aligned}$$

Then the number of moves of degree two for $r = 4$ is

$$18\binom{n}{4} + 960\binom{n}{5} + 10620\binom{n}{6} + 30240\binom{n}{7} + 17640\binom{n}{8}.$$

By the similar calculation we obtain the following polynomial which represents the number of moves of degree two for the case $r = 5$:

$$1050\binom{n}{5} + 40050\binom{n}{6} + 485100\binom{n}{7} + 2444400\binom{n}{8} + 3969000\binom{n}{9} + 1701000\binom{n}{10}.$$

The number of moves of degree two in minimal Markov bases are summarized as Table 1. The authors confirmed that the numbers above the horizontal lines in Table 1 coincide with the numbers obtained by the output of the computational software 4ti2([2]).

Table 1: Number of moves of degree two.

$r \backslash n$	2	3	4	5
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0
4	6	18	18	0
5	30	360	1050	1050
6	90	2160	16650	46350
7	210	8190	125370	787500
8	420	23940	611940	7505400
9	756	58968	2262708	46928700
10	1260	128520	6898500	218276100

3.2 Moves of degree three

The structure of fibers for datasets which consists of three votes is more complicated. Let

$$M = (\{a_1, b_1, c_1\}, \dots, \{a_r, b_r, c_r\})$$

be the observed sufficient statistic where $a_j, b_j, c_j \in [n]$, $j = 1, \dots, r$ and each $k \in [n]$ appears in the multiset $\{a_1, \dots, a_r, b_1, \dots, b_r, c_1, \dots, c_r\}$ at most three times. Similarly to the case of two votes, a graph G_M on $[r]$ can be defined: an edge $\{j, j'\}$ of G_M exists if and only if $\{a_j, b_j, c_j\} \cap \{a_{j'}, b_{j'}, c_{j'}\} \neq \emptyset$. For example, consider the case $n = 6$, $r = 3$ and let the set of candidates be $\{a, b, c, d, e, f\}$. Let

$$M = (\{a, a, b\}, \{c, c, d\}, \{d, e, f\})$$

be the observed sufficient statistic. The vertex set of G_M is $[3]$ and the connected components are $\{1\}$ and $\{2, 3\}$. The possible assignment in the first connected component is $\{(a), (a), (b)\}$. In the second connected component there are two kinds of assignments, $\{(c, d), (c, e), (d, f)\}$ and $\{(c, d), (c, f), (d, e)\}$. In this case the number of elements of the corresponding fiber is six.

Now we discuss the detailed structure of fibers arising from the sufficient statistic $M = (\{a_1, b_1, c_1\}, \dots, \{a_r, b_r, c_r\})$ such that the associated graph G_M is connected. Thanks to the symmetry in permutation of ranking orders and of labels of the candidates, we consider the equivalence classes of such sufficient statistics. Figures 1–7 show the graph G_M 's for all the representatives of the equivalence classes for $r = 2, 3$ whose corresponding fiber needs a move of degree three for its connectivity. The moves of degree three arising from these figures except Figure 6 are indispensable. On the other hand, to guarantee the connectivity of the fiber associated with Figure 6, the Markov basis needs to include a dispensable move of degree three.

Let us consider the case $r = 4$. There are 241 different equivalence classes of the sufficient statistic M 's with connected G_M . For 38 classes among them, the corresponding fibers need moves of degree three for their connectivity. Table 2 summarizes the structure of the equivalence classes. In this table, 38 equivalence classes are classified by the associated graph G_M , the number n_M of candidates appearing in M , and whether a move of degree three needed for the connectivity of the corresponding fiber is indispensable or not. Figure 8 shows an example of a

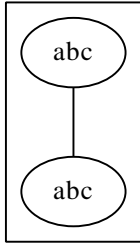


Figure 1:

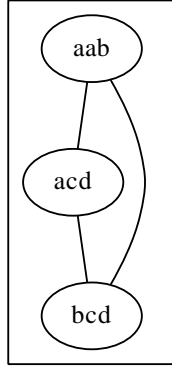


Figure 2:

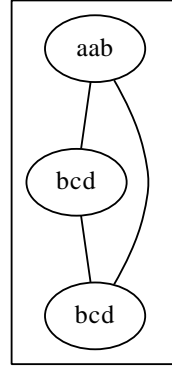


Figure 3:

