MATHEMATICAL ENGINEERING
TECHNICAL REPORTS

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METR 2013–08 May 2013

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Packing Non-zero A-paths via Matroid Matching

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May 2013

Abstract

A Γ-labeled graph is a directed graph G in which each edge is associated with an element of a group Γ by a label function $\psi : E(G) \to \Gamma$. For a vertex subset $A \subseteq V(G)$, a path (of the underlying undirected graph) is called an A-path if its start and end vertices belong to A and does not intersect A in between, and an A-path is called *non-zero* if the product of the labels along the path is not equal to the identity. Chudnovsky, et al. (2006) introduced the problem of packing non-zero A-paths and gave a min-max formula for characterizing the maximum number of vertex-disjoint non-zero A-paths. In this paper we show that the problem of packing non-zero A-paths can be reduced to the matroid matching problem on a certain combinatorial matroid, and discuss how to derive the min-max formula based on Lovász's idea of reducing Mader's $\mathcal{S}$-path problem to a matroid matching problem.

1 Introduction

Let G be a directed graph which may contain multiple edges and loops, and let Γ be a group. A function $\psi : E(G) \to \Gamma$ on the edge set is called a *label function* if the following condition holds: we can change orientation of each edge as we like by inverting its label, i.e., if ψ assigns a label $\gamma \in \Gamma$ to an edge in one direction, then ψ assigns γ^{-1} to the edge in the other direction. A pair (G, ψ) of a directed graph G and a label function ψ is called a Γ-*labeled graph*.

Let (G, ψ) be a Γ-labeled graph and $A \subseteq V(G)$. An A-*path* is a path (of the underlying undirected graph) which starts and ends in A and does not intersect A in between. An A-path P is called *non-zero* if the product of the labels through the path is not equal to the identity, where we multiply the inverse $\psi(e)^{-1}$ if e is traversed in the reverse direction. Let $\mu(G, \psi, A)$ be the maximum number of (fully) vertex-disjoint non-zero A-paths in (G, ψ). Chudnovsky, et al. [1] gave a min-max formula for $\mu(G, \psi, A)$. An efficient algorithm for computing $\mu(G, \psi, A)$ is presented in [2]. Later, Pap [12] gave a simpler proof for the min-max theorem in a slightly generalized setting.

The problem of packing non-zero A-paths generalizes Mader's $\mathcal{S}$-path problem, where we are given a partition $\mathcal{S}$ of a terminal set A and A-paths connecting distinct classes in $\mathcal{S}$ are allowed to be packed. Mader's $\mathcal{S}$-path problem includes the problem of packing A-paths both in vertex-disjoint setting and in edge-disjoint setting, and, as a result, include several fundamental problems in combinatorial optimization such as the maximum matching problem and Menger's disjoint path problem. Another important example of packing non-zero A-paths is packing odd-length A-paths. More examples are given in [1].

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In [7] Lovász showed that the problem of packing $\mathcal{S}$-paths can be reduced to the matroid matching problem on a certain combinatorial matroid, and Schrijver [13] (implicitly) showed that the matroid that appears in this reduction has a linear representation, which means that the problem can be solved by a linear matroid matching algorithm. In this paper, extending Lovász's idea, we show that the problem of packing non-zero A-paths can also be reduced to the matroid matching problem of a combinatorial matroid.

To explain our contribution more specifically, let us briefly review the matroid matching problem by listing preliminary facts on 2-polymatroids, which will be used throughout the paper. Let $\mathbf{M}$ be a matroid with ground set S. A set of pairs $X \subseteq \binom{S}{2}$ is called a *matching* if the rank of the set of elements in X is equal to $2|X|$ in $\mathbf{M}$. In the *matroid matching problem* for a matroid $\mathbf{M}$ on a finite set S, we are given $X \subseteq \binom{S}{2}$ and are asked to find a matching of maximum size in X.

A pair (S, f) of a finite set S and a set function $f : 2^S \to \mathbb{Z}$ is called a *d-polymatroid* if f is a monotone submodular function with $f(\emptyset) = 0$ and $f(\{e\}) \leq d$ for every $e \in S$. We say that $F \subseteq S$ is a *feasible set* in a 2-polymatroid (S, f) if $f(F) = 2|F|$. The size of a maximum feasible set in (S, f) is denoted by $\nu(S, f)$. It is known that the matroid matching problem is equivalent to the problem of finding a maximum feasible set in a 2-polymatroid (see, e.g., [9]).

For a 2-polymatroid (S, f) and a subset $Z \subseteq S$, the *contraction of* Z is defined as 2-polymatroid $(S \setminus Z, f_Z)$ with $f_Z(X) = f(X \cup Z) - f(Z)$ for each $X \subseteq S \setminus Z$. For $X \subseteq S$, let $\mathrm{sp}_f(X) := \{e \in S \mid f(X + e) = f(X)\}$.

The matroid matching problem is a hard problem in general, and there is no polynomial time algorithm for finding an optimal solution [6, 8]. However, there are a number of special classes of matroids for which the min-max formula of $\nu(S, f)$ is known or even the problem can be solved in polynomial time. In particular, Lovász [8] showed that the matroid matching problem can be solved in polynomial time for linearly represented matroids. Lovász [7] (or formally Dress and Lovász [3]) also showed that, if the (poly)matroid satisfies a certain property (discussed below), then the min-max formula can be derived.

A subset $C \subseteq S$ is called a *circuit* in a 2-polymatroid (S, f) if $f(C) = 2|C| - 1$ and $S - e$ is a feasible set for every $e \in S$. A subset $D \subseteq S$ is called a *double circuit* if $f(D) = 2|D| - 2$ and $f(D - e) = 2|D| - 3$ for every $e \in D$. It is known that, if D is a double circuit, then there is a unique partition $D_1, \dots, D_d$ of D into nonempty subsets such that $D \setminus D_i$ is a circuit for every $1 \leq i \leq d$ and every circuit in D is written in this form. A double circuit is called *nontrivial* if $d \geq 3$. The *kernel* of D is defined to be $\bigcap_{1 \leq i \leq d} \mathrm{sp}_f(D \setminus D_i)$.

Let k be a positive integer. A subset $F \subseteq S$ is called a *k-flower* if there exist a circuit C and a feasible set M such that $F = C \cup M$ and $f(F) = f(C) + f(M) = 2k + 1$. A subset F is called a *k-double-flower* if there exist a double circuit D and a feasible set M such that $F = D \cup M$ and $f(F) = f(D) + f(M) = 2k + 2$. One can easily observe that F is a k-flower if and only if $f(F) = 2k + 1$ and $|F| = k + 1$, and that F is a k-double-flower if and only if $f(F) = 2k + 2$, $|F| = k + 2$, and F contains no feasible set of size $k + 1$ [7].

Lovász showed the following deep theorem for general 2-polymatroids.

Theorem 1.1 ([7], see also Theorem 11.2.7 in [9]). *Let (S, f) be a 2-polymatroid and let $\nu = \nu(S, f)$. Then at least one of the following holds.*

(i) $f(S) = 2\nu + 1$.

(ii) *S has a partition $\{S_1, S_2\}$ into nonempty subsets such that $\nu = \nu(S_1, f) + \nu(S_2, f)$.*

(iii) *S has an element e which is contained in the span of every maximum feasible set.*

(iv) *(S, f) contains a nontrivial ν-double-flower.*

Theorem 1.1 suggests an algorithmic approach to the matroid matching problem as follows. If we encounter **(i)** or **(ii)** when applying Theorem 1.1, then we can identify a solution or reduce the problem to smaller ones. Similarly, if we encounter **(iii)**, we can consider the contraction of $\{e\}$, which reduces the problem size. So, the only difficult situation is the case **(iv)**. However, as shown in [7], if the rank of the kernel is not equal to zero for every nontrivial double circuit of any contraction of (S, f), then the problem can be reduced to a smaller one by contracting an element in the kernel of the double circuit in a ν-double-flower.

As an application of his theory, Lovász showed how to derive Mader's min-max formula from Theorem 1.1, where he reduced the problem to a matroid matching problem on a 2-polymatroid defined by combinatorial conditions. Within his reduction framework, the class of 2-polymatroid is not closed under contractions, and hence he gave a slightly involved argument to derive Mader's min-max formula. However, it turns out that the 2-polymatroid admits a linear representation as (implicitly) pointed out by Schrijver [13].

