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Yusuke Kobayashi*

Abstract

For an undirected graph and a fixed integer k, a 2-matching is said to be C_k-free if it has no cycle of length k or less. The problem of finding a maximum cardinality C_k-free 2-matching is polynomially solvable when $k \leq 3$, and NP-hard when $k \geq 5$. It is known that the polynomial solvability of this problem is closely related to jump systems. Indeed, the degree sequences of the C_k-free 2-matchings form a jump system for $k \leq 4$, and do not always form a jump system for $k \geq 5$.

As a quantitative extension of these results, we investigate a relationship between weighted C_k-free 2-matchings and M-concave functions on constant-parity jump systems. It is known that the weighted C_k-free 2-matchings induce an M-concave function on a constant-parity jump system for $k \leq 2$, and it is not always true for $k \geq 4$, which is consistent with the polynomial solvability of the maximization problem. In this paper, we show that the weighted C_3-free 2-matchings induce an M-concave function on a constant-parity jump system.

Keywords: Triangle-free 2-matching, Degree sequence, Jump system, M-concave function

1 Introduction

In an undirected graph, an edge set M is said to be a 2-*matching* if each vertex is incident to at most two edges in M (it is usually called a *simple* 2-matching in the literature). We say that a 2-matching M is C_k-*free* if M contains no cycle of length k or less. The condition "C_3-free" is sometimes referred to as "triangle-free". The C_k-*free* 2-*matching problem* is to find a C_k-free 2-matching of maximum size in a given graph. Note that the case $k \leq 2$ is exactly the classical simple 2-matching problem, which can be solved efficiently. Papadimitriou showed that the C_k-free 2-matching problem is NP-hard for $k \geq 5$ (see [5]). On the other hand, Hartvigsen [7] proved that the problem is polynomial-time solvable for $k = 3$. The case $k = 4$ is left open.

The relationship between jump systems and C_k-free 2-matchings was investigated in [4, 10]. A jump system, introduced by Bouchet and Cunningham [3], is a set of integer lattice points with an exchange property (see Section 2). It is a generalization of a matroid, a delta-matroid, and a base polyhedron of an integral polymatroid (or a submodular system). Many efficiently solvable combinatorial optimization problems closely relate to these structures. Cunningham [4] proved that the degree sequences of the C_k-free 2-matchings form a jump system for $k \leq 3$, and do not always form a jump system for $k \geq 5$. Later, it was shown in [10] that the degree sequences of the C_4-free 2-matchings form a jump system.

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In this paper, we consider the edge-weighted version. The concept of M-concave (M-convex) functions on constant-parity jump systems is a general framework of optimization problems on jump systems [13] (see Section 2 for a definition), and it is a generalization of valuated matroids, valuated delta-matroids, and M-convex functions on base polyhedra. It is known that the C_k-free 2-matchings induce an M-concave function on a constant-parity jump system for $k \le 2$ [13], and it is not always true for $k \ge 4$ [10]. These results are consistent with the polynomial-time solvability of the weighted C_k-free 2-matching problem in the sense that this problem is NP-hard when $k \ge 4$ (see [2, 6]) and polynomial-time solvable when $k = 2$. The case when $k = 3$ is still open.

In this paper, we show that the weighted C_3-free 2-matchings induce an M-concave function on a constant-parity jump system. This generalizes the result in [8], which shows the same result in subcubic graphs.

We note that a polynomial-time algorithm for the (resp. weighted) C_k-free 2-matching problem does not imply that the (resp. weighted) C_k-free 2-matchings induce a jump system (resp. an M-concave function on a jump system), and vice versa. On the other hand, the fact that the degree sequences form a matroidal structure such as a jump system has a potential to be used in a polynomial-time algorithm (see e.g. [2, 8]). Besides the theoretical interest on discrete structures, this motivates us to consider the relationship between C_k-free 2-matchings and a jump system (or an M-concave function on a jump system).

2 Preliminaries

2.1 Triangle-free 2-matchings

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E. Let $\delta(v)$ denote the set of edges incident to $v \in V$, and the *degree* of v is $|\delta(v)|$. The *degree sequence* of an edge set $F \subseteq E$ is the vector $d_F \in \mathbf{Z}^V$ defined by $d_F(v) = |\delta(v) \cap F|$. An edge set $M \subseteq E$ is said to be a *2-matching* if $d_M(v) \le 2$ for every $v \in V$. In other words, a 2-matching is a vertex-disjoint collection of paths and cycles. An edge set $M \subseteq E$ is said to be *triangle-free* (or *C_3-free*) if M contains no cycle of length three or less as a subgraph. In a graph with a weight function w on the edge set, the *weighted triangle-free 2-matching problem* is to find a triangle-free 2-matching M maximizing $w(M) := \sum_{e \in M} w(e)$. It is unknown whether or not the weighted triangle-free 2-matching problem can be solved in polynomial time.

