MATHEMATICAL ENGINEERING
TECHNICAL REPORTS

A General Framework for Finding Energy Dissipative/Conservative H^1-Galerkin Schemes and Their Underlying H^1-Weak Forms for Nonlinear Evolution Equations

Yuto MIYATAKE and Takayasu MATSUO

METR 2013–09 June 2013

DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

WWW page: http://www.keisu.t.u-tokyo.ac.jp/research/techrep/index.html

A General Framework for Finding Energy Dissipative/Conservative H^1-Galerkin Schemes and Their Underlying H^1-Weak Forms for Nonlinear Evolution Equations

Yuto MIYATAKE and Takayasu MATSUO

Department of Mathematical Informatics
Graduate School of Information Science and Technology
The University of Tokyo
{yuto_miyatake,matsuo}@mist.i.u-tokyo.ac.jp

June 27th, 2013

Abstract

A general framework for constructing energy dissipative or conservative Galerkin schemes for time dependent partial differential equations is presented. The framework targets wide variety of dissipative or conservative PDEs with variational structure and has a welcome feature that the resulting scheme can be implemented only with P1 elements. The concept of formal weak form and an L^2-projection technique are used to derive schemes and their underlying H^1-weak forms.

Keyword
Discrete partial derivative method, Dissipation, Conservation, Galerkin method, Structure-preserving integration, L^2-projection, Discrete gradient method

1 Introduction

For PDEs that enjoy the energy dissipation or conservation property, numerical schemes that inherit the property are often advantageous, in that the schemes give qualitatively better numerical solutions in practice. For example, the Cahn–Hilliard equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}\left(pu + ru^3 + q\frac{\partial^2 u}{\partial x^2}\right), \qquad 0 < x < L, \quad t > 0, \tag{1}$$

where $p < 0$, $q < 0$, $r > 0$, has the energy dissipation property

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \left(\frac{p}{2}u^2 + \frac{r}{4}u^4 - \frac{q}{2}\left(\frac{\partial u}{\partial x}\right)^2\right)\mathrm{d}x \leq 0, \qquad t > 0,$$

under appropriate boundary conditions. The Korteweg–de Vries (KdV) equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{2}u^2 + \frac{\partial^2 u}{\partial x^2}\right), \qquad 0 < x < L, \quad t > 0, \tag{2}$$

has the energy conservation property

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \left(\frac{1}{6}u^3 - \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2\right)\mathrm{d}x = 0, \qquad t > 0,$$

again under appropriate boundary conditions.

In the last two decades, much effort has been devoted in this topic to finally find out several frameworks to derive dissipative or conservative schemes. For example, Furihata proposed the discrete variational derivative method (DVDM) [11] (see also Furihata–Matsuo [12]) in finite difference context. In the method, originally spatial and temporal discretizations have been done simultaneously. However, in order to get a completely systematic framework and extend the method in various ways, it is convenient to treat the spatial and temporal discretizations separately. Celledoni et al. [4] explicitly pointed out this and applied the average vector field method, one of the most commonly used discrete gradient method, for the temporal discretization (see also the discrete gradient method [4, 11, 13, 14, 23]), and Dahlby–Owren [7] established a linearly implicit technique (see also [19]). On the other hand, for the spatial discretization, the DVDM has been developed on uniform meshes only. However, especially in multi-dimensional cases, the use of nonuniform meshes is of importance, because the restriction to uniform meshes forces the domains to be rectangles. Furthermore, even in one dimensional cases, nonuniform meshes are often useful when solutions exhibit locally complicated behavior. To extend the DVDM to nonuniform meshes, some efforts have been devoted recently. For example, Yaguchi–Matsuo–Sugihara extended the DVDM to nonuniform meshes by using either the mapping method [26] or discrete differential forms [27], and Matsuo [17] extended the DVDM to Galerkin (finite element) context.

In this paper, we consider the Galerkin framework, which we refer to as the discrete partial derivative method (DPDM). The goal of this paper is to solve a big drawback of the DPDM to make the method a completely systematic framework. Blow, we first briefly review the DPDM to clarify the drawback.

Matsuo [17] considered real-valued equations of the form

$$\frac{\partial u}{\partial t} = (-1)^{s+1}\left(\frac{\partial}{\partial x}\right)^{2s}\frac{\delta G}{\delta u}, \qquad s = 0, 1, 2, \ldots, \tag{3}$$

where $\delta G/\delta u$ is the variational derivative of $G(u, u_x)$ with respect to $u(t, x)$. Under appropriate boundary conditions, these PDEs become dissipative. For example, the Cahn–Hilliard equation (1) belongs to this class with $s = 1$ and $G(u, u_x) = pu^2/2 + ru^4/4 - qu_x^2/2$ (where $u_x = \partial u/\partial x$). Real-valued conservative PDEs of the form

$$\frac{\partial u}{\partial t} = \left(\frac{\partial}{\partial x}\right)^{2s-1}\frac{\delta G}{\delta u}, \qquad s = 1, 2, 3, \ldots, \tag{4}$$

were also targeted. The KdV equation (2) is an example of this class with $s = 1$ and $G(u, u_x) = u^3/6 - u_x^2/2$. Matsuo [17] succeeded in designing dissipative or conservative Galerkin schemes for above equations and then the method has been applied to several specific PDEs [16, 18, 20]. Moreover, it turned out that various dissipative/conservative schemes in the literature can be reformulated as special cases of the method, such as the famous Du–Nicolaides scheme for the Cahn–Hilliard equation [9]. In this sense, the DPDM has been successful to a certain extent.

There remained, however, a big drawback. In the DPDM, we firstly define H^1-weak forms which explicitly has dissipation or conservation properties, and then we discretize them appropriately with P1 elements. P1 elements are preferable for less computational complexity, in particular in two or three dimensional problems. It becomes, however, surprisingly difficult to find an appropriate H^1-weak forms for PDEs as the variational structures become more complicated than those in (3) and (4). Such PDEs can be categorized into the following two types.

Type 1: PDEs whose energy functional contains higher order derivatives, i.e., (3) or (4) with $G = G(u, u_x, u_{xx}, \ldots)$. For such PDEs, the definition of the energy in H^1 space is not obvious in the first place. One example of such dissipative PDEs is the Swift–Hohenberg (SH) equation [24]

$$\frac{\partial u}{\partial t} = -\left(-2u + u^3 + 2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4}\right), \qquad 0 < x < L, \quad t > 0. \tag{5}$$

This equation belongs to the class (3) with $s = 0$ and $G(u, u_x, u_{xx}) = -u^2 + u^4/4 - u_x^2 + u_{xx}^2/2$. One example of the conservative PDEs is the Kawahara equation (fifth-order KdV-type equation) [15]

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(-\frac{1}{2}u^2 - \alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^4 u}{\partial x^4}\right), \qquad 0 < x < L, \quad t > 0. \tag{6}$$

This equation belongs to the class (4) with $s = 1$ and $G(u, u_x, u_{xx}) = -u^3/6 + \alpha u_x^2/2 + \beta u_{xx}^2/2$.

Type 2: PDEs whose differential operator in front of the variational derivative (namely, $\mathcal{B}$ in $u_t = \mathcal{B}\delta G/\delta u$) is complicated. The dissipation or conservation property is due to the skew-symmetric or negative semi-definite property of the differential operator. When the operator cannot operate on a function in H^1, we need some special treatments. The Camassa–Holm equation [2, 3, 10]

$$u_t - u_{xxt} = uu_{xxx} + 2u_x u_{xx} - 3uu_x, \qquad 0 < x < L, \quad t > 0, \tag{7}$$

and the Degasperis–Procesi (DP) equation [8]

$$u_t - u_{xxt} = uu_{xxx} + 3u_x u_{xx} - 4uu_x, \qquad 0 < x < L, \quad t > 0,$$

are typical examples. The Camassa–Holm equation can be written in the variational (Hamiltonian) form with $\mathcal{B} = (1 - \partial_x{}^2)^{-1}(m\partial_x + \partial_x m)(1 - \partial_x{}^2)^{-1}$ and $G(u, u_x) = -(u^2 + u_x^2)/2$. Similarly, the DP equation can be written with $\mathcal{B} = (1 - \partial_x{}^2)^{-1}\partial_x(4 - \partial_x{}^2)$ and $G(u) = -u^3/6$.

As far as the authors know, there is no systematic procedure to find dissipative or conservative H^1-weak forms for above two types. This makes it difficult to apply the DPDM, unless we use smoother function spaces. But we do not hope that for the following two reasons.

- The H^1-formulation can be implemented by cheap P1 elements, which is crucial in multi-dimensional problems.
- There are some high-order PDEs with H^1 solutions. The Camassa–Holm equation which has peaked soliton solutions is a typical example. For such cases, H^1-formulations are preferable from the theoretical point of view.

Taking these backgrounds into account, in this paper we propose a new framework for constructing H^1 schemes for Types 1 and 2 PDEs. The proposed method utilizes the variational structure of the PDEs like the original DPDM, but it takes a different approach in that it finds intended schemes without finding underlying dissipative or conservative H^1-weak forms. This nontrivial approach is made possible by the idea of L^2-projection operators. As will be pointed out in Section 5, the proposed method is not a superset of the original DPDM.

Remark 1. *A part of the ideas presented in this paper has been already reported in our recent papers [21, 22] for specific PDEs without detailed discussions. In this paper we establish a completely systematic framework which can be applicable to wide variety of PDEs.*

This paper is organized as follows. In Section 2, we briefly describe the idea and difficulty of the discrete partial derivative method [17]. In Section 3, we propose a new method for one dimensional problems. After describing a general framework, we show some applications. In Section 4, we extend the proposed method to multi-dimensional problems. Concluding remarks are given in Section 5.

We use the following notation. The numerical solution is denoted by $u^{(n)} \simeq u(n\Delta t, \cdot)$ where Δt is the time mesh size. H^j denotes the jth order Sobolev space. For one-dimensional cases, the interval (domain) is set to $[0, L]$, and the inner product is defined by $(f, g) = \int_0^L fg \mathrm{d}x$. S_i's and W_i's denote the trial and test spaces, respectively. When we consider the periodic boundary conditions, we often use the notation $H^1(\mathbb{T})$ ($\mathbb{T}$ denotes the torus of length L). For multi-dimensional cases, $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) denotes the domain. The inner product is defined by $(f, g) = \int_\Omega fg \mathrm{d}x$ when f and g are scalar-valued functions or $(\boldsymbol{f}, \boldsymbol{g}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{g} \mathrm{d}x$ when $\boldsymbol{f}$ and $\boldsymbol{g}$ are vector-valued functions (the dot means $\boldsymbol{f} \cdot \boldsymbol{g} = \boldsymbol{f}^\top \boldsymbol{g}$). $\boldsymbol{L}^2$ and $\boldsymbol{H}^j$ denote $(L^2)^d$ and $(H^j)^d$, respectively. $\Gamma = \partial\Omega$ and $\boldsymbol{n}$ denote the boundary of Ω and the normal vector at the boundary. The Green theorem

$$\int_\Omega (f \triangle g + \nabla f \cdot \nabla g) \mathrm{d}x = \int_\Gamma f \boldsymbol{n} \cdot \nabla g \ \mathrm{d}\Gamma \tag{8}$$

is used instead of the integration-by-parts formula.

2 Idea of the discrete partial derivative method and its difficulty

In this section, we briefly describe the idea of the original discrete partial derivative method (DPDM) [17] and its essential difficulty. The procedure of the DPDM can be divided into the following three steps.

Step 1 Construct an H^1-weak form that explicitly has the desired dissipation/conservation property.

Step 2 Discretize the weak form in space to get a semi-discrete scheme so that it is consistent in some finite dimensional approximation spaces of H^1 and it keeps the dissipation/conservation property.

Step 3 Discretize the semi-discrete scheme in time so that the desired property remains kept (this step is essentially the same as the discrete gradient method).

We demonstrate the above steps taking the the dissipative case with $s = 0$ as an example.

Step 1

With

$$\frac{\delta G}{\delta u} = \frac{\partial G}{\partial u} - \frac{\partial}{\partial x}\frac{\partial G}{\partial u_x}$$

in mind, a dissipative H^1-weak form is defined as follows.

Weak form 1 (Dissipative H^1-weak form for (3) when $s = 0$ [17])**.** *Suppose $u(0, \cdot)$ is given in $H^1(0, L)$. Find $u(t, \cdot) \in H^1(0, L)$ such that, for any $v \in H^1(0, L)$,*

$$(u_t, v) = -\left(\frac{\partial G}{\partial u}, v\right) - \left(\frac{\partial G}{\partial u_x}, v_x\right) + \left[\frac{\partial G}{\partial u_x} v\right]_0^L. \tag{9}$$

Proposition 1 (Weak form 1: Dissipation property [17]). *Assume that the boundary conditions satisfy*

$$\left[\frac{\partial G}{\partial u_x} u_t\right]_0^L = 0, \tag{10}$$

and $u_t(t,\cdot) \in H^1(0,L)$. Then the solution of Weak form 1 satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L G(u,u_x)\mathrm{d}x \leq 0.$$

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L G(u,u_x)\mathrm{d}x = \left(\frac{\partial G}{\partial u}, u_t\right) + \left(\frac{\partial G}{\partial u_x}, u_{xt}\right) = -\left\|u_t\right\|^2 + \left[\frac{\partial G}{\partial u_x} u_t\right]_0^L \leq 0.$$

The first equality is just the chain rule, and the second follows from (9) with $v = u_t \in H^1(0,L)$. The last inequality is shown by the assumption (10). □

Note that the partial derivatives $\partial G/\partial u$ and $\partial G/\partial u_x$ play an important role in constructing the above weak form ("DPDM" was named after this fact and its discrete version below).

Step 2

In this step, we replace the function space $H^1(0,L)$ in Weak form 1 with finite dimensional approximation spaces S_1 and $W_1 \subset H^1(0,L)$ to get the following semi discrete scheme.

Semi-discrete scheme 1 (Semi-discrete dissipative scheme for (3) when $s = 0$ [17]). *Suppose $u(0,\cdot)$ is given in S_1. Find $u(t,\cdot) \in S_1$ such that, for any $v \in W_1$,*

$$(u_t, v) = -\left(\frac{\partial G}{\partial u}, v\right) - \left(\frac{\partial G}{\partial u_x}, v_x\right) + \left[\frac{\partial G}{\partial u_x} v\right]_0^L.$$

Proposition 2 (Semi-discrete scheme 1: Dissipation property [17]). *Assume that boundary conditions and the trial and test spaces are set such that $\left[\frac{\partial G}{\partial u_x} u_t\right]_0^L = 0$ and $u_t \in W_1$ hold. Then the solution of Semi-discrete scheme 1 satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L G(u,u_x)\mathrm{d}x \leq 0.$$

Proof. The proof is similar to that of Proposition 1:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L G(u,u_x)\mathrm{d}x = \left(\frac{\partial G}{\partial u}, u_t\right) + \left(\frac{\partial G}{\partial u_x}, u_{xt}\right) = -\left\|u_t\right\|^2 + \left[\frac{\partial G}{\partial u_x} u_t\right]_0^L \leq 0.$$

The substitution $v = u_t$ is allowed by the assumption $u_t \in W_1$. □

Step 3

To discretize Semi discrete scheme 1 in time, we introduce the concept of discrete partial derivatives.