In this paper, we shall show that the problem of packing non-zero A-paths can be reduced to a matroid matching problem on a combinatorial 2-polymatroid. To demonstrate the meaning of the reduction, we show how to derive the min-max formula described in [1] from Theorem 1.1 by using Lovász's idea. A companion paper [15] by the second author shows the applicability of Schrijver's idea, which leads to more efficient solvability for a special case. It turns out that Schrijver's reduction technique can be adapted only when the underlying group Γ has a two-dimensional linear representation, but it can be adapted even to a slightly generalized problem introduced by Pap [12]. This suggests the importance of Lovász's combinatorial argument.

Our reduction is based on a recent work on polymatroid constructions on group-labeled graphs by the first author. A *frame matroid (or bias matroid)* is a well-known matroid on the edge set of a group-labeled graph, which plays an important role in the matroid representation theory (see, e.g. [11].) Motivated by recent works in combinatorial rigidity theory, the first author [14] proposed a polymatroid construction on the edge set of group-labeled graphs, where the rank formula of frame matroids is generalized based on submodular functions on the power set of the underlying group. The 2-polymatroid proposed in this paper is a special case of this construction, and our construction and reduction exhibit a clear connection of the problem of packing non-zero A-paths to well-investigated frame matroids.

2 Preliminaries

In this section we shall review preliminary facts on group-labeled graphs in Section 2.1 and then introduce a polymatroid construction given in [14] in Section 2.2.

2.1 Group-labeled graphs

Let (G, ψ) be a Γ-labeled graph. A *walk* is a sequence $W = (v_0, e_1, v_1, e_2, v_2, \dots, v_{k-1}, e_k, v_k)$ of vertices and edges such that v_{i-1} and v_i are endvertices of e_i for $1 \leq i \leq k$. A walk is called *closed* if the start vertex and the end vertex coincide. The *label* of a walk W is defined as $\psi(W) = \psi(e_k) \cdots \psi(e_2) \cdot \psi(e_1)$ if each edge is oriented in the forward direction through W, and for a backward edge e_i we replace $\psi(e_i)$ with $\psi(e_i)^{-1}$ in the formulation.

Let (G, ψ) be a Γ-labeled graph for a group Γ. For $v \in V(G)$, we denote by $\pi_1(G, v)$ the set of closed walks starting at v. Similarly, for $X \subseteq E(G)$ and $v \in V(G)$, $\pi_1(X, v)$ denotes the set of closed walks starting at v and using only edges of X, where $\pi_1(X, v) = \emptyset$

if $v \notin V(X)$. For $X \subseteq E(G)$, the subgroup induced by X relative to v is defined as $\langle X \rangle_{\psi,v} = \{\psi(W) \mid W \in \pi_1(X,v)\}$.

Proposition 2.1. *For any connected $X \subseteq E(G)$ and two vertices $u, v \in V(X)$, $\langle X \rangle_{\psi,u}$ is conjugate to $\langle X \rangle_{\psi,v}$.*

For $v \in V(G)$ and $\gamma \in \Gamma$, a *switching* at v with γ changes the label function ψ on $E(G)$ to $\psi' : E(G) \to \Gamma$ defined as

$$\psi'(e) = \begin{cases} \psi(e) \cdot \gamma^{-1} & \text{if } e \text{ is directed from } v \\ \gamma \cdot \psi(e) & \text{if } e \text{ is directed to } v \\ \psi(e) & \text{otherwise.} \end{cases}$$

By definition, $\psi'(e) = \gamma \cdot \psi(e) \cdot \gamma^{-1}$ if e is a loop attached at v. We say that a label function ψ' on $E(G)$ is *equivalent* to another label function ψ on $E(G)$ if ψ' is obtained from ψ by a sequence of switchings.

The following properties of label functions are well-known and are easily shown.

Proposition 2.2. *If ψ' is equivalent to ψ, then, for any $X \subseteq E(G)$ and any $v \in V(G)$, $\langle X \rangle_{\psi',v}$ is conjugate to $\langle X \rangle_{\psi,v}$.*

Proposition 2.3. *Let (G, ψ) be a group-labeled graph. Then, for any forest $F \subseteq E(G)$, there is a label function ψ' equivalent to ψ such that $\psi'(e)$ is identity for every $e \in F$.*

Proposition 2.3 suggests a simple way to compute $\langle F \rangle_{\psi,v}$ up to congruence, in analogy with the fact that a cycle space of a graph is spanned by fundamental cycles.

Proposition 2.4. *For a connected $X \subseteq E(G)$ and a spanning tree T of graph $(V(X), X)$, suppose that $\psi(e)$ is identity for all $e \in T$. Then, $\langle X \rangle_{\psi,v} = \langle \psi(e) : e \in X \setminus T \rangle$, the subgroup generated by $\psi(e)$ for $e \in X \setminus T$.*

A connected edge subset F in a group-labeled graph (G, ψ) is called *balanced* if $\langle F \rangle_{\psi,v}$ is the trivial group for some $v \in V(F)$. F is called *unbalanced* if it is not balanced. By Proposition 2.1, this property is invariant under the choice of the base vertex $v \in V(F)$, and F is unbalanced if and only if F contains an unbalanced cycle. Thus, we can extend this notion to any $F \subseteq E(G)$ (possibly disconnected sets) such that F is *unbalanced* if and only if F contains an unbalanced cycle.

For the analysis of disjoint non-zero A-paths, we shall extend these notions as follows. Let $A \subseteq V(G)$. We say that ψ' is *A-equivalent to ψ* if ψ' is obtained from ψ by a sequence of switchings at vertices in $V(G) \setminus A$. An edge set F is called *A-balanced* if F is balanced and contains no non-zero A-path. This property is invariant up to the A-equivalence of label functions. As in Propositions 2.3 and 2.4, we have the following.

Proposition 2.5. *Let (G, ψ) be a group-labeled graph and $A \subseteq V(G)$. If an edge set F is A-balanced, then there is a label function ψ' A-equivalent to ψ such that $\psi'(e)$ is identity for every $e \in F$.*

2.2 Frame matroids and their extensions

Let (G, ψ) be a Γ-labeled graph for a group Γ. The *frame matroid (or bias matroid)* of (G, ψ) is defined such that $F \subseteq E(G)$ is independent if and only if each connected component of F contains no cycle or just one cycle, which is unbalanced [16, 17]. The rank function $g_\Gamma : 2^{E(G)} \to \mathbb{Z}$ is written by

$$g_\Gamma(F) = \sum_{X \in C(F)} \{|V(X)| - 1 + \alpha_\Gamma(X)\} \qquad (F \subseteq E(G)),$$

where $C(F)$ denotes the partition of F into the edge sets of the connected components induced by F and

$$\alpha_\Gamma(X) = \begin{cases} 1 & \text{if } X \text{ is unbalanced} \\ 0 & \text{otherwise.} \end{cases}$$

In [14], Tanigawa extended the construction of the union of frame matroids by using structures of the underlying group. The idea is to replace the term α_Γ by a function taking fractional values.

For a group Γ, we consider a function $\rho : 2^\Gamma \to \mathbb{R}_+$ satisfying the following properties:

(Normalized) $\rho(\emptyset) = 0$;

(Monotonicity) $\rho(X) \leq \rho(Y)$ for any $X \subseteq Y \subseteq \Gamma$;

(Submodularity) $\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$ for any $X, Y \subseteq \Gamma$;

(Invariance under closure) $\rho(X) = \rho(\langle X \rangle)$ for any nonempty $X \subseteq \Gamma$;

(Invariance under conjugate) $\rho(X) = \rho(\gamma X \gamma^{-1})$ for any nonempty $X \subseteq \Gamma$ and $\gamma \in \Gamma$.

We say that $\rho : 2^\Gamma \to \mathbb{R}_+$ is a *symmetric polymatroidal function* over Γ if ρ satisfies these five conditions.