2.2 Jump systems

Let V be a finite set. For $u \in V$, we denote by χ_u the *characteristic vector* of u, with $\chi_u(u) = 1$ and $\chi_u(v) = 0$ for $v \in V \setminus \{u\}$. For $x, y \in \mathbf{Z}^V$, a vector $s \in \mathbf{Z}^V$ is called an *(x, y)-increment* if $x(u) < y(u)$ and $s = \chi_u$ for some $u \in V$, or $x(u) > y(u)$ and $s = -\chi_u$ for some $u \in V$. We say that a nonempty set $J \subseteq \mathbf{Z}^V$ is a *jump system* if it satisfies the following [3]:

> For any $x, y \in J$ and for any (x, y)-increment s with $x + s \notin J$, there exists an $(x + s, y)$-increment t such that $x + s + t \in J$.

A set $J \subseteq \mathbf{Z}^V$ is a *constant-parity system* if $x(V) - y(V)$ is even for any $x, y \in J$. Here $x(S) = \sum_{v \in S} x(v)$ for $x \in \mathbf{Z}^V$ and $S \subseteq V$. For constant-parity jump systems, Geelen showed that a nonempty set J is a constant-parity jump system if and only if it satisfies the following (see [13] for details):

(EXC) For any $x, y \in J$ and for any (x, y)-increment s, there exists an $(x + s, y)$-increment t such that $x + s + t \in J$ and $y - s - t \in J$.

A constant-parity jump system is a generalization of the base family of a matroid, an even delta-matroid, and a base polyhedron of an integral polymatroid (or a submodular system).

The degree sequences of all subgraphs in an undirected graph are a typical example of a constant-parity jump system [3, 12]. Cunningham [4] showed that the set of degree sequences of all C_k-free 2-matchings is a jump system for $k \leq 3$, but not a jump system for $k \geq 5$. Kobayashi, Szabó, and Takazawa [10] showed that it is also a jump system when $k = 4$.

2.3 M-concave functions

An *M-concave function on a constant-parity jump system* is a quantitative extension of a jump system, which is a generalization of valuated matroids, valuated delta-matroids, and M-concave functions on base polyhedra.

Definition 1 (M-concave function on a constant-parity jump system [13])**.** For $J \subseteq \mathbf{Z}^V$, we call $f : J \to \mathbf{R}$ an *M-concave function on a constant-parity jump system* if it satisfies the following exchange axiom:

(M-EXC) For any $x, y \in J$ and for any (x, y)-increment s, there exists an $(x+s, y)$-increment t such that $x+s+t \in J$, $y-s-t \in J$, and $f(x)+f(y) \leq f(x+s+t)+f(y-s-t)$.

It directly follows from (M-EXC) that J satisfies (EXC), and hence J is a constant-parity jump system. For simplicity, we identify $f : J \to \mathbf{R}$ with a function $f' : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ defined by $f'(x) = f(x)$ for $x \in J$ and $f'(x) = -\infty$ for $x \in \mathbf{Z}^V \setminus J$.

M-concave functions on constant-parity jump systems appear in many combinatorial optimization problems such as the weighted matching problem, the minsquare factor problem [1], the weighted even factor problem in odd-cycle-symmetric digraphs [11], and the weighted square-free 2-matching problem [10]. Some maximization algorithms are proposed in [14, 15] and some properties of M-concave functions are investigated in [9]. In particular, it is shown in [9] that M-concave functions are closed under an operation called *convolution*, which is a quantitative extension of sum. For two functions $f_1 : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ and $f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$, we define their *convolution* as a function $f_1 \Box f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty, -\infty\}$ given by

$$(f_1 \Box f_2)(x) = \sup\{f_1(x_1) + f_2(x_2) \mid x_1 + x_2 = x,\ x_1 \in \mathbf{Z}^V,\ x_2 \in \mathbf{Z}^V\}.$$

The following theorem plays an important role in this paper.

Theorem 1 (Kobayashi et al. [9])**.** *If f_1 and f_2 are M-concave functions then their convolution $f_1 \Box f_2$ is M-concave, provided $f_1 \Box f_2 < +\infty$.*

3 Our result

For a graph $G = (V, E)$, let $J_{\mathrm{tri}}(G) \subseteq \mathbf{Z}^V$ denote the set of all degree sequences of triangle-free 2-matchings in G, that is,

$$J_{\mathrm{tri}}(G) = \{d_M \mid M \text{ is a triangle-free 2-matching in } G\}.$$

It is shown in [4] that $J_{\mathrm{tri}}(G)$ is always a constant-parity jump system. For a weighted graph (G, w), define a function f_{tri} on $J_{\mathrm{tri}}(G)$ by

$$f_{\mathrm{tri},G}(x) = \max\left\{ w(M) \;\middle|\; M \text{ is a triangle-free 2-matching, } d_M = x \right\},$$

where $w(F) = \sum_{e \in F} w(e)$ for an edge set $F \subseteq E$. Our main theorem is stated as follows.

Theorem 2. *For a graph $G = (V, E)$ with a weight function $w : E \to \mathbf{R}$, $f_{\rm tri}$ is an M-concave function on the constant-parity jump system $J_{\rm tri}(G)$.*

Note that the same result was shown in [8] for subcubic graphs. We also note that since we do not have a polynomial-time algorithm to compute $f_{{\rm tri},G}(x)$, this result does not imply a polynomial-time algorithm for the weighted C_3-free 2-matching problem. However, our result has a potential to be used in a polynomial-time algorithm (see e.g. [2, 8]).