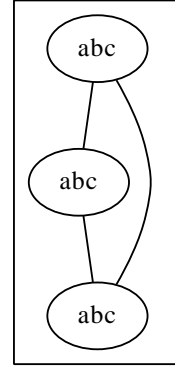


Figure 4:

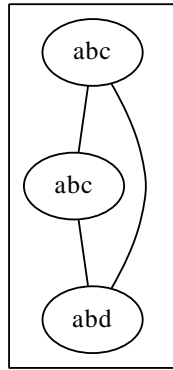


Figure 5:

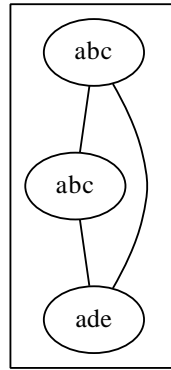


Figure 6:

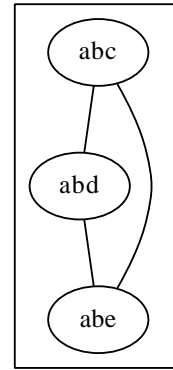







Figure 7:

fiber connected by an indispensable move of degree three. On the other hand, the fiber in Figure 9 needs a dispensable move of degree three for its connectivity.

The rest of this section is devoted to the evaluation of the number of moves of degree three in a minimal Markov basis. We first evaluate the sizes of equivalence classes of the sufficient statistic M 's with connected G_M . For the fiber \mathcal{F}_M associated with a given sufficient statistic M , let $\mathcal{G}_{\mathcal{F}_M}$ be a graph on the vertex set \mathcal{F}_M defined as follows: an edge $\{x, y\}$ for $x, y \in \mathcal{F}_M$ exists if and only if x and y are connected by a move of degree two. Table 3 counts the sufficient statistic M 's in each equivalence classes classified by the length r of ranking, the number n_M of the candidates appearing in M , and the number N_M of connected components of the graph $\mathcal{G}_{\mathcal{F}_M}$.

Using Table 3, the number of moves of degree three in a minimal Markov basis can be calculated. To illustrate the process of this calculation we define some notations. Let $[M]_{r',n',N'}$ be the equivalence class whose length of ranking is r' , the number of candidates is n' , and the number of connected component of $\mathcal{G}_{\mathcal{F}_M}$ is N' . Let $n_{r',n',N'} = n_M, M \in [M]_{r',n',N'}$. Let $N_{r',n',N'} = N_M, M \in [M]_{r',n',N'}$. For example, $n_{2,3,1} = 3$ and $N_{2,3,1} = 1$. Denote the size of equivalence class $[M]_{r',n',N'}$ by $\#[r',n',N']$. For example, $\#[1,1,1] = 1$ and $\#[2,4,1] = 60$.

Table 2: Classification of the equivalence classes for $r = 4$.

G_M	n_M	indispensability	# of equiv. classes
	5	yes	2
	5	yes	2
		no	2
	6	yes	1
		no	6
	7	no	2
	5	yes	7
		no	1
	6	no	5
	7	no	1
	4	yes	1
	5	yes	2
		no	4
	6	yes	1
		no	1

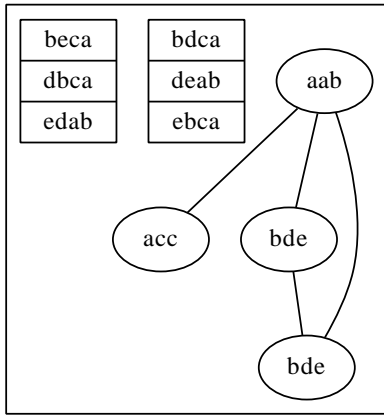


Figure 8:

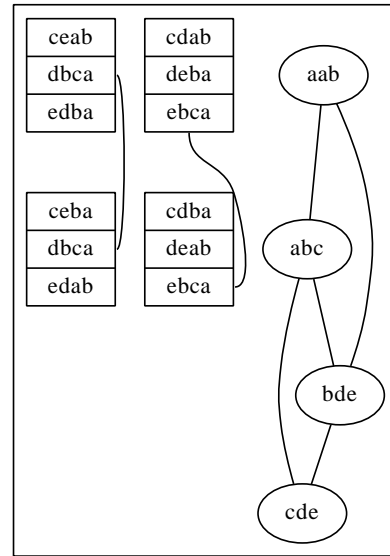


Figure 9:

Furthermore, for simplicity, we set

$$I[n, r', n', N'] = \binom{n}{n'} \#[r', n', N'].$$

Consider the case $r = 2$. The number of moves of degree three is

$$I[n, 2, 3, 2](N_{2,3,2} - 1) = \binom{n}{3} \times 1 \times (2 - 1) = \binom{n}{3}.$$

Table 3: Sizes of equivalence classes of sufficient statistics.

r	n_M	N_M	size of equiv. class	r	n_M	N_M	size of equiv. class
1	1	1	1	3	6	1	8820
	2	1	2		7	1	4410
	3	1	1		4	4	1
2	2	1	2	2		144	
	3	1	30	5		1	82080
		2	1			2	23040
	4	1	60	4		600	
	5	1	30	6		1	885240
3	3	1	30			2	60480
		2	1			7	1
	4	1	1128	2		37800	
5	1	5760	8	1		3749760	
		150	9	1	1814400		
	2	150					

Consider the case $r = 3$. The number of moves of degree three for the partition (3) of three is

$$\begin{aligned}
 & I[n, 3, 3, 2](N_{3,3,2} - 1) + I[n, 3, 4, 2](N_{3,4,2} - 1) + I[n, 3, 5, 2](N_{3,5,2} - 1) \\
 &= \binom{n}{3} + 144\binom{n}{4} + 150\binom{n}{5}.
 \end{aligned}$$

The number of moves of degree three for the partition (2, 1) of three is

$$\begin{aligned}
 & \binom{3}{2} I[n, 2, 3, 2] \times (I[n - n_{2,3,2}, 1, 1, 1](N_{2,3,2}N_{1,1,1} - 1) \\
 & + I[n - n_{2,3,2}, 1, 2, 1](N_{2,3,2}N_{1,2,1} - 1) + I[n - n_{2,3,2}, 1, 3, 1](N_{2,3,2}N_{1,3,1} - 1)) \\
 &= 12\binom{n}{4} + 60\binom{n}{5} + 60\binom{n}{6}.
 \end{aligned}$$

Then the number of moves of degree three for $r = 3$ is

$$\binom{n}{3} + 156\binom{n}{4} + 210\binom{n}{5} + 60\binom{n}{6}.$$

Consider the case $r = 4$. The number of moves of degree three for the partition (4) of four is

$$\begin{aligned}
 & I[n, 4, 4, 2](N_{4,4,2} - 1) + I[n, 4, 5, 2](N_{4,5,2} - 1) + I[n, 4, 5, 4](N_{4,5,4} - 1) \\
 & + I[n, 4, 6, 2](N_{4,6,2} - 1) + I[n, 4, 7, 2](N_{4,7,2} - 1) \\
 &= 144\binom{n}{4} + 24840\binom{n}{5} + 60480\binom{n}{6} + 37800\binom{n}{7}.
 \end{aligned}$$

The number of moves of degree three for the partition (3, 1) of four is

$$\begin{aligned}
& \binom{4}{3} \left(I[n, 3, 3, 2] \times \left(I[n - n_{3,3,2}, 1, 1, 1](N_{3,3,2}N_{1,1,1} - 1) \right. \right. \\
& + I[n - n_{3,3,2}, 1, 2, 1](N_{3,3,2}N_{1,2,1} - 1) + I[n - n_{3,3,2}, 1, 3, 1](N_{3,3,2}N_{1,3,1} - 1) \\
& + I[n, 3, 4, 2] \times \left(I[n - n_{3,4,2}, 1, 1, 1](N_{3,4,2}N_{1,1,1} - 1) \right. \\
& + I[n - n_{3,4,2}, 1, 2, 1](N_{3,4,2}N_{1,2,1} - 1) + I[n - n_{3,4,2}, 1, 3, 1](N_{3,4,2}N_{1,3,1} - 1) \\
& + I[n, 3, 5, 2] \times \left(I[n - n_{3,5,2}, 1, 1, 1](N_{3,5,2}N_{1,1,1} - 1) \right. \\
& + I[n - n_{3,5,2}, 1, 2, 1](N_{3,5,2}N_{1,2,1} - 1) + I[n - n_{3,5,2}, 1, 3, 1](N_{3,5,2}N_{1,3,1} - 1) \left. \left. \right) \right) \\
& = 16 \binom{n}{4} + 2960 \binom{n}{5} + 20960 \binom{n}{6} + 45360 \binom{n}{7} + 33600 \binom{n}{8}.
\end{aligned}$$