Let (G, ψ) be a Γ-labeled graph. We consider $\rho(\langle F \rangle_{\psi,v})$ for a connected $F \subseteq E(G)$ and $v \in V(F)$. By Proposition 2.1, $\langle F \rangle_{\psi,v}$ is conjugate to $\langle F \rangle_{\psi,u}$ for any $u, v \in V(F)$ for $F \subseteq E(G)$, and hence $\rho(\langle F \rangle_{\psi,u}) = \rho(\langle F \rangle_{\psi,v})$ for any $u, v \in V(F)$. Also, by Proposition 2.2, $\rho(\langle F \rangle_{\psi,v})$ is invariant with respect to the choice of equivalent label functions ψ. We hence simply denote $\rho(\langle F \rangle_{\psi,v})$ by $\rho\langle F \rangle$, implicitly assuming a label function ψ and the base vertex $v \in V(F)$. We can then define a set function $g_\rho : 2^{E(G)} \to \mathbb{R}$ by

$$g_\rho(F) = \sum_{X \in C(F)} \{|V(X)| - 1 + \rho\langle X \rangle\} \qquad (F \subseteq E(G)). \tag{1}$$

Theorem 2.6 ([14]). *Let $\rho : 2^\Gamma \to [0, 1]$ be a symmetric polymatroidal function over a group Γ, and let (G, ψ) be a Γ-labeled graph. Then, g_ρ is a monotone submodular function over $E(G)$.*

For the reader's convenience we put a copy of the proof in the appendix.

Suppose that a symmetric polymatroidal function ρ takes fractional values, that is, $\rho : 2^\Gamma \to \{0, \frac{1}{d}, \ldots, \frac{d-1}{d}, 1\}$ for some finite positive integer d. Then, if we define $f_\rho : 2^{E(G)} \to \mathbb{Z}$ by

$$f_\rho(F) := d g_\rho(F) \qquad (F \subseteq E(G)), \tag{2}$$

f_ρ is a normalized integer-valued monotone submodular function, and $(E(G), f_\rho)$ is a d-polymatroid.

Notice that the frame matroid is a special case, where ρ is defined by $\rho(X) = 0$ for $X = \emptyset$ or $X = \{1_\Gamma\}$, and otherwise $\rho(X) = 1$ for $X \subseteq \Gamma$.

Now we construct a symmetric polymatroidal function of rank two. For nontrivial groups Γ_1 and Γ_2, the free product $\Gamma_1 * \Gamma_2$ is the group consisting of all words $\gamma_1 \gamma_2 \ldots \gamma_m$

of arbitrary finite length $m \geq 0$, where each letter γ_i is non-identity element of Γ_1 or Γ_2 and adjacent letters γ_i and γ_{i+1} belong to different groups. The identity element of $\Gamma_1 * \Gamma_2$ is defined to be the empty word. See, e.g., [5] for more details on the free product.

Lemma 2.7. *Let Γ_1 and Γ_2 be distinct nontrivial groups, and let Γ be the free product of Γ_1 and Γ_2. If we define $\rho : 2^\Gamma \to \mathbb{Z}$ by*

$$\rho(X) := \begin{cases} 0 & \textit{if } X \textit{ is trivial (i.e., } X = \emptyset \textit{ or } X = \{1_\Gamma\}) \\ 1 & \textit{if } X \textit{ is nontrivial and } X \subseteq \gamma^{-1}\Gamma_i\gamma \textit{ for some } i \in \{1,2\} \textit{ and some } \gamma \in \Gamma \\ 2 & \textit{otherwise.} \end{cases}$$

Then ρ is symmetric polymatroidal.

Proof. Clearly ρ is normalized and monotone. Also, it is closed under conjugate and under closure since $\gamma^{-1}\Gamma_i\gamma$ is a subgroup.

Let $\mathcal{G} = \{\gamma^{-1}\Gamma_i\gamma \colon i \in \{1,2\}, \gamma \in \Gamma\}$. Then we have

$$X \cap Y = \{1_\Gamma\} \qquad \text{for any distinct } X, Y \in \mathcal{G} \tag{3}$$

since Γ is the free product of Γ_1 and Γ_2.

We show $\rho(X) + \rho(Y) \geq \rho(X \cap Y) + \rho(X \cup Y)$ for any X and Y in 2^Γ. If $\rho(Y) = 0$ (or symmetrically $\rho(X) = 0$), then we have $\rho(X \cap Y) = \rho(Y)$ and $\rho(X) = \rho(X \cup Y)$. Thus, the modularity holds.

If $\rho(Y) = 2$ (or symmetrically $\rho(X) = 2$), then we have $\rho(Y) = \rho(X \cup Y)$ and $\rho(X) \geq \rho(X \cap Y)$ since ρ is monotone and the range of the value of ρ is $\{0,1,2\}$. Thus the submodularity holds.

Finally, assume $\rho(X) = 1$ and $\rho(Y) = 1$. Note that there is a unique $\Gamma_X \in \mathcal{G}$ with $X \subseteq \Gamma_X$ and a unique $\Gamma_Y \in \mathcal{G}$ with $Y \subseteq \Gamma_Y$. By (3), if $\Gamma_X \neq \Gamma_Y$, then $\rho(X \cup Y) = 2$ and $\rho(X \cap Y) = 0$; otherwise $\rho(X \cup Y) = 1$ and $\rho(X \cap Y) \leq 1$. Thus we have proved the submodularity for this case. ☐

3 Packing Non-zero A-paths

Let (G, ψ) be a Γ-labeled graph and $A \subseteq V(G)$. Recall that an A-path P is called non-zero if $\psi(P) \neq 1_\Gamma$. In Section 3.1 we shall show that finding a maximum packing of size $\mu(G, \psi, A)$ is reduced to finding a maximum feasible set in a 2-polymatroid. Then, in Section 3.2, we shall show how the min-max formula of Chudnovsky, et al. [1] can be derived from Theorem 1.1.

3.1 Construction of the corresponding 2-polymatroid

Let (G, ψ) be a Γ-labeled graph and $A \subseteq V(G)$. Let Γ' be a group consisting of $\{1_{\Gamma'}, \bullet\}$ and isomorphic to $\mathbb{Z}/2\mathbb{Z}$ (i.e., $\bullet^2$ is identity), and let $\Gamma^\bullet$ be the free product of Γ' and the underlying group Γ of (G, ψ). We then define a function $\rho : 2^{\Gamma^\bullet} \to \mathbb{Z}$ by

$$\rho(X) := \begin{cases} 0 & \text{if } X \text{ is trivial} \\ 1 & \text{if } X \text{ is nontrivial and } X \subseteq \gamma^{-1}\Gamma\gamma \text{ or } X = \{1_{\Gamma^\bullet}, \gamma^{-1} \bullet \gamma\} \text{ for some } \gamma \in \Gamma^\bullet \\ 2 & \text{otherwise} \end{cases},$$

for each $X \subseteq \Gamma$. By Lemma 2.7, ρ is a symmetric polymatroidal function over $\Gamma^\bullet$.

Let (H, ψ) be a $\Gamma^\bullet$-labeled graph obtained from (G, ψ) by attaching a new loop $\ell_{v,\gamma}$ at each vertex $v \in V(G)$ with label $\psi(\ell_{v,\gamma}) = \gamma^{-1} \bullet \gamma$ for each $\gamma \in \Gamma$. (Hence we added

$|V(G)||\Gamma|$ loops in total, if Γ is finite.) Let L be the set of the new loops and, for each $U \subseteq V(G)$, let $L_U^\psi := \{\ell \in L \mid \ell \text{ is incident to } U \text{ and } \psi(\ell) = \bullet\}$. By Theorem 2.6, the function $f : 2^{E(H)} \to \mathbb{Z}$ defined by

$$f(F) = \sum_{F_i \in C(F)} (2|V(F_i)| - 2 + \rho\langle F_i \rangle) \qquad (F \subseteq E(H)) \tag{4}$$

is a normalized monotone submodular function, and thus $(E(H), f)$ is a 2-polymatroid.

We consider the contraction of $(E(H), f)$ by L_A^ψ and then restrict it to $E(G)$. The resulting 2-polymatroid on $E(G)$ is denoted by $(E(G), f_A^\psi)$. (Note that L_A^ψ depends on ψ, and hence so does f_A^ψ.) Then $(E(G), f_A^\psi)$ is a 2-polymatroid on $E(G)$ whose rank and feasible sets are characterized as follows.