The rest of this paper is devoted to the proof of Theorem 2. Since we consider triangle-free 2-matchings, we may assume that the given graph G is simple. In what follows in this paper, when we deal with a graph with parallel edges, we call it a *multigraph.*

4 Applying induction

To show Theorem 2, we use the induction on the number of edges of G. That is, for a graph $G = (V, E)$, we show that $f_{{\rm tri},G}$ is an M-concave function under the assumption that $f_{{\rm tri},G'}$ is an M-concave function for any graph G' that has fewer edges than G.

4.1 Induction step: applying the convolution

In this subsection, we consider the case when there exists a partition (E_1, E_2) of the edge set E such that any triangle $C \subseteq E$ is contained in either E_1 or E_2. In this case, an edge set $M \subseteq E$ is triangle-free if and only if $M \cap E_i$ is triangle-free in $G_i = (V, E_i)$ for $i = 1, 2$. With this observation, we have

$$f_{{\rm tri},G} = f_{{\rm tri},G_1} \Box f_{{\rm tri},G_2}.$$

Since $f_{{\rm tri},G_i}$ is an M-concave function by induction hypothesis, $f_{{\rm tri},G}$ is also an M-concave function by Theorem 1.

Therefore, in what follows, we assume that for any partition (E_1, E_2) of E, there exists a triangle $C \subseteq E$ that intersects both E_1 and E_2.

4.2 Finding an $(x + s, y)$-increment

For given degree sequences $x, y \in J_{\rm tri}(G)$, take triangle-free 2-matchings $M, N \subseteq E$ such that $d_M = x$, $d_N = y$, $f_{{\rm tri},G}(x) = w(M)$, and $f_{{\rm tri},G}(y) = w(N)$. In order to prove Theorem 2, we show that for any (x, y)-increment s, there exists an $(x + s, y)$-increment t satisfying the conditions in (M-EXC).

4.2.1 Induction step: removing irrelevant edges

Assume that there exists an edge $e \in E \setminus (M \cup N)$. In such a case, for a graph $G' := G - e$, we have $f_{{\rm tri},G'}(x) = w(M) = f_{{\rm tri},G}(x)$ and $f_{{\rm tri},G'}(y) = w(N) = f_{{\rm tri},G}(y)$. Since $f_{{\rm tri},G'}$ is an M-concave function by induction hypothesis, for any (x, y)-increment s, there exists an $(x+s, y)$-increment t such that $x + s + t, y - s - t \in J_{\rm tri}(G') \subseteq J_{\rm tri}(G)$ and

$$\begin{aligned} f_{{\rm tri},G}(x) + f_{{\rm tri},G}(y) &= f_{{\rm tri},G'}(x) + f_{{\rm tri},G'}(y) \\ &\le f_{{\rm tri},G'}(x + s + t) + f_{{\rm tri},G'}(y - s - t) \\ &\le f_{{\rm tri},G}(x + s + t) + f_{{\rm tri},G}(y - s - t). \end{aligned}$$

This shows that there exists an $(x + s, y)$-increment t satisfying the conditions in (M-EXC).

4.2.2 Enumeration of base cases

If $\|x-y\|_1 = 2$, then $t = y-x-s$ is an $(x+s, y)$-increment satisfying the conditions in (M-EXC). Thus, by the arguments in Sections 4.1 and 4.2.1, it suffices to consider the case when G, x, y, M and N satisfy the following conditions.

(1) For any partition (E_1, E_2) of E, there is a triangle $C \subseteq E$ that intersects both E_1 and E_2,

(2) $E = M \cup N$, and

(3) $\|x - y\|_1 \geq 4$.

By the second and third conditions, the following condition is also satisfied:

(4) the degree of each vertex of G is at most four and $\|4\chi_V - d_E\|_1 \geq 4$.

Here, χ_V is a vector in $\mathbf{Z}^V$ whose every element is one.

Now, we enumerate all graphs satisfying the conditions (1) and (4). We begin with the following lemma.

Lemma 3. *If a graph with at least two edges and no isolated vertices satisfies the condition (1), then it can be obtained from a triangle by applying the following operations repeatedly.*

(I) Add an edge e to a graph $G' = (V', E')$ so that e is contained in a triangle in $G' + e$.

(II) Add a new vertex u and two new edges uv_1 and uv_2 to a graph $G' = (V', E')$ such that $v_1, v_2 \in V'$ and $v_1v_2 \in E'$.

(III) Add two new edges uv_1 and uv_2 to a graph $G' = (V', E')$, where $u, v_1, v_2 \in V'$, $v_1v_2 \in E'$, and $uv_1, uv_2 \notin E'$.