Definition 1 (Discrete partial derivatives [17]). *We call the discrete quantities*

$$\frac{\partial G_{\mathrm{d}}}{\partial(u^{(n+1)}, u^{(n)})}, \qquad \frac{\partial G_{\mathrm{d}}}{\partial(u_x^{(n+1)}, u_x^{(n)})}, \tag{11}$$

the "discrete partial derivatives," which correspond to $\partial G/\partial u$ *and* $\partial G/\partial u_x$*, respectively, if they satisfy the following identity:*

$$\begin{aligned}&\frac{1}{\Delta t}\int_0^L \Big(G(u^{(n+1)}, u_x^{(n+1)}) - G(u^{(n)}, u_x^{(n)})\Big)\mathrm{d}x \\ &= \left(\frac{\partial G_{\mathrm{d}}}{\partial(u^{(n+1)}, u^{(n)})}, \frac{u^{(n+1)} - u^{(n)}}{\Delta t}\right) + \left(\frac{\partial G_{\mathrm{d}}}{\partial(u_x^{(n+1)}, u_x^{(n)})}, \frac{u_x^{(n+1)} - u_x^{(n)}}{\Delta t}\right).\end{aligned} \tag{12}$$

Since (12) corresponds to the continuous chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L G(u, u_x)\mathrm{d}x = \left(\frac{\partial G}{\partial u}, u_t\right) + \left(\frac{\partial G}{\partial u_x}, u_{xt}\right),$$

we refer to (12) as the discrete chain rule.

There are several approaches of calculating the discrete partial derivatives (readers may refer to the discrete gradient method [4, 11, 13, 14, 23]). Here we show two of them. The first one is based on the average vector field method [23]. The discrete partial derivatives defined by

$$\frac{\partial G_{\mathrm{d}}}{\partial(u^{(n+1)}, u^{(n)})} := \int_0^1 G_u((1-\xi)u^{(n+1)} + \xi u^{(n)}, (1-\xi)u_x^{(n+1)} + \xi u_x^{(n)})\mathrm{d}\xi, \tag{13}$$

$$\frac{\partial G_{\mathrm{d}}}{\partial(u_x^{(n+1)}, u_x^{(n)})} := \int_0^1 G_{u_x}((1-\xi)u^{(n+1)} + \xi u^{(n)}, (1-\xi)u_x^{(n+1)} + \xi u_x^{(n)})\mathrm{d}\xi, \tag{14}$$

where

$$G_u(u, u_x) = \frac{\partial G}{\partial u}, \quad G_{u_x}(u, u_x) = \frac{\partial G}{\partial u_x},$$

satisfy the discrete chain rule (12). The second one is based on the decomposition introduced by Furihata [11]. Consider generalized energy functions of the form

$$G(u, u_x) = \sum_{l=1}^{M} f_l(u) g_l(u_x),$$

where $M \in \{1, 2, \dots\}$, and f_l, g_l are sufficiently smooth real-valued functions. Then

$$\frac{\partial G_{\mathrm{d}}}{\partial(u^{(n+1)}, u^{(n)})} := \sum_{l=1}^{M}\left(\frac{f_l(u^{(n+1)}) - f_l(u^{(n)})}{u^{(n+1)} - u^{(n)}}\right)\left(\frac{g_l(u_x^{(n+1)}) + g_l(u_x^{(n)})}{2}\right), \tag{15}$$

$$\frac{\partial G_{\mathrm{d}}}{\partial(u_x^{(n+1)}, u_x^{(n)})} := \sum_{l=1}^{M}\left(\frac{f_l(u^{(n+1)}) + f_l(u^{(n)})}{2}\right)\left(\frac{g_l(u_x^{(n+1)}) - g_l(u_x^{(n)})}{u_x^{(n+1)} - u_x^{(n)}}\right) \tag{16}$$

satisfy the discrete chain rule (12) [17]. This approach is particularly useful when G is polynomial with respect to u and u_x.

Let us return to the temporal discretization of Semi discrete scheme 1. Replacing the time derivative u_t with $(u^{(n+1)} - u^{(n)})/\Delta t$, and the partial derivatives $\partial G/\partial u$ and $\partial G/\partial u_x$ with the discrete partial derivatives (11) leads to the following fully discrete scheme.

Scheme 1 (Dissipative Galerkin scheme for (3) when $s = 0$ [17]). *Suppose $u^{(0)}$ is given in S_1. Find $u^{(n+1)} \in S_1$ $(n = 0, 1, \dots)$ such that, for any $v \in W_1$,*

$$\left(\frac{u^{(n+1)} - u^{(n)}}{\Delta t}, v\right)$$
$$= -\left(\frac{\partial G_\mathrm{d}}{\partial(u^{(n+1)}, u^{(n)})}, v\right) - \left(\frac{\partial G_\mathrm{d}}{\partial(u_x^{(n+1)}, u_x^{(n)})}, v_x\right) + \left[\frac{\partial G_\mathrm{d}}{\partial(u_x^{(n+1)}, u_x^{(n)})} v\right]_0^L. \tag{17}$$

Theorem 3 (Scheme 1: Dissipation property [17]). *Assume that boundary conditions and the trial and test spaces are set such that*

$$\left[\frac{\partial G_\mathrm{d}}{\partial(u_x^{(n+1)}, u_x^{(n)})}\left(\frac{u^{(n+1)} - u^{(n)}}{\Delta t}\right)\right]_0^L = 0 \tag{18}$$

and $(u^{(n+1)} - u^{(n)})/\Delta t \in W_1$ hold. Then the solution of Scheme 1 satisfies

$$\frac{1}{\Delta t}\int_0^L \left(G(u^{(n+1)}, u_x^{(n+1)}) - G(u^{(n)}, u_x^{(n)})\right)\mathrm{d}x \le 0, \qquad n = 0, 1, 2, \dots.$$

Proof.

$$\frac{1}{\Delta t}\int_0^L \left(G(u^{(n+1)}, u_x^{(n+1)}) - G(u^{(n)}, u_x^{(n)})\right)\mathrm{d}x$$
$$= \left(\frac{\partial G_\mathrm{d}}{\partial(u^{(n+1)}, u^{(n)})}, \frac{u^{(n+1)} - u^{(n)}}{\Delta t}\right) + \left(\frac{\partial G_\mathrm{d}}{\partial(u_x^{(n+1)}, u_x^{(n)})}, \frac{u_x^{(n+1)} - u_x^{(n)}}{\Delta t}\right)$$
$$= -\left\|\frac{u^{(n+1)} - u^{(n)}}{\Delta t}\right\|^2 + \left[\frac{\partial G_\mathrm{d}}{\partial(u_x^{(n+1)}, u_x^{(n)})}\left(\frac{u^{(n+1)} - u^{(n)}}{\Delta t}\right)\right]_0^L \le 0.$$

The first equality is just the discrete chain rule (12), and the second follows from (17) with $v = (u^{(n+1)} - u^{(n)})/\Delta t \in W_1$. The last inequality is shown by the assumption (18). □

It is notable that the procedure of Steps 2 and 3 is completely automatic. In other words, once we find a dissipative/conservative H^1-weak form, an intended fully discrete Galerkin scheme can be systematically obtained. In the existing works, dissipative/conservative H^1-weak forms for (3) or (4) were already found [17].

Unfortunately, however, it does not seem straightforward to find desired weak forms if the PDEs are complicated as pointed out in Section 1. For Type 1 PDEs, the partial derivatives do not always live in H^1 space. For Type 2 PDEs, it is not easy to treat the complicated operator in H^1 space when the operator demands smoother function spaces.

3 New framework for one-dimensional problems

In order to tackle the above difficulty, we propose a new framework. The new procedure is summarized as follows (see Fig. 1).

PHASE 1

Step 1′ Construct a *formal* weak form which is not necessarily formulated within H^1 space, but whose dissipation/conservation property can be explicitly obtained by formal calculations. The meaning of "*formal*" will become clear soon.

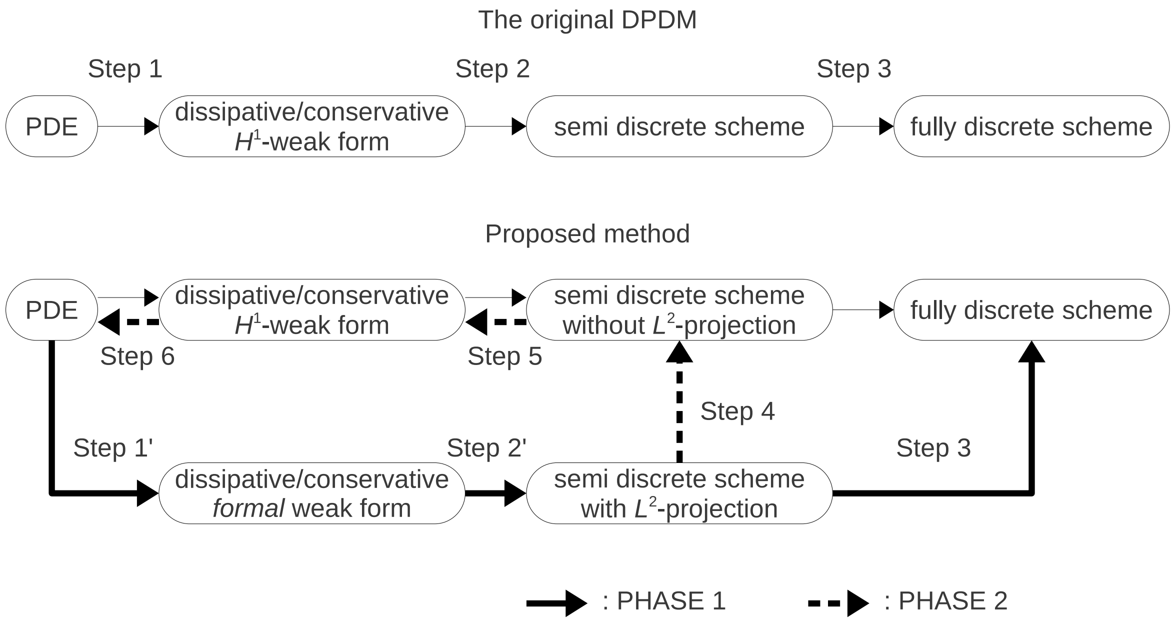

Figure 1: Standard versus proposed strategies.

Step 2′ Discretize the *formal* weak form in space to get a semi discrete scheme so that it is consistent in some finite dimensional approximation spaces of H^1 and it keeps the dissipation/conservation property. In this step, L^2-projection operators play an important role.

Step 3 Discretize the semi discrete scheme in time so that the desired property remains kept.

If we are interested in theoretical aspects of the schemes such as convergence issues, the following steps help us to find the underlying dissipative/conservative H^1-weak forms.

PHASE 2

Step 4 In the semi discrete scheme derived in Step 2′, L^2-projection operators are explicitly used. Expand them to get a more familiar form.

Step 5 Pull the semi discrete scheme back to a weak form which is consistent in H^1. In this step, we only rewrite finite dimensional approximation spaces to infinite dimensional subspaces of H^1.

Step 6 Check the relation between the PDE and the obtained H^1-weak form.

The key of the proposed method is to avoid proper dissipative/conservative H^1-weak forms which are difficult to be found. L^2-projection operators enable this approach. Note that our approach is completely automatic except for some degrees of freedom in Step 2′ (see Remark 6).

Below in Section 3.1, we introduce the L^2-projection operators. In Sections 3.2 and 3.3, we define the proposed method for Type 1 and Type 2 PDEs, respectively. In Section 3.4, we show some applications.

3.1 L^2-projection operators

We introduce the L^2-projection operators which are the key devices of the proposed method. Although we explain the concept of the L^2-projection operators in relation to one-dimensional case, it can be extended to multi-dimensional cases. We will come back to the extension in Section 4.

We define the L^2-projection operator $\mathcal{P}_X : L^2 \to X \subseteq H^1$ (X is a closed (finite dimensional approximation) space of H^1) satisfying

$$(u, v) = (\mathcal{P}_X u, v), \tag{19}$$

for any $v \in X$. We also denote $\mathcal{P}_X u_x$ by $\mathcal{D}_X u$ for convenience, namely $\mathcal{D}_X := \mathcal{P}_X \partial_x : H^1 \to X$. Roughly speaking, $\mathcal{D}_X^p (:= (\mathcal{D}_X)^p)$ $(p \geq 1)$ is the operator that approximates ${\partial_x}^p$. The following formula is straightforward.

Lemma 4. *For any $u \in H^1$ and $v \in X$, it holds*

$$\left(\mathcal{D}_X^p u, v\right) = \left((\mathcal{D}_X^{p-1} u)_x, v\right) \quad (p \geq 1). \tag{20}$$

Proof. Eq. (20) can be shown by (19):

$$\left(\mathcal{D}_X^p u, v\right) = \left(\mathcal{P}_X (\mathcal{D}_X^{p-1} u)_x, v\right) = \left((\mathcal{D}_X^{p-1} u)_x, v\right).$$

□

Note that the operator $\mathcal{D}_X$ can operate on any functions in H^1 any number of times (note that $\mathcal{D}_X^p : H^1 \to X$). From (20), we get the following equalities which show that $\mathcal{D}_X$ is skew-symmetric and $\mathcal{D}_X^2$ is symmetric corresponding to the skew-symmetry of ∂_x and symmetry of ${\partial_x}^2$, respectively.

Corollary 5. *For any $u \in X$ and $v \in X$, if $[uv]_0^L = 0$, it holds*

$$(\mathcal{D}_X u, v) = -(u, \mathcal{D}_X v).$$

Also assume that $[(\mathcal{D}_X u)v]_0^L = [u(\mathcal{D}_X v)]_0^L = 0$. Then it holds

$$\left(\mathcal{D}_X^2 u, v\right) = \left(u, \mathcal{D}_X^2 v\right).$$

Proof. These properties can be proved by (20) and the integration-by-parts formula. □

As long as we consider periodic boundary conditions, the above operators are enough for the new method. However, as will be shown soon, these operators are not sufficient to deal with several different boundary conditions. As an extension of $\mathcal{P}_X$, we define an operator $\mathcal{P}_{X(Y)} : L^2 \to X \subseteq H^1$ satisfying

$$\left(\mathcal{P}_{X(Y)} u, v\right) = (u, v) \tag{21}$$

for any $v \in Y \subseteq H^1$. Accordingly, we define an operator $\mathcal{D}_{X(Y)}$ by $\mathcal{D}_{X(Y)} := \mathcal{P}_{X(Y)} \partial_x : H^1 \to X$.

Lemma 6. *It follows that for any $u \in H^1$ and $v \in Y$,*

$$\left(\mathcal{D}_{X(Y)} u, v\right) = (u_x, v). \tag{22}$$

Although the operators $\mathcal{P}_{X(Y)}$ and $\mathcal{D}_{X(Y)}$ have no longer the meaning of "projection," we also refer to them as "L^2-projection" operators regarding them as extensions of $\mathcal{P}_X$ and $\mathcal{D}_X$.

3.2 Proposed method for Type 1 PDEs

We here present the new method taking the dissipative equation (3) with $G = G(u, u_x, u_{xx})$ and $s = 0$:

$$u_t = -\frac{\delta G}{\delta u}, \qquad G = G(u, u_x, u_{xx}) \tag{23}$$

as a working example, which is sufficient to show the essential idea. Note that in this case the variational derivative $\delta G/\delta u$ is defined as

$$\frac{\delta G}{\delta u} := \frac{\partial G}{\partial u} - \partial_x \frac{\partial G}{\partial u_x} + {\partial_x}^2 \frac{\partial G}{\partial u_{xx}}.$$

3.2.1 Design of dissipative schemes: PHASE 1

We define the procedure of Phase 1 (derivation of dissipative schemes) for (23).