Lemma 3.1. *For $F \subseteq E(G)$,*

$$f_A^\psi(F) = \sum_{F_i \in C(F)} \{2|V(F_i)| - 2 + \rho_A^\psi(F_i) - |V(F_i) \cap A|\} \tag{5}$$

where ρ_A^ψ is written by

$$\rho_A^\psi(F) = \begin{cases} 2 & \text{if } F \text{ has a non-zero } A\text{-path, or } F \text{ is unbalanced with } |V(F) \cap A| \geq 1 \\ 1 & \begin{array}{l}\text{if } F \text{ is } A\text{-balanced with } |V(F) \cap A| \geq 1, \text{ or} \\ \quad F \text{ is unbalanced with } |V(F) \cap A| = 0\end{array} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For $F \subseteq E(G)$, let $L_A^\psi(F)$ be the set of loops incident to F with label $\bullet$ in (H, ψ). Let us first check

$$\rho_A^\psi(F) = \rho\langle F \cup L_A^\psi(F) \rangle \tag{6}$$

for any connected $F \subseteq E(G)$.

If F is balanced with $V(F) \cap A = \emptyset$, then $\langle F \cup L_A^\psi(F) \rangle$ is trivial, which implies $\rho\langle F \cup L_A^\psi(F) \rangle = 0 = \rho_A^\psi(F)$.

If F is unbalanced with $V(F) \cap A = \emptyset$, then $\langle F \rangle$ is a nontrivial subgroup of Γ, so $\rho\langle F \cup L_A^\psi(F) \rangle = 1$, which is equal to $\rho_A^\psi(F)$.

Suppose next that F is A-balanced with $|V(F) \cap A| \geq 1$. Take a terminal $v \in V(F) \cap A$. Then the label of every path in F from v to any other terminal in $V(F) \cap A$ is identity as F is A-balanced, and hence $\langle F \cup L_A^\psi(F) \rangle_{v,\psi} = \{1_{\Gamma^\bullet}, \bullet\}$. This means $\rho\langle F \cup L_A^\psi(F) \rangle = 1 = \rho_A^\psi(F)$.

Finally, suppose that F has a non-zero A-path or is unbalanced with $|V(F) \cap A| \geq 1$. If F has a non-zero A-path from one terminal u to another terminal v, then $\langle F \cup L_A^\psi(F) \rangle_{v,\psi}$ contains $\bullet$ and $\gamma^{-1} \bullet \gamma$, where γ is the label of the non-zero A-path. Since γ is not the identity, $\rho\langle F \cup L_A^\psi(F) \rangle_{v,\psi} = 2 = \rho_A^\psi(F)$.

Similarly, if F is unbalanced, then $\langle F \cup L_A^\psi(F) \rangle_{v,\psi}$ contains $\bullet$ and γ, where $v \in V(G) \cap A$ and γ is the non-identity label of a closed walk in F. Thus, we have $\rho\langle F \cup L_A^\psi(F) \rangle_{v,\psi} = 2 = \rho_A^\psi(F)$, and we have confirmed (6).

Note that, for any $U \subseteq V(G)$, we have $f(L_U^\psi) = |L_U^\psi|$. Therefore, since f_A^ψ is defined as the contraction of f by L_A^ψ, we have, for any $F \subseteq E(G)$,

$$\begin{aligned} f_A^\psi(F) &= f(F \cup L_A^\psi) - f(L_A^\psi) \\ &= f(F \cup L_A^\psi(F)) - f(L_A^\psi(F)) \\ &= \sum_{F_i \in C(F)} \{2|V(F_i)| - 2 + \rho\langle F_i \cup L_A^\psi(F_i)\rangle - f(L_A^\psi(F_i))\} \\ &= \sum_{F_i \in C(F)} \{2|V(F_i)| - 2 + \rho_A^\psi(F_i) - |V(F_i) \cap A|\}. \end{aligned}$$

This completes the proof. □

Lemma 3.1 implies that f_A^ψ is invariant up to the A-equivalence of ψ.

Lemma 3.2. *An edge set $F \subseteq E(G)$ is feasible in $(E(G), f_A^\psi)$ if and only if*

- *F contains no cycle, and*
- *for each $F' \in C(F)$, either $|V(F') \cap A| \leq 1$, or $|V(F') \cap A| = 2$ and the A-path between the two terminals is non-zero.*

Proof. By Lemma 3.1 it is sufficient to check the statement for connected F.

Suppose that F contains a cycle. Since one can easily observe $\rho_A^\psi(F) - |V(F) \cap A| \leq 1$, $f_A^\psi(F) \leq 2|V(F)| - 1 < 2|F|$ by Lemma 3.1. Hence F is not feasible.

Suppose that F contains no cycle. If $|V(F) \cap A| \leq 1$, then $\rho_A^\psi(F) = |V(F) \cap A|$ and hence $f_A^\psi(F) = 2|V(F)| - 2 = 2|F|$, implying that F is feasible. If $|V(F) \cap A| = 2$, then $\rho_A^\psi(F) = |V(F) \cap A|$ if and only if the A-path between the two terminals is non-zero, and hence F is a feasible set if and only if the A-path is non-zero. If $|V(F) \cap A| \geq 3$, then $\rho_A^\psi(F) < |V(F) \cap A|$ and hence $f_A^\psi(F) \leq 2|V(F)| - 1$. Thus F is not feasible. □

Recall that $\nu(E(G), f_A^\psi)$ denotes the size of a maximum feasible set in $(E(G), f_A^\psi)$.

Theorem 3.3. *If G is connected and $A \neq \emptyset$, then $\nu(E(G), f_A^\psi) = |V(G)| - |A| + \mu(G, \psi, A)$.*

Proof. Let us simply denote $\nu = \nu(E(G), f_A^\psi)$ and $\mu = \mu(G, \psi, A)$. Let B be a maximal feasible set in $(E(G), f_A^\psi)$. We denote by c_i the number of connected components of the graph $(V(G), B)$ containing exactly i terminals. By Lemma 3.2 and the maximality of B, we have $1 \leq |V(F) \cap A| \leq 2$ for each $F \in C(B)$, $V(B) \cup A = V(G)$, and $c_2 \leq \mu$. Therefore, $\nu = |B| = |V(G)| - (c_1 + c_2) = |V(G)| - |A| + c_2 \leq |V(G)| - |A| + \mu$.

The reverse direction can be easily seen as follows. Let F be the edge set of a maximum family of vertex-disjoint non-zero A-paths. By Lemma 3.2, F is feasible and can be extended to a feasible set F' so that $V(F') \cup A = V(G)$ and $1 \leq |V(F'') \cap A| \leq 2$ for each $F'' \in C(F')$. Then, $\nu \geq |F'| = |V(G)| - |A| + \mu$. □

3.2 Application of Lovász's theorem

First, we describe the min-max formula of Chudnovsky, et al. [1]. For $A, X \subseteq V(G)$ and $\psi : E(G) \to \Gamma$, we define

$$t(G, \psi, A; X) := |X| + \sum_{H \in \mathrm{comp}(G')} \left\lfloor \frac{|A \cap V(H)|}{2} \right\rfloor,$$

where $G' := G - X - \{uv \in E \mid u, v \in A,\ \psi(uv) = 1_\Gamma\}$ and $\mathrm{comp}(G')$ denotes the set of the connected components of G'.

Theorem 3.4 (Chudnovsky, et al. [1]). *Let (G, ψ) be a Γ-labeled graph with terminal set A. Then*

$$\mu(G, \psi, A) = \min_{\psi', A', X} t(G, \psi', A'; X) \tag{7}$$

where the minimum is taken over all $X \subseteq V(G)$, all A' with $A \setminus X \subseteq A' \subseteq V(G) \setminus X$, and all label functions ψ' A-equivalent to ψ.

The direction of $\leq$ is easy to see as follows. For any $X \subseteq V(G)$, any A' with $A \setminus X \subseteq A' \subseteq V(G) \setminus X$, and any label function ψ' A-equivalent to ψ, we have

$$\mu(G, \psi, A) = \mu(G, \psi', A) \leq \mu(G, \psi', A' \cup X) \leq t(G, \psi', A'; X).$$

The first equality holds since a sequence of switchings at non-terminals does not change the label of each A-path. The next inequality holds since each non-zero A-path contains at least one non-zero $(A' \cup X)$-path as its subpath. The final inequality holds, since at most $|X|$ paths in a family of vertex-disjoint $(A' \cup X)$-paths can intersect X and each connected component of $G - X$ contains a family of vertex-disjoint A'-paths of size at most a half of the number of terminals in it.