Proof. If a graph $G = (V, E)$ has at least two edges and satisfies the condition (1), then it must contain a triangle. For any subgraph $G' = (V', E')$ of G, since there exists a triangle $C \subseteq E$ that intersects both E' and $E \setminus E'$, we can apply one of the the operations (I), (II), and (III) to G'. By using this argument repeatedly, we can construct G from a triangle. □

Let $G'' = (V'', E'')$ be the graph obtained from $G' = (V', E')$ by applying one of the operations (I), (II), and (III) in Lemma 3, and suppose that the degree of each vertex is at most four in G' and G''. Then, we can see that

- if we apply the operation (I), then $\|4\chi_{V''} - d_{E''}\|_1 = \|4\chi_{V'} - d_{E'}\|_1 - 2$,
- if we apply the operation (II), then $\|4\chi_{V''} - d_{E''}\|_1 = \|4\chi_{V'} - d_{E'}\|_1$, and
- if we apply the operation (III), then $\|4\chi_{V''} - d_{E''}\|_1 = \|4\chi_{V'} - d_{E'}\|_1 - 4$.

With this observation, we have the following.

Lemma 4. *If a graph with at least two edges and no isolated vertices satisfies the conditions (1) and (4), then it is obtained from a triangle by applying the operations (I) and (II), repeatedly. Furthermore, the operation (I) is executed at most once.*

Proof. By Lemma 3, we can construct G from a triangle by applying the operations (I), (II), and (III), repeatedly. For a triangle $C = (V_0, E_0)$, it holds that $\|4\chi_{V_0} - d_{E_0}\|_1 = 6$. Since $\|4\chi_V - d_E\|_1 \geq 4$ by the condition (4), we cannot apply (I) more than once and we cannot apply (III). □

By this lemma, we can enumerate all graphs satisfying the conditions (1) and (4) as in Figure 1, which shows the following proposition.

Proposition 5. *If a graph with at least two edges and no isolated vertices satisfies the conditions (1) and (4), then it is one of the followings.*

- $G_1, G_2, \ldots, G_7$, *where each graph is shown in Figure 1.*
- H_i *for some* $i \geq 1$, *where* H_i *is the graph consisting of* i *triangles as in Figure 1.*

Since we may assume that the graph G has at least two edges and no isolated vertices, it suffices to consider the case when G is one of the graphs in this proposition. Here, we note that M and N might share an edge. By the conditions (2) and (3), if M and N share an edge, then $G = H_i$ for some $i \geq 1$ and exactly one edge is contained in $M \cap N$. In such a case, we suppose that M and N are edge-disjoint by regarding G as a multigraph H'_i ($i \geq 1$), where H'_i is obtained from H_i by duplicating one edge as in Figure 2. Thus, we may assume that M and N are edge-disjoint, and our remaining task is to find an $(x+s, y)$-increment t when the given (multi)graph is one of $G_1, \ldots, G_7$, H_i ($i \geq 1$), and H'_i ($i \geq 1$), which is discussed in the next section.

5 Base cases

In this section, we find an $(x+s, y)$-increment t when the given (multi)graph is one of $G_1, \ldots, G_7$, H_i ($i \geq 1$), and H'_i ($i \geq 1$), and we are given x, y, M, N, and s satisfying the conditions (1), (2), (3), and (4). We first show that $G_1, \ldots, G_7$, H_i ($i \geq 1$), and H'_i ($i \geq 1$) have a nice property called *universality*, and then we show that there exists an $(x+s, y)$-increment in every universal multigraph. We say that a multigraph $G = (V, E)$ is *universal with respect to triangle-free* 2*-matchings* if for any $x, y \in J_{\text{tri}}(G)$ with $x + y = d_E$, there exist edge-disjoint triangle-free 2-matchings M and N such that $d_M = x$, $d_N = y$, and $M \cup N = E$.

5.1 Universality of G_i

In this subsection, we show the universality of G_i.

Lemma 6. *Graphs* $G_1, \ldots, G_7$ *are universal with respect to triangle-free* 2*-matchings.*

Proof. We consider each graph separately.

1. Consider the graph $G_1 = (V, E)$. If $x, y \in J_{\text{tri}}(G_1)$ satisfy $x + y = d_E$, then $\{x, y\} = \{(1,1,1,1), (2,2,2,2)\}$ or $\{(1,1,2,2), (2,2,1,1)\}$ (by relabeling vertices if necessary). Thus, G_1 is universal with respect to triangle-free 2-matchings by Figure 3.

2. Consider the graph $G_2 = (V, E)$. Let v_1, v_2, v_3, v_4 and v_5 be vertices of G_2 as in Figure 3. If $x, y \in J_{\text{tri}}(G_2)$ satisfy $x + y = d_E$, then $\{x, y\} = \{(1,1,2,2,0), (2,2,2,2,2)\}$, $\{(1,1,2,2,2), (2,2,2,2,0)\}$, or $\{(1,2,2,2,1), (2,1,2,2,1)\}$. Thus, G_2 is universal with respect to triangle-free 2-matchings by Figure 3.

3. Consider the graph $G_3 = (V, E)$. Let v_1, v_2, v_3, v_4 and v_5 be vertices of G_3 as in Figure 3. If $x, y \in J_{\text{tri}}(G_3)$ satisfy $x + y = d_E$, then $\{x, y\} = \{(2,2,2,2,0,0), (2,2,2,2,2,2)\}$, $\{(2,2,2,2,0,2), (2,2,2,2,2,0)\}$, or $\{(2,2,2,2,1,1), (2,2,2,2,1,1)\}$. Thus, G_3 is universal with respect to triangle-free 2-matchings by Figure 3.

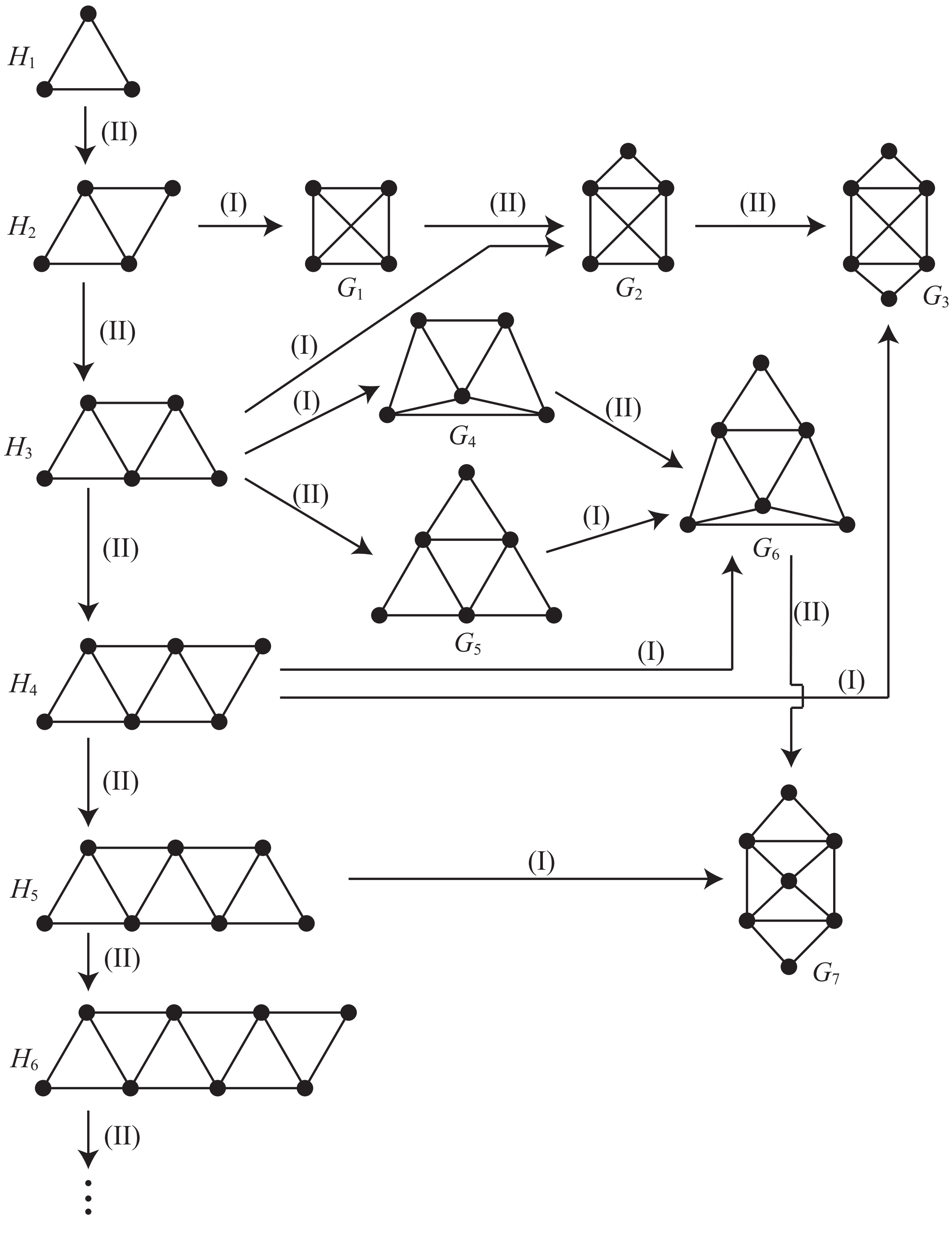

Figure 1: Graphs satisfying the conditions (1) and (4)

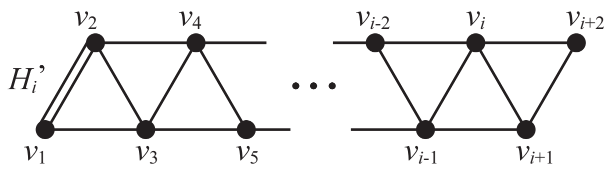

Figure 2: Definition of H_i'

4. Consider the graph $G_4 = (V, E)$. Let v_1, v_2, v_3, v_4 and v_5 be vertices of G_4 as in Figure 3. If $x, y \in J_{\rm tri}(G_4)$ satisfy $x + y = d_E$, then $\{x, y\} = \{(1,1,1,1,2),(2,2,2,2,2)\}$, $\{(1,1,2,2,2),(2,2,1,1,2)\}$, or $\{(1,2,1,2,2),(2,1,2,1,2)\}$ (by relabeling vertices if necessary). Thus, G_4 is universal with respect to triangle-free 2-matchings by Figure 3.

5. Consider the graph $G_5 = (V, E)$. Let v_1, v_2, v_3, v_4, v_5 and v_6 be vertices of G_5 as in Figure 3. If $x, y \in J_{\rm tri}(G_5)$ satisfy $x+y = d_E$, then $\{x, y\} = \{(0,1,1,2,2,2),(2,1,1,2,2,2)\}$ or $\{(0,2,2,2,2,2),(2,0,0,2,2,2)\}$ (by relabeling vertices if necessary). Note that $(0,0,0,2,2,2)$ is not in $J_{\rm tri}(G_5)$. Thus, G_5 is universal with respect to triangle-free 2-matchings by Figure 3.

6. Consider the graph $G_6 = (V, E)$. Let v_1, v_2, v_3, v_4, v_5 and v_6 be vertices of G_6 as in Figure 3. If $x, y \in J_{\rm tri}(G_6)$ satisfy $x + y = d_E$, then $\{x, y\} = \{(1,1,2,2,2,2),(2,2,2,2,2,0)\}$, $\{(1,1,2,2,2,0),(2,2,2,2,2,2)\}$, or $\{(1,2,2,2,2,1),(2,1,2,2,2,1)\}$. Thus, G_6 is universal with respect to triangle-free 2-matchings by Figure 3.

7. Consider the graph $G_7 = (V, E)$. Let $v_1, v_2, v_3, v_4, v_5, v_6$ and v_7 be vertices of G_7 as in Figure 3. If $x, y \in J_{\rm tri}(G_7)$ satisfy $x+y = d_E$, then $\{x, y\} = \{(2,2,2,2,2,0,2),(2,2,2,2,2,2,0)\}$, $\{(2,2,2,2,2,1,1),(2,2,2,2,2,1,1)\}$, or $\{(2,2,2,2,2,0,0),(2,2,2,2,2,2,2)\}$. Thus, G_7 is universal with respect to triangle-free 2-matchings by Figure 3.

By the above cases, we obtain the lemma. □

5.2 Universality of H_i

In this subsection, we show the universality of H_i.

Lemma 7. *For $i \geq 1$, H_i is universal with respect to triangle-free 2-matchings.*

Proof. We can easily see that H_1 is universal with respect to triangle-free 2-matchings. Consider the graph $H_i = (V, E)$ for some fixed $i \geq 2$. Let $v_1, v_2, \dots, v_i, v_{i+1}$, and v_{i+2} be vertices of H_i as in Figure 4. If $x, y \in J_{\rm tri}(H_i)$ satisfy $x + y = d_E$, then $\{x, y\}$ is equal to one of the followings by relabeling vertices if necessary:

$$\begin{array}{ll} \{(0,1,\dots,1,0),(2,2,\dots,2,2)\}, & \{(0,1,\dots,1,2),(2,2,\dots,2,0)\}, \\ \{(0,1,\dots,2,1),(2,2,\dots,1,1)\}, & \{(0,2,\dots,1,1),(2,1,\dots,2,1)\}, \\ \{(0,2,\dots,2,0),(2,1,\dots,1,2)\}, & \{(1,1,\dots,1,1),(1,2,\dots,2,1)\}. \end{array}$$

Note that $x(v_j) = y(v_j) = 2$ for $j = 3, 4, \dots, i$. We also note that $(0,2,2,0),(2,2,2,0) \notin J_{\rm tri}(H_2)$ and $(0,2,2,2,0) \notin J_{\rm tri}(H_3)$, which means that we do not have to consider the cases with these degree sequences. For each case, there exist edge-disjoint triangle-free 2-matchings M and N such that $d_M = x$, $d_N = y$, and $M \cup N = E$ as in Figure 4, which shows that H_i is universal with respect to triangle-free 2-matchings. □