Step 1′

We first note that it is not straightforward to find a dissipative H^1-weak form for (23). In fact, if we simply try to find a weak form, it would require H^2 elements, due to the term $\partial G/\partial u_{xx}$. Instead, motivated by the construction of Weak form 1, let us consider the following formulation which is obtained by integrating-by-part *only up to once* each term: Find u such that, for any v,

$$\begin{aligned}(u_t, v) &= -\left(\frac{\partial G}{\partial u}, v\right) + \left(\partial_x \frac{\partial G}{\partial u_x}, v\right) - \left({\partial_x}^2 \frac{\partial G}{\partial u_{xx}}, v\right) \\ &= -\left(\frac{\partial G}{\partial u}, v\right) - \left(\frac{\partial G}{\partial u_x}, v_x\right) + \left[\frac{\partial G}{\partial u_x} v\right]_0^L + \left(\partial_x \frac{\partial G}{\partial u_{xx}}, v_x\right) - \left[\left(\partial_x \frac{\partial G}{\partial u_{xx}}\right) v\right]_0^L.\end{aligned}$$

Note that, by the restriction of the integration-by-part, the fourth term in the most right hand side is not

$$-\left(\frac{\partial G}{\partial u_{xx}}, v_{xx}\right)$$

as in the standard finite-element formulation. This formulation still makes sense only in H^2 (or smoother spaces) but not in H^1. Next follow the rules below to define the following *formal* weak form, so that only the test functions are in H^1.

Rules for defining formal weak forms

(R1′ a) Eliminate all the derivatives in front of the partial derivatives by introducing intermediate functions

$$q = \partial_x \frac{\partial G}{\partial u_{xx}}, \quad r = {\partial_x}^2 \frac{\partial G}{\partial u_{xxx}}, \ \ldots,$$

and associated equations (weak forms), such that only first order derivatives appear in test functions (this step should be done *recursively*, when needed; see Remark 2).

(R1′ b) Leave other derivatives untouched.

For our working example, applying the rules to the above formulation, we obtain the following formal weak form. We replace $\partial_x(\partial G/\partial u_{xx})$ with q and add (24b) by the rule (R1′ a), and leave anything other by (R1′ b).

Formal weak form 1. *Suppose $u(0,\cdot)$ is given. Find u, q such that, for any v_1, v_2,*

$$(u_t, v_1) = -\left(\frac{\partial G}{\partial u}, v_1\right) - \left(\frac{\partial G}{\partial u_x}, (v_1)_x\right) + \left[\frac{\partial G}{\partial u_x} v_1\right]_0^L + (q, (v_1)_x) - [q v_1]_0^L, \tag{24a}$$

$$(q, v_2) = -\left(\frac{\partial G}{\partial u_{xx}}, (v_2)_x\right) + \left[\frac{\partial G}{\partial u_{xx}} v_2\right]_0^L. \tag{24b}$$

As is obvious from the construction, the above formulation is still not formulated in H^1 space, and thus it makes sense only formally (this is the reason why we call it a *formal* weak form). The important point there is that if we ignore this defect, the dissipation property can be explicitly obtained by formal calculations. Under the assumptions: $[q u_t]_0^L = 0$, $\left[\frac{\partial G}{\partial u_x} u_t\right]_0^L = 0$ and $\left[\frac{\partial G}{\partial u_{xx}} u_{xt}\right]_0^L = 0$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L G(u, u_x, u_{xx})\mathrm{d}x &= \left(\frac{\partial G}{\partial u}, u_t\right) + \left(\frac{\partial G}{\partial u_x}, u_{xt}\right) + \left(\frac{\partial G}{\partial u_{xx}}, u_{xxt}\right) \\
&= \left(\frac{\partial G}{\partial u}, u_t\right) + \left(\frac{\partial G}{\partial u_x}, u_{xt}\right) - (q, u_{xt}) + \left[\frac{\partial G}{\partial u_{xx}} u_{xt}\right]_0^L \\
&= -\left\|u_t\right\|^2 + \left[\frac{\partial G}{\partial u_x} u_t\right]_0^L - [q u_t]_0^L \le 0.
\end{aligned} \tag{25}$$

Here, the first equality is just the chain rule. The second follows from (24b) with $v_2 = u_{xt}$ and the third from (24a) with $v_1 = u_t$.

Remark 2. *To help the readers' understanding, we also mention on the case $G = G(u, u_x, u_{xx}, u_{xxx})$. Based on the formulation (obtained by integrating-by-part up to once):*

$$\begin{aligned}
(u_t, v) &= -\left(\frac{\partial G}{\partial u}, v\right) + \left(\partial_x \frac{\partial G}{\partial u_x}, v\right) - \left({\partial_x}^2 \frac{\partial G}{\partial u_{xx}}, v\right) + \left({\partial_x}^3 \frac{\partial G}{\partial u_{xxx}}, v\right) \\
&= -\left(\frac{\partial G}{\partial u}, v\right) - \left(\frac{\partial G}{\partial u_x}, v_x\right) + \left[\frac{\partial G}{\partial u_x} v\right]_0^L + \left(\partial_x \frac{\partial G}{\partial u_{xx}}, v_x\right) - \left[\left(\partial_x \frac{\partial G}{\partial u_{xx}}\right) v\right]_0^L \\
&\quad - \left({\partial_x}^2 \frac{\partial G}{\partial u_{xxx}}, v_x\right) + \left[\left({\partial_x}^2 \frac{\partial G}{\partial u_{xxx}}\right) v\right]_0^L,
\end{aligned}$$

and introducing intermediate functions suggested by the rule (R1′ a), we can formulate a formal weak form as follows. Suppose $u(0,\cdot)$ is given. Find u, q, r, r_1 such that, for any v_1, v_2, v_3, v_4,

$$\begin{aligned}
(u_t, v_1) &= -\left(\frac{\partial G}{\partial u}, v_1\right) - \left(\frac{\partial G}{\partial u_x}, (v_1)_x\right) + \left[\frac{\partial G}{\partial u_x} v_1\right]_0^L + (q, (v_1)_x) - [q v_1]_0^L \\
&\quad - (r, (v_1)_x) + [r v_1]_0^L,
\end{aligned}$$

$$(q, v_2) = -\left(\frac{\partial G}{\partial u_{xx}}, (v_2)_x\right) + \left[\frac{\partial G}{\partial u_{xx}} v_2\right]_0^L, \tag{26}$$

$$(r, v_3) = -(r_1, (v_3)_x) + [r_1, v_3]_0^L, \tag{27}$$

$$(r_1, v_4) = -\left(\frac{\partial G}{\partial u_{xxx}}, (v_4)_x\right) + \left[\frac{\partial G}{\partial u_{xxx}} v_4\right]_0^L. \tag{28}$$

The intermediate functions q, r and their associated equations (26), (27) are introduced by the rule (R1′ a). The function r_1 and its associated equation (28) are recursively introduced by (R1′ a) in connection with r.

Step 2′

In this step, we discretize the above formal weak form in space so that the semi discrete scheme is consistent in some finite dimensional approximation spaces of H^1. Follow the rules below.

Rules for constructing semi-discrete schemes

(R2′ a) Set finite dimensional trial and test function spaces for solutions of the formal weak form;

(R2′ b) Replace derivatives in G and in partial derivatives with $\mathcal{D}_{S_j(W_j)}$ by introducing function spaces S_j's, W_j's as necessary such that

$$u_x \to \mathcal{D}_{S_j(W_j)}u, \quad u_{xx} \to \mathcal{D}_{S_{j+1}(W_{j+1})}\mathcal{D}_{S_j(W_j)}u, \ \ldots$$

(i.e. we introduce new function spaces for each additional derivatives);

(R2′ c) Place projection operators in front of partial derivatives, by introducing new function spaces for each partial derivative;

(R2′ d) Leave other derivatives (mainly in test functions) untouched.

Semi-discrete scheme 2 (with L^2-projection operators)**.** *Suppose $u(0,\cdot)$ is given in S_1. Find $u(t,\cdot) \in S_1, q \in S_2$ such that, for any $v_1 \in W_1, v_2 \in W_2$,*

$$(u_t, v_1) = -\left(\frac{\partial G}{\partial u}, v_1\right) - \left(\mathcal{P}_{S_5(W_5)}\frac{\partial G}{\partial(\mathcal{D}_{S_3(W_3)}u)}, (v_1)_x\right) + \left[\left(\mathcal{P}_{S_5(W_5)}\frac{\partial G}{\partial(\mathcal{D}_{S_3(W_3)}u)}\right)v_1\right]_0^L$$
$$+ (q, (v_1)_x) - [qv_1]_0^L, \tag{29a}$$
$$(q, v_2) = -\left(\mathcal{P}_{S_6(W_6)}\frac{\partial G}{\partial(\mathcal{D}_{S_4(W_4)}\mathcal{D}_{S_3(W_3)}u)}, (v_2)_x\right) + \left[\left(\mathcal{P}_{S_6(W_6)}\frac{\partial G}{\partial(\mathcal{D}_{S_4(W_4)}\mathcal{D}_{S_3(W_3)}u)}\right)v_2\right]_0^L, \tag{29b}$$

where $G = G(u, \mathcal{D}_{S_3(W_3)}u, \mathcal{D}_{S_4(W_4)}\mathcal{D}_{S_3(W_3)}u)$.

Following (R2′ a), we introduce trial function spaces S_1 and S_2 for $u(t,\cdot)$ and q, respectively, and corresponding test function spaces W_1 and W_2. Following (R2′ b), we introduce S_3, W_3, S_4 and W_4 to replace derivatives such that

$$u_x \to \mathcal{D}_{S_3(W_3)}u, \quad u_{xx} \to \mathcal{D}_{S_4(W_4)}\mathcal{D}_{S_3(W_3)}u.$$

Following (R2′ c), we introduce S_5, W_5, S_6 and W_6 to place $\mathcal{P}_{S_5(W_5)}$ and $\mathcal{P}_{S_6(W_6)}$ in front of partial derivatives. As long as we obey the rules, the numbering is arbitrary. The following proposition reveals sufficient conditions for the dissipation property for each given numbering.

Remark 3. *In the proposed method, (R2′ a), (R2′ b) and (R2′ d) make the semi discrete scheme consistent in H^1. The rule (R2′ c) is necessary for the dissipation property.*

Remark 4. *The proposed method forces us to use many function (test and trial) spaces. We have to select each space so that they are consistent with the boundary conditions and satisfy the assumptions in the following proposition. But this can be done based on the standard theory of finite element methods. See Section 3.4.1 for a concrete example.*

Remark 5. *Some remarks on the notation:* $\partial G/\partial(\mathcal{D}_{S_j(W_j)}u)$ *denotes the substitution of* $\mathcal{D}_{S_j(W_j)}u, \mathcal{D}_{S_{j+1}(W_{j+1})}\mathcal{D}_{S_j(W_j)}u, \dots$ *into* $u_x, u_{xx}, \dots$ *in* $\partial G/\partial u_x$*, and same notation is used for* $\partial G/\partial u_{xx}, \partial G/\partial u_{xxx}, \dots$. *For example, for* $G(u, u_x, u_{xx}) = uu_xu_{xx}$*,*

$$\frac{\partial G}{\partial u} = (\mathcal{D}_{S_j(W_j)}u)(\mathcal{D}_{S_{j+1}(W_{j+1})}\mathcal{D}_{S_j(W_j)}u), \quad \frac{\partial G}{\partial(\mathcal{D}_{S_j(W_j)}u)} = u(\mathcal{D}_{S_{j+1}(W_{j+1})}\mathcal{D}_{S_j(W_j)}u),$$
$$\frac{\partial G}{\partial(\mathcal{D}_{S_{j+1}(W_{j+1})}\mathcal{D}_{S_j(W_j)}u)} = u(\mathcal{D}_{S_j(W_j)}u).$$

Proposition 7 (Semi-discrete scheme 2: Dissipation property)**.** *Assume that the boundary conditions satisfy*

$$\left[\left(\mathcal{P}_{S_6(W_6)}\frac{\partial G}{\partial(\mathcal{D}_{S_4(W_4)}\mathcal{D}_{S_3(W_3)}u)}\right)\mathcal{D}_{S_3(W_3)}u_t\right]_0^L = 0,$$
$$\left[\left(\mathcal{P}_{S_5(W_5)}\frac{\partial G}{\partial(\mathcal{D}_{S_3(W_3)}u)}\right)u_t\right]_0^L = 0, \quad [qu_t]_0^L = 0.$$

Also assume that $S_5 \subseteq W_3$*,* $S_6 \subseteq W_4$*,* $S_2 \subseteq W_3$*,* $\mathcal{D}_{S_3(W_3)}u_t \in W_5$*,* $\mathcal{D}_{S_4(W_4)}\mathcal{D}_{S_3(W_3)}u_t \in W_6$*,* $\mathcal{D}_{S_3(W_3)}u_t \in W_2$*,*$u_t \in W_1$ *and* $u_x, \mathcal{D}_{S_3(W_3)}u \in C^1(\mathbb{R}^+; L^2(0, L))$*. Then the solution of Semi discrete scheme 2 satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L G(u, \mathcal{D}_{S_3(W_3)}u, \mathcal{D}_{S_4(W_4)}\mathcal{D}_{S_3(W_3)}u)\mathrm{d}x \le 0.$$

Proof. The proof is basically similar to the formal calculation (25). We carefully check each equality.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L G(u, \mathcal{D}_{S_3(W_3)}u, \mathcal{D}_{S_4(W_4)}\mathcal{D}_{S_3(W_3)}u)\mathrm{d}x \\
&= \left(\frac{\partial G}{\partial u}, u_t\right) + \left(\frac{\partial G}{\partial(\mathcal{D}_{S_3(W_3)}u)}, \mathcal{D}_{S_3(W_3)}u_t\right) + \left(\frac{\partial G}{\partial(\mathcal{D}_{S_4(W_4)}\mathcal{D}_{S_3(W_3)}u)}, \mathcal{D}_{S_4(W_4)}\mathcal{D}_{S_3(W_3)}u_t\right) \\
&= \left(\frac{\partial G}{\partial u}, u_t\right) + \left(\mathcal{P}_{S_5(W_5)}\frac{\partial G}{\partial(\mathcal{D}_{S_3(W_3)}u)}, \mathcal{D}_{S_3(W_3)}u_t\right) \\
&\quad + \left(\mathcal{P}_{S_6(W_6)}\frac{\partial G}{\partial(\mathcal{D}_{S_4(W_4)}\mathcal{D}_{S_3(W_3)}u)}, \mathcal{D}_{S_4(W_4)}\mathcal{D}_{S_3(W_3)}u_t\right) \\
&= \left(\frac{\partial G}{\partial u}, u_t\right) + \left(\mathcal{P}_{S_5(W_5)}\frac{\partial G}{\partial(\mathcal{D}_{S_3(W_3)}u)}, u_{xt}\right) + \left(\mathcal{P}_{S_6(W_6)}\frac{\partial G}{\partial(\mathcal{D}_{S_4(W_4)}\mathcal{D}_{S_3(W_3)}u)}, (\mathcal{D}_{S_3(W_3)}u_t)_x\right) \\
&= \left(\frac{\partial G}{\partial u}, u_t\right) + \left(\mathcal{P}_{S_5(W_5)}\frac{\partial G}{\partial(\mathcal{D}_{S_3(W_3)}u)}, u_{xt}\right) - (q, \mathcal{D}_{S_3(W_3)}u_t) \\
&\quad + \left[\left(\mathcal{P}_{S_6(W_6)}\frac{\partial G}{\partial(\mathcal{D}_{S_4(W_4)}\mathcal{D}_{S_3(W_3)}u)}\right)\mathcal{D}_{S_3(W_3)}u_t\right]_0^L \\
&= \left(\frac{\partial G}{\partial u}, u_t\right) + \left(\mathcal{P}_{S_5(W_5)}\frac{\partial G}{\partial(\mathcal{D}_{S_3(W_3)}u)}, u_{xt}\right) - (q, u_{xt}) \\
&= -\|u_t\|^2 + \left[\left(\mathcal{P}_{S_5(W_5)}\frac{\partial G}{\partial(\mathcal{D}_{S_3(W_3)}u)}\right)u_t\right]_0^L - [qu_t]_0^L \le 0.
\end{aligned}$$

The first equality is just the chain rule. In the second equality, we have used (21) for the second and third terms. This is allowed by the assumption $\mathcal{D}_{S_3(W_3)}u_t \in W_5$ for the second term and $\mathcal{D}_{S_4(W_4)}\mathcal{D}_{S_3(W_3)}u_t \in W_6$ for the third term. Note that the terms included in the partial derivatives stay in L^2: by the Sobolev embedding theorem, $S_3,\ S_4 \subset H^1 \subset C^0$ (see, for example, [1, Theorem 7.3.8]).