We now show the reverse direction by using Lovász's theorem (Theorem 1.1) for the corresponding 2-polymatroid $(E(G), f_A^{\psi})$. The key observation for applying Theorem 1.1 is to enumerate all nontrivial double circuits of $(E(G), f_A^{\psi})$. The following Lemma 3.5 is an extension of [7, Lemma 3.4][1], and it shows that $(E(G), f_A^{\psi})$ is a relatively simple 2-polymatroid.

Lemma 3.5. *For a Γ-labeled graph (G, ψ) with terminal set $A \subseteq V(G)$, every nontrivial double circuit $D \subseteq E(G)$ of $(E(G), f_A^{\psi})$ is a tree all of whose leaves are terminals, and is one of the following forms (see Figure 1):*

D1 *$V(D) \cap A = \{a_1, a_2, a_3\}$, and all three A-paths contained in D have the identity label and intersect at a non-terminal $v \in V(D) \setminus A$.*

D2 *$V(D) \cap A = \{a_1, a_2, a_3, a_4\}$, and D contains six A-paths. At most one of the six A-paths has the identity label, and all A-paths intersect at a non-terminal $v \in V(D) \setminus A$.*

D3 *$V(D) \cap A = \{a_1, a_2, a_3, a_4\}$, A-paths between a_1 and a_2 and between a_3 and a_4 are fully disjoint, and there exists a nontrivial path between $u, v \in V(D) \setminus A$ connecting these two A-paths. The A-path between a_1 and a_2 has the identity label and intersects v, and the other A-paths are non-zero.*

D4 *$V(D) \cap A = \{a_1, a_2, a_3, a_4\}$, and D contains only four A-paths. Three of the four connecting a_1, a_2, a_3 intersect at a non-terminal $v \in V(D) \setminus A$, and the rest connects a_3 and a_4. The A-path between a_1 and a_2 has the identity label, and the other A-paths are non-zero.*

D5 *$V(D) \cap A = \{a_1, a_2, a_3, a_4\}$ and D contains only three non-zero A-paths, which intersect at a_1.*

[1] We should remark that **D4** is a missing case in the list of double circuits given in [7, Lemma 3.4] even in the case of Mader's $\mathcal{S}$-path problem.

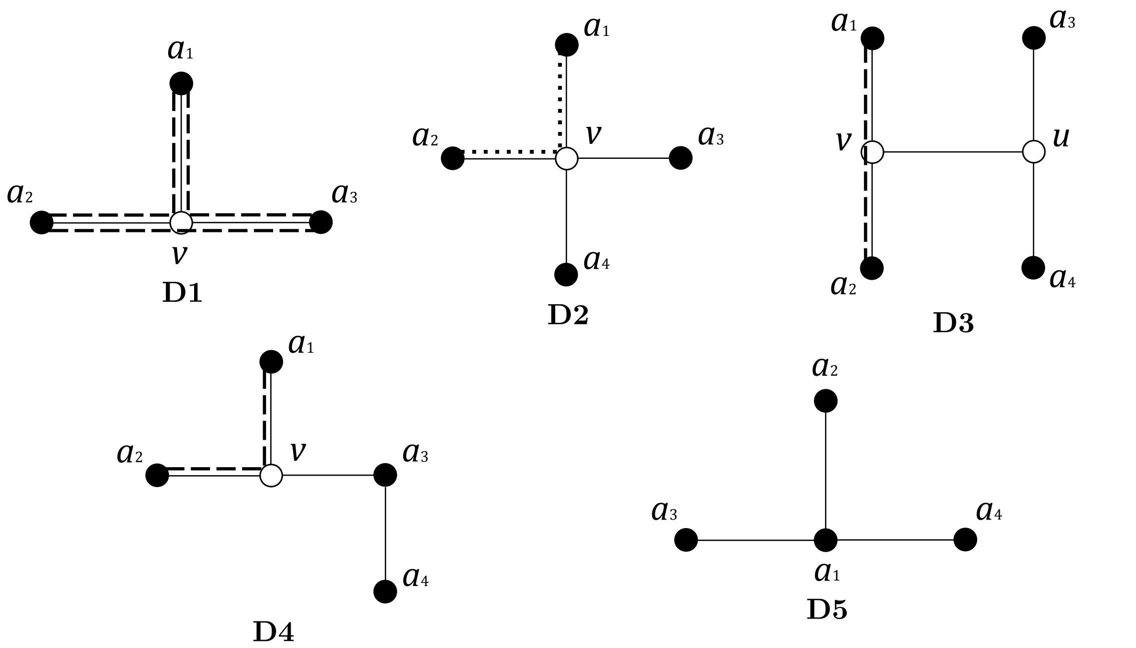

Figure 1: Nontrivial double circuits, where the dashed A-paths have the identity label, the dotted A-path has an arbitrary label, and the others are non-zero A-paths.

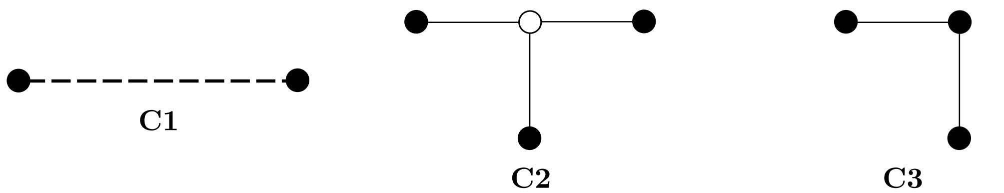

Figure 2: Circuits with no cycle, where the dashed A-path has the identity label and the others are non-zero A-paths.

Proof. We first enumerate all the patterns of circuits in $(E(G), f_A^\psi)$ which contains no cycle. Let $C \subseteq E(G)$ be such a circuit. From the definition of f_A^ψ, one can easily check that C is connected. Hence we have $|V(C)| = |C| + 1$, and $\rho_A^\psi(C) = |V(C) \cap A| - 1$ by Lemma 3.1. This means $(\rho_A^\psi(C), |V(C) \cap A|) = (0, 1)$, $(1, 2)$ or $(2, 3)$. Since $|V(C) \cap A| \geq 1$ implies $\rho_A^\psi(C) \geq 1$, it suffices to check the latter two cases. Note that, in both cases, each leaf of C is a terminal, since otherwise the deletion of the edge incident to a non-terminal leaf decreases $|V(C)|$ by 1 and hence decreases the value of f_A^ψ by 2, a contradiction.

If $(\rho_A^\psi(C), |V(C) \cap A|) = (1, 2)$, then C forms a zero A-path, i.e., C is of type **C1** in Figure 2. Suppose $(\rho_A^\psi(C), |V(C) \cap A|) = (2, 3)$. Then C forms either a path or three paths joined at a non-terminal (see Figure 2), since F has at most three leaves. In both cases, if there exists a zero A-path P in C, then C remains infeasible after the deletion of an edge $e \in C \setminus E(P)$, a contradiction. Therefore, all A-paths in C is non-zero, i.e., C is of type **C2** or **C3** in Figure 2. It follows from Lemma 3.1 that C is indeed a circuit if it is of type **C2** or **C3**.

Let $D \subseteq E(G)$ be a nontrivial double circuit of $(E(G), f_A^\psi)$. We shall prove that D is connected and contains no cycle. If D is not connected, then D forms two connected components each of which is a circuit of $(E(G), f_A^\psi)$, and hence D is trivial.

Thus $2|D| - 2 = f_A^\psi(D) = 2|V(D)| - 2 + \rho_A^\psi(D) - |V(D) \cap A|$ by Lemma 3.1. Therefore, by the definition of ρ_A^ψ, $2(|D| - |V(D)|) = \rho_A^\psi(D) - |V(D) \cap A| \leq 1$, and hence we have

$|D| \leq |V(D)|$. This implies that D has at most one cycle. Suppose that D contains exactly one cycle C. Then we have $\rho_A^\psi(D) = |V(D) \cap A|$, which implies $|V(D) \cap A| \leq 2$. If D contains at least two A-paths, then at least one of them is non-zero since D is unbalanced. Therefore, if D contains no A-path or at least two A-paths, then $D - e$ is feasible for some $e \in C$. If D contains exactly one A-path, then D is a form like the letter 'Q' consisting of an unbalanced cycle and a zero A-path joined at exactly one vertex. In this case, D is trivial since the unbalanced cycle and the zero A-path are circuits.