The number of moves of degree three for the partition (2, 2) of four is

$$\begin{aligned}
& \frac{1}{2!} \binom{4}{2} I[n, 2, 3, 2] \times \left(I[n - n_{2,3,2}, 2, 2, 1](N_{2,3,2}N_{2,2,1} - 1) \right. \\
& + I[n - n_{2,3,2}, 2, 3, 1](N_{2,3,2}N_{2,3,1} - 1) + I[n - n_{2,3,2}, 2, 3, 2](N_{2,3,2}N_{2,3,2} - 1) \\
& + I[n - n_{2,3,2}, 2, 4, 1](N_{2,3,2}N_{2,4,1} - 1) + I[n - n_{2,3,2}, 2, 5, 1](N_{2,3,2}N_{2,5,1} - 1) \left. \right) \\
& = 120 \binom{n}{5} + 3780 \binom{n}{6} + 12600 \binom{n}{7} + 10080 \binom{n}{8}.
\end{aligned}$$

The number of moves of degree three for the partition (2, 1, 1) of four is

$$\begin{aligned}
& \frac{1}{2!} \frac{4!}{2!1!1!} I[n, 2, 3, 2] \\
& \times \left(I[n - n_{2,3,2}, 1, 1, 1] \left(I[n - n_{2,3,2} - n_{1,1,1}, 1, 1, 1](N_{2,3,2}N_{1,1,1}N_{1,1,1} - 1) \right. \right. \\
& + I[n - n_{2,3,2} - n_{1,1,1}, 1, 2, 1](N_{2,3,2}N_{1,1,1}N_{1,2,1} - 1) \\
& + I[n - n_{2,3,2} - n_{1,1,1}, 1, 3, 1](N_{2,3,2}N_{1,1,1}N_{1,3,1} - 1) \\
& + I[n - n_{2,3,2}, 1, 2, 1] \left(I[n - n_{2,3,2} - n_{1,2,1}, 1, 1, 1](N_{2,3,2}N_{1,2,1}N_{1,1,1} - 1) \right. \\
& + I[n - n_{2,3,2} - n_{1,2,1}, 1, 2, 1](N_{2,3,2}N_{1,2,1}N_{1,2,1} - 1) \\
& + I[n - n_{2,3,2} - n_{1,2,1}, 1, 3, 1](N_{2,3,2}N_{1,2,1}N_{1,3,1} - 1) \left. \right) \\
& + I[n - n_{2,3,2}, 1, 3, 1] \left(I[n - n_{2,3,2} - n_{1,3,1}, 1, 1, 1](N_{2,3,2}N_{1,3,1}N_{1,1,1} - 1) \right. \\
& + I[n - n_{2,3,2} - n_{1,3,1}, 1, 2, 1](N_{2,3,2}N_{1,3,1}N_{1,2,1} - 1) \\
& + I[n - n_{2,3,2} - n_{1,3,1}, 1, 3, 1](N_{2,3,2}N_{1,3,1}N_{1,3,1} - 1) \left. \right) \left. \right) \\
& = 120 \binom{n}{5} + 1440 \binom{n}{6} + 6720 \binom{n}{7} + 13440 \binom{n}{8} + 10080 \binom{n}{9}.
\end{aligned}$$

Then the number of moves of degree three for $r = 4$ is

$$160 \binom{n}{4} + 28040 \binom{n}{5} + 86660 \binom{n}{6} + 102480 \binom{n}{7} + 57120 \binom{n}{8} + 10080 \binom{n}{9}.$$

Table 4: Number of moves of degree three.

$n \backslash r$	2	3	4	5
1	0	0	0	0
2	0	0	0	0
3	1	1	0	0
4	4	160	160	0
5	10	1000	28840	28840
6	20	3680	257300	7056240
7	35	10325	1303540	84797440
8	56	24416	4884880	565736640
9	84	51240	15046080	2735910240
10	120	98400	40267080	10678207680

By the similar calculation we obtain the following polynomial which represents the number of moves of degree two for the case $r = 5$:

$$28840 \binom{n}{5} + 6883200 \binom{n}{6} + 36009400 \binom{n}{7} + 83316800 \binom{n}{8} + 107898000 \binom{n}{9} \\ + 76104000 \binom{n}{10} + 27720000 \binom{n}{11} + 3696000 \binom{n}{12}.$$

The number of moves of degree three in minimal Markov bases are summarized as Table 4. The authors confirmed that the numbers above the horizontal lines in Table 4 coincides with the numbers obtained by the output of the computational software 4ti2([2]).