The third equality is from (22) which is allowed by the assumptions $S_5 \subseteq W_3$ and $S_6 \subseteq W_4$. The fourth follows from (29a) with $v_2 = \mathcal{D}_{S_3(W_3)} u_t \in W_2$. The fifth is again from (22) which is allowed by the assumption $S_2 \subseteq W_3$ and the sixth from (29b) with $v_1 = u_t \in W_1$. □

Remark 6. *The procedure of this step was defined so that it can be carried out completely automatically; but it can be slightly modified if necessary. In the above illustration, we consider the energy of the form*

$$\int_0^L G(u, \mathcal{D}_{S_3(W_3)} u, \mathcal{D}_{S_4(W_4)} \mathcal{D}_{S_3(W_3)} u) \mathrm{d}x.$$

However, other definitions are sometimes possible. For example, with the energy of the form

$$\int_0^L G(u, u_x, \mathcal{D}_{S_4(W_4)} \mathcal{D}_{S_3(W_3)} u) \mathrm{d}x,$$

we can derive an intended semi-discrete scheme as well.

Step 3

In this step, we discretize Semi-discrete scheme 2 in time, so that the dissipation property remains kept. Although this step is just the discrete gradient method, it is helpful to use the following notation. Hereafter we also call the discrete quantities

$$\frac{\partial G_{\mathrm{d}}}{\partial(u^{(n+1)}, u^{(n)})}, \quad \frac{\partial G_{\mathrm{d}}}{\partial(\mathcal{D}_{S_j(W_j)} u^{(n+1)}, \mathcal{D}_{S_j(W_j)} u^{(n)})},$$
$$\frac{\partial G_{\mathrm{d}}}{\partial(\mathcal{D}_{S_{j+1}(W_{j+1})} \mathcal{D}_{S_j(W_j)} u^{(n+1)}, \mathcal{D}_{S_{j+1}(W_{j+1})} \mathcal{D}_{S_j(W_j)} u^{(n)})}$$

the discrete partial derivatives, which correspond to $\partial G/\partial u$, $\partial G/\partial u_x$ and $\partial G/\partial u_{xx}$, respectively, if they satisfy the following discrete chain rule:

$$\begin{aligned}
&\frac{1}{\Delta t} \int_0^L \Big(G(u^{(n+1)}, \mathcal{D}_{S_j(W_j)} u^{(n+1)}, \mathcal{D}_{S_{j+1}(W_{j+1})} \mathcal{D}_{S_j(W_j)} u^{(n+1)}) \\
&\qquad\qquad - G(u^{(n)}, \mathcal{D}_{S_j(W_j)} u^{(n)}, \mathcal{D}_{S_{j+1}(W_{j+1})} \mathcal{D}_{S_j(W_j)} u^{(n)}) \Big) \mathrm{d}x \\
&= \left(\frac{\partial G_{\mathrm{d}}}{\partial(u^{(n+1)}, u^{(n)})}, \frac{u^{(n+1)} - u^{(n)}}{\Delta t} \right) \\
&\quad + \left(\frac{\partial G_{\mathrm{d}}}{\partial(\mathcal{D}_{S_j(W_j)} u^{(n+1)}, \mathcal{D}_{S_j(W_j)} u^{(n)})}, \frac{\mathcal{D}_{S_j(W_j)} u^{(n+1)} - \mathcal{D}_{S_j(W_j)} u^{(n)}}{\Delta t} \right) \\
&\quad + \Bigg(\frac{\partial G_{\mathrm{d}}}{\partial(\mathcal{D}_{S_{j+1}(W_{j+1})} \mathcal{D}_{S_j(W_j)} u^{(n+1)}, \mathcal{D}_{S_{j+1}(W_{j+1})} \mathcal{D}_{S_j(W_j)} u^{(n)})}, \\
&\qquad\qquad \frac{\mathcal{D}_{S_{j+1}(W_{j+1})} \mathcal{D}_{S_j(W_j)} u^{(n+1)} - \mathcal{D}_{S_{j+1}(W_{j+1})} \mathcal{D}_{S_j(W_j)} u^{(n)}}{\Delta t} \Bigg).
\end{aligned}$$

Actually, such quantities can be explicitly defined in a similar manner as (13), (14) or (15), (16). Then using the discrete partial derivatives, we define a dissipative scheme as follows.

Scheme 2 (Dissipative H^1-Galerkin schemes for (23))**.** *Suppose $u^{(0)}$ is given in S_1. Find $u^{(n+1)} \in S_1$ and $q^{(n+\frac{1}{2})} \in S_2$ $(n = 0, 1, \dots)$ such that, for any $v_1 \in W_1$ and $v_2 \in W_2$,*

$$\left(\frac{u^{(n+1)} - u^{(n)}}{\Delta t}, v_1\right) = -\left(\frac{\partial G_{\mathrm{d}}}{\partial(u^{(n+1)}, u^{(n)})}, v_1\right)$$
$$-\left(\mathcal{P}_{S_5(W_5)}\frac{\partial G_{\mathrm{d}}}{\partial(\mathcal{D}_{S_3(W_3)}u^{(n+1)}, \mathcal{D}_{S_3(W_3)}u^{(n)})}, (v_1)_x\right)$$
$$+\left[\mathcal{P}_{S_5(W_5)}\frac{\partial G_{\mathrm{d}}}{\partial(\mathcal{D}_{S_3(W_3)}u^{(n+1)}, \mathcal{D}_{S_3(W_3)}u^{(n)})}v_1\right]_0^L + \left(q^{(n+\frac{1}{2})}, (v_1)_x\right) - \left[q^{(n+\frac{1}{2})}v_1\right]_0^L,$$

$$\left(q^{(n+\frac{1}{2})}, v_2\right) = -\left(\mathcal{P}_{S_6(W_6)}\frac{\partial G_{\mathrm{d}}}{\partial(\mathcal{D}_{S_4(W_4)}\mathcal{D}_{S_3(W_3)}u^{(n+1)}, \mathcal{D}_{S_4(W_4)}\mathcal{D}_{S_3(W_3)}u^{(n)})}, (v_2)_x\right)$$
$$+\left[\left(\mathcal{P}_{S_6(W_6)}\frac{\partial G_{\mathrm{d}}}{\partial(\mathcal{D}_{S_4(W_4)}\mathcal{D}_{S_3(W_3)}u^{(n+1)}, \mathcal{D}_{S_4(W_4)}\mathcal{D}_{S_3(W_3)}u^{(n)})}\right)v_2\right]_0^L.$$

By the construction of the scheme, the following theorem which indicates that the scheme is dissipative immediately follows.

Theorem 8 (Scheme 2: Dissipation property)**.** *Assume that the boundary conditions satisfy*

$$\left[\left(\mathcal{P}_{S_6(W_6)}\frac{\partial G_{\mathrm{d}}}{\partial(\mathcal{D}_{S_4(W_4)}\mathcal{D}_{S_3(W_3)}u^{(n+1)}, \mathcal{D}_{S_4(W_4)}\mathcal{D}_{S_3(W_3)}u^{(n)})}\right)\left(\frac{\mathcal{D}_{S_3(W_3)}u^{(n+1)} - \mathcal{D}_{S_3(W_3)}u^{(n)}}{\Delta t}\right)\right]_0^L = 0,$$

$$\left[\left(\mathcal{P}_{S_5(W_5)}\frac{\partial G_{\mathrm{d}}}{\partial(\mathcal{D}_{S_3(W_3)}u^{(n+1)}, \mathcal{D}_{S_3(W_3)}u^{(n)})}\right)\left(\frac{u^{(n+1)} - u^{(n)}}{\Delta t}\right)\right]_0^L = 0,$$

$$\left[q^{(n+\frac{1}{2})}\left(\frac{u^{(n+1)} - u^{(n)}}{\Delta t}\right)\right]_0^L = 0.$$

Also assume that $S_5 \subseteq W_3$, $S_6 \subseteq W_4$, $S_2 \subseteq W_3$, $(\mathcal{D}_{S_3(W_3)}u^{(n+1)} - \mathcal{D}_{S_3(W_3)}u^{(n)})/\Delta t \in W_5$, $(\mathcal{D}_{S_4(W_4)}\mathcal{D}_{S_3(W_3)}u^{(n+1)} - \mathcal{D}_{S_4(W_4)}\mathcal{D}_{S_3(W_3)}u^{(n)})/\Delta t \in W_6$, $(\mathcal{D}_{S_3(W_3)}u^{(n+1)} - \mathcal{D}_{S_3(W_3)}u^{(n)})/\Delta t \in W_2$ and $(u^{(n+1)} - u^{(n)})/\Delta t \in W_1$. Then the solution of Scheme 2 satisfies

$$\frac{1}{\Delta t}\int_0^L \Big(G(u^{(n+1)}, \mathcal{D}_{S_3(W_3)}u^{(n+1)}, \mathcal{D}_{S_4(W_4)}\mathcal{D}_{S_3(W_3)}u^{(n+1)})$$
$$-G(u^{(n)}, \mathcal{D}_{S_3(W_3)}u^{(n)}, \mathcal{D}_{S_4(W_4)}\mathcal{D}_{S_3(W_3)}u^{(n)})\Big)\,\mathrm{d}x \le 0, \qquad n = 0, 1, 2, \dots.$$

3.2.2 Design of dissipative schemes: PHASE 2

In Phase 1 (Steps 1-3), we have succeeded in designing the dissipative H^1-Galerkin scheme. In the remaining steps, we find an underlying H^1-weak form of Scheme 2 to confirm the validity of the scheme.

Step 4

In this step, we change the expression of Semi discrete scheme 2 to a more familiar form, eliminating the L^2-projection operators. Here we do not introduce any new concepts, but just rewrite the expression of the semi discrete scheme. Hence, the following Semi discrete scheme 2′ is mathematically equivalent to Semi discrete scheme 2. The procedure of this step is based on the definition of the L^2-projection operators. For instance, introducing an intermediate function $a_1 \in S_3$, we rewrite the term $\partial G/\partial(\mathcal{D}_{S_3(W_3)}u)$ into $\partial G/\partial a_1$ with adding a new equation $(a_1, v) = (u_x, v)$ for any $v \in W_3$ (see Lemma 6).

Semi discrete scheme 2′ (without L^2-projection operators)**.** *Find $u(t,\cdot) \in S_1$, $q \in S_2$, $a_1 \in S_3$, $a_2 \in S_4$, $r_1 \in S_5$, $r_2 \in S_6$ such that, for any $v_1 \in W_1$, $v_2 \in W_2$, $v_3 \in W_3$, $v_4 \in W_4$, $v_5 \in W_5$, $v_6 \in W_6$,*

$$
\begin{aligned}
(u_t, v_1) &= -\left(\frac{\partial G}{\partial u}, v_1\right) - (r_1, (v_1)_x) + [r_1 v_1]_0^L + (q, (v_1)_x) - [q v_1]_0^L \,, \\
(q, v_2) &= -(r_2, (v_2)_x) + [r_2 v_2]_0^L \,, \\
(a_1, v_3) &= (u_x, v_3)\,, \\
(a_2, v_4) &= ((a_1)_x, v_4)\,, \\
(r_1, v_5) &= \left(\frac{\partial G}{\partial a_1}, v_5\right), \\
(r_2, v_6) &= \left(\frac{\partial G}{\partial a_2}, v_6\right).
\end{aligned}
$$

Proposition 9 (Semi discrete scheme 2′: Dissipation property)**.** *Assume that the boundary conditions satisfy*

$$[r_2 (a_1)_t]_0^L = 0, \quad [r_1 u_t]_0^L = 0, \quad [q u_t]_0^L = 0.$$

Also assume that $S_5 \subseteq W_3$, $S_6 \subseteq W_4$, $S_2 \subseteq W_3$, $(a_1)_t \in W_5$, $(a_2)_t \in W_6$, $(a_1)_t \in W_2$, $u_t \in W_1$ and $u_x, (a_1)_x \in C^1(\mathbb{R}^+; L^2(0, L))$. Then the solution of Semi discrete scheme 2′ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^L G(u, a_1, a_2)\mathrm{d}x \leq 0.$$

Proof. Since Proposition 7 was already proved, we skip this proof. Of course, the claim can be directly shown for Semi discrete scheme 2′. □

Step 5

In this step, we replace all finite dimensional function spaces S_i's and W_i's to the corresponding infinite dimensional function spaces S_i^{c}'s and W_i^{c}'s which are subspaces of $H^1(0, L)$ to get the following weak form.

Weak form 2. *Find $u(t,\cdot) \in S_1^{\mathrm{c}}$, $q \in S_2^{\mathrm{c}}$, $a_1 \in S_3^{\mathrm{c}}$, $a_2 \in S_4^{\mathrm{c}}$, $r_1 \in S_5^{\mathrm{c}}$, $r_2 \in S_6^{\mathrm{c}}$ such that, for any $v_1 \in W_1^{\mathrm{c}}$, $v_2 \in W_2^{\mathrm{c}}$, $v_3 \in W_3^{\mathrm{c}}$, $v_4 \in W_4^{\mathrm{c}}$, $v_5 \in W_5^{\mathrm{c}}$, $v_6 \in W_6^{\mathrm{c}}$,*

$$
\begin{aligned}
(u_t, v_1) &= -\left(\frac{\partial G}{\partial u}, v_1\right) - (r_1, (v_1)_x) + [r_1 v_1]_0^L + (q, (v_1)_x) - [q v_1]_0^L \,, \\
(q, v_2) &= -(r_2, (v_2)_x) + [r_2 v_2]_0^L \,, \\
(a_1, v_3) &= (u_x, v_3)\,, \\
(a_2, v_4) &= ((a_1)_x, v_4)\,, \\
(r_1, v_5) &= \left(\frac{\partial G}{\partial a_1}, v_5\right), \\
(r_2, v_6) &= \left(\frac{\partial G}{\partial a_2}, v_6\right).
\end{aligned}
$$

Obviously Weak form 2 is consistent in H^1, and has the dissipation property.

Proposition 10 (Weak form 2: Dissipation property)**.** *Assume that the boundary conditions satisfy*

$$[r_2 (a_1)_t]_0^L = 0, \quad [r_1 u_t]_0^L = 0, \quad [q u_t]_0^L = 0.$$

Also assume that $S_5^{\mathrm{c}} \subseteq W_3^{\mathrm{c}}$, $S_6^{\mathrm{c}} \subseteq W_4^{\mathrm{c}}$, $S_2^{\mathrm{c}} \subseteq W_3^{\mathrm{c}}$, $(a_1)_t \in W_5^{\mathrm{c}}$, $(a_2)_t \in W_6^{\mathrm{c}}$, $(a_1)_t \in W_2^{\mathrm{c}}$, $u_t \in W_1^{\mathrm{c}}$ *and* $u_x, (a_1)_x \in C^1(\mathbb{R}^+; L^2(0,L))$. *Then the solution of Weak form 2 satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L G(u, a_1, a_2)\mathrm{d}x \leq 0.$$

Thus we found the desired underlying weak form, which is consistent in H^1 and keeps the dissipation property. Obviously, it is quite difficult to find this directly from the original PDE (23); even determining the required intermediate functions (five in the above case) is a hard task. But by simply following the proposed approach, we can automatically reach the desired dissipative scheme and the underlying weak form. We would like to emphasize here again that this is the point of the new method.

Step 6

Finally, we have to check the relation between (23) and Weak form 2. Actually, Weak form 2 can be seen as the natural weak formulation of the system of equations

$$u_t = -\frac{\partial G}{\partial u} + (r_1)_x - q_x, \quad q = (r_2)_x, \quad a_1 = u_x,$$
$$a_2 = (a_1)_x, \quad r_1 = \frac{\partial G}{\partial a_1}, \quad r_2 = \frac{\partial G}{\partial a_2},$$

which is equivalent to (23).