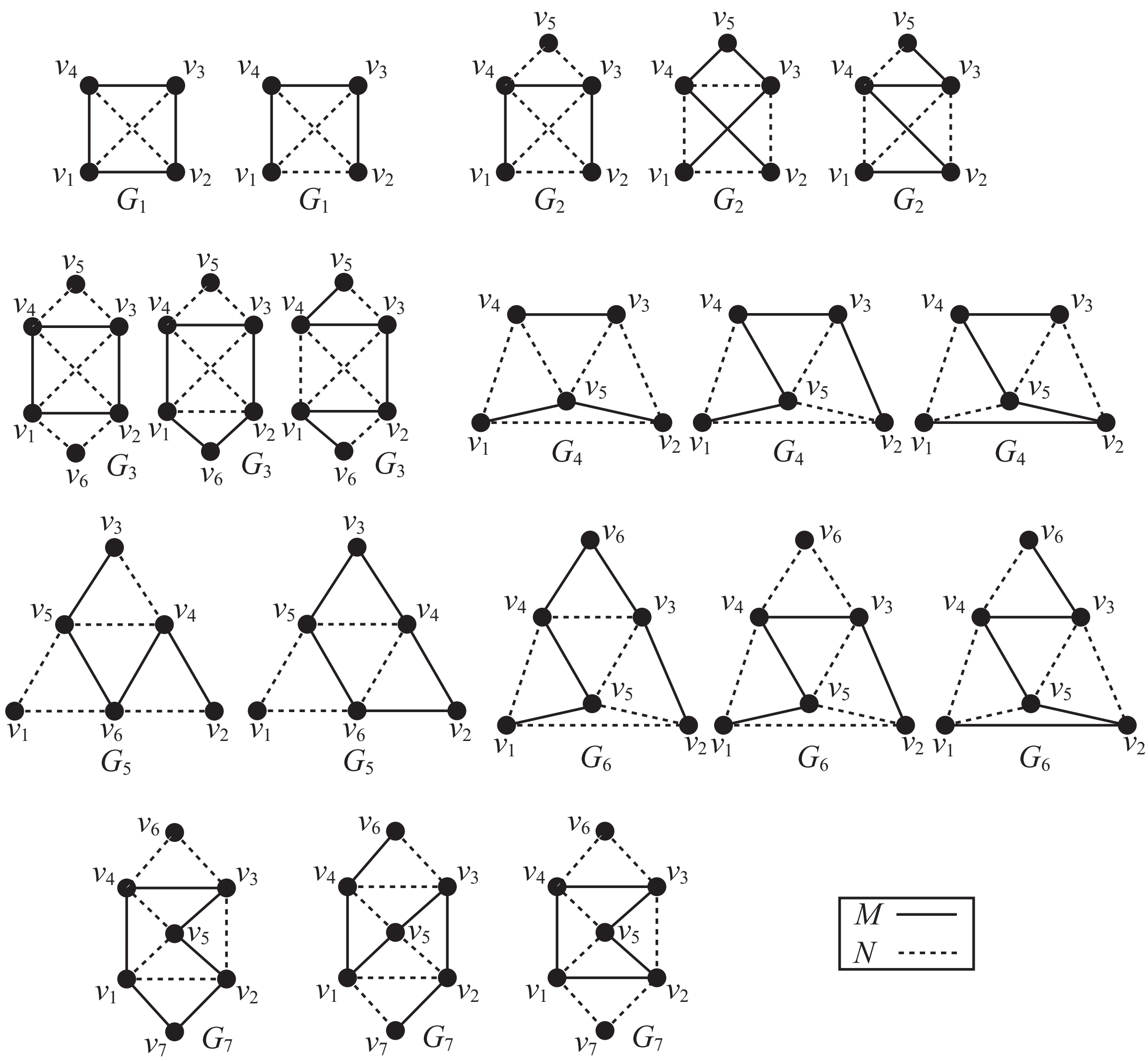

Figure 3: Universality of $G_1, \dots, G_7$

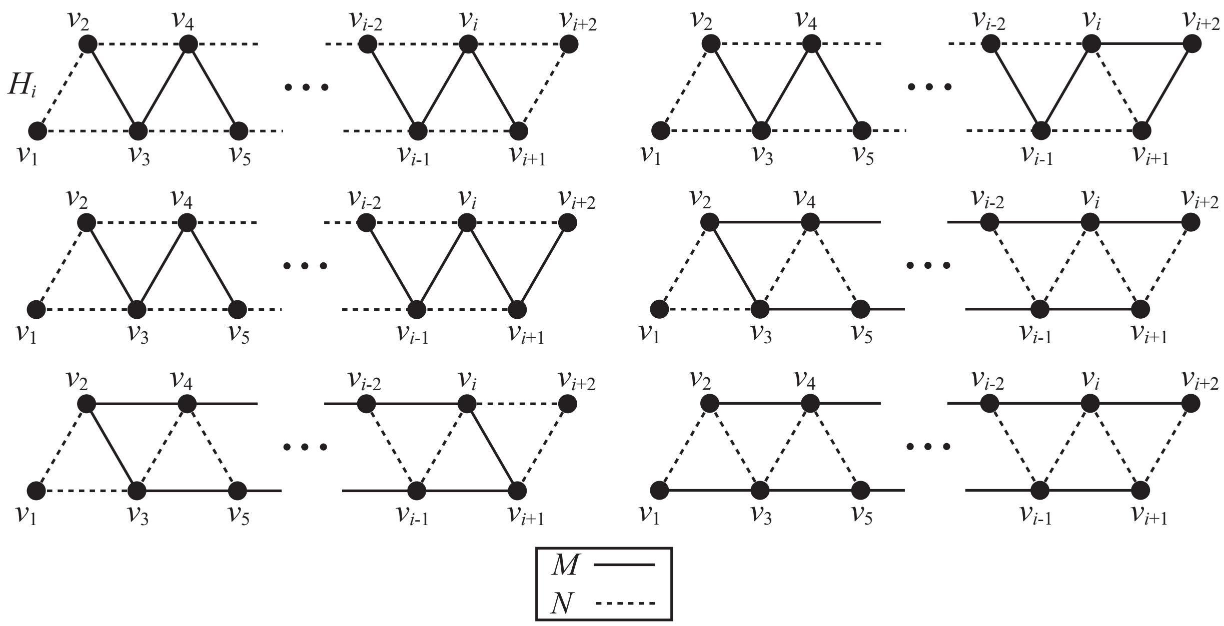

Figure 4: Universality of H_i

5.3 Universality of H_i'

In this subsection, we show the universality of H_i'.

Lemma 8. *For $i \geq 1$, H_i' is universal with respect to triangle-free 2-matchings.*

Proof. We can easily see that H_1' is universal with respect to triangle-free 2-matchings. Consider the multigraph $H_i' = (V, E)$ for some fixed $i \geq 2$. Let $v_1, v_2, \ldots, v_i, v_{i+1}$, and v_{i+2} be vertices of H_i' as in Figure 5. If $x, y \in J_{\text{tri}}(H_i')$ satisfy $x + y = d_E$, then $\{x, y\}$ is equal to one of the followings:

$$\{(1,2,\ldots,1,0),(2,2,\ldots,2,2)\}, \qquad \{(1,2,\ldots,1,2),(2,2,\ldots,2,0)\},$$
$$\{(1,2,\ldots,2,1),(2,2,\ldots,1,1)\}.$$

Note that $x(v_j) = y(v_j) = 2$ for $j = 2, 3, \ldots, i$. We also note that $(2,2,2,0) \notin J_{\text{tri}}(H_2)$, which means that we do not have to consider the case with this degree sequence. For each case, there exist triangle-free 2-matchings M and N such that $d_M = x$, $d_N = y$, and $M \cup N = E$ as in Figure 5, which shows that H_i' is universal with respect to triangle-free 2-matchings. □

5.4 Finding an $(x + s, y)$-increment in universal multigraphs

In this subsection, we show that there exists an $(x + s, y)$-increment in every universal multigraph.

Lemma 9. *Suppose that a multigraph $G = (V, E)$ is universal with respect to triangle-free 2-matchings. For any edge-disjoint triangle-free 2-matchings M and N in G with $M \cup N = E$, $d_M = x$, and $d_N = y$ and for any (x, y)-increment s, there exists an $(x + s, y)$-increment t such that $w(M) + w(N) \leq f_{\text{tri},G}(x + s + t) + f_{\text{tri},G}(y - s - t)$.*

Proof. Since $J_{\text{tri}}(G)$ is a jump system [4], there exists an $(x + s, y)$-increment t such that $x + s + t, y - s - t \in J_{\text{tri}}(G)$. Since G is universal with respect to triangle-free 2-matchings, there

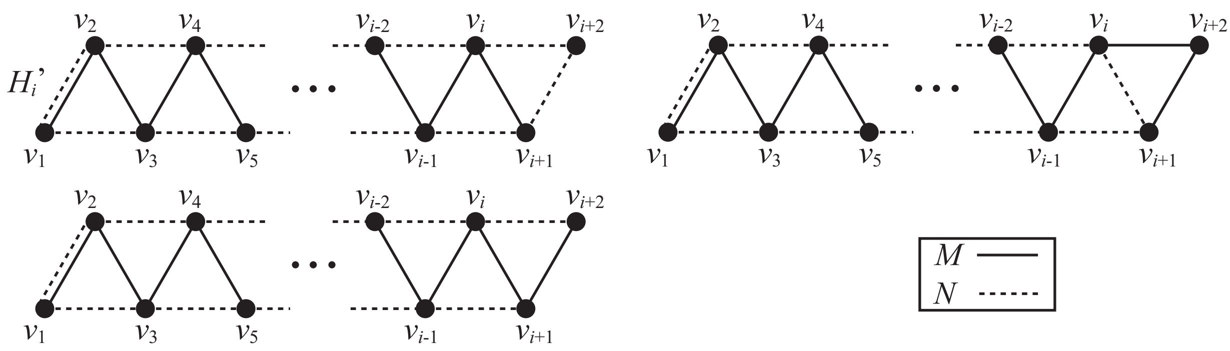

Figure 5: Universality of H'_i

exist edge-disjoint triangle-free 2-matchings M' and N' such that $d_{M'} = x+s+t$, $d_{N'} = y-s-t$, and $M' \cup N' = E$. This shows that

$$w(M) + w(N) = w(M') + w(N') \leq f_{\mathrm{tri},G}(x+s+t) + f_{\mathrm{tri},G}(y-s-t),$$

which completes the proof. □

Suppose that we are given a multigraph G that is universal with respect to triangle-free 2-matchings and degree sequences $x, y \in J_{\mathrm{tri}}(G)$. Let $M, N \subseteq E$ be triangle-free 2-matchings such that $d_M = x$, $d_N = y$, $f_{\mathrm{tri},G}(x) = w(M)$, and $f_{\mathrm{tri},G}(y) = w(N)$, and assume that M and N are edge-disjoint and they satisfy the conditions (1), (2), (3), and (4). In this case, for any (x, y)-increment s, there exists an $(x+s, y)$-increment t such that

$$f_{\mathrm{tri},G}(x) + f_{\mathrm{tri},G}(y) = w(M) + w(N) \leq f_{\mathrm{tri},G}(x+s+t) + f_{\mathrm{tri},G}(y-s-t)$$

by Lemma 9. Therefore, by Lemmas 6, 7, and 8, we can see that there exists an $(x+s, y)$-increment t when the given (multi)graph G is one of $G_1, \ldots, G_7$, H_i $(i \geq 1)$, and H'_i $(i \geq 1)$.

By combining these base cases and the arguments in Section 4, we obtain Theorem 2.

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