Thus D is a tree, and hence we have $|V(D)| = |D| + 1$ and $\rho_A^\psi(D) - |V(D) \cap A| = -2$. Since $|V(D) \cap A| \geq 1$ implies $\rho_A^\psi(D) \geq 1$, it suffices to consider two cases: $|V(D) \cap A| = 3$ or $|V(D) \cap A| = 4$. Note that, similarly to the case of circuits, each leaf of D is terminal.

Case 1. Suppose $|V(D) \cap A| = 3$. Then we have $\rho_A^\psi(D) = 1$, and hence D contains no non-zero A-path. If D contains only two A-paths, then D is trivial. Therefore, D contains three A-paths, and hence it is of type **D1**.

Case 2. Suppose $|V(D) \cap A| = 4$. Then we have $\rho_A^\psi(D) = 2$, and hence D contains at least one non-zero A-path. Every tree with at most four leaves is one of the following forms: a path, three paths joined at one vertex (Figure 1, **D4** and **D5**), four paths joined at one vertex (Figure 1, **D2**), and two fully disjoint paths connected by a nontrivial path between thier internal vertices (Figure 1, **D3**).

If some A-paths in D have labels violating the rule given in Figure 1, then one can easily check from Lemma 3.2 and the list of circuits given in Figure 2 that (i) D is trivial, (ii) there is an edge e such that $D - e$ is feasible, or (iii) $f_A^\psi(D) \neq 2|D| - 2$. $\square$

As we reviewed in Section 1, Theorem 1.1 leads to a min-max formula for the matroid matching problem if the kernel of every double circuit has the rank at least one. By contracting an element in the kernel, one can reduce the problem to a smaller one. In our situation, we observe that the kernel of a double circuit of type **D1**, **D2**, **D3**, or **D4** of Lemma 3.5 contains a loop of L in $(E(H), f)$, and hence, contracting it, we can reduce the problem size appropriately. For the completeness, we shall formalize this fact in terms of packing non-zero A-paths as follows. In the following Lemmas 3.6 and 3.7, let $\nu := \nu(E(G), f_A^\psi)$.

Lemma 3.6. *Let D be a ν-double-flower containing a double circuit of type* **D1***,* **D2***,* **D3***, or* **D4***. Then there exists a label function ψ' A-equivalent to ψ such that $\nu > \nu(E(G), f_{A+v}^{\psi'})$, where v is the vertex specified in the definition of each type.*

Proof. Let D' be the double circuit in D. By Proposition 2.5, there exists a label function ψ' A-equivalent to ψ such that all edges along each zero A-path have the identity label. Then, observe that every circuit in D spans the loop l in L incident to v with $\psi'(l) = \bullet$, i.e., the kernel of D' contains l, and hence $f_{A+v}^{\psi'}(C) = f_A^\psi(C) - 1$ for every circuit C in D.

Now suppose contrary that $(E(G), f_{A+v}^{\psi'})$ has a feasible set F of size ν. Let us choose such F with $|F \cap D|$ maximum. We then have $f_{A+v}^{\psi'}(D) = f_A^\psi(D) - 1 = 2\nu + 1 > 2\nu = f_{A+v}^{\psi'}(F)$. Thus there exists $e \in D \setminus \mathrm{sp}_{f_{A+v}^{\psi'}}(F)$. Since $F + e$ cannot be feasible in $(E(G), f_A^\psi)$, we have $f_A^\psi(F + e) \leq 2\nu + 1$. Furthermore, since $f_{A+v}^{\psi'}(F + e) \leq f_A^\psi(F + e)$ and $e \notin \mathrm{sp}_{f_{A+v}^{\psi'}}(F)$, we obtain $f_A^\psi(F + e) = f_{A+v}^{\psi'}(F + e) = 2\nu + 1$. This implies that $F + e$ is a ν-flower in both $(E(G), f_A^\psi)$ and $(E(G), f_{A+v}^{\psi'})$.

Let C' be the circuit of $F + e$ in $(E(G), f_{A+v}^{\psi'})$. Then C' is also the circuit of $F + e$ in $(E(G), f_A^\psi)$ since otherwise $F + e$ becomes feasible. If $C' \not\subseteq D$, then, for any $e' \in C' \setminus D$, $F' := F + e - e'$ is a feasible set of size ν with $|F' \cap D| > |F \cap D|$. This contradicts the choice of F. If $C' \subseteq D$, then $f_{A+v}^{\psi'}(C') = f_A^{\psi}(C') - 1$, contradicing the fact that C' is a circuit in both $(E(G), f_A^{\psi})$ and $(E(G), f_{A+v}^{\psi'})$. $\square$

In the redution of Mader's $\mathcal{S}$-path problem to a matroid matching problem, Lovász introduced a notion of a *regular set* to solve the case when we encounter a double circuit of type **D5**. He claimed that the set of edges incident to a_1 forms a regular set in [7, Lemma 3.5(b)]. This claim turns out to be false, but at least we can apply the proof idea of [7, Lemma 1.6] to accomplish our purpose as follows.

Lemma 3.7. *Let D be a ν-double-flower containing a double circuit of type* **D5**. *Then $\nu > \nu(E(G - a_1), f_{A-a_1}^{\psi})$, where a_1 is the vertex specified in the definition of type* **D5**.

Proof. Suppose contrary that $(E(G - a_1), f_{A-a_1}^{\psi})$ has a feasible set F of size ν. Let us choose such F with $|F \cap D|$ maximum, and let $\overline{F} := \mathrm{sp}_{f_A^{\psi}}(F)$. Let D' be the double circuit in D, and let E_1 be the set of edges in D' incident to a_1. Observe that $E_1 \cap C \neq \emptyset$ for every circuit C in D since D' is of type **D5**.

Suppose $D \not\subseteq \overline{F} \cup E_1$. Take $e \in D \setminus (\overline{F} \cup E_1)$. Then $F + e$ is ν-flower and the circuit C in $F + e$ is not contained in D since every circuit in D intersects E_1. Therefore, for an edge $e' \in C \setminus D$, $F' := F + e - e'$ is a feasible set of size ν with $|F' \cap D| > |F \cap D|$, contradicting the choice of F.

Suppose $D \subseteq \overline{F} \cup E_1$. Recall $E_1 \subseteq D'$ and that every A-path in D' is non-zero. Therefore, if there is an edge $e \in E_1$ incident to a connected component $F_i \in \mathcal{C}(\overline{F})$ that does not contain a non-zero A-path, then $\rho_A^{\psi}(F_i + e) = \rho_A^{\psi}(F_i) + 1$ holds and hence $f_A^{\psi}(\overline{F} + e) = f_A^{\psi}(\overline{F}) + 2$ holds. This however implies $f_A^{\psi}(F + e) = f_A^{\psi}(F) + 2$, and hence $F + e$ is a larger feasible set than F, a contradiction. Otherwise (i.e., each connected component of $\overline{F}$ around a_1 contains a non-zero A-path), we have $f_A^{\psi}(\overline{F} \cup E_1) = f_A^{\psi}(\overline{F}) + 1 = f_A^{\psi}(F) + 1 = 2\nu + 1$. However, since $D \subseteq \overline{F} \cup E_1$, we also have $f_A^{\psi}(\overline{F} \cup E_1) \geq f_A^{\psi}(D) = 2\nu + 2$, a contradiction. $\square$

Proof of Theorem 3.4. We have already seen the direction of $\leq$, and here we prove the reverse direction. The proof is done by induction on $|V(G) \setminus A| + |E(G)|$. We may assume that G is connected, that A is nonempty, and that there is no A-path consisting of a single edge with the identity label. By Theorem 1.1, we split the proof into four cases.

Case 1. Suppose **(i)** of Theorem 1.1 holds. Then $f_A^{\psi}(E(G)) = 2\nu(E(G), f_A^{\psi}) + 1$. By Theorem 3.3,

$$\begin{aligned}
\mu(G, \psi, A) &= \nu(E(G), f_A^{\psi}) - |V(G)| + |A| \\
&= \frac{f_A^{\psi}(E(G)) - 1}{2} - |V(G)| + |A| \\
&= \frac{2|V(G)| - 3 + \rho_A^{\psi}(E(G)) - |A|}{2} - |V(G)| + |A| \\
&= \frac{\rho_A^{\psi}(E(G)) + |A| - 3}{2}\,.
\end{aligned} \tag{8}$$

If $\rho_A^{\psi}(E(G)) = 1$, then $|A| \geq 2$ holds and $E(G)$ is A-balanced, and hence there exists a label function ψ' A-equivalent to ψ such that $\psi'(e) = 1_\Gamma$ for every $e \in E(G)$. This implies that $t(G, \psi', V(G); \emptyset) = 0$, which is no more than $\mu(G, \psi, A)$ by (8). If $\rho_A^{\psi}(E(G)) = 2$, then $|A|$ is odd, and hence by (8), we have

$$\mu(G, \psi, A) = \frac{|A| - 1}{2} = \left\lfloor \frac{|A|}{2} \right\rfloor = t(G, \psi, A; \emptyset).$$

Case 2. Suppose **(ii)** of Theorem 1.1 holds. Then there exists a partition E_1, E_2 of $E(G)$ such that $\nu(E(G), f_A^\psi) = \nu(E_1, f_A^\psi) + \nu(E_2, f_A^\psi)$. By Theorem 3.3,

$$\begin{aligned}
\mu(G, \psi, A) &= \nu(E(G), f_A^\psi) - |V(G)| + |A| \\
&= \nu(E_1, f_A^\psi) + \nu(E_2, f_A^\psi) - |V(G) \setminus A| \\
&= \sum_{i=1}^{2} \big(\nu(E_i, f_A^\psi) - |V(E_i) \setminus A|\big) + |(V(E_1) \cap V(E_2)) \setminus A| \\
&= \sum_{i=1}^{2} \mu(G[E_i], \psi, A) + |(V(E_1) \cap V(E_2)) \setminus A|. \qquad (9)
\end{aligned}$$

By the induction hypothesis, for each $i \in \{1, 2\}$, there exist A_i, X_i, ψ_i such that $X_i \subseteq V(E_i)$, $(V(E_i) \cap A) \setminus X_i \subseteq A_i \subseteq V(E_i) \setminus X_i$, $\psi_i : E_i \to \Gamma$ is A-equivalent to the restriction of ψ to E_i, and $\mu(G[E_i], \psi, A) = t(G[E_i], \psi_i, A_i; X_i)$. Let $X := X_1 \cup X_2 \cup (V(E_1) \cap V(E_2) \setminus A)$ and $A' := A_1 \cup A_2 \setminus X$. For each $i \in \{1, 2\}$, let E_i' be the set of edges in E_i which are contained in $G - X$. Since $V(E_1') \cap V(E_2') \subseteq A$ by $(V(E_1) \cap V(E_2)) \setminus A \subseteq X$, there exists a label function $\psi' : E(G) \to \Gamma$ A-equivalent to ψ such that the restriction of ψ' to E_i' coincides with the restriction of ψ_i to E_i' for each $i \in \{1, 2\}$. Then we have $\mu(G, \psi, A) \geq t(G, \psi', A'; X)$ by (9).

Case 3. Suppose **(iii)** of Theorem 1.1 holds. Then there exists an edge $e = uv \in E(G)$ contained in the span of every maximum feasible set.

Suppose that there is a maximum feasible set F with $e \notin F$. We first show that

$$\begin{array}{l}\text{if } e \text{ connects distinct connected components, say } F_1 \text{ and } F_2, \text{ of } F, \\ \text{then } F_i \text{ contains a non-zero } A\text{-path for each } i = 1, 2.\end{array} \qquad (10)$$

To see this, observe first $\sum_{i=1,2}\{\rho_A^\psi(F_i) - |V(F_i) \cap A|\} = 2 + \rho_A^\psi(F_1 \cup F_2 + e) - |V(F_1 \cup F_2 + e) \cap A|$ by $f_A^\psi(F) = f_A^\psi(F + e)$ and Lemma 3.1. Moreover, since F is feasible, we have $\rho_A^\psi(F_i) = |V(F_i) \cap A|$, which means $2 + \rho_A^\psi(F_1 \cup F_2 + e) = |V(F_1 \cup F_2 + e) \cap A|$. Therefore, $(\rho_A^\psi(F_1 \cup F_2 + e), |V(F_1 \cup F_2 + e) \cap A|) = (0, 2), (1, 3)$ or $(2, 4)$. However, if $|V(F_1 \cup F_2 + e) \cap A| \geq 1$ then $\rho_A^\psi(F_1 \cup F_2 + e) \geq 1$, and if $|V(F_1 \cup F_2 + e) \cap A| \geq 3$ then $\rho_A^\psi(F_1 \cup F_2 + e) \geq 2$ since F_1 or F_2 contains a non-zero A-path in this case. It thus follows that $|V(F_1 \cup F_2 + e)| = 4$ holds and F_i contains a non-zero A-path for each $i = 1, 2$.

Suppose $u, v \in A$. We may assume that $\psi(e) \neq 1_\Gamma$ since otherwise we can delete e and use induction. Let $E_{\mathcal{P}}$ be the edge set of a maximum family of vertex-disjoint non-zero A-paths, and suppose $u \notin V(E_{\mathcal{P}})$. Then we must have $v \in V(E_{\mathcal{P}})$. If we extend $E_{\mathcal{P}}$ to a maximum feasible set F, then u and v belong to distinct connected components of F and moreover the component to which u belongs does not contain a non-zero A-path. This however contradicts (10). Thus every maximum family of vertex-disjoint non-zero A-paths uses terminal u, and hence we can delete u (by adding u to X) and use induction to complete the proof.

Suppose $u \notin A$. If the addition of u to A does not increase the value of μ, then we can use induction since $|V(G) \setminus A|$ decreases. Otherwise, there are $\mu + 1$ vertex-disjoint non-zero $(A + u)$-paths $P_0, \ldots, P_\mu$ such that u is an end of P_0. Let $a \in A$ be the other end of P_0. If G contains no A-path traversing the edge e, then we can delete e and use induction to complete the proof. Hence we assume that there exists an A-path Q traversing e in G.

Let $A_{\mathcal{P}} := A \cup (\bigcup_{i=1}^{\mu} V(P_i))$. We take the subpath Q' of Q such that Q' is an $A_{\mathcal{P}}$-path traversing the edge e. We walk along P_0 from a until we hit Q' first and then continue walking along Q' so that we traverse e until the end of Q'. The resulting path, denoted by P_0', is an $A_{\mathcal{P}}$-path which starts at a and traverses e. Then an edge set $(E(P_0') - e) \cup (\bigcup_{i=1}^{\mu} E(P_i))$ is feasible and can be extended to a maximum feasible set F with $e \notin F$. Furthermore e connects distinct connected components in $C(F)$, at least one of which does not contain a non-zero A-path. This again contradicts (10), and we complete the proof.

Case 4. Suppose **(iv)** of Theorem 1.1 holds. Then there exists a nontrivial ν-double-flower $D \subseteq E(G)$. By Lemma 3.5, D contains a double circuit of one of the five types. If D contains **D1**, **D2**, **D3**, or **D4**, then, by Theorem 3.3, Lemma 3.6 and the induction hypothesis, we have

$$\begin{aligned} \mu(G, \psi, A) &= \nu(E(G), f_A^{\psi}) - |V(G)| + |A| \\ &\geq \nu(E(G), f_{A+v}^{\psi'}) + 1 - |V(G)| + |A+v| - 1 \\ &= \mu(G, \psi', A+v) \\ &= t(G, \psi'', A'; X), \end{aligned} \tag{11}$$

for some $X \subseteq V(G)$, $(V(G) \setminus X) \cap (A+v) \subseteq A' \subseteq V(G) \setminus X$, a label function ψ' A-equivalent to ψ, and a label function ψ'' $(A+v)$-equivalent to ψ'. Notice that, for the same X, A' and ψ'', we have $X \subseteq V(G)$, $(V(G) \setminus X) \cap A \subseteq A' \subseteq V(G) \setminus X$, and that ψ'' is A-equivalent to ψ. Thus we complete the proof by (11).

If D contains **D5**, then, by Theorem 3.3 and Lemma 3.7, we have

$$\begin{aligned} \mu(G, \psi, A) &= \nu(E(G), f_A^{\psi}) - |V(G)| + |A| \\ &\geq \nu(E(G - a_1), f_{A-a_1}^{\psi}) + 1 - |V(G) - a_1| + |A - a_1| \\ &= \mu(G - a_1, \psi, A - a_1) + 1 \\ &= t(G - a_1, \psi', A'; X) + 1, \end{aligned}$$

for some $X \subseteq V(G) - a_1$, $(V(G) \setminus X) \cap (A - a_1) \subseteq A' \subseteq (V(G) - a_1) \setminus X$, and a label function ψ' A-equivalent to ψ. Let $X' := X + a_1$. Then we have $t(G - a_1, \psi', A'; X) + 1 = t(G, \psi', A'; X')$, where $X' \subseteq V(G)$, $(V(G) \setminus X') \cap A \subseteq A' \subseteq V(G) \setminus X'$, and ψ' is A-equivalent to ψ. This completes the proof. $\square$

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A Proof of Theorem 2.6

Proof. For each $X \subseteq E$ and $e = ij \in E \setminus X$, let $\Delta(X, e) = g_\rho(X + e) - g_\rho(X)$. We denote by X_i the connected component of X for which $i \in V(X_i)$. If such a component does not exist, let $X_i = \emptyset$. Similarly, we denote by X_j the component of X for which $j \in V(X_j)$.

By a simple calculation, we have the following relation:

$$\Delta(X, e) = \begin{cases} \rho\langle X_i + e\rangle - \rho\langle X_i\rangle & \text{if } e \text{ is a loop or } X_i = X_j \neq \emptyset \\ \rho\langle X_i \cup X_j + e\rangle + 1 - \rho\langle X_i\rangle - \rho\langle X_j\rangle & \text{otherwise.} \end{cases} \tag{12}$$

Let us check the monotonicity. Suppose that e is a loop or $X_i = X_j \neq \emptyset$. Due to the monotonicity of ρ over Γ, $\rho\langle X_i + e\rangle - \rho\langle X_i\rangle \geq 0$. On the other hand, suppose not. Since X_i and $X_i \cup X_j + e$ are connected, we have $\rho\langle X_i\rangle \leq \rho\langle X_i \cup X_j + e\rangle$ by the monotonicity of ρ over Γ. Also, by the upper bound of ρ, $\rho\langle X_j\rangle \leq 1$. We thus have $\Delta(X, e) = \rho\langle X_i \cup X_j + e\rangle + 1 - (\rho\langle X_i\rangle + \rho\langle X_j\rangle) \geq 0$. This completes the proof of the monotonicity.

For the submodularity, we check

$$\Delta(X, e) \geq \Delta(Y, e) \tag{13}$$

for any $X \subseteq Y \subseteq E$ and $e \in E \setminus Y$. We split the proof into two cases.

Case 1. Suppose that e is a loop or $X_i = X_j \neq \emptyset$. We then have $X_i \subseteq Y_i = Y_j$. We take a tree $T \subseteq Y_i$ spanning $V(Y_i)$ such that $T \cap X_i$ forms a tree spanning $V(X_i)$. By using switching operations, we may assume by Proposition 2.3 that $\psi(f) = \mathrm{id}$ for every $f \in T$.

Observe then that $\langle Y_i + e\rangle_{\psi,i} = \langle \langle Y_i\rangle_{\psi,i} \cup \{\psi(e)\}\rangle$ and $\langle X_i + e\rangle_{\psi,i} = \langle \langle X_i\rangle_{\psi,i} \cup \{\psi(e)\}\rangle$ by Proposition 2.4. We thus have

$$\begin{aligned}
\Delta(X,e) &= \rho\langle X_i + e\rangle - \rho\langle X_i\rangle \\
&= \rho(\langle \langle X_i\rangle_{\psi,i} \cup \{\psi(e)\}\rangle) - \rho(\langle X_i\rangle_{\psi,i}) \\
&= \rho(\langle X_i\rangle_{\psi,i} \cup \{\psi(e)\}) - \rho(\langle X_i\rangle_{\psi,i}) \\
&\geq \rho(\langle Y_i\rangle_{\psi,i} \cup \{\psi(e)\}) - \rho(\langle Y_i\rangle_{\psi,i}) \\
&= \rho(\langle \langle Y_i\rangle_{\psi,i} \cup \{\psi(e)\}\rangle) - \rho(\langle Y_i\rangle_{\psi,i}) \\
&= \rho\langle Y_i + e\rangle - \rho\langle Y_i\rangle = \Delta(Y,e),
\end{aligned}$$

where we used (12), the submodularity, and the invariance of ρ under closures.

Case 2. Suppose that e is a non-loop edge and at least one of $X_i \neq X_j$ or $X_i = X_j = \emptyset$ holds. We further split the proof into subcases.

(2-i) If $Y_i = Y_j \neq \emptyset$, then, by (12), we have $\Delta(X,e) - \Delta(Y,e) = \rho\langle X_i \cup X_j + e\rangle + 1 + \rho\langle Y_i\rangle - \rho\langle X_i\rangle - \rho\langle X_j\rangle - \rho\langle Y_i + e\rangle$. Since all these sets are connected or empty, $\rho\langle X_i \cup X_j + e\rangle \geq \rho\langle X_j\rangle$, $\rho\langle Y_i\rangle \geq \rho\langle X_i\rangle$, and $1 \geq \rho\langle Y_i + e\rangle$. Thus, $\rho\langle X_i \cup X_j + e\rangle + \rho\langle Y_i\rangle + 1 \geq \rho\langle X_i\rangle + \rho\langle X_j\rangle + \rho\langle Y_i + e\rangle$, implying (13).

(2-ii) If $Y_i \neq Y_j$ or $Y_i = Y_j = \emptyset$ holds, then e is a bridge connecting X_i and X_j in $X_i \cup X_j + e$ and is also a bridge connecting Y_i and Y_j in $Y_i \cup Y_j + e$. By a switch operation, we may assume that $\psi(e)$ is identity. Then, $\langle X_i \cup X_j + e\rangle_{\psi,i} = \langle \langle X_i\rangle_{\psi,i} \cup \langle X_j\rangle_{\psi,j}\rangle$. This implies $\rho\langle X_i \cup X_j + e\rangle = \rho(\langle X_i\rangle_{\psi,i} \cup \langle X_j\rangle_{\psi,j})$ by the invariance under closure. Symmetrically, we have $\rho\langle Y_i \cup Y_j + e\rangle = \rho(\langle Y_i\rangle_{\psi,i} \cup \langle Y_j\rangle_{\psi,j})$. By using the submodularity and the monotonicity of ρ over Γ, along with $X_k \subseteq Y_k$ for $k = 1, 2$, we have

$$\begin{aligned}
&\rho\langle X_i \cup X_j + e\rangle + \rho\langle Y_i\rangle + \rho\langle Y_j\rangle \\
&= \rho(\langle X_i\rangle_{\psi,i} \cup \langle X_j\rangle_{\psi,j}) + \rho(\langle Y_i\rangle_{\psi,i}) + \rho(\langle Y_j\rangle_{\psi,j}) \\
&\geq \rho(\langle X_i\rangle_{\psi,i} \cup \langle X_j\rangle_{\psi,j} \cup \langle Y_i\rangle_{\psi,i}) + \rho((\langle X_i\rangle_{\psi,i} \cup \langle X_j\rangle_{\psi,j}) \cap \langle Y_i\rangle_{\psi,i}) + \rho(\langle Y_j\rangle_{\psi,j}) \\
&\geq \rho(\langle Y_i\rangle_{\psi,i} \cup \langle X_j\rangle_{\psi,j}) + \rho(\langle X_i\rangle_{\psi,i}) + \rho(\langle Y_j\rangle_{\psi,j}) \\
&\geq \rho(\langle Y_i\rangle_{\psi,i} \cup \langle X_j\rangle_{\psi,j} \cup \langle Y_j\rangle_{\psi,j}) + \rho((\langle Y_i\rangle_{\psi,i} \cup \langle X_j\rangle_{\psi,j}) \cap \langle Y_j\rangle_{\psi,j}) + \rho(\langle X_i\rangle_{\psi,i}) \\
&\geq \rho(\langle Y_i\rangle_{\psi,i} \cup \langle Y_j\rangle_{\psi,j}) + \rho(\langle X_j\rangle_{\psi,j}) + \rho(\langle X_i\rangle_{\psi,i}) \\
&= \rho\langle Y_i \cup Y_j + e\rangle + \rho\langle X_j\rangle + \rho\langle X_i\rangle.
\end{aligned}$$

This implies (13) by (12). □