3.3 Proposed method for Type 2 PDEs

We show the new method for the conservative equation:

$$u_t = \mathcal{B}\frac{\delta G}{\delta u}, \tag{30}$$

where $\mathcal{B} = \mathcal{B}(u, u_x, u_{xx}, \ldots, \partial_x, {\partial_x}^2, \ldots)$ is skew-symmetric and polynomial with respect to $u, u_x, u_{xx}, \ldots, \partial_x, {\partial_x}^2, \ldots$. For simplicity, we assume that $G = G(u, u_x)$.

In order to clarify the essential idea of the proposed method, we restrict our attention to the conservative case and assume that the boundary conditions are periodic (see Remark 8). Moreover we will describe the framework with the toy problem $\mathcal{B} = (u_{xx}\partial_x + \partial_x u_{xx}) + {\partial_x}^3$ (as usual in this research field this operator operates on a function f in such a way that $(g\partial_x + \partial_x g)f = gf_x + \partial_x(gf)$), which contains two typical types of complicated differential operators:

- $\mathcal{B}$ containing not only ${\partial_x}^s$ but some functions of $u, u_x, \ldots$,
- $\mathcal{B}$ expressed as the summation of some differential operators.

Nowadays, we are often concerned with much more complicated operators such as the inverse of differential operators. We will also demonstrate the treatment of such a case in the next subsection with a concrete example.

3.3.1 Design of conservative schemes: PHASE 1

In this subsection, we show the procedure of Phase 1 (derivation of dissipative schemes) for (30).

Step 1′

Let us simply consider the following formulation. Since this does not necessarily make sense in H^1 (depending on the operator $\mathcal{B}$), we call the following the formal weak form.

Formal weak form 2. *Suppose $u(0,\cdot)$ is given. Find u, p such that, for any v_1, v_2,*

$$\begin{aligned}(u_t, v_1) &= (\mathcal{B}p, v_1),\\ (p, v_2) &= \left(\frac{\partial G}{\partial u}, v_2\right) + \left(\frac{\partial G}{\partial u_x}, (v_2)_x\right).\end{aligned}$$

If the operator $\mathcal{B}$ is skew-symmetric, the conservation property can be obtained by formal calculations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} G(u, u_x)\mathrm{d}x = \left(\frac{\partial G}{\partial u}, u_t\right) + \left(\frac{\partial G}{\partial u_x}, u_{xt}\right) = (p, u_t) = (\mathcal{B}p, p) = 0. \tag{31}$$

Step 2′

In this step, we discretize the above formal weak form in space so that the semi discrete scheme is consistent in an approximation space X_{p} of $H^1(\mathbb{T})$. This step is completely automatic and can be done just by replacing differential operators with $\mathcal{D}_{X_{\mathrm{p}}}$; i.e., replace $\mathcal{B}(u_{xx}, \partial_x, {\partial_x}^3)$ with $\mathcal{B}_{\mathrm{sd}} = \mathcal{B}(\mathcal{D}^2_{X_{\mathrm{p}}}u, \mathcal{D}_{X_{\mathrm{p}}}, \mathcal{D}^3_{X_{\mathrm{p}}})$. Note that for $\mathcal{B}(u_{xx}, \partial_x, {\partial_x}^3)$ defined above, $\mathcal{B}(\mathcal{D}^2_{X_{\mathrm{p}}}u, \mathcal{D}_{X_{\mathrm{p}}}, \mathcal{D}^3_{X_{\mathrm{p}}})$ is skew-symmetric.

Semi-discrete scheme 3 (with L^2-projection operators)**.** *Suppose $u(0,\cdot)$ is given in X_{p}. Find $u(t,\cdot)$ and $p \in X_{\mathrm{p}}$ such that, for any v_1 and $v_2 \in X_{\mathrm{p}}$,*

$$\begin{aligned}(u_t, v_1) &= (\mathcal{B}_{\mathrm{sd}}p, v_1),\\ (p, v_2) &= \left(\frac{\partial G}{\partial u}, v_2\right) + \left(\frac{\partial G}{\partial u_x}, (v_2)_x\right)\end{aligned}$$

where $G = G(u, u_x)$.

Note that replacing the operator $\mathcal{B}$ with $\mathcal{B}_{\mathrm{sd}}$ makes the scheme consistent in H^1 space.

Proposition 11 (Semi discrete scheme 3: Conservation property)**.** *The solution of Semi discrete scheme 3 satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} G(u, u_x)\mathrm{d}x = 0.$$

Proof. Since $\mathcal{B}_{\mathrm{sd}}$ makes sense in H^1 space, the calculation (31) is not formal but mathematically rigorous. □

Step 3

In this step, we discretize the above semi-discrete scheme in time. Since this step is nothing but the discrete gradient method, we show only the result.

Scheme 3 (Conservative H^1-Galerkin schemes for (30))**.** *Suppose $u^{(0)}$ is given in X_{p}. Find $u^{(n+1)} \in X_{\mathrm{p}}$ and $q^{(n+\frac{1}{2})} \in X_{\mathrm{p}}$ $(n = 0, 1, \dots)$ such that, for any $v_1 \in X_{\mathrm{p}}$ and $v_2 \in X_{\mathrm{p}}$,*

$$\left(\frac{u^{(n+1)} - u^{(n)}}{\Delta t}, v_1\right) = (\mathcal{B}_{\mathrm{d}}p^{(n+\frac{1}{2})}, v_1),$$

$$(p^{(n+\frac{1}{2})}, v_2) = \left(\frac{\partial G_{\mathrm{d}}}{\partial(u^{(n+1)}, u^{(n)})}, v_2\right) + \left(\frac{\partial G_{\mathrm{d}}}{\partial(u_x^{(n+1)}, u_x^{(n)})}, (v_2)_x\right),$$

where $\mathcal{B}_{\mathrm{d}} = \mathcal{B}(\mathcal{D}^2_{X_{\mathrm{p}}}u^{(n+\frac{1}{2})}, \mathcal{D}_{X_{\mathrm{p}}}, \mathcal{D}^3_{X_{\mathrm{p}}})$ and $u^{(n+\frac{1}{2})} = (u^{(n+1)} + u^{(n)})/2$.

Theorem 12 (Scheme 3: Conservation property)**.** *The solution of Scheme 2 satisfies*

$$\frac{1}{\Delta t}\int_{\mathbb{T}} (G(u^{(n+1)}, u_x^{(n+1)}) - G(u^{(n)}, u_x^{(n)}))\mathrm{d}x = 0, \qquad n = 0, 1, 2, \dots.$$

3.3.2 Design of conservative schemes: PHASE 2

Step 4

Recall that $\mathcal{B}_{\mathrm{sd}} p = \mathcal{B}(\mathcal{D}^2_{X_\mathrm{p}} u, \mathcal{D}_{X_\mathrm{p}}, \mathcal{D}^3_{X_\mathrm{p}}) p = (\mathcal{D}^2_{X_\mathrm{p}} u)(\mathcal{D}_{X_\mathrm{p}} p) + \mathcal{D}_{X_\mathrm{p}}((\mathcal{D}^2_{X_\mathrm{p}} u)p) + \mathcal{D}^3_{X_\mathrm{p}} p$. Introducing new variables

$$\begin{aligned}
&u_1 := \mathcal{D}_{X_\mathrm{p}} u, \quad u_2 := \mathcal{D}_{X_\mathrm{p}} u_1,\\
&p_1 := \mathcal{D}_{X_\mathrm{p}} p, \quad p_2 := \mathcal{D}_{X_\mathrm{p}} p_1, \quad p_3 := \mathcal{D}_{X_\mathrm{p}} p_2,\\
&q := \mathcal{D}_{X_\mathrm{p}} (u_2 p),
\end{aligned}$$

we can rewrite Semi discrete scheme 3 to a more familiar form as follows.

Semi discrete scheme 3′ (without L^2-projection operators)**.** *Find* $u(t,\cdot), u_1, u_2, p, p_1, p_2, p_3, q \in X_\mathrm{p}$ *such that, for any* $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8 \in X_\mathrm{p}$,

$$\begin{aligned}
&(u_t, v_1) = (u_2 p_1 + q + p_3, v_1),\\
&(p, v_2) = \left(\frac{\partial G}{\partial u}, v_2\right) + \left(\frac{\partial G}{\partial u_x}, (v_2)_x\right),\\
&(u_1, v_3) = (u_x, v_3), \quad (u_2, v_4) = ((u_1)_x, v_4),\\
&(p_1, v_5) = (p_x, v_5), \quad (p_2, v_6) = ((p_1)_x, v_6), \quad (p_3, v_7) = ((p_2)_x, v_7),\\
&(q, v_8) = ((u_2 p)_x, v_8).
\end{aligned}$$

Steps 5 and 6

One can easily obtain a conservative H^1-weak form by changing X_p in Semi discrete scheme 3′ into $H^1(\mathbb{T})$, and by confirming it is consistent with the original equation (30). We omit the straightforward (but tedious) calculation here. Once again, like as the comment at the end of Section 3.2, we like to emphasize that the weak form (with seven intermediate functions) is not obvious at all.

Remark 7. *We explained the proposed method with the definition* $\mathcal{B}_{\mathrm{sd}} = \mathcal{B}(\mathcal{D}^2_{X_\mathrm{p}} u, \mathcal{D}_{X_\mathrm{p}}, \mathcal{D}^3_{X_\mathrm{p}})$*. But we can adopt other definitions such as* $\mathcal{B}(\mathcal{D}^2_{X_\mathrm{p}} u, \partial_x, \mathcal{D}^3_{X_\mathrm{p}})$*, as long as it is consistent with* H^1 *and skew-symmetric.*

Remark 8. *For PDEs whose* G *has higher order derivatives and* $\mathcal{B}$ *is complicated, one can derive the desired Galerkin schemes by the combination of the proposed methods for Types 1 and 2. Similarly, general boundary conditions can be treated by a similar discussions in the Type 1 procedure.*

3.4 Some applications

In this subsection, we show some applications of the proposed method. In Section 3.4.1, we consider the Swift–Hohenberg equation, which is a dissipative equation of Type 1, and discuss the treatment of boundary conditions. In Section 3.4.2, we consider the Kawahara equation, which is a conservative equation of Type 1, discuss the implementation issue, and show some numerical experiments. In Section 3.4.3, we consider the Camassa–Holm equation, which is a conservative PDE of Type 2.

3.4.1 Type 1: The Swift–Hohenberg equation

The Swift–Hohenberg equation is an example of (23) with $G(u, u_x, u_{xx}) = -u^2 + u^4/4 - u_x^2 + u_{xx}^2/2$, which is usually solved subject to the boundary conditions

$$u_x = u_{xxx} = 0 \quad \text{at } x = 0, L, \tag{32}$$

or

$$u = u_{xx} = 0 \quad \text{at } x = 0, L. \tag{33}$$

For either case, it can be easily confirmed that a classical solution has an energy-dissipation property. By applying the procedure in Section 3.2, we automatically obtain the formal weak form, semi discrete scheme and fully discrete scheme. Below we only show examples of the choice of function spaces (as mentioned in Remark 4, the choice is based on the standard theory of finite element methods). Let $S_h \subset H^1(0, L)$ be the piecewise linear function space over the grids. Let $S_{h,0} = \{v \mid v \in S_h, v(0) = v(L) = 0\}$. Obviously, $S_{h,0}$ corresponds to $H_0^1 = \{v \mid v \in H^1, v(0) = v(L) = 0\}$. For (32), it is natural to chose $S_1 = W_1 = S_4 = W_4 = S_6 = W_6 = S_{h,0}$ and $S_2 = W_2 = S_3 = W_3 = S_5 = W_5 = S_h$. Correspondingly, $S_1^{\mathrm{c}} = W_1^{\mathrm{c}} = S_4^{\mathrm{c}} = W_4^{\mathrm{c}} = S_6^{\mathrm{c}} = W_6^{\mathrm{c}} = H_0^1$ and $S_2^{\mathrm{c}} = W_2^{\mathrm{c}} = S_3^{\mathrm{c}} = W_3^{\mathrm{c}} = S_5^{\mathrm{c}} = W_5^{\mathrm{c}} = H^1$. For (33), it is natural to chose $S_2 = W_2 = S_3 = W_3 = S_6 = W_6 = S_{h,0}$ and $S_1 = W_1 = S_4 = W_4 = S_5 = W_5 = S_h$. Correspondingly, $S_2^{\mathrm{c}} = W_2^{\mathrm{c}} = S_3^{\mathrm{c}} = W_3^{\mathrm{c}} = S_6^{\mathrm{c}} = W_6^{\mathrm{c}} = H_0^1$ and $S_1^{\mathrm{c}} = W_1^{\mathrm{c}} = S_4^{\mathrm{c}} = W_4^{\mathrm{c}} = S_5^{\mathrm{c}} = W_5^{\mathrm{c}} = H^1$. These relations are consistent with the assumptions in Propositions 7 and 10.

3.4.2 Type 1: The Kawahara equation

Let us consider the PDE of the form

$$u_t = \partial_x \frac{\delta G}{\delta u}, \qquad G = G(u, u_x, u_{xx}).$$

We illustrate how we can apply the proposed method taking the Kawahara equation (6) as an example. We assume the periodic boundary conditions for simplicity. We set $S_1 = S_2 = \cdots = W_1 = W_2 = \cdots =: X_{\mathrm{p}} \subset H^1(\mathbb{T})$ ($\mathbb{T}$ denotes the torus of length L). Since it is lengthy to show all steps (Steps 1-6), we only show a formal weak form, the resulting Galerkin scheme, and its underlying weak form.

Firstly, let us define a formal weak form. Introducing an intermediate function p, we can translate the equation into the system

$$\begin{cases} u_t = (p_1)_x, \\ p_1 = \dfrac{\delta G}{\delta u} \end{cases}$$

as suggested in the original DPDM [17]. Let us consider the following formulation which is obtained by integrating-by-part up to once each term: Find u, p_1 such that, for any v_1, v_2,

$$\begin{aligned} &(u_t, v_1) = ((p_1)_x, v_1), \\ &(p_1, v_2) = \left(\frac{\partial G}{\partial u}, v_2\right) + \left(\frac{\partial G}{\partial u_x}, (v_2)_x\right) - \left(\partial_x \frac{\partial G}{\partial u_{xx}}, (v_2)_x\right). \end{aligned}$$

We then follow the same procedure as in Section 3.2 to find the following formal weak form: Find u, p, q such that, for any v_1, v_2, v_3,

$$(u_t, v_1) = ((p_1)_x, v_1),$$

$$(p_1, v_2) = \left(\frac{\partial G}{\partial u}, v_2\right) + \left(\frac{\partial G}{\partial u_x}, (v_2)_x\right) - (q, (v_2)_x),$$
$$(q, v_3) = -\left(\frac{\partial G}{\partial u_{xx}}, (v_3)_x\right).$$

Note that the new intermediate function q is introduced by the rule (R1$'$ a). The conservation property can be obtained by formal calculations:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} G(u, u_x, u_{xx})\mathrm{d}x &= \left(\frac{\partial G}{\partial u}, u_t\right) + \left(\frac{\partial G}{\partial u_x}, u_{xt}\right) + \left(\frac{\partial G}{\partial u_{xx}}, u_{xxt}\right) \\ &= \left(\frac{\partial G}{\partial u}, u_t\right) + \left(\frac{\partial G}{\partial u_x}, u_{xt}\right) - (q, u_{xt}) \\ &= (p_1, u_t) = ((p_1)_x, p_1) = 0.\end{aligned}$$

Secondly, we derive a semi-discrete scheme by using the L^2-projection operators. Suppose $u(0, \cdot)$ is given in X_{p}. Find $u(t, \cdot), p_1, q \in X_{\mathrm{p}}$ such that, for any $v_1, v_2, v_3 \in X_{\mathrm{p}}$,

$$(u_t, v_1) = ((p_1)_x, v_1),$$
$$(p_1, v_2) = \left(\frac{\partial G}{\partial u}, v_2\right) + \left(\mathcal{P}_{X_{\mathrm{p}}}\frac{\partial G}{\partial(\mathcal{D}_{X_{\mathrm{p}}}u)}, (v_2)_x\right) - (q, (v_2)_x),$$
$$(q, v_3) = -\left(\mathcal{P}_{X_{\mathrm{p}}}\frac{\partial G}{\partial(\mathcal{D}^2_{X_{\mathrm{p}}}u)}, (v_3)_x\right),$$

where

$$\frac{\partial G}{\partial u} = -\frac{u^2}{3}, \quad \frac{\partial G}{\partial(\mathcal{D}_{X_{\mathrm{p}}}u)} = \alpha\mathcal{D}_{X_{\mathrm{p}}}u, \quad \frac{\partial G}{\partial(\mathcal{D}^2_{X_{\mathrm{p}}}u)} = \beta\mathcal{D}^2_{X_{\mathrm{p}}}u.$$

Here u_x and u_{xx} in partial derivatives are replaced by $\mathcal{D}_{X_{\mathrm{p}}}u$ and $\mathcal{D}^2_{X_{\mathrm{p}}}u$ based on (R2$'$ b), and $\mathcal{P}_{X_{\mathrm{p}}}$'s are placed in front of partial derivatives based on (R2$'$ c). We implicitly obey (R2$'$ a), but since in this case all function spaces are X_{p} due to the periodic boundary conditions, we skip the discussion about the function spaces. This scheme is consistent in $X_{\mathrm{p}} \subset H^1(\mathbb{T})$, and has the conservation property which is not just formal but rigorous:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} G(u, \mathcal{D}_{X_{\mathrm{p}}}u, \mathcal{D}^2_{X_{\mathrm{p}}}u)\mathrm{d}x = 0.$$

Thirdly, we discretize the above semi discrete scheme in time to get the following fully discrete scheme. Suppose $u^{(0)}$ is given in X_{p}. Find $u^{(n+1)}, p_1^{(n+\frac{1}{2})}, q^{(n+\frac{1}{2})} \in X_{\mathrm{p}}$ $(n = 0, 1, \dots)$ such that, for any $v_1, v_2, v_3 \in X_{\mathrm{p}}$,

$$\left(\frac{u^{(n+1)} - u^{(n)}}{\Delta t}, v_1\right) = \left((p_1^{(n+\frac{1}{2})})_x, v_1\right), \tag{34a}$$

$$\begin{aligned}\left(p_1^{(n+\frac{1}{2})}, v_2\right) = &\left(\frac{\partial G_{\mathrm{d}}}{\partial(u^{(n+1)}, u^{(n)})}, v_2\right) + \left(\mathcal{P}_{X_{\mathrm{p}}}\frac{\partial G_{\mathrm{d}}}{\partial(\mathcal{D}_{X_{\mathrm{p}}}u^{(n+1)}, \mathcal{D}_{X_{\mathrm{p}}}u^{(n)})}, (v_2)_x\right) \\ &- \left(q^{(n+\frac{1}{2})}, (v_2)_x\right),\end{aligned} \tag{34b}$$

$$\left(q^{(n+\frac{1}{2})}, v_3\right) = -\left(\mathcal{P}_{X_{\mathrm{p}}}\frac{\partial G_{\mathrm{d}}}{\partial(\mathcal{D}^2_{X_{\mathrm{p}}}u^{(n+1)}, \mathcal{D}^2_{X_{\mathrm{p}}}u^{(n)})}, (v_3)_x\right), \tag{34c}$$

where

$$\frac{\partial G_{\mathrm{d}}}{\partial(u^{(n+1)},u^{(n)})} = -\frac{(u^{(n+1)})^2+u^{(n+1)}u^{(n)}+(u^{(n)})^2}{6},$$
$$\frac{\partial G_{\mathrm{d}}}{\partial(\mathcal{D}_{X_{\mathrm{p}}}u^{(n+1)},\mathcal{D}_{X_{\mathrm{p}}}u^{(n)})} = \alpha\left(\frac{\mathcal{D}_{X_{\mathrm{p}}}u^{(n+1)}+\mathcal{D}_{X_{\mathrm{p}}}u^{(n)}}{2}\right),$$
$$\frac{\partial G_{\mathrm{d}}}{\partial(\mathcal{D}^2_{X_{\mathrm{p}}}u^{(n+1)},\mathcal{D}^2_{X_{\mathrm{p}}}u^{(n)})} = \beta\left(\frac{\mathcal{D}^2_{X_{\mathrm{p}}}u^{(n+1)}+\mathcal{D}^2_{X_{\mathrm{p}}}u^{(n)}}{2}\right).$$

These discrete partial derivatives correspond to $\partial G/\partial u = -u^2/2$, $\partial G/\partial u_x = \alpha u_x$ and $\partial G/\partial u_{xx} = \beta u_{xx}$ (recall that $G(u,u_x,u_{xx}) = -u^3/6+\alpha u_x^2/2+\beta u_{xx}^2/2$) and satisfy the discrete chain rule.

Theorem 13. *The solution of the scheme* (34a)*,* (34b)*,* (34c) *satisfies*

$$\frac{1}{\Delta t}\int_{\mathbb{T}}\Big(G(u^{(n+1)},\mathcal{D}_{X_{\mathrm{p}}}u^{(n+1)},\mathcal{D}^2_{X_{\mathrm{p}}}u^{(n+1)}) - G(u^{(n)},\mathcal{D}_{X_{\mathrm{p}}}u^{(n)},\mathcal{D}^2_{X_{\mathrm{p}}}u^{(n)})\Big)\,\mathrm{d}x = 0, \qquad n=0,1,2,\ldots.$$

Finally, following the procedure of Phase 2, we obtain the underlying H^1-weak form: Find $u(t,\cdot),p_1,q,a_1,a_2\in H^1(\mathbb{T})$ such that, for any $v_1,v_2,v_3,v_4,v_5\in H^1(\mathbb{T})$,

$$\begin{aligned}
&(u_t,v_1) = ((p_1)_x,v_1)\,,\\
&(p_1,v_2) = \left(-\frac{u^2}{2},v_2\right)+\alpha\,(a_1,(v_2)_x)-(q,(v_2)_x)\,,\\
&(q,v_3) = -\beta\,((a_2,(v_3)_x)\,,\\
&(a_1,v_4) = (u_x,v_4)\,,\\
&(a_2,v_5) = ((a_1)_x,v_5)\,.
\end{aligned}$$

Let us turn to numerical experiments. The implementation of the scheme (34a), (34b), (34c) is straightforward. The basis functions of X_{p} are denoted by $\psi_i(x)$ ($i=0,1,\ldots,N-1$). The concrete form of the scheme is

$$\mathsf{A}\left(\frac{\boldsymbol{u}^{(n+1)}-\boldsymbol{u}^{(n)}}{\Delta t}\right) = \mathsf{B}\boldsymbol{p}_1^{(n+\frac{1}{2})}, \tag{35a}$$

$$\mathsf{A}\boldsymbol{p}_1^{(n+\frac{1}{2})} = \boldsymbol{f}\Big(\boldsymbol{u}^{(n+1)},\boldsymbol{u}^{(n)},\boldsymbol{q}_1^{(n+\frac{1}{2})}\Big), \tag{35b}$$

$$\mathsf{A}\boldsymbol{q}_1^{(n+\frac{1}{2})} = \boldsymbol{g}\Big(\boldsymbol{u}^{(n+1)},\boldsymbol{u}^{(n)}\Big), \tag{35c}$$

where $\boldsymbol{u}^{(n)} := (u_0^{(n)},u_1^{(n)},\ldots,u_{N-1}^{(n)})$ are the coefficient vectors of $u^{(n)}(x)=\sum_{i=0}^{N-1}u_i^{(n)}\psi_i(x)$ (the same notation is used for $p_1^{(n+\frac{1}{2})}(x)$ and $q_1^{(n+\frac{1}{2})}(x)$), and $\boldsymbol{f}$ and $\boldsymbol{g}$ are the vectors arising from the right hand side of (34b) and (34c), respectively ($\boldsymbol{f}$ is nonlinear and $\boldsymbol{g}$ is linear in terms of $\boldsymbol{u}^{(n+1)}$). The matrix A is the mass matrix whose elements are $\mathsf{A}_{ij}=(\psi_i,\psi_j)$, and B's elements are $\mathsf{B}_{ij}=((\psi_i)_x,\psi_j)$. Since the matrix A is invertible, (35a), (35b) and (35c) immediately reduce to

$$\mathsf{A}\left(\frac{\boldsymbol{u}^{(n+1)}-\boldsymbol{u}^{(n)}}{\Delta t}\right) = \mathsf{B}\mathsf{A}^{-1}\boldsymbol{f}\Big(\boldsymbol{u}^{(n+1)},\boldsymbol{u}^{(n)},\mathsf{A}^{-1}\boldsymbol{g}\Big(\boldsymbol{u}^{(n+1)},\boldsymbol{u}^{(n)}\Big)\Big).$$

Thus, the computation of the intermediate variables $\boldsymbol{p}_1^{(n+\frac{1}{2})}$ and $\boldsymbol{q}_1^{(n+\frac{1}{2})}$ can be skipped, and the dimension of the nonlinear systems to be solved is N, instead of $3N$.

The operator $\mathcal{D}_{X_\mathrm{p}}$ can be implemented as follows. If we denote $\mathcal{D}_{X_\mathrm{p}} u^{(n)} = \sum_{i=0}^{N-1} d_i^{(n)} \psi_i(x)$, the coefficient $\boldsymbol{d}^{(n)} = (d_0^{(n)}, d_1^{(n)}, \ldots, d_{N-1}^{(n)})^\top$ is calculated by

$$\mathsf{A}\boldsymbol{d}^{(n)} = \mathsf{B}\boldsymbol{u}^{(n)},$$

which is equivalent to $\left(\mathcal{D}_X u^{(n)}, \psi_i\right) = (u_x, \psi_i)$ $(i = 0, 1, \ldots, N-1)$.

Next we check the qualitative behavior and discrete conservation law of the numerical solution. For simplicity, we use a uniform mesh and employed the P1 elements. The parameters were set to $\alpha = \beta = 1$, $t = [0, 400]$, $x \in [0, 50]$, $\Delta x = 50/101$ $(N = 101)$, $\Delta t = 0.1$. Motivated by the fact that the Kawahara equation has a solitary wave solution [25]

$$u(t,x) = \frac{105\alpha^2}{169\beta} \mathrm{sech}^4 \left[\frac{1}{2} \sqrt{\frac{\alpha}{13\beta}} \left(x - x_0 - \frac{36\alpha^2}{169\beta} t \right) \right], \qquad x \in \mathbb{R},$$

we set the initial value to $u(0,x) = (105/169)\mathrm{sech}^4((1/2)\sqrt{1/13}(x - 25))$. Fig. 2 shows the numerical solution obtained by the scheme (34a), (34b), (34c) and error in the discrete energy. The scheme can well capture the solitary solutions over a long time and the error well agrees with the discrete conservation law (Theorem 13).

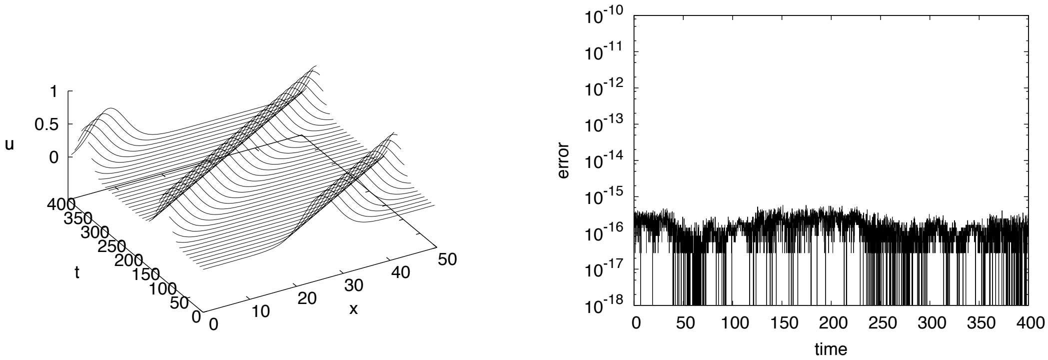

Figure 2: Scheme 3: (left) the numerical solution and (right) error in the discrete energy.

Finally let us look at the scheme from a different viewpoint. One of the simplest H^1-weak formulations for the Kawahara equation (6) would be the following. Find $u(t,\cdot), p, q \in H^1(\mathbb{T})$ such that, for any $v_1, v_2, v_3 \in H^1(\mathbb{T})$,

$$\begin{aligned}
(u_t, v_1) &= \left(\frac{u^2}{2}, (v_1)_x \right) + \alpha \left(p, (v_1)_x \right) - \beta \left(q, (v_1)_x \right), \\
(p, v_2) &= -\left(u_x, (v_2)_x \right), \\
(q, v_3) &= -\left(p_x, (v_3)_x \right).
\end{aligned}$$

We consider a standard spatial discretization of the above naive weak form (which is not conservative), and the conservative semi discrete scheme derived by the proposed method, and then discretize them in time by means of the fourth order explicit Runge–Kutta method. Thus, the conservation is destroyed for both cases by the temporal discretization. Nevertheless, the results are truly different. The time mesh size was set to $\Delta t = 0.0025$ (other parameters were set to those employed in the above experiment). From Fig. 3, we observe that numerical solution based on the conservative weak form was stable in $t \in [0, 10]$, while numerical solution based on the naive weak form blew up in first four steps. This example implies that the conservative weak form itself obtained as a by-product of the proposed method is meaningful.

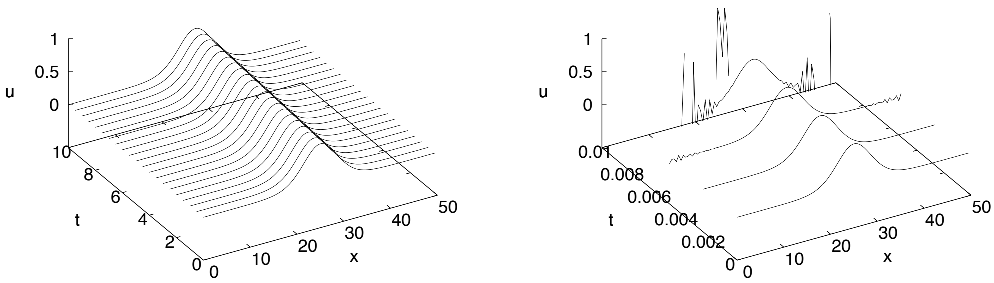

Figure 3: The numerical solutions obtained by 4th order explicit Runge–Kutta method based on (left) conservative weak form and (right) naive weak form.

3.4.3 Type 2: The Camassa–Holm equation

In this example, we illustrate the derivation of conservative schemes for the Camassa–Holm equation which is a conservative PDE of Type 2. Although one of the schemes was already published in [21], we give additional discussions especially as for the weak formulation.

Let us start by emphasizing the difficulty of finding a conservative H^1-weak form for the Camassa–Holm equation (7). As is well known, the Camassa–Holm equation has a characteristic peakon (peaked soliton) solution: $u(t,x) = c\exp(-|x-ct|)$ which lives in H^1 but not in C^1. One H^1-weak formulation has been already given in [5] (see also [6]):

$$u_t + \frac{1}{2}\left(u^2 + \mathcal{K}\left(u^2 + \frac{u_x^2}{2}\right)\right)_x = 0, \quad \text{where } \mathcal{K} = (1-{\partial_x}^2)^{-1}.$$

But this formulation is not convenient in our project, since it seems that the conservation law cannot be directly established. On the other hand, Matsuo [18] proposed the following weak form: Find $m(t,\cdot), p \in H^1(\mathbb{T})$ such that, for any $v_1, v_2 \in H^1(\mathbb{T})$,

$$\begin{aligned}
(m_t, v_1) &= ((m\partial_x + \partial_x m)p, v_1)\,,\\
(p, v_2) &= \left(\frac{\partial G}{\partial(\mathcal{K}m)}, \mathcal{K}v_2\right) + \left(\frac{\partial G}{\partial(\mathcal{K}m_x)}, \mathcal{K}(v_2)_x\right),
\end{aligned}$$

which directly leads to the conservation law. But this formulation has a drawback that it can capture only H^3 (when $m \in H^1$) or smoother solutions in terms of the original variable u. Note that when u represents peakon solutions, m becomes delta functions.

The Camassa–Holm equation has the Hamiltonian structure

$$m_t = (m\partial_x + \partial_x m)\frac{\delta G}{\delta m},$$

where

$$G = -\frac{u^2 + u_x^2}{2}, \quad \text{and} \quad m = (1-{\partial_x}^2)u,$$

or equivalently

$$u_t = (1-{\partial_x}^2)^{-1}(m\partial_x + \partial_x m)(1-{\partial_x}^2)^{-1}\frac{\delta G}{\delta u}. \tag{36}$$

Introducing an intermediate variable p, we can further translate (36) into the system

$$\begin{cases} (1-{\partial_x}^2)u_t = (m\partial_x + \partial_x m)p, \\ (1-{\partial_x}^2)p = \dfrac{\delta G}{\delta u}. \end{cases} \tag{37}$$

Firstly, we consider the following formal weak form: Find u, p such that, for any v_1, v_2,

$$\big((1-{\partial_x}^2)u_t, v_1\big) = \left((m\partial_x + \partial_x m)p, v_1\right), \tag{38a}$$

$$\big((1-{\partial_x}^2)p, v_2\big) = \left(\frac{\partial G}{\partial u}, v_2\right) + \left(\frac{\partial G}{\partial u_x}, (v_2)_x\right). \tag{38b}$$

The conservation law can be explicitly obtained by formal calculations:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} G(u,u_x)\mathrm{d}x &= \left(\frac{\partial G}{\partial u}, u_t\right) + \left(\frac{\partial G}{\partial u_x}, u_{xt}\right) = \big((1-{\partial_x}^2)p, u_t\big) \\ &= \big(p, (1-{\partial_x}^2)u_t\big) = \left((m\partial_x + \partial_x m)p, p\right) = 0. \end{aligned}$$

The first equality is just the chain rule. The second equality follows from (38b) with $v_2 = u_t$, the third from the symmetry of $(1-{\partial_x}^2)$, and the fourth from (38a) with $v_1 = p$. The last is from the skew-symmetry of $(m\partial_x + \partial_x m)$.

Remark 9. *The translation from* (36) *to* (37) *is automatic in the following sense. In the translation, we note the two points:*

- *symmetric operators should be kept (in this case, $(m\partial_x + \partial_x m)$);*
- *variational derivative should be treated separately (such as the second equation of* (37)*).*

Then we can easily find a system which is formally conservative. Such a system is not unique in general, for example for the Camassa–Holm equation, we can also find

$$\begin{cases} u_t = (1-{\partial_x}^2)^{-1}(m\partial_x + \partial_x m)(1-{\partial_x}^2)^{-1}p, \\ p = \dfrac{\delta G}{\delta u}; \end{cases}$$

but for all of them the subsequent procedure can be applied.

Secondly, we derive a semi-discrete scheme by using the L^2-projection operators. Suppose $u(0,\cdot) \in X_{\mathrm{p}}$ is given. Find $u(t,\cdot), p(t,\cdot) \in X_{\mathrm{p}}$ such that, for any $v_1, v_2 \in X_{\mathrm{p}}$,

$$\begin{aligned} \big((1-(\mathcal{D}_{X_{\mathrm{p}}})^2)u_t, v_1\big) &= \left((m\partial_x + \partial_x m)p, v_1\right), \\ \big((1-(\mathcal{D}_{X_{\mathrm{p}}})^2)p, v_2\big) &= \left(\frac{\partial G}{\partial u}, v_2\right) + \left(\frac{\partial G}{\partial u_x}, (v_2)_x\right), \end{aligned}$$

where $m = (1-(\mathcal{D}_{X_{\mathrm{p}}})^2)u$. This semi-discrete scheme is consistent in $X_{\mathrm{p}} \subset H^1(\mathbb{T})$, and has the conservation property which is not formal but rigorous:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} G(u,u_x)\mathrm{d}x = 0.$$

Remark 10. *If we strictly obey the rules of the proposed method, we should replace $(m\partial_x + \partial_x m)$ with $(m\mathcal{D}_{X_{\mathrm{p}}} + \mathcal{D}_{X_{\mathrm{p}}}m)$. But as mentioned in Remark 7, we can leave the operator ∂_x because the operator $(m\partial_x + \partial_x m)$ with the definition $m = (1-\mathcal{D}_{X_{\mathrm{p}}}^2)u$ makes sense in H^1 in this case. The above semi-discrete scheme is exactly the same as the one already proposed in our recent report [21].*

Thirdly, we discretize the above semi-discrete scheme in time to get the following fully discrete scheme. Suppose $u^{(0)} \in X_{\mathrm{p}}$ is given. Find $u^{(n+1)}, p^{(n+\frac{1}{2})} \in X_{\mathrm{p}}$ $(n = 0, 1, \dots)$ such that, for any $v_1, v_2 \in X_{\mathrm{p}}$,

$$\left((1 - (\mathcal{D}_{X_{\mathrm{p}}})^2)\frac{u^{(n+1)} - u^{(n)}}{\Delta t}, v_1\right) = \left((m^{(n+\frac{1}{2})}\partial_x + \partial_x m^{(n+\frac{1}{2})})p^{(n+\frac{1}{2})}, v_1\right), \tag{39}$$

$$\left((1 - (\mathcal{D}_{X_{\mathrm{p}}})^2)p^{(n+\frac{1}{2})}, v_2\right) = \left(\frac{\partial G_{\mathrm{d}}}{\partial(u^{(n+1)}, u^{(n)})}, v_2\right) + \left(\frac{\partial G_{\mathrm{d}}}{\partial(u_x^{(n+1)}, u_x^{(n)})}, (v_2)_x\right), \tag{40}$$

where $m^{(n+\frac{1}{2})} = (1 - (\mathcal{D}_{X_{\mathrm{p}}})^2)(u^{(n+1)} + u^{(n)})/2$,

$$\frac{\partial G_{\mathrm{d}}}{\partial(u^{(n+1)}, u^{(n)})} = -\frac{u^{(n+1)} + u^{(n)}}{2}, \qquad \frac{\partial G_{\mathrm{d}}}{\partial(u_x^{(n+1)}, u_x^{(n)})} = -\frac{u_x^{(n+1)} + u_x^{(n)}}{2}.$$

Theorem 14. *The solution of the scheme* (39), (40) *satisfies*

$$\frac{1}{\Delta t}\int_{\mathbb{T}} \left(G(u^{(n+1)}, u_x^{(n+1)}) - G(u^{(n)}, u_x^{(n)})\right) \mathrm{d}x = 0, \qquad n = 0, 1, 2, \dots.$$

Finally, following the procedure of Phase 2, we obtain the underlying H^1-weak form. Find $u, m, p, q_1, q_2, q_3 \in H^1(\mathbb{T})$ such that, for any $v_1, \dots, v_6 \in H^1(\mathbb{T})$,

$$\begin{aligned}
&(m_t, v_1) = ((m\partial_x + \partial_x m)p, v_1),\\
&(m, v_2) = (u, v_2) + (q_1, (v_2)_x),\\
&(q_1, v_3) = (u_x, v_3),\\
&(q_2, v_4) = \left(\frac{\partial G}{\partial u}, v_4\right) + \left(\frac{\partial G}{\partial u_x}, (v_4)_x\right),\\
&(q_2, v_5) = (p, v_5) + (q_3, (v_5)_x),\\
&(q_3, v_6) = (p_x, v_6).
\end{aligned}$$

The solution of this weak form is conservative in the following sense:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} G(u, u_x)\mathrm{d}x = 0.$$

4 Extension to multi-dimensional problems

In this section, we extend the proposed method to multi-dimensional cases. We explain the extension through the following example. We consider the dissipative equation of the form

$$\frac{\partial u}{\partial t} = (-1)^{s+1}\triangle^s \frac{\delta G}{\delta u}, \qquad G = G(u, \nabla u, \triangle u), \tag{41}$$

where the variational derivative in two or three dimensional cases is defined by

$$\frac{\delta G}{\delta u} := \frac{\partial G}{\partial u} - \nabla \cdot \frac{\partial G}{\partial \nabla u} + \triangle \frac{\partial G}{\partial \triangle u}.$$

Firstly we introduce the notation of the L^2-projection operators for high dimensional cases, and show their properties. Next we illustrate the derivation of dissipative schemes for (41) with $s = 0$ taking the two-dimensional Swift–Hohenberg (2D-SH) equation

$$u_t = -u^3 + 2u - 2\nabla u - \triangle^2 u, \tag{42}$$

on the torus $\mathbb{T}^2$, as an example, whose energy functional is $G(u, \nabla u, \triangle u) = u^4/4 - u^2 - |\nabla u|^2 + (\triangle u)^2/2$. To save space, we show only a formal weak form and the resulting scheme, but its underlying weak form can be also derived by the procedure of Phase 2.

4.1 L^2-projection operators in multi-dimensional cases

We define the L^2-projection operators $\mathcal{P}_X : L^2 \to X \subseteq H^1(\Omega) \subset L^2(\Omega)$ satisfying

$$(\mathcal{P}_X u, v) = (u, v)$$

for any $v \in X$, and $\mathcal{P}_{\boldsymbol{X}} : \boldsymbol{L}^2(\Omega) \to \boldsymbol{X} \subseteq \boldsymbol{H}^1(\Omega) \subset \boldsymbol{L}^2(\Omega)$ satisfying

$$(\mathcal{P}_{\boldsymbol{X}} \boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{v})$$

for any $\boldsymbol{v} \in \boldsymbol{X}$. We also denote $\mathcal{P}_{\boldsymbol{X}} \nabla u$ and $\mathcal{P}_X \nabla \cdot \boldsymbol{u}$ by $\mathcal{D}_{\boldsymbol{X}} u$ and $\mathcal{D}_X \boldsymbol{u}$. That is, $\mathcal{D}_{\boldsymbol{X}} := \mathcal{P}_{\boldsymbol{X}} \nabla : H^1(\Omega) \to \boldsymbol{X}$ and $\mathcal{D}_X := \mathcal{P}_X \nabla \cdot : \boldsymbol{H}^1(\Omega) \to X$. As for these operators, the following formulas corresponding to Lemma 4 and Corollary 5 are straightforward.

Lemma 15. *For any $u \in H^1(\Omega)$ and $\boldsymbol{v} \in \boldsymbol{X}$, it holds*

$$(\mathcal{D}_{\boldsymbol{X}} u, \boldsymbol{v}) = (\nabla u, \boldsymbol{v}),$$

and for any $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ and $v \in X$, it holds

$$(\mathcal{D}_X \boldsymbol{u}, v) = (\nabla \cdot \boldsymbol{u}, v).$$

Corollary 16. *For any $u \in X$ and $\boldsymbol{v} \in \boldsymbol{X}$ such that $\int_\Gamma u(\boldsymbol{n} \cdot \boldsymbol{v})\, \mathrm{d}\Gamma = 0$, it holds*

$$(\mathcal{D}_{\boldsymbol{X}} u, \boldsymbol{v}) = -(u, \mathcal{D}_X \boldsymbol{v}). \tag{43}$$

For any $u \in X$ and $v \in X$ such that $\int_\Gamma (\boldsymbol{n} \cdot \mathcal{D}_{\boldsymbol{X}} u) v\, \mathrm{d}\Gamma = \int_\Gamma u(\boldsymbol{n} \cdot \mathcal{D}_{\boldsymbol{X}} v)\, \mathrm{d}\Gamma = 0$, it holds

$$(\mathcal{D}_X \mathcal{D}_{\boldsymbol{X}} u, v) = (u, \mathcal{D}_X \mathcal{D}_{\boldsymbol{X}} v). \tag{44}$$

For any $\boldsymbol{u} \in \boldsymbol{X}$ and $\boldsymbol{v} \in \boldsymbol{X}$ such that $\int_\Gamma (\mathcal{D}_X \boldsymbol{u})(\boldsymbol{n} \cdot \boldsymbol{v})\, \mathrm{d}\Gamma = \int_\Gamma (\boldsymbol{n} \cdot \boldsymbol{u})(\mathcal{D}_X \boldsymbol{v})\, \mathrm{d}\Gamma = 0$, it holds

$$(\mathcal{D}_X \mathcal{D}_{\boldsymbol{X}} u, v) = (u, \mathcal{D}_X \mathcal{D}_{\boldsymbol{X}} v). \tag{45}$$

Proof. Eq. (43) directly follows from the Green theorem (8). We immediately obtain (44) and (45) from (43). □

We can also define operators corresponding to $\mathcal{P}_{X(Y)}$ and $\mathcal{D}_{X(Y)}$, but we here omit them.

4.2 Application to the 2D-SH equation

We derive a dissipative scheme for the 2D-SH equation. We assume the periodic boundary conditions for simplicity. We first show a formal weak form and fully discrete scheme, and then show numerical results.

We start with the following formal weak form. Find u and $\boldsymbol{q}$, such that, for all v_1 and $\boldsymbol{v}_2$,

$$\begin{aligned}
(u_t, v_1) &= -\left(\frac{\partial G}{\partial u}, v_1\right) - \left(\frac{\partial G}{\partial \nabla u}, \nabla v_1\right) + (\boldsymbol{q}, \nabla v_1), \\
(\boldsymbol{q}, \boldsymbol{v}_2) &= -\left(\frac{\partial G}{\partial \triangle u}, \nabla \cdot \boldsymbol{v}_2\right).
\end{aligned}$$

Since this formal weak form completely corresponds to Formal weak form 1, the subsequent procedures are straightforward. Thus we here show the fully discrete scheme only. Suppose

$u^{(0)}$ is given in X_{p}. Find $u^{(n+1)} \in X_{\mathrm{p}}$ and $\boldsymbol{q}^{(n+\frac{1}{2})} \in \boldsymbol{X}_{\mathrm{p}}$ $(n = 0, 1, \dots)$ such that, for any $v_1 \in X_{\mathrm{p}}$ and $\boldsymbol{v}_2 \in \boldsymbol{X}_{\mathrm{p}}$,

$$\left(\frac{u^{(n+1)} - u^{(n)}}{\Delta t}, v_1\right) = -\left(\frac{\partial G}{\partial(u^{(n+1)}, u^{(n)})}, v_1\right)$$
$$- \left(\mathcal{P}_{\boldsymbol{X}_{\mathrm{p}}} \frac{\partial G}{\partial(\mathcal{D}_{\boldsymbol{X}_{\mathrm{p}}} u^{(n+1)}, \mathcal{D}_{\boldsymbol{X}_{\mathrm{p}}} u^{(n)})}, \nabla v_1\right) + \left(\boldsymbol{q}^{(n+\frac{1}{2})}, \nabla v_1\right), \qquad \text{(46a)}$$

$$\left(\boldsymbol{q}^{(n+\frac{1}{2})}, \boldsymbol{v}_4\right) = -\left(\mathcal{P}_{X_{\mathrm{p}}} \frac{\partial G}{\partial(\mathcal{D}_{X_{\mathrm{p}}} \mathcal{D}_{\boldsymbol{X}_{\mathrm{p}}} u^{(n+1)}, \mathcal{D}_{X_{\mathrm{p}}} \mathcal{D}_{\boldsymbol{X}_{\mathrm{p}}} u^{(n)})}, \nabla \cdot \boldsymbol{v}_2\right), \qquad \text{(46b)}$$

where

$$\frac{\partial G}{\partial(u^{(n+1)}, u^{(n)})} = \frac{((u^{(n+1)})^2 + (u^{(n)})^2)(u^{(n+1)} + u^{(n)})}{4} - (u^{(n+1)} + u^{(n)}),$$
$$\frac{\partial G}{\partial(\mathcal{D}_{\boldsymbol{X}_{\mathrm{p}}} u^{(n+1)}, \mathcal{D}_{\boldsymbol{X}_{\mathrm{p}}} u^{(n)})} = -(\mathcal{D}_{\boldsymbol{X}_{\mathrm{p}}} u^{(n+1)} + \mathcal{D}_{\boldsymbol{X}_{\mathrm{p}}} u^{(n)}),$$
$$\frac{\partial G}{\partial(\mathcal{D}_{X_{\mathrm{p}}} \mathcal{D}_{\boldsymbol{X}_{\mathrm{p}}} u^{(n+1)}, \mathcal{D}_{X_{\mathrm{p}}} \mathcal{D}_{\boldsymbol{X}_{\mathrm{p}}} u^{(n)})} = \frac{\mathcal{D}_{X_{\mathrm{p}}} \mathcal{D}_{\boldsymbol{X}_{\mathrm{p}}} u^{(n+1)} + \mathcal{D}_{X_{\mathrm{p}}} \mathcal{D}_{\boldsymbol{X}_{\mathrm{p}}} u^{(n)}}{2},$$

which correspond to $\partial G / \partial u = u^3 - 2u$, $\partial G / \partial \nabla u = -2\nabla u$, and $\partial G / \partial \triangle u = \triangle u$.

Theorem 17. *The solution of Scheme* (46a) *and* (46b) *satisfies*

$$\frac{1}{\Delta t} \int_\Omega \Big(G(u^{(n+1)}, \mathcal{D}_{\boldsymbol{X}_{\mathrm{p}}} u^{(n+1)}, \mathcal{D}_{X_{\mathrm{p}}} \mathcal{D}_{\boldsymbol{X}_{\mathrm{p}}} u^{(n+1)})$$
$$- G(u^{(n)}, \mathcal{D}_{\boldsymbol{X}_{\mathrm{p}}} u^{(n)}, \mathcal{D}_{X_{\mathrm{p}}} \mathcal{D}_{\boldsymbol{X}_{\mathrm{p}}} u^{(n)}) \Big) \, \mathrm{d}x \le 0. \qquad n = 0, 1, 2, \dots.$$

Finally we show the numerical results. Suppose that $\Omega = (0, L_x) \times (0, L_y)$. For computation, the parameters were set to $L_x = L_y = 10$, $t \in [0, 10]$, $\Delta t = 0.01$, and the initial value was set to $u(0, x) = 0.5 + 0.5 \sin(2\pi x / L_x) \sin(2\pi y / L_y)$. We employed the P1 elements. The mesh is shown in Fig. 4. Fig. 5 shows the numerical solutions, which seem good, and Fig. 6 shows the evolution of the energy which well agree with Theorem 17.

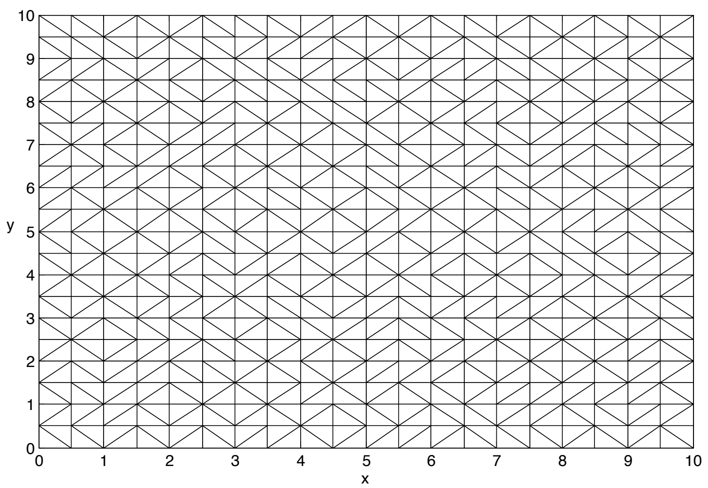

Figure 4: The mesh used for computation.

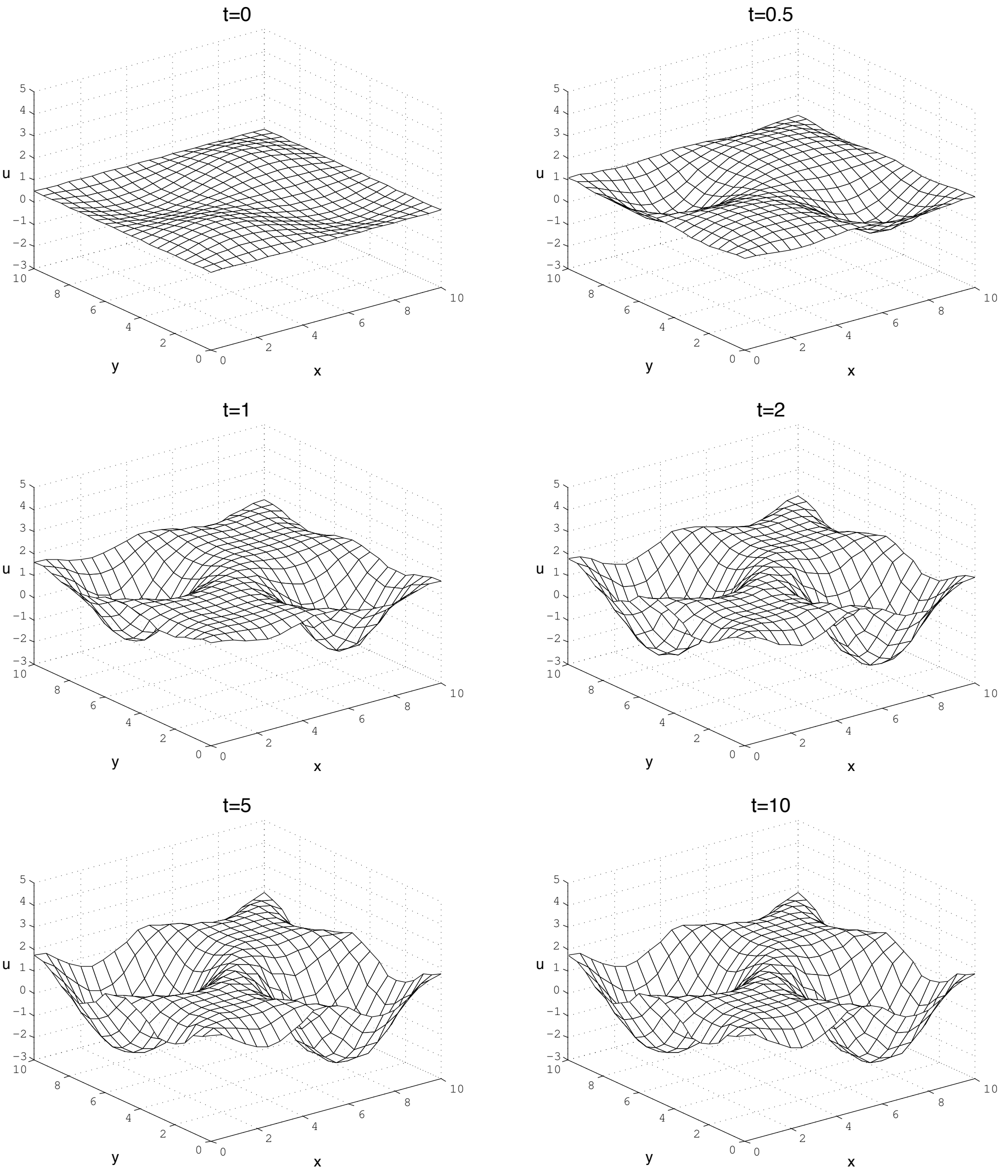

Figure 5: Numerical solutions for the 2D-SH equation (42) at $t = 0$, 0.5, 1, 2, 5 and 10.

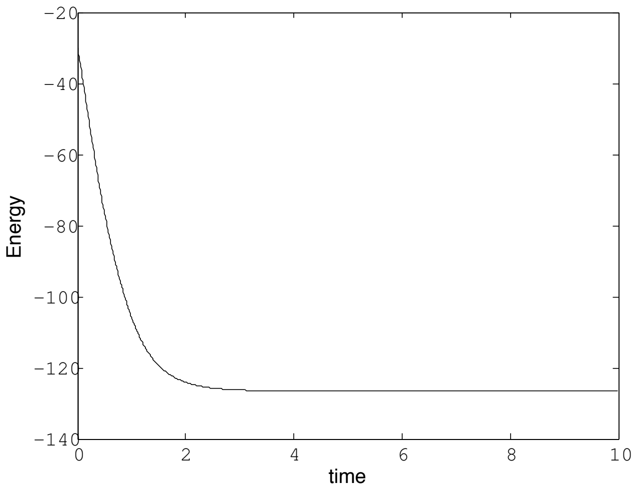

Figure 6: Evolution of the energy.

5 Concluding remarks

In this paper, we constructed a general framework for finding energy dissipative/conservative H^1-Galerkin schemes and their underlying H^1-weak forms for evolution equations. In the proposed method, the intended schemes and their underlying H^1-weak forms can be automatically derived via the formal weak form, which is made possible by the use of the L^2 projection operators.

Our future works include the followings.

- There are many dissipative or conservative PDEs for which the proposed method can be applied. Such PDEs include, for example, the modified Camassa–Holm equation and the phase-field-crystal equation. Numerical studies of such PDEs will be presented in the near future elsewhere.
- In this paper, we did not discuss mathematical analyses such as unique solvability (or existence) and convergence analysis of the derived schemes. Such analyses should be studied for individual PDE and we are now working on this topic as well.
- Obviously PDEs of the form (3) or (4) can be also regarded as Type 2 PDEs. But the proposed method sometimes finds different dissipative/conservative schemes and their underlying H^1-weak forms from the ones in [17]. This means that the new method is not a superset of the original discrete partial derivative method (DPDM). It would deserve trying to construct a more general framework which contains both the proposed method and the original DPDM. Moreover, this fact also indicates that dissipative/conservative H^1-weak forms are not always unique. Comparison of such formulations from numerical and theoretical points of view would be interesting.

Acknowledgements

The authors are grateful to Takaharu Yaguchi and Norikazu Saito for many useful discussions. This work was partly supported by JSPS KAKENHI Grant Numbers 25287030, 23560063. It was also partly supported by the Aihara Project, the FIRST program from JSPS, initiated by CSTP. The first author is supported by the Research Fellowship of the Japan Society for the Promotion of Science for Young Scientists.

References

[1] K. Atkinson and W. Han. *Theoretical Numerical Analysis: A Functional Analysis Framework (3rd ed.).* Springer-Verlag, Hidelberg, 2009.

[2] R. Camassa and D. D. Holm. An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.*, 71:1661–1664, 1993.

[3] R. Camassa, D. D. Holm, and J. M. Hyman. A new integrable shallow water equation. *Adv. Appl. Mech.*, 31, 1994.

[4] E. Celledoni, V. Grimm, R. I. McLachlan, D. I.McLaren, D. OfNeale, B. Owren, and G. R. W. Quispel. Preserving energy resp. dissipation in numerical PDEs using the "Average Vector Field" method. *J. Comput. Phys.*, 231:6770–6789, 2012.

[5] A. Constantin and J. Escher. Global existence and blow-up for a shallow water equation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. XXVI*, 25:303–328, 1998.

[6] A. Constantin and W. A. Strauss. Stability of peakons. *Comm. Pure Appl. Math.*, 53:603–610, 2000.

[7] M. Dahlby and B. Owren. A general framework for deriving integral preserving numerical methods for PDEs. *SIAM J. Sci. Comput.*, 33:2318–2340, 2011.

[8] A. Degasperis and M. Procesi. *Asymptotic Integrability in Symmetry and Perturbation theory.* World Scientific, Singapore, 1999, 23–37.

[9] Q. Du and R. A. Nicolaides. Numerical analysis of a continuum model of phase transition. *SIAM J. Numer. Anal.*, 28:1310–1322, 1991.

[10] B. Fuchssteiner and A. S. Fokas. Symplectic structures, their bäcklund transformations and hereditary symmetries. *Physica D*, 4:47–66, 1981.

[11] D. Furihata. Finite difference schemes for $\frac{\partial u}{\partial t} = \left(\frac{\partial}{\partial x}\right)^{\alpha} \frac{\delta G}{\delta u}$ that inherit energy conservation or dissipation property. *J. Comput. Phys.*, 156:181–205, 1999.

[12] D. Furihata and T. Matsuo. *Discrete Variational Derivative Method: A Structure-Preserving Numerical Method for Partial Differential Equations.* Chapman & Hall/CRC, Boca Raton, 2011.

[13] O. Gonzalez. Time integration and discrete Hamiltonian systems. *J. Nonlinear Sci.*, 6:449–467, 1996.

[14] T. Itoh and K. Abe. Hamiltonian-conserving discrete canonical equations based on variational difference quotients. *J. Comput. Phys.*, 76:85–102, 1988.

[15] T. Kawahara. Oscillatory solitary waves in dispersive media. *J. Phys. Soc. Japan*, 33:260–264, 1972.

[16] H. Kuramae. An alternating discrete variational derivative method and its applications (in Japanese). Master's thesis, University of Tokyo, 2012.

[17] T. Matsuo. Dissipative/conservative Galerkin method using discrete partial derivatives for nonlinear evolution equations. *J. Comput. Appl. Math.*, 218:506–521, 2008.

[18] T. Matsuo. A Hamiltonian-conserving Galerkin scheme for the Camassa–Holm equation. *J. Comput. Appl. Math.*, 234:1258–1266, 2010.

[19] T. Matsuo and D. Furihata. Dissipative or conservative finite-difference schemes for complex-valued monlinear partial differential equations. *J. Comput. Phys.*, 171:425–447, 2001.

[20] T. Matsuo and H. Yamaguchi. An energy-conserving Galerkin scheme for a class of nonlinear dispersive equations. *J. Comput. Phys.*, 228:4346–4358, 2009.

[21] Y. Miyatake and T. Matsuo. Energy-preserving H^1-Galerkin schemes for shallow water wave equations with peakon solutions. *Phys. Lett. A*, 376:2633–2639, 2012.

[22] Y. Miyatake, T. Yaguchi, and T. Matsuo. Numerical integration of the Ostrovsky equation based on its geometric structures. *J. Comput. Phys.*, 231:4542–4559, 2012.

[23] G. R. W. Quispel and D. I. McLaren. A new class of energy-preserving numerical integration methods. *J. Phys. A*, 41:045206, 2008.

[24] J. Swift and P. C. Hohenberg. Hydrodynamic fluctuations at the convective instability. *Phys. Rev. A*, 15:319–328, 1977.

[25] A. M. Wazwaz. New solitary wave solutions to the Kuramoto–Sivashinsky and the Kawahara equations. *Appl. Math. Comput.*, 182:1642–1650, 2006.

[26] T. Yaguchi, T. Matsuo, and M. Sugihara. An extension of the discrete variational method to nonuniform grids. *J. Comput. Phys.*, 229:4382–4423, 2010.

[27] T. Yaguchi, T. Matsuo, and M. Sugihara. The discrete variational derivative method based on discrete differential forms. *J. Comput. Phys.*, 231:3963–3986, 2012.