4 Some discussions

In this section we discuss some topics related to our main result.

4.1 Extension of fibers by allowing one negative element

As discussed after the proof of Lemma 2.8 we have shown the following result by our proof of Theorem 2 (cf. (23)).

Proposition 4.1. *Let P and P' be any two proper datasets with the same sufficient statistic. If we allow incomplete datasets, P and P' are connected by swap operations among two rows at each step.*

Note an improper dataset has one -1 in its orthogonal array representation. Then Proposition 4.1 seems to suggest that every fiber \mathcal{F}_t for the configuration $A_{n,r}$ becomes connected by degree two moves if we extend \mathcal{F}_t by allowing one negative element $x(\sigma) = -1$ in \mathbf{x} which satisfies $\mathbf{t} = A\mathbf{x}$. However this is not correct. In fact allowing -1 in the orthogonal representation and allowing -1 in \mathcal{F}_t are two different things.

This can be confirmed by the following basic example. For $n = 3$ and $r = 2$ with candidates labeled as a, b, c , it is easily seen that $\dim \ker A_{3,2} = 1$ and $I_{A_{3,2}}$ is a principal ideal generated by a single binomial $p(ab)p(bc)p(ca) - p(ac)(cb)p(ba)$. Hence there is no degree two move in $\ker_{\mathbb{Z}} A_{3,2}$. Yet, we can connect two datasets

$$P = \begin{bmatrix} a & b \\ b & c \\ c & a \end{bmatrix}, \quad P' = \begin{bmatrix} a & c \\ c & b \\ b & a \end{bmatrix}$$

by applying $\{2, 3\} : a \xleftrightarrow{1} b$, $\{1, 2\} : b \xleftrightarrow{2} c$ and $\{2, 3\} : a \xleftrightarrow{1} c$ in this order:

$$\begin{bmatrix} a & b \\ b & c \\ c & a \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ a & c \\ b+c-a & a \end{bmatrix} \rightarrow \begin{bmatrix} a & c \\ a & b \\ b+c-a & a \end{bmatrix} \rightarrow \begin{bmatrix} a & c \\ c & b \\ b & a \end{bmatrix}.$$

Note that the middle two datasets can be interpreted either as

adding $(ab), (ac), (bc), (ca)$ and subtracting (ac)

or as

adding $(ab), (ac), (ba), (cb)$ and subtracting (ab) .

However the middle two datasets do not correspond to an element of a fiber for $A_{3,2}$.

4.2 Normality

Here we discuss the normality of the semigroup generated by $A_{n,r}$ and its relation to Lemma 2.2.

Each column of the configuration matrix $A_{n,r}$ such as $A_{4,3}$ in (2) can be considered as the stacked form of an $n \times r$ 0-1 matrix. Consider the set Q of $n \times r$ real matrices $X = \{x_{ij}\}$ satisfying

$$0 \leq x_{ij} \leq 1, \forall i, j, \quad \sum_{j=1}^r x_{ij} \leq 1, \forall i, \quad \sum_{i=1}^n x_{ij} = 1, \forall j.$$

Q is a polytope in $\mathbb{R}^{n \times r}$. By [5] the set of vertices of Q is exactly the same as the set of columns of $A_{n,r}$. Then by the results of [8] and [10] the semigroup generated by $A_{n,r}$ is normal. Lemma 2.2 is a consequence of this normality, because by the normality each fiber indexed by M in (26) has a solution consisting of two valid votes. These two valid votes can be obtained from the two rows of a swap operation in Lemma 2.2.

However the normality is not useful in proving Lemmas 2.9 and 2.10.

4.3 Generation of moves for running a Markov chain

Our detailed investigation of Markov basis for $r \leq 5$ suggests that it will be difficult to obtain an exhaustive list of various types of elements of Markov bases for $r \geq 6$. However based

on Theorem 2.1 we can run a Markov chain for general r as follows. We randomly generate two or three valid votes of r candidates out of n candidates. Once these votes are obtained, we randomly perform permutations of candidates in the same position. We do this for each position. If no collision occurs, then we have two or three valid votes. If the obtained set of votes is different from the initial set, then the difference is a move. In this way, we obtain a random move of degree two or three and then run a Markov chain over a given fiber.

Acknowledgment

This work is partially supported by Grant-in-Aid for JSPS Fellows (No. 12J07561) from Japan Society for the Promotion of Science (JSPS).

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