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Abstract

Many of practical design specifications are provided by finite frequency properties described by inequalities over restricted finite frequency intervals. In this paper, we consider a characterization of the finite frequency domain inequalities (FFDIs) for n-dimensional systems from a view point of dissipation theory using quadratic differential forms (QDFs), which are useful algebraic tools for the dissipation theory based on the behavioral approach. The QDFs allow us to derive a clear characterization of the FFDIs using some inequality analogous to dissipation inequality with a compensating rate and an inequality of an integral of the supply rate with a matrix integral quadratic constraint as a main result. This characterization leads to a physical interpretation in terms of dissipativity for subbehavior with some rate constraints. We also show how we resolve a difficulty on the expression of a compensating rate peculiar to n-dimensional systems. The results of this paper can be regarded as a finite frequency version for the characterizations of frequency properties over the entire frequency domain due to Pillai and Willems (2002).

1 Introduction

Many of practical design specifications are provided by sets of finite frequency properties which are expressed as inequalities over restricted finite frequency intervals. The properties play important role for dynamical system design including control and signal processing. In n-dimensional systems [1][6][10][14] [1], the finite frequency properties appear in many context such as filter design [3][4][15], image processing [2][20], and so on including Fornasini-Marchesini [7][8] and Roessor [20] (discrete-time) state-space systems.

Dissipativity is one of the most important properties which captures a dynamical system from the view point of energy and power interactions with its external environment [21][22][23]. It is well-known that the dissipativity can be equivalently transformed to the matrix inequality over the imaginary axis [21]. Hence, it may be important to articulate the relationship between finite frequency properties and dissipativity. This claim

[1] We call a system that depend on n independent variables ($n \geq 2$) as an *n-dimensional system*.

can also be validated by the fact that a stability condition for a feedback system is given in terms of integrals over entire frequencies, called integral quadratic constraint [16].

A quadratic differential form (QDF) is a useful algebraic tool in dissipation theory based on the behavioral approach [24][18], because it has a one-to-one correspondence to a two-variable polynomial matrix. The behavioral approach is a theoretic framework which does not assume an input-output relationship, a particular representation and causality in advance. Since an n-dimensional system has an infinite-dimensional state space and no causality in the space coordinates, we can naturally analyze and design an n-dimensional system based on the approach. Using QDFs, Willems and Trentelman [25] proved that a dissipativity of a behavior is equivalent to a certain frequency domain inequalities on the entire frequency range. This equivalence is characterized by the dissipation inequality in terms of QDFs. This characterization was extended to n-dimensional systems by Pillai and Willems [19]. On the other hand, for a characterization of finite frequency properties, the authors of this paper clarified that the properties are equivalent to a dissipativity of some rate constrained subbehavior in time domain based on QDFs for one-dimensional systems [12]. A key point was to prove the existence of a compensating rate which appears in the inequality corresponding to the dissipation inequality for the finite frequency case. Since the most of physical systems are described by partial differential-algebraic equations at the beginning of analysis and synthesis, we have a great interest in how the finite frequency properties can be expressed from a theoretical viewpoint of dissipativity in n-dimensional systems. However, there has never been derived a characterization for the n-dimensional case. Hence, we conceived to derive a characterization of the properties using from this viewpoint in n-dimensional systems.

In this paper, motivated by the observation in the above paragraphs, we will characterize the finite frequency properties of n-dimensional systems based on QDFs. As a main result, we derive a characterization of the properties using some inequality and an integral of a supplied rate. We also characterize the properties in terms of dissipativity of some subbehavior of the original behavior. These characterizations are obtained by generalizing the idea of [12] for the one-dimensional system to the n-dimensional case. Although a nonnegative property of a compensating rate played an important role in [12], we find that a straightforward extension of [12] is not easy in n-dimensional systems, since the expression of a compensating rate satisfying the property is not clear in this case. Hence, we need a further theoretical consideration for the characterization in the generalization. We show how this theoretical difficulty can be resolved in n-dimensional systems. The results of this paper allow us to understand the significance of the properties directly. Figure 1 illustrates a series of these results comparing with the previous works [25][19][12].

The organization of this paper is as follows. In Section 2, we review some basic definitions and results on the behavioral system theory and QDFs. We give the problem formulation of finite frequency characterization for n-dimensional systems in Section 3. In Section 4, we derive a characterization of the finite frequency properties using QDFs as a main result. A numerical example demonstrates our characterization in Section 5.

We adopt the following notations in this paper.

The set of $p \times q$ real and complex matrices are denoted by $\mathbb{R}^{p\times q}$ and $\mathbb{C}^{p\times q}$, respectively. We also denote $\mathbb{S}^{q\times q}$ and $\mathbb{H}^{q\times q}$ as the set of $q \times q$ real symmetric and Hermitian matrices, respectively. The set of $p \times q$ real coefficient n-variable polynomial matrices are defined by $\mathbb{R}^{p\times q}[\xi]$ [2]. The set of $p \times q$ complex coefficient n- and

[2] We denote the indeterminates $\xi := (\xi_1, \cdots, \xi_n)$ and $\zeta := (\zeta_1, \cdots, \zeta_n)$, $\eta := (\eta_1, \cdots, \eta_n)$ when we consider n- and $2n$-variable polynomial matrices, respectively.

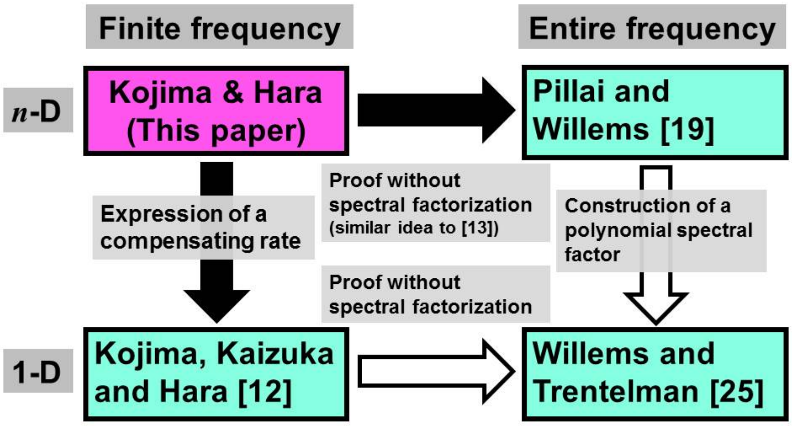

Figure 1: The relationship between our results and the previous works. The notations"1-D" and "n-D" means that one-dimensional and n-dimensional systems, respectively. Contributions of this paper are illustrated with black arrows.

$2n$-variable polynomial matrices are denoted by $\mathbb{C}^{p\times q}[\xi]$ and $\mathbb{C}^{p\times q}[\zeta,\eta]$, respectively. We also denote $\mathbb{H}^{q\times q}[\zeta,\eta]$ as the set of Hermitian $2n$-variable polynomial matrices, i.e. $\Phi(\zeta,\eta)=\Phi(\bar{\eta},\bar{\zeta})^*$ for any $\Phi\in\mathbb{H}^{q\times q}[\zeta,\eta]$.

We denote $\mathbb{W}^{\mathbb{T}}$ as the set of maps from $\mathbb{T}$ to $\mathbb{W}$. Define $\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{V})$ as the set of infinitely differentiable functions from $\mathbb{R}^n$ to the vector space $\mathbb{V}$, and denote $\mathcal{D}^\infty(\mathbb{R}^n,\mathbb{V})$ as the set $\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{V})$ with compact support.

Finally, the row dimension of the matrix A is denoted by $\mathrm{rowdim}(A)$. We define the rank of polynomial matrix $R(\xi)$ and constant matrix $R(\lambda)$ are denoted by $\mathrm{rank}R$ and $\mathrm{rank}R(\lambda)$, respectively. We denote the matrix $\begin{bmatrix}A_1^\top & A_2^\top & \cdots & A_n^\top\end{bmatrix}^\top$ by $\mathrm{col}\,(A_1,A_2,\cdots,A_n)$. We also define $\mathrm{He}\,(A):=\frac{1}{2}\,(A+A^*)$.

2 Preliminaries

In this section, we review the basic definitions and results from the behavioral system and dissipation theory for n-dimensional behaviors from the references [18][19].

2.1 Linear Continuous-time Systems

In the behavioral system theory, a dynamical system is defined as a triple $\Sigma=(\mathbb{T},\mathbb{W},\mathfrak{B})$, where $\mathbb{T}$ is the set of independent variables, and $\mathbb{W}$ is the signal space in which the trajectories take their values on. The behavior $\mathfrak{B}\subseteq\mathbb{W}^{\mathbb{T}}$ is the set of all possible trajectories.

In this paper, we consider an *n-dimensional linear time-invariant continuous-time system* $\Sigma=(\mathbb{R}^n,\mathbb{C}^q,\mathfrak{B})$ with the independent variable $x:=(x_1,\cdots,x_n)\in\mathbb{R}^n$. Such a Σ is typically represented by the linear partial differential-algebraic equation expressed as

$$\sum_{i_1=0}^{L_1}\cdots\sum_{i_n=0}^{L_n}R_{i_1,\cdots,i_n}\frac{\partial^{i_1}}{\partial x_1^{i_1}}\cdots\frac{\partial^{i_n}}{\partial x_n^{i_n}}w(x_1,\cdots,x_n)=0, \tag{1}$$

where $R_{i_1,\cdots,i_n}\in\mathbb{C}^{p\times q}$ $(i_k=0,1,\cdots,L_k;k=1,\cdots,n)$ and $L_k\geq 0$. The variable $w\in\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{C}^q)$ is called the *manifest variable*.

For a simplicity of the description, we introduce the multi-index notation [19][17] by $i := (i_1, \cdots, i_n) \in \mathbb{Z}^n$ and $\xi := (\xi_1, \cdots, \xi_n)$, where i_k $(k = 1, \cdots, n)$ is a nonnegative integer. By using this notation, we define the n-variable polynomial matrix $R \in \mathbb{C}^{p\times q}[\xi]$ by

$$R(\xi) := \sum_{i=0}^{L} R_i \xi^i = \sum_{i_1=0}^{L_1} \cdots \sum_{i_n=0}^{L_n} R_{i_1,\cdots,i_n} \xi_1^{i_1} \cdots \xi_n^{i_n}, \tag{2}$$

where ξ^i is defined by $\xi^i := (\xi^{i_1}, \cdots, \xi^{i_n})$. For the multi-index $i = (i_1, \cdots, i_n) \in \mathbb{Z}^n$ $(i_k \in \mathbb{Z};\ k = 1, \cdots, n)$, we define the corresponding partial differential operator $\frac{\mathrm{d}^i}{\mathrm{d}x^i}$ as

$$\frac{\mathrm{d}^i}{\mathrm{d}x^i} := \frac{\partial^{i_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_n}}{\partial x_n^{i_n}}.$$

Then, (1) is expressed as

$$R\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) w = \sum_{i=1}^{L} R_i \frac{\mathrm{d}^i}{\mathrm{d}x^i} w = 0 \tag{3}$$

in short hand, where $\frac{\mathrm{d}}{\mathrm{d}x}$ denotes

$$\frac{\mathrm{d}}{\mathrm{d}x} := \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right).$$

Then, the behavior $\mathfrak{B} \subset \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}^q)$ is defined as the kernel of the operator $R\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)$ given by

$$\mathfrak{B} = \left\{ w \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}^q) \,\middle|\, R\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) w = 0 \right\}. \tag{4}$$

For this reason, (3) is called the *kernel representation of* $\mathfrak{B}$. The representation (3) is said to be the *minimal representation of* $\mathfrak{B}$ if $\mathrm{rowdim} R \leq \mathrm{rowdim} R'$ holds for any other $R' \in \mathbb{R}^{p'\times q}[\xi]$ which induces a kernel representation of $\mathfrak{B}$.

The behavior $\mathfrak{B} \subset \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}^q)$ is called *controllable* if for any $w_1, w_2 \in \mathfrak{B}$ and $X_1, X_2 \subset \mathbb{R}^n$ with a disjoint closure, there exists a $w \in \mathfrak{B}$ such that

$$w|_{X_1} = w_1|_{X_1} \text{ and } w|_{X_2} = w_2|_{X_2},$$

where $w|_X$ denotes the restriction of the trajectory $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^q)$ to the domain $X \subset \mathbb{R}^n$. The behavior $\mathfrak{B}$ is controllable if and only if $\mathrm{rank} R(\lambda)$ is constant for all $\lambda \in \mathbb{C}^n$ [18].

Whenever $\mathfrak{B}$ is controllable, it can be described by an *image representation*

$$w = M\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) \ell, \tag{5}$$

where $M \in \mathbb{R}^{q\times m}[\xi]$ and the variable $\ell \in \mathcal{C}^\infty(\mathbb{R}^q, \mathbb{C}^m)$ is called the *latent variable*. Then, $\mathfrak{B}$ is given by

$$\mathfrak{B} = \{ w \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}^q) \mid \exists\, \ell \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}^m) \text{ s.t. (5)} \}.$$

When $\mathfrak{B}$ is represented by an image representation, $\mathfrak{B}$ is called *observable* if $w = M\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) \ell = 0$ implies $\ell = 0$. The representation (5) is observable if and only if the constant matrix $M(\lambda)$ is of full column rank for all $\lambda \in \mathbb{C}^n$ [18].

As we have mentioned in the above, every controllable behavior admits an image representation. However, for n-dimensional behaviors, it should be noted that every controllable behavior does not necessarily have an observable image representation contrary to the one-dimensional case [18].

2.2 Quadratic Differential Forms

We review the definition and basic results of QDFs [25][19] for n-dimensional behaviors, which play a central role in this paper.

We consider a $2n$-variable polynomial matrix in $\mathbb{C}^{q_1\times q_2}[\zeta,\eta]$ and, similarly to Section 2.1, we use the multi-index notation [19][17] by $i := (i_1,\cdots,i_n) \in \mathbb{Z}^n$, $j := (j_1,\cdots,j_n) \in \mathbb{Z}^n$ and $\zeta := (\zeta_1,\cdots,\zeta_n)$, $\eta := (\eta_1,\cdots,\eta_n)$, where i_k and j_k $(k = 1,\cdots,n)$ are nonnegative integers. We also denote $\zeta^i := (\zeta^{i_1},\cdots,\zeta^{i_n})$ and $\eta^i := (\eta^{j_1},\cdots,\eta^{j_n})$. We can describe any matrix in $\mathbb{C}^{q_1\times q_2}[\zeta,\eta]$ as

$$\Phi(\zeta,\eta) = \sum_{i\in\mathbb{Z}^n}\sum_{j\in\mathbb{Z}^n} \Phi_{i,j}\zeta^i\eta^j,\ \Phi_{i,j}\in\mathbb{C}^{q_1\times q_2}, \tag{6}$$

where the above sum ranges over all nonnegative multi-indices $i\in\mathbb{Z}^n$, $j\in\mathbb{Z}^n$, and is assumed to be finite. The degree of $\Phi(\zeta,\eta)$ with respect to ζ_k and η_k $(k = 1,\cdots,n)$ are defined as

$$\deg_{\zeta_k}\Phi := \max_{(i,j)\in\mathcal{I}} i_k \text{ and } \deg_{\eta_k}\Phi := \max_{(i,j)\in\mathcal{I}} j_k,$$

respectively, where $\mathcal{I}\subset\mathbb{Z}^{2n}$ is the set defined by

$$\mathcal{I} := \left\{ (i,j)\in\mathbb{Z}^{2n} \;\middle|\; \Phi_{i,j}\neq 0_{q_1\times q_2}\right\}.$$

For $\Phi(\zeta,\eta)$ in (6), we define the mapping

$$\begin{aligned}&\partial : \mathbb{C}^{q_1\times q_2}[\zeta,\eta]\to\mathbb{C}^{q_1\times q_2}[\xi],\\ &\partial\Phi(\xi) := \Phi(-\xi,\xi).\end{aligned}$$

We define a bilinear differential form for n-dimensional behaviors in the following [19].

A *bilinear differential form (BDF)* is induced by the $2n$-variable polynomial matrix $\Phi(\zeta,\eta)$ in (6). The BDF is represented by

$$\begin{aligned}&\mathsf{L}_\Phi : \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{C}^{q_1})\times\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{C}^{q_2})\to\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}),\\ &\mathsf{L}_\Phi(\ell_1,\ell_2) := \sum_{i=0}^{K_1}\sum_{j=0}^{K_2}\left(\frac{\mathrm{d}^i\ell_1}{\mathrm{d}x^i}\right)^*\Phi_{i,j}\frac{\mathrm{d}^j\ell_2}{\mathrm{d}x^j},\end{aligned} \tag{7}$$

where $K_i := (K_{i,1},\cdots,K_{i,n})\in\mathbb{Z}^n$ $(i = 1,2)$, $K_{1,k} := \deg_{\zeta_k}\Phi$ and $K_{2,k} := \deg_{\eta_k}\Phi$ $(k = 1,\cdots,n)$. The definition (7) means that ζ and η correspond to the partial differentiations on ℓ^* and ℓ, respectively.

We call $\Phi(\zeta,\eta)$ *Hermitian* if $\Phi(\bar\zeta,\bar\eta)^* = \Phi(\eta,\zeta)$ holds, which implies $q_1 = q_2 =: q$ and $K_1 = K_2 =: K$. Then, $\Phi(\zeta,\eta)$ is expressed as

$$\Phi(\zeta,\eta) = \sum_{i=0}^{K}\sum_{j=0}^{K}\Phi_{i,j}\zeta^i\eta^j. \tag{8}$$

In this case, $\Phi(\zeta,\eta)$ in (8) induces a *quadratic differential form (QDF)* represented by

$$\begin{aligned}&\mathsf{Q}_\Phi : \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{C}^q)\to\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}),\\ &\mathsf{Q}_\Phi(\ell) := \sum_{i=0}^{K}\sum_{j=0}^{K}\left(\frac{\mathrm{d}^i\ell}{\mathrm{d}x^i}\right)^*\Phi_{i,j}\frac{\mathrm{d}^j\ell}{\mathrm{d}x^j}.\end{aligned}$$

Consider the $2n$-variable polynomial matrix

$$\Psi(\zeta,\eta) = \begin{bmatrix} \Psi_1(\zeta,\eta) \\ \vdots \\ \Psi_n(\zeta,\eta) \end{bmatrix} \in \mathbb{C}^{nq\times q}[\zeta,\eta],$$

where $\Psi_k \in \mathbb{H}^{q\times q}[\zeta,\eta]$ $(k = 1,\cdots,n)$. This induces a *vector of QDFs (VQDFs)*

$$\mathsf{Q}_\Psi : \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{C}^q) \to \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}^n),$$
$$\mathsf{Q}_\Psi(\ell) := \begin{bmatrix} \mathsf{Q}_{\Psi_1}(\ell) \\ \vdots \\ \mathsf{Q}_{\Psi_n}(\ell) \end{bmatrix}.$$

The *divergence* of the VQDF $\mathsf{Q}_\Psi(\ell)$ is defined by

$$\mathrm{div}\mathsf{Q}_\Psi(\ell) := \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\mathsf{Q}_{\Psi_k}(\ell).$$

This is also a QDF. Let $\nabla\Psi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ induce $\mathrm{div}\mathsf{Q}_\Psi(\ell)$, i.e. $\mathrm{div}\mathsf{Q}_\Psi(\ell) = \mathsf{Q}_{\nabla\Psi}(\ell)$. Then, it is given by

$$\nabla\Psi(\zeta,\eta) = \sum_{k=1}^{n}(\zeta_k + \eta_k)\Psi_k(\zeta,\eta).$$

2.3 Dissipation Theory

In this section, we review the basic definitions and properties of dissipativity for n-dimensional behaviors using QDFs [19].

We assume that $\mathfrak{B}$ in (4) is controllable in this section. Then, $\mathfrak{B}$ has an observable image representation (5). Let $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ in (8) be given.

We give the definition of dissipativity of a behavior.

Definition 1 [19] Let $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ in (8) be given. Then, a behavior $\mathfrak{B}$ is called *dissipative with respect to the supply rate* $\mathsf{Q}_\Phi(w)$ if the inequality

$$\int_{\mathbb{R}^n} \mathsf{Q}_\Phi(w)\mathrm{d}x \geq 0 \tag{9}$$

holds for all $w \in \mathfrak{B} \cap \mathcal{D}^\infty(\mathbb{R}^n,\mathbb{C}^q)$. □

We may think of $\mathsf{Q}_\Phi(w)$ as the power delivered to the behavior $\mathfrak{B}$. The dissipativity implies that the net flow of energy into the system is nonnegative. This shows the system dissipates energy. Hence, due to this dissipation, the rate of increase of the energy stored inside of the system does not exceed the power supplied to it. This interaction between supply, storage, and dissipation is now formalized in Definition 2 and Proposition 1 below.

We give the definitions of storage function and dissipation rate.

Definition 2 [19] Assume that $\mathfrak{B}$ is controllable. Let $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ be given.

(i) The VQDF $\mathsf{Q}_\Psi(\ell)$ induced by

$$\Psi(\zeta,\eta) = \begin{bmatrix} \Psi_1(\zeta,\eta) \\ \vdots \\ \Psi_n(\zeta,\eta) \end{bmatrix} \in \mathbb{C}^{nm\times m}[\zeta,\eta],\ \Psi_k \in \mathbb{H}^{m\times m}[\zeta,\eta]\ (k=1,\cdots,n) \tag{10}$$

is called a *storage function for* $\mathfrak{B}$ *with respect to the supply rate* $\mathsf{Q}_\Phi(w)$ if

$$\mathrm{div}\mathsf{Q}_\Psi(\ell) \le \mathsf{Q}_\Phi(w) \tag{11}$$

holds for all $w \in \mathfrak{B}$ with the image representation (5). We call (11) the *dissipation inequality*.

(ii) The QDF $\mathsf{Q}_\Delta(\ell)$ induced by $\Delta \in \mathbb{H}^{m\times m}[\zeta,\eta]$ is called a *dissipation rate for* $\mathsf{Q}_\Phi(w)$ if

$$\mathsf{Q}_\Delta(\ell) \ge 0,\ \forall\, w \in \mathfrak{B}$$

and

$$\int_{\mathbb{R}^n} \mathsf{Q}_\Phi(w)\mathrm{d}x = \int_{\mathbb{R}^n} \mathsf{Q}_\Delta(\ell)\mathrm{d}x,\ \forall\, w \in \mathfrak{B} \cap \mathcal{D}^\infty(\mathbb{R}^n,\mathbb{C}^q)$$

hold with the image representation (5).

There is a one-to-one relation between a storage function $\mathsf{Q}_\Psi(w)$ and a dissipation rate $\mathsf{Q}_\Delta(w)$ defined by

$$\mathrm{div}\mathsf{Q}_\Psi(w) = \mathsf{Q}_\Phi(w) - \mathsf{Q}_\Delta(w). \tag{12}$$

The equation (12) is called the *dissipation equality*.

The next proposition gives a characterization of dissipativity in terms of a storage function and a dissipation rate.

Proposition 1 [19] *Let* $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ *be given. The following statements (i), (ii) and (iii) are equivalent.*

(i) The behavior $\mathfrak{B}$ *is dissipative with respect to the supply rate* $\mathsf{Q}_\Phi(w)$.

*(ii) There exists a 2*n*-variable polynomial matrix* $\Psi \in \mathbb{C}^{nm\times m}[\zeta,\eta]$ *in* (10) *satisfying the dissipation inequality* (11) *for all* $\ell \in \mathcal{D}^\infty(\mathbb{R}^n,\mathbb{R}^m)$ *with the image representation* (5).

*(iii) There exist 2*n*-variable polynomial matrices* $\Psi \in \mathbb{H}^{nm\times m}[\zeta,\eta]$ *in* (10) *and* $\Delta \in \mathbb{H}^{m\times m}[\zeta,\eta]$ *satisfying the dissipation equality* (12) *and* $Q_\Delta(\ell) \ge 0$ *for all* $\ell \in \mathcal{D}^\infty(\mathbb{R}^n,\mathbb{R}^m)$ *with the image representation* (5).

Remark 1 As we have remarked in Section 2.1, the observability of (5) does not hold for n-dimensional behaviors. This implies that the storage function does not necessarily becomes a function of manifest variable [19]. Hence, the uniqueness of the storage function does not hold, i.e. there will be many possible storage functions.

Remark 2 We give an interpretation of the inequality (24) in terms of flux [19] in this remark. This enables us to further clarify the above physical interpretation.

Suppose that the independent variable $x_1 = t$ represents the time variable and the remaining variables $x_2, \cdots, x_n$ are the space variables. Then, the dissipation equality (12) can be rewritten as

$$\frac{\partial}{\partial t}\mathsf{Q}_{\Psi_1}(\ell) = \mathsf{Q}_{\Phi}(w) - \sum_{i=2}^{n} \frac{\partial}{\partial x_i}\mathsf{Q}_{\Psi_i}(\ell) - \mathsf{Q}_{\Delta}(\ell).$$

The interpretation of the above equality is described as follows. The change in the stored energy $\frac{\partial}{\partial t}\mathsf{Q}_{\Psi_1}(\ell)$ in an infinitesimal volume exactly equals to the difference between the supply rate $\mathsf{Q}_{\Phi}(w)$ into the infinitesimal volume, the energy lost $\sum_{i=2}^{n} \frac{\partial}{\partial x_i}\mathsf{Q}_{\Psi_i}(\ell)$ by the volume, which is called flux, and the dissipation $\mathsf{Q}_{\Delta}(\ell)$ within the volume. Hence, the rate of change of the stored energy does not exceed the power supplied the system due to this dissipation and flux.

In the remainder of this section, we explain how the dissipativity can be equivalently described in the frequency domain.

Suppose that (5) is an image representation of $\mathfrak{B}$. Consider the frequency domain inequality (FDI) expressed as

$$M(j\omega)^* \partial\Phi(j\omega) M(j\omega) \geq 0, \quad \forall\, \omega \in \mathbb{R}^n. \tag{13}$$

The FDI (13) is an interpretation of dissipativity of $\mathfrak{B}$ in entire frequency domain.

Proposition 2 [19] *Suppose that $\mathfrak{B}$ is represented by an image representation* (5)*. Let $\Phi \in \mathbb{H}^{q\times q}[\zeta, \eta]$ in* (8) *be given. Then, the following statements (i) and (ii) are equivalent.*

(i) The behavior $\mathfrak{B}$ is dissipative with respect to the supply rate $\mathsf{Q}_{\Phi}(w)$.

(ii) The FDI (13) *holds for all $\omega \in \mathbb{R}^n$.*

The above proposition shows that (13) is an inequality which interprets dissipativity in the frequency domain.

3 Problem Formulation

We characterize finite frequency properties for a linear time-invariant system $\Sigma = (\mathbb{R}^n, \mathbb{C}^q, \mathfrak{B})$ using QDFs. We give the problem formulation in this section for this purpose.

We consider the behavior $\mathfrak{B}$ typically represented by the kernel representation (3), where $w \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}^q)$ is the manifest variable and $R \in \mathbb{C}^{p\times q}[\xi]$ is the polynomial matrix. Then, $\mathfrak{B}$ is given by (4). We set the following assumption on $\mathfrak{B}$ throughout this paper.

Assumption 1

(i) The behavior $\mathfrak{B}$ in (4) is controllable.

(ii) The kernel representation (3) is minimal.

(iii) An image representation of $\mathfrak{B}$ is described by (5), which is possibly unobservable.

Let $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ in (8) be given. Suppose that this $\Phi(\zeta,\eta)$ induces the supply rate for $\mathfrak{B}$. Let $\omega \in \mathbb{R}^n$ be the frequency vector given by $\omega := (\omega_1, \cdots, \omega_n)$. Define the frequency domain $\Omega \subset \mathbb{R}^n$ as a product of finite intervals by

$$\Omega := \prod_{k=1}^{n} \Omega_k = \Omega_1 \times \cdots \times \Omega_n, \tag{14}$$
$$\Omega_k := \{\omega_k \in \mathbb{R} \mid \tau_k(\omega_k - \varpi_{k,1})(\omega_k - \varpi_{k,2}) \leq 0\} \ (k = 1, \cdots, n),$$

where $\varpi_{k,1}, \varpi_{k,2} \in \mathbb{R}$, $\varpi_{k,1} \leq \varpi_{k,2}$ are given and $\tau_k \in \mathbb{Z}$ is either $+1$ or -1.

The domain Ω can represent the various type of finite frequency domain by the choice of τ_k and $\varpi_{k,1}, \varpi_{k,2} \in \mathbb{R}$. For $\tau_k = +1, \forall\, k = 1, \cdots, n$, Ω becomes the middle frequency domain

$$\Omega_{\rm m} := \prod_{k=1}^{n} \Omega_{{\rm m},k},\ \Omega_{{\rm m},k} := \{\omega_k \in \mathbb{R} \mid \ \varpi_{k,1} \leq \omega_k \leq \varpi_{k,2}\}\ (k = 1, \cdots, n).$$

We can also consider the low frequency domain

$$\Omega_{\rm l} := \prod_{k=1}^{n} \Omega_{{\rm l},k},\ \Omega_{{\rm l},k} := \{\omega_k \in \mathbb{R} \mid \ |\omega_k| \leq \varpi_k\}\ (k = 1, \cdots, n) \tag{15}$$

by putting $\varpi_{k,1} = -\varpi_k$ and $\varpi_{k,2} = \varpi_k$ for $k = 1, \cdots, n$, where $\varpi := (\varpi_1, \cdots, \varpi_n) \in \mathbb{R}^n$ is a given vector satisfying

$$\varpi_k \geq 0,\ \forall\, k = 1, \cdots, n. \tag{16}$$

On the other hand, Ω expresses the high frequency domain

$$\Omega_{\rm h} := \prod_{k=1}^{n} \Omega_{{\rm h},k},\ \Omega_{{\rm h},k} := \{\omega_k \in \mathbb{R} \mid \ \omega_k \leq \varpi_{k,1},\ \varpi_{k,2} \leq \omega_k\}\ (k = 1, \cdots, n)$$

for $\tau_k = -1, \forall\, k = 1, \cdots, n$. The domain Ω also becomes the entire real vectors, i.e. $\Omega = \mathbb{R}^n$, by choosing the parameters $\varpi_{k,1} = \varpi_{k,2} = 0$ in addition. Of course, we can represent other frequency domains by choosing the values of τ_k and $\varpi_{k,1}, \varpi_{k,2}$, appropriately.

Consider the finite frequency property described by the following *finite frequency domain inequality (FFDI)*

$$M^*(j\omega)\partial\Phi(j\omega)M(j\omega) \geq 0, \quad \forall\, \omega \in \Omega. \tag{17}$$

Our goal is to find a characterization of the above FFDI using QDFs from the viewpoint of dissipativity introduced in Section 2.3. Especially, we want to give clear answers to the following two questions from the viewpoint of dissipativity under the restriction of the frequency domain to the product of finite intervals, which is formulated mathematically in this section.

Question

(i) What power function newly appears in the dissipation inequality (11), or equivalently dissipation equality (12), for compensating the restriction of the frequency domain? Specifically, what is the different point comparing with the finite frequency characterization for one-dimensional behaviors [12].

(ii) What additional property of $\mathfrak{B}$ to the dissipativity is equivalent to the FFDI (17)?

An interpretation of the FFDI (17) from the behavioral approach is the following. Consider the QDF $\mathsf{Q}_\Phi(w)$ induced by $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ in (8). Fourier transform of $\mathsf{Q}_\Phi(w)$ is computed as

$$\hat{w}(j\omega)^*\partial\Phi(j\omega)\hat{w}(j\omega) = \hat{\ell}(j\omega)^*M(j\omega)^*\partial\Phi(j\omega)M(j\omega)\hat{\ell}(j\omega),$$

where $\hat{w} \in \mathcal{L}_2(\mathbb{C}^n,\mathbb{C}^q)$ and $\hat{\ell} \in \mathcal{L}_2(\mathbb{C}^n,\mathbb{C}^m)$ are Fourier transforms of $w \in \mathfrak{B}\cap\mathcal{D}^\infty(\mathbb{R}^n,\mathbb{C}^q)$ and $\ell \in \mathcal{D}^\infty(\mathbb{R}^n,\mathbb{C}^m)$, respectively. Since ℓ can be taken an arbitrarily trajectory in $\mathcal{D}^\infty(\mathbb{C}^n,\mathbb{C}^m)$, the inequality

$$\hat{w}(j\omega)^*\partial\Phi(j\omega)\hat{w}(j\omega) \geq 0,\ \forall\, w \in \mathfrak{B}\cap\mathcal{D}^\infty(\mathbb{R}^n,\mathbb{C}^m),\ \omega \in \Omega$$

is equivalent to the FFDI (17). We can regard the above inequality imposes a weighted frequency constraint on $w \in \mathfrak{B}$ over the restricted frequency domain Ω. Hence, it expresses the weighted rate limitation on the trajectories contained in $\mathfrak{B}$, although the FFDI (17) is described by using $M(\xi)$.

Remark 3 Chakrabarti et.al. [3] considered the two-dimensional frequency domain $\Omega \subset \mathbb{R}^2$ which is a compact subset of $[0,\infty)\times[0,\infty)$ containing $\mathrm{col}(0,0)$ in the design of a two-dimensional low pass filter. For example, the set is expressed by a linear combination of ϖ_1 and ϖ_2 as

$$\Omega = \left\{\omega \in \mathbb{R}^2 \;\middle|\; \omega_1 \geq 0,\ \ \omega_2 \geq 0 \text{ and } \omega_1+\omega_2 = \varpi_0\right\},$$

where $\varpi_0 \in \mathbb{R}$ is a nonnegative constant. This set cannot be represented in the form of (14), since the domain is described by the sum of the frequency variables. It remains as a future work to characterize the finite frequency properties for such frequency domains.

4 Characterization of Finite Frequency Properties

This section derives a characterization of the finite frequency properties using QDFs for n-dimensional behaviors as a main result. We give a finite frequency property characterization in Section 4.1. A physical interpretation of the characterization is provided in Section 4.2. Finally, we give a characterization of the property in terms of $\mathfrak{B}$-canonical polynomial matrices [13] in Section 4.3.

4.1 Main Theorem

In this subsection, we derive a characterization of the FFDI (17) using QDFs as a main result.

We first point out what issues should be solved in this paper before we provide our main result. In order to generalize the previous characterizations [19][12] to the n-dimensional and finite frequency case, we should examine the following two points from a theoretical view point, which are also illustrated in Figure 1.

- We cannot originally consider a spectral factorization of the polynomial matrix $\partial\Phi(\xi)$ constructed by $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ in (8), which played an important role to construct a dissipation rate in the characterization of [19] [3].

[3] In n-dimensional behaviors, there exists a problem that a spectral factor $F(\xi)$ does not always becomes a polynomial matrix in the spectral factorization $M(\xi)^\sim\partial\Phi(\xi)M(\xi) = F(\xi)^\sim F(\xi)$, i.e. $F(\xi)$ can be a rational function matrix. Pillai and Willems [19] have solved the problem by developing a constructive proof for the existence of a polynomial spectral factor. However, since we avoid a spectral factorization based on [12] in this paper, we need not to deal with the problem.

- It is not clear how a compensating rate can be expressed in the n-dimensional case. In [12], the characterization was derived by using a property that a compensating rate is induced by a polynomial matrix which is nonnegative definite in the (finite) frequency domain.

We can resolve the first point on the spectral factorization by naturally generalizing the idea of [12] to the n-dimensional case. See the proof of Lemma B.1 (i)⇒(ii) for the detail. Thus, it can be the main focus to tackle the second point on a compensating rate. We explain it in detail after we show the main result (Theorem 1) of this paper. This relates to an answer to the latter part of Question (i).

For the purpose stated in the above paragraph, we introduce some notions to construct a compensating rate. Define $\varpi_{k,-}, \varpi_{k,+} \in \mathbb{R}$ $(k = 1, \cdots, n)$ by

$$\varpi_{k,-} := \frac{\varpi_{k,2} - \varpi_{k,1}}{2} \text{ and } \varpi_{k,+} := \frac{\varpi_{k,1} + \varpi_{k,2}}{2}. \tag{18}$$

and the set $\mathcal{G} \subset \mathbb{H}^{m\times m}[\zeta,\eta]$ by

$$\mathcal{G} := \left\{ \Gamma \in \mathbb{H}^{m\times m}[\zeta,\eta] \;\middle|\; \begin{array}{l} \Gamma(\zeta,\eta) := \sum_{k=1}^{n} \chi_k(\zeta,\eta)\Upsilon_k(\zeta,\eta) \\ \text{for some } \Upsilon_k \in \mathbb{H}^{m\times m}[\zeta,\eta] \\ (k = 1,\cdots,n) \text{ such that (21)} \end{array} \right\}, \tag{19}$$

$$\chi_k(\zeta,\eta) = \begin{bmatrix} 1 \\ \zeta_k \end{bmatrix}^* \begin{bmatrix} -\varpi_{k,1}\varpi_{k,2} & -j\varpi_{k,+} \\ j\varpi_{k,+} & -1 \end{bmatrix} \begin{bmatrix} 1 \\ \eta_k \end{bmatrix}, \tag{20}$$

$$\tau_k \mathsf{Q}_{\Upsilon_k}(\ell) \geq 0, \ \forall\, \ell \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}^m), \tag{21}$$

where τ_k is equal to either $+1$ or -1 for $k = 1, \cdots, n$. We see that $\Gamma \in \mathcal{G}$ satisfies the inequality

$$\begin{aligned} \partial\Gamma(j\omega) &= -\sum_{k=1}^{n} \tau_k \left(\omega_k - \varpi_{k,1}\right)\left(\omega_k - \varpi_{k,2}\right) \cdot \tau_k \partial\Upsilon_k(j\omega) \\ &\geq 0, \quad \forall\, \omega \in \Omega. \end{aligned} \tag{22}$$

We have seen from Proposition 1 that the FDI (13) is equivalent to the dissipation inequality (11). Since we consider the case where the FDI (13) is restricted to the domain Ω, we can imagine that an analogous inequality to (11) holds from Proposition 1. This is explained as follows.

Assume that there exist $2n$-variable polynomial matrix

$$\Psi(\zeta,\eta) := \begin{bmatrix} \Psi_1(\zeta,\eta) \\ \vdots \\ \Psi_n(\zeta,\eta) \end{bmatrix}, \ \Psi_k \in \mathbb{H}^{m\times m}[\zeta,\eta] \ (k = 1,\cdots,n) \tag{23}$$

and $\Gamma \in \mathcal{G}$ satisfying the inequality

$$\operatorname{div}\mathsf{Q}_\Psi(\ell) \leq \mathsf{Q}_\Phi(w) - \mathsf{Q}_\Gamma(\ell) \tag{24}$$

for all $w \in \mathfrak{B}$ with the image representation (5). The above inequality corresponds to the dissipation inequality (11) in the finite frequency case. This is equivalent to the existence of $\Delta \in \mathbb{H}^{m\times m}[\zeta,\eta]$ satisfying the $2n$-variable polynomial matrix equation

$$\nabla\Psi(\zeta,\eta) = M(\zeta)^*\Phi(\zeta,\eta)M(\eta) - \Gamma(\zeta,\eta) - \Delta(\zeta,\eta) \tag{25}$$

and $\mathsf{Q}_\Delta(\ell) \geq 0, \forall\, \ell \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}^m)$. Substituting $\zeta = -j\omega$ and $\eta = j\omega$ into (25), we obtain the FFDI

$$\begin{aligned} M(j\omega)^* \partial\Phi(j\omega) M(j\omega) &= \partial\Gamma(j\omega) + \partial\Delta(j\omega) \\ &\geq 0, \quad \forall\, \omega \in \Omega \end{aligned}$$

from (22). The above inequality guarantees that the FFDI (17) holds.

The inequality (24) also gives a necessary condition for the finite frequency property. Thus, we obtain the following main result which equivalently characterizes the property in terms of QDFs. This theorem gives answers to Questions (i) and (ii) in Section 3.

Theorem 1 *Let $\mathfrak{B}$ in* (4) *and $\Phi \in \mathbb{H}^{q\times q}[\zeta, \eta]$ in* (8) *be given. Suppose that Assumption 1 holds. Define Ω by* (14) *and $\mathcal{G}$ by* (19). *Then, the following statements (i), (ii) and (iii) are equivalent.*

(i) The FFDI (17) *holds for all $\omega \in \Omega$.*

(ii) There exist $2n$-variable polynomial matrices $\Psi \in \mathbb{C}^{nm\times m}[\zeta, \eta]$ in (23) *and $\Gamma \in \mathcal{G}$ satisfying the inequality* (24) *with the image representation* (5).

(iii) The inequality

$$\int_{\mathbb{R}^n} \mathsf{Q}_\Phi(w) \mathrm{d}x \geq 0 \tag{26}$$

holds for all $w \in \mathfrak{B}$ with the image representation (5) *and $\ell \in \mathcal{D}^\infty(\mathbb{R}^n, \mathbb{C}^m)$ satisfying*

$$\tau_k \int_{\mathbb{R}^n} \mathrm{He}\left[\left(\frac{\partial z_k}{\partial x_k} - j\varpi_{k,1} z_k\right)\left(\frac{\partial z_k}{\partial x_k} - j\varpi_{k,2} z_k\right)^*\right] \mathrm{d}x \leq 0 \tag{27}$$

for $k = 1, \cdots, n$, where $z_k \in \mathcal{D}^\infty(\mathbb{R}^n, \mathbb{C}^{\{\prod_{l=1}^n (N_{k,l}+1)\}m})$ is defined by

$$z_k := Z_{N_k}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\ell, \tag{28}$$

$$Z_{N_k}(\xi) := \begin{bmatrix} I_m \\ \xi I_m \\ \vdots \\ \xi^{N_k} I_m \end{bmatrix} \in \mathbb{R}^{\{\prod_{l=1}^n (N_{k,l}+1)\}m\times m}[\xi]$$

for some multi-index $N_k := (N_{k,1}, \cdots, N_{k,n}) \in \mathbb{Z}^n$.

Proof See Appendix B.1 for the proof. □

We describe the answers to Questions in Section 3 corresponding to their statements.

Answer 1

(i) In the inequality (24), the QDF $\mathsf{Q}_\Gamma(\ell)$ is called a *compensation rate for $\mathfrak{B}$ with respect to the supply rate $\mathsf{Q}_\Phi(w)$ and the frequency domain Ω.* This QDF is the new function which appears in the dissipation inequality (11). Since $\mathfrak{B}$ is not dissipative with respect to the supply rate $\mathsf{Q}_\Phi(w)$, $\mathsf{Q}_\Gamma(\ell)$ guarantees dissipativity of some rate constrained subbehavior related to $\mathfrak{B}$ and Ω. This claim gives an answer to the former part of

Question (i). See also Answer 2 (i) in Section 4.2 for the further property of this function. It clarifies this point using a dissipation rate for the subbehavior.

In the following, we give an answer to the latter part of Question (i). The authors [12] proved that a compensating rate is induced by a polynomial matrix which is nonnegative definite in the frequency domain. Although we can use the idea due to [12], the straightforward extension of [12] is not clear. For the difficulty, this paper clarifies that a compensating rate $\mathsf{Q}_\Gamma(\ell)$ is induced by a $2n$-variable polynomial matrix $\Gamma(\zeta,\eta)$ expressed as a summation of $2n$-variable polynomial matrices which are nonnegative definite on each frequency domain. This is described in (19) and (22).

The above resolution is completed by proving that a compensating rate satisfies the inequality

$$\begin{aligned}
&v^*\partial\Gamma(j\omega)v \\
&= -\sum_{k=1}^{n}\chi_k(j\omega_k)v^*\partial\Upsilon_k(j\omega)v \\
&= \mathrm{tr}\left(\begin{bmatrix} vv^* & 0 & \cdots & 0 \\ 0 & vv^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & vv^* \end{bmatrix}\right. \\
&\cdot\left.\begin{bmatrix} -\partial\chi_1(j\omega_1)\partial\Upsilon_1(j\omega) & 0 & \cdots & 0 \\ 0 & -\partial\chi_2(j\omega_2)\partial\Upsilon_2(j\omega) & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & -\partial\chi_n(j\omega_n)\partial\Upsilon_n(j\omega) \end{bmatrix}\right) \\
&\geq 0,\ \forall\,\omega\in\Omega
\end{aligned}$$

for some $v\neq 0$ and, if there exists an $\Upsilon_k(\zeta,\eta)$ which does not satisfy the inequality

$$-\partial\chi_k(j\omega_k)\partial\Upsilon_k(j\omega)\geq 0,\ \forall\,\omega\in\Omega,$$

the nonnegativity of the compensating rate is violated in the frequency domain. We omit the detail description due to a space limitation. See the proof of Lemma B.1 (i)⇒(ii) and (ii)⇒(iii) for the detail.

(ii) The statement (iii) in Theorem 1 gives an answer to Question (ii). We see that the matrix integral quadratic constraint (27) is opposed to the inequality (9) as the additional property. See Answer 2 (ii) in Section 4.2 for the further description. This point is explained exactly in terms of dissipativity of some rate constrained subbehavior of $\mathfrak{B}$.

Remark 4 In n-dimensional behaviors, there does not always exist an observable image representation [18]. Hence, the QDFs $\mathsf{Q}_\Psi(\ell)$ and $\mathsf{Q}_\Gamma(\ell)$ do not necessarily become the functions of the manifest variable w contrary to the one-dimensional case [12].

Remark 5 We should remark that $\chi_k(\zeta,\eta)$ $(k=1,\cdots,n)$ in (20) is a real coefficient polynomial if Ω is symmetric about the origin, e.g. the low frequency domain Ω_l in (15). If $M(\xi)$ and $\Phi(\zeta,\eta)$ are all real polynomial matrices, we can restrict $\Psi(\zeta,\eta)$ and $\Gamma(\zeta,\eta)$ in Theorem 1 to real symmetric $2n$-variable polynomial matrices without loss of generality.

Remark 6 Similarly to Remark 2, if we regard the variable $x_1 = t$ and $x_2, \cdots, x_n$ as the time and space variables, respectively, then the inequality (24) is rewritten as

$$\frac{\partial}{\partial t}\mathsf{Q}_{\Psi_1}(\ell) \leq \mathsf{Q}_{\Phi}(w) - \left\{\frac{\partial}{\partial x_2}\mathsf{Q}_{\Psi_2}(\ell) + \cdots + \frac{\partial}{\partial x_n}\mathsf{Q}_{\Psi_n}(\ell)\right\} - \mathsf{Q}_{\Gamma}(\ell).$$

This inequality can be interpreted as follows from the viewpoint of dissipativity. The rate of change of the stored energy $\frac{\partial}{\partial t}\mathsf{Q}_{\Psi_1}(\ell)$ does not exceed the power $\mathsf{Q}_{\Phi}(w)$ supplied the system with energy lost due to flux $\sum_{i=2}^{n}\frac{\partial}{\partial x_i}\mathsf{Q}_{\Psi_i}(\ell)$ and with the compensating power $\mathsf{Q}_{\Gamma}(\ell)$.

4.2 Physical Interpretation

In this subsection, we clarify a physical interpretation of Theorem 1 from the view point of the dissipation theory.

Define the subbehavior $\mathfrak{B}_{\Omega} \subset \mathfrak{B}$ by

$$\mathfrak{B}_{\Omega} := \left\{ w \in \mathcal{D}^{\infty}(\mathbb{R}^n, \mathbb{C}^q) \;\middle|\; \begin{array}{l} w = M\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\ell, \\ \forall\, \ell \in \mathcal{D}^{\infty}(\mathbb{R}^n, \mathbb{C}^q) \text{ s.t. (27)} \end{array} \right\}. \tag{29}$$

The trajectories of $\mathfrak{B}_{\Omega}$ vary in the frequency in Ω. This implies that $\mathfrak{B}_{\Omega}$ is the rate constrained subbehavior of $\mathfrak{B}$. Then, we have the following corollary.

Corollary 1 *Let* $\mathfrak{B}$ *in* (4) *and* $\Phi \in \mathbb{H}^{q\times q}[\zeta, \eta]$ *in* (8) *be given. Suppose that Assumption 1 holds. Define* $\mathcal{G}$ *by* (19) *and define* $\mathfrak{B}_{\Omega}$ *by* (29) *for* Ω *in* (14)*. Then, the following statements (i), (ii) and (iii) are equivalent.*

(i) The FFDI (17) *holds for all* $\omega \in \Omega$*.*

*(ii) There exist 2*n*-variable polynomial matrices* $\Psi \in \mathbb{C}^{nm\times m}[\zeta, \eta]$ *in* (23)*,* $\Delta \in \mathbb{H}^{m\times m}[\zeta, \eta]$ *and* $\Gamma \in \mathcal{G}$ *satisfying*

$$\mathrm{div}\mathsf{Q}_{\Psi}(\ell) = \mathsf{Q}_{\Phi}(w) - \mathsf{Q}_{\Delta+\Gamma}(\ell), \tag{30}$$

$$\mathsf{Q}_{\Delta+\Gamma}(\ell) \geq 0 \tag{31}$$

for all $w \in \mathfrak{B}_{\Omega}$*.*

(iii) The behavior $\mathfrak{B}_{\Omega}$ *in* (29) *is dissipative with respect to the supply rate* $\mathsf{Q}_{\Phi}(w)$*.*

Proof See Appendix B.2 for the proof. □

Corollary 1 provides a physical interpretation of Theorem 1 from the view point of dissipativity. This gives a clear answer to the latter part of Question (ii) in Section 3.

Answer 2

(i) Answer 1 (i) states that the compensating rate $\mathsf{Q}_{\Gamma}(\ell)$ is the new function which newly appears in the dissipation inequality (11). We further clarify the role of the function in this answer.

From Corollary 1 (ii), if we concentrate ourselves to the subbehavior $\mathfrak{B}_{\Omega}$, the QDF $\mathsf{Q}_{\Delta+\Gamma}(\ell)$ becomes the dissipation rate for $\mathfrak{B}_{\Omega}$ with respect to the supply rate $\mathsf{Q}_{\Psi}(w)$. This can be verified as follows.

Since (31) and

$$\int_{\mathbb{R}^n} \mathsf{Q}_{\Delta+\Gamma}(\ell)\mathrm{d}x = \int_{\mathbb{R}^n} \mathsf{Q}_\Phi(w)\mathrm{d}x,\ \forall\, w \in \mathfrak{B}_\Omega \cap \mathcal{D}^\infty(\mathbb{R}^n, \mathbb{C}^q) \text{ s.t. } (5)$$

hold, we observe that the QDF $\mathsf{Q}_{\Delta+\Gamma}(\ell)$ in (30) and (31) becomes the dissipation rate for $\mathfrak{B}_\Omega$ with respect to the supply rate $\mathsf{Q}_\Phi(w)$. Since the QDF $\mathsf{Q}_\Gamma(\ell)$ guarantees the dissipativity, $\mathsf{Q}_\Gamma(\ell)$ can be considered as a compensating power. This shows that the compensating rate $\mathsf{Q}_\Gamma(\ell)$ plays a role which guarantees dissipativity of $\mathfrak{B}_\Omega$. This is the reason why we call $\mathsf{Q}_\Gamma(\ell)$ as the compensation rate for $\mathfrak{B}$ with respect to the supply rate $\mathsf{Q}_\Phi(w)$. Then, we should also stress that the QDF $\mathsf{Q}_\Psi(\ell)$ becomes the storage function for $\mathfrak{B}_\Omega$ with respect to the supply rate $\mathsf{Q}_\Phi(w)$.

(ii) It is not difficult to see that $\mathfrak{B}$ is not necessarily dissipative with respect to the supply rate $\mathsf{Q}_\Phi(w)$ from Proposition 1. However, Corollary 1 (iii) states that, if we concentrate ourselves to the subbehavior $\mathfrak{B}_\Omega$, then $\mathfrak{B}_\Omega$ becomes dissipative with respect to the supply rate $\mathsf{Q}_\Phi(w)$. This corresponds to an answer to the latter part of Question (ii).

4.3 Characterization Using $\mathfrak{B}$-canonical Polynomial Matrices

In Theorem 1, the degree of $\Psi(\zeta,\eta)$ and $\Gamma(\zeta,\eta)$ in the statement (ii) are not specified explicitly. However, thanks to $\mathfrak{B}$-canonical polynomial matrices [13], we can determine the bounds by the degree of the polynomial matrix which induces a kernel representation of $\mathfrak{B}$. See Appendix A and the reference [13] for the definition and basic properties of $\mathfrak{B}$-canonical polynomial matrices.

We set some assumptions to characterize the upper bound of the degree of $\Psi(\zeta,\eta)$ and $\Gamma(\zeta,\eta)$.

Assumption 2

(i) The polynomial matrix $R \in \mathbb{C}^{p\times q}[\xi]$ in (2) is row reduced [11][13].

(ii) The $2n$-variable polynomial matrix $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ in (8) is $\mathfrak{B}$-canonical.

(iii) The behavior $\mathfrak{B}$ in (4) is represented by an observable image representation (5).

Assumption 2 (i) does not lose any generality, because there always exists a unimodular polynomial matrix $U \in \mathbb{C}^{p\times p}[\xi]$ satisfying $R_{\mathrm{red}}(\xi) = U(\xi)R(\xi)$, where $R_{\mathrm{red}} \in \mathbb{C}^{p\times q}[\xi]$ is row reduced. It should be noted that $R_{\mathrm{red}}(\xi)$ may be obtained by using the package Singular [9]. Assumption 2 (ii) implies that the following degree constraint holds.

$$\deg_{\xi_k} R \geq \deg_{\zeta_k} \Phi - 1 = \deg_{\eta_k} \Phi - 1,\ \forall\, k = 1,\cdots,n \tag{32}$$

This assumption does not lose the generality. If (32) does not hold, i.e. $\deg R_{\xi_k} < \deg_{\zeta_k} \Phi - 1 = \deg_{\eta_k} \Phi -$ for some $k \in \{1,\cdots,n\}$, we can reduce it to (32) by taking $R_i = 0_{p\times q}$ for all multi-indices $i \in \mathbb{Z}^n$ which have at least one element greater than $\deg_{\xi_k} R$. Hence, it is sufficient to prove under the assumption (32). Finally, we should remark that the discussions in Section 4.3 does not hold without Assumption 2 (iii), i.e. the observability assumption.

In order to derive a characterization for the $\mathfrak{B}$-canonical case, we redefine $\mathcal{G} \subset \mathbb{H}^{m\times m}[\zeta,\eta]$ in (19) under Assumption 2 (iii). Define the set $\mathcal{G}' \subset \mathbb{H}^{q\times q}[\zeta,\eta]$ by

$$\mathcal{G}' := \left\{ \Gamma' \in \mathbb{H}^{q\times q}[\zeta,\eta] \;\middle|\; \begin{array}{l} \Gamma'(\zeta,\eta) := \sum_{k=1}^{n} \chi_k(\zeta,\eta)\Upsilon'_k(\zeta,\eta) \\ \text{for some } \Upsilon'_k \in \mathbb{H}^{q\times q}[\zeta,\eta] \ (k=1,\cdots,n) \text{ such that (34)} \end{array} \right\}, \tag{33}$$

$$\tau_k \mathsf{Q}_{\Upsilon'_k}(w) \geq 0,\ \forall\, w \in \mathfrak{B}, \tag{34}$$

where τ_k is equal to either $+1$ or -1 and $\chi_k \in \mathbb{C}[\zeta,\eta]$ is defined by (20). From Theorem 1 and Lemma A.3, we obtain a characterization for the finite frequency property using $\mathfrak{B}$-canonical polynomial matrices.

Proposition 3 *Let $\mathfrak{B}$ in* (4) *and let $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ in* (8) *be given. Suppose that Assumptions 1 and 2 hold. Define Ω by* (14) *and $\mathcal{G}'$ by* (33)*. Then, the following statements (i), (ii) and (iii) are equivalent.*

(i) The FFDI (17) *holds for all $\omega \in \Omega$.*

(ii) There exist unique $2n$-variable polynomial matrices

$$\Psi'(\zeta,\eta) = \begin{bmatrix} \Psi'_1(\zeta,\eta) \\ \vdots \\ \Psi'_n(\zeta,\eta) \end{bmatrix} \in \mathbb{H}^{nq\times q}[\zeta,\eta]$$

and $\Gamma' \in \mathcal{G}'$ with $\mathfrak{B}$-canonical $\Psi'_k, \Upsilon'_k \in \mathbb{H}^{q\times q}[\zeta,\eta]$ $(k=1,\cdots,n)$ satisfying

$$\mathrm{div}\mathsf{Q}_{\Psi'}(w) \leq \mathsf{Q}_{\Phi}(w) - \mathsf{Q}_{\Gamma'}(w). \tag{35}$$

(iii) The inequality (26) *holds for all $w \in \mathfrak{B}$ satisfying*

$$\tau_k \int_{\mathbb{R}^n} \mathrm{He}\left[\left(\frac{\partial z'_k}{\partial x_k} - j\varpi_{k,1} z'_k \right) \left(\frac{\partial z'_k}{\partial x_k} - j\varpi_{k,2} z'_k \right)^* \right] \mathrm{d}x \leq 0 \tag{36}$$

for $k = 1,\cdots,n$, where $z'_k \in \mathcal{D}^\infty(\mathbb{R}^n, \mathbb{C}^{\{\prod_{l=1}^n (N'_{k,l}+1)\}q})$ is defined by

$$z'_k = Z_{N'_k}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) w$$

for some multi-index $N'_k := (N'_{k,1},\cdots,N'_{k,n}) \in \mathbb{Z}^n$, $N'_{k,l} \leq \deg_{\xi_l} R - 1$ $(l=1,\cdots,n)$.

Proof See Appendix B.3 for the proof. □

Proposition 3 shows that the upper bounds of the degree of $\Psi(\zeta,\eta)$ and $\Gamma(\zeta,\eta)$ are determined by that of $R(\xi)$.

5 A Numerical Example

In this section, we apply Theorem 1 and Corollary 1 to a numerical example.

Consider a two-dimensional behavior $\mathfrak{B} \subset \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2)$ whose kernel representation is described by

$$R\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) w = 0,\ R(\xi) := \begin{bmatrix} -1 & \xi_1 + \xi_2^2 + 1 \end{bmatrix},$$

where $w := \mathrm{col}\,(w_1, w_2)$ is the manifest variable. Define the frequency domain and the symmetric matrix by

$$\Omega := \left\{ \omega \in \mathbb{R}^2 \mid \omega_1 \in \mathbb{R},\ |\omega_2| \le 1 \right\} \text{ and } \Phi := \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

respectively. The domain Ω is restricted to the low frequency with respect to ω_2, however it represents an entire interval with respect to ω_1.

Since $\mathfrak{B}$ is controllable, $\mathfrak{B}$ can be represented by the image representation

$$w = M\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\ell,\ M(\xi) := \begin{bmatrix} \xi_1 + \xi_2^2 + 1 \\ 1 \end{bmatrix},$$

which is an observable image representation. Then, we have

$$M^\star(\zeta)\Phi(\zeta, \eta)M(\eta) = \zeta_1 + \zeta_2^2 + \eta_1 + \eta_2^2 + 3$$

From the above equation, we obtain

$$M^*(j\omega)\partial\Phi(j\omega)M(j\omega) = 3 - 2\omega_2^2,$$

which implies that the FFDI (17) holds for all $\omega \in \Omega$. We observe that $M^\star(\zeta)\Phi(\zeta, \eta)M(\eta)$ can be decomposed to

$$M^\star(\zeta)\Phi(\zeta, \eta)M(\eta) = \begin{bmatrix} \zeta_1 + \eta_1 & \zeta_2 + \eta_2 \end{bmatrix} \Psi(\zeta, \eta) + \Gamma(\zeta, \eta) + \Delta(\zeta, \eta).$$

where $\Psi \in \mathbb{R}^{2\times 1}[\zeta, \eta]$, $\Gamma \in \mathbb{R}[\zeta, \eta]$ and $\Delta \in \mathbb{R}[\zeta, \eta]$ are given by

$$\Psi(\zeta, \eta) := \begin{bmatrix} 1 \\ \zeta_2 + \eta_2 \end{bmatrix},\ \Gamma(\zeta, \eta) := 2(1 - \zeta_2\eta_2) \text{ and } \Delta(\zeta, \eta) := 1,$$

respectively. Then, we have the inequality

$$\begin{aligned} \mathrm{div}\mathrm{Q}_\Psi(\ell) &= \mathrm{Q}_\Phi(w) - \mathrm{Q}_\Gamma(\ell) - \mathrm{Q}_\Delta(\ell) \\ &\le \mathrm{Q}_\Phi(w) - \mathrm{Q}_\Gamma(\ell), \quad \forall\, \ell \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2), \end{aligned} \tag{37}$$

which satisfies Theorem 1 (ii). The inequality (37) shows that $\mathfrak{B}$ dissipates a power with the compensating power $\mathrm{Q}_\Gamma(w)$.

From Corollary 1, (37) is equivalently rewritten by the dissipation inequality

$$\mathrm{div}\mathrm{Q}_\Psi(\ell) \le \mathrm{Q}_\Phi(w), \quad \forall\, w \in \mathfrak{B}_\Omega,$$

where $\mathfrak{B}_\Omega \subset \mathcal{D}^\infty(\mathbb{R}^2, \mathbb{R}^2)$ is the subbehavior of $\mathfrak{B}$ defined by

$$\mathfrak{B}_\Omega := \left\{ w \in \mathcal{D}^\infty(\mathbb{R}^2, \mathbb{R}^2) \,\middle|\, \begin{array}{l} w = M\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\ell, \\ \forall\, \ell \in \mathcal{D}^\infty(\mathbb{R}^2, \mathbb{R}^2) \text{ s.t. (38) and (39)} \end{array} \right\},$$

$$\int_{\mathbb{R}^2} \frac{\partial \ell}{\partial x_1} \frac{\partial \ell^\top}{\partial x_1} \mathrm{d}x \geq 0, \tag{38}$$

$$\int_{\mathbb{R}^2} \left(\frac{\partial z}{\partial x_2} \frac{\partial z^\top}{\partial x_2} - z z^\top \right) \mathrm{d}x \leq 0,\ z := \begin{bmatrix} \ell \\ \frac{\partial \ell}{\partial x_1} \end{bmatrix}. \tag{39}$$

This shows that $\mathfrak{B}_\Omega$ is dissipative with respect to the supply rate $Q_\Phi(w)$. □

6 Conclusions

In this paper, we have characterized the finite frequency properties using some inequality with the compensating rate and an inequality of an integral of the supply rate with a matrix integral quadratic constraint based on QDFs as a main result. We have resolved a problem of an expression of a compensating rate in n-dimensional behaviors. The characterization has led to a physical interpretation in terms of the dissipation inequality, equivalently dissipativity, for the subbehavior with some rate constraints. These results can be regarded as a generalization of the previous one-dimensional results [12] to the n-dimensional behaviors. Such an interpretation has not been clarified by the previous studies of finite frequency properties. The aforementioned characterization also yields a characterization in terms of $\mathfrak{B}$-canonical polynomial matrices.

As a future direction, an LMI characterization should be derived, which is a tractable condition for a numerical checking of the finite frequency properties. For this problem, Yang et.al. [26] derived the generalized KYP lemma to the two-dimensional discrete-time Roessor state-space system as a sufficient characterization. It is desired to derive a necessary and sufficient characterization which should be tackled in our future work.

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Appendix A Background Materials

In this appendix, we collect the background materials which are used in the proofs.

A.1 Coefficient Matrices

We define the coefficient matrix of a polynomial matrix. For this purpose, we first introduce an ordering on the multi-index $i = (i_1, \cdots, i_n)$ using the multi-index notation [19]. Of course, many orderings are possible. We choose the ordering based on *anti-lexicographic ordering* [5].

The ordering is defined as follows. For given multi-indices $i := (i_1, \cdots, i_n),\ j := (j_1, \cdots, j_n) \in \mathbb{Z}^n$ $(i_k \geq 0; k = 1, \cdots, n)$, we define the ordering $i < j$ if and only if the rightmost nonzero entry of $(i_1 - j_1, i_2 - j_2 \cdots, i_n - j_n)$ is negative.

We give the definition of the coefficient matrix of the $2n$-variable polynomial matrices based on the ordering of the multi-index defined in the above paragraph. With every $\Phi \in \mathbb{H}^{q\times q}[\zeta, \eta]$ in (8), we define its *coefficient* *matrix* $\widetilde{\Phi} \in \mathbb{H}^{\{\prod_{l=1}^{n}(K_l+1)\}q \times \{\prod_{l=1}^{n}(K_l+1)\}q}$ by

$$\widetilde{\Phi} := \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots & \Phi_{0,K} \\ \Phi_{1,0} & \Phi_{1,1} & \cdots & \Phi_{1,K} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{K,0} & \Phi_{K,1} & \cdots & \Phi_{K,K} \end{bmatrix}, \tag{A.1}$$

where the (i,j)th block matrix $\Phi_{i,j}$ $(i,j = 0,1,\cdots,K)$ are aligned based on the ordering of multi-indices and $K = (K_1,\cdots,K_n)$. See pp. 1116-1118 of [17] for the more detailed construction of $\widetilde{\Phi}$. Then, $\Phi(\zeta,\eta)$ is expressed as $\Phi(\zeta,\eta) = Z_K(\zeta)^\top \widetilde{\Phi} Z_K(\eta)$.

The nonnegativity of a QDF is characterized by the nonnegativity of its coefficient matrix as seen in the following lemma.

Lemma A.1 [17] *Let* $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ *in* (8) *be given. Define* $\widetilde{\Phi} \in \mathbb{H}^{\{\prod_{l=1}^{n}(K_l+1)\}q \times \{\prod_{l=1}^{n}(K_l+1)\}q}$ *by* (A.1). *Then, we have* $\mathsf{Q}_\Phi(\ell) \geq 0$ *for all* $\ell \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}^q)$ *if and only if* $\widetilde{\Phi} \geq 0$ *holds.*

A.2 $\mathfrak{B}$-canonical Polynomial Matrices

We introduce $\mathfrak{B}$-canonicity of polynomial matrices in this appendix, which are taken from the references [11][13].

We assume that $R \in \mathbb{C}^{p\times q}[\xi]$ in (3) is row reduced [11][13] in this section. The assumption does not lose the generality as we have explained in Section 4.1.

Definition A.1 [13] *Let* $\mathfrak{B}$ *be represented by a kernel representation* (3) *for* $R \in \mathbb{C}^{p\times q}[\xi]$. *Assume that* $R(\xi)$ *is row reduced. Let* $D \in \mathbb{C}^{p\times q}[\xi]$ *be given. Let* $r_i \in \mathbb{C}^{1\times q}[\xi]$ *and* $d_i \in \mathbb{C}^{1\times q}[\xi]$ $(i = 1,\cdots,p)$ *denote the ith rows of* $R(\xi)$ *and* $D(\xi)$, *respectively. A polynomial matrix* $D(\xi)$ *is called* $\mathfrak{B}$-canonical *if* $\deg d_i \leq \deg r_i - 1$, $\forall\, i = 1,\cdots,p$ *holds.*

The next lemma ensures the uniqueness of an R-canonical polynomial matrix up to $\mathfrak{B}$-equivalence.

Lemma A.2 [13] *Let* $\mathfrak{B}$ *be represented by a kernel representation* (3) *for* $R \in \mathbb{C}^{p\times q}[\xi]$. *Assume that* $R(\xi)$ *is row reduced. For any* $D \in \mathbb{C}^{p\times q}[\xi]$, *there exists a unique* $\mathfrak{B}$*-canonical* $D' \in \mathbb{C}^{p\times q}[\xi]$ *satisfying* $D\left(\frac{d}{dt}\right)w = D'\left(\frac{d}{dt}\right)w$, $\forall\, w \in \mathfrak{B}$.

For $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ in (8), there exist $\widetilde{F} \in \mathbb{C}^{\mathrm{rank}\widetilde{\Phi} \times \{\prod_{l=1}^{n}(K_l+1)\}q}$ satisfying $\widetilde{\Phi} = \widetilde{F}^* \Sigma_\Phi \widetilde{F}$, where $\Sigma_\Phi \in \mathbb{S}^{\mathrm{rank}\widetilde{\Phi}\times\mathrm{rank}\widetilde{\Phi}}$, $\widetilde{F}$ is of full row rank, and $\det\Sigma_\Phi \neq 0$. In this case, we get $\mathrm{rank}\Sigma_\Phi = \mathrm{rank}\widetilde{\Phi}$. With such a factorization of $\widetilde{\Phi}$, we obtain a *canonical factorization* of $\Phi(\zeta,\eta)$ as

$$\Phi(\zeta,\eta) = F(\zeta)^* \Sigma_\Phi F(\eta), \tag{A.2}$$

where $F \in \mathbb{C}^{\mathrm{rank}\widetilde{\Phi}\times q}[\xi]$ is defined by $F(\xi) := \widetilde{F} Z_K(\xi)$.

Definition A.2 [13] *Let* $\mathfrak{B}$ *be represented by a kernel representation* (3) *for* $R \in \mathbb{C}^{p\times q}[\xi]$. *Assume that* $R(\xi)$ *is row reduced. Let* $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ *be given by* (8). *Let* $F \in \mathbb{C}^{\mathrm{rank}\widetilde{\Phi}\times q}[\xi]$ *be defined by the canonical factorization* (A.2). *Then,* $\Phi(\zeta,\eta)$ *is called* $\mathfrak{B}$-canonical *if* $F(\xi)$ *is* $\mathfrak{B}$*-canonical.*

The following result is an immediate consequence of the uniqueness of the canonical factorization of $\Phi(\zeta,\eta)$ and of Lemma A.2.

Lemma A.3 [13] *Let $\mathfrak{B}$ be represented by a kernel representation* (3) *for $R \in \mathbb{C}^{p\times q}[\xi]$. Assume that $R(\xi)$ is row reduced. Let $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ be given by* (8)*. Then, for any $\Phi(\zeta,\eta)$, there exists a unique $\mathfrak{B}$-canonical $\Phi' \in \mathbb{H}^{q\times q}[\zeta,\eta]$ satisfying $\mathsf{Q}_{\Phi'}(w) = \mathsf{Q}_{\Phi}(w)$, $\forall\, w \in \mathfrak{B}$.*

Appendix B Proofs

In this appendix, we summarize the proofs of the results obtained in this paper.

Appendix B.1 Proof of Theorem 1

We give the proof of Theorem 1 in this subsection. The proof consists of three parts. We first show a characterization for the low frequency property in Appendix B.1.1. We generalize the discussion in Appendix B.1.1 to the case where the frequency domain is given as a combination of low and high frequency domains in Appendix B.1.2. Finally, we conclude the proof in Appendix B.1.3 for the general frequency property. The most part of the proof is devoted to the low frequency case in Appendix B.1.1.

Appendix B.1.1 Low Frequency Case

In this subsection, we restrict our attention to the low frequency property and derive a characterization of the property as preliminary lemma.

Define the low frequency domain $\Omega_l \subset \mathbb{R}^n$ in the rectangular domain by (15). Then, $\mathcal{G}$ in (19) and $\chi_k \in \mathbb{H}[\zeta,\eta]$ in (20) become

$$\mathcal{G}_l := \left\{ \Gamma \in \mathbb{H}^{m\times m}[\zeta,\eta] \;\middle|\; \begin{array}{l} \Gamma(\zeta,\eta) := \sum_{k=1}^{n} \chi_k(\zeta,\eta)\Upsilon_k(\zeta,\eta) \text{ for some} \\ \Upsilon_k \in \mathbb{H}^{m\times m}[\zeta,\eta]\ (k=1,\cdots,n) \text{ such that (21)} \end{array} \right\} \tag{B.1}$$

and

$$\chi_k(\zeta,\eta) := \varpi_k^2 - \zeta_k\eta_k,$$

respectively. We see that

$$\begin{aligned} \partial\Gamma(j\omega) &= \sum_{k=1}^{n} \left(\varpi_k^2 - \omega_k^2\right) \partial\Upsilon_k(j\omega) \\ &\geq 0,\ \forall\, \omega \in \Omega_l. \end{aligned} \tag{B.2}$$

holds for any $\Gamma \in \mathcal{G}_l$.

We obtain a necessary and sufficient condition which is equivalent to the low frequency property using QDFs.

Lemma B.1 *Let $\mathfrak{B}$ in* (4) *and $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ in* (8) *be given. Suppose the Assumption 1 holds. Define Ω_l by* (15) *and $\mathcal{G}_l$ by* (B.1)*. Then, the following statements (i), (ii) and (iii) are equivalent.*

(i) The FFDI (17) *holds for all* $\omega \in \Omega_1$.

(ii) There exist $2n$*-variable polynomial matrices* $\Psi \in \mathbb{C}^{nm\times m}[\zeta,\eta]$ *in* (23) *and* $\Gamma \in \mathcal{G}_1$ *satisfying* (24) *with the image representation* (5).

(iii) The inequality (26) *holds for all* $w \in \mathfrak{B}$ *with the image representation* (5) *and* $\ell \in \mathcal{D}^\infty(\mathbb{R}^n,\mathbb{C}^m)$ *satisfying*

$$\int_{\mathbb{R}^n} \left(\frac{\partial z_k}{\partial x_k}\frac{\partial z_k^*}{\partial x_k} - \varpi_k^2 z_k z_k^* \right) \mathrm{d}x \le 0 \ (k = 1,\cdots,n), \tag{B.3}$$

where $z_k \in \mathcal{D}^\infty(\mathbb{R}^n, \mathbb{C}^{\{\prod_{l=1}^n (N_{k,l}+1)\}m})$ *is defined by* (28) *for some multi-index* $N_k := (N_{k,1},\cdots,N_{k,n}) \in \mathbb{Z}^n$.

Proof (ii)⇒(iii) By integrating (24) from $x_k = -\infty$ to $x_k = +\infty$ for $k = 1,\cdots,n$, we obtain the inequality

$$\int_{\mathbb{R}^n} \mathsf{Q}_\Phi(w)\mathrm{d}x \ge \int_{\mathbb{R}^n} \mathsf{Q}_\Gamma(\ell)\mathrm{d}x.$$

It follows from the definition of $\Gamma(\zeta,\eta)$ that the inequality

$$\int_{\mathbb{R}^n} \mathsf{Q}_\Phi(w)\mathrm{d}x \ge \sum_{k=1}^n \int_{\mathbb{R}^n} \left\{ \varpi_k^2 \mathsf{Q}_{\Upsilon_k}(\ell) - \mathsf{Q}_{\Upsilon_k}\left(\frac{\partial \ell}{\partial x_k}\right) \right\} \mathrm{d}x \tag{B.4}$$

holds. Since $\mathsf{Q}_{\Upsilon_k}(\ell)$ is expressed as $\mathsf{Q}_{\Upsilon_k}(\ell) = z_k^* \widetilde{\Upsilon}_k z_k$, where $\widetilde{\Upsilon}_k \in \mathbb{H}^{\{\prod_{l=1}^n (N_{k,l}+1)\}m\times\{\prod_{l=1}^n (N_{k,l}+1)\}m}$ is the coefficient matrix of $\Upsilon_k(\zeta,\eta)$ and $z_k \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}^{\{\prod_{l=1}^n (N_{k,l}+1)\}m})$ is defined by (28) for $N_{k,l} := \deg_{\zeta_l} \Upsilon_k$ $(l = 1,\cdots,n)$. In the right hand side of (B.4), the inner term of the integral can be rewritten by

$$\begin{aligned}
\varpi_k^2 \mathsf{Q}_{\Upsilon_k}(\ell) - \mathsf{Q}_{\Upsilon_k}\left(\frac{\partial \ell}{\partial x_k}\right) &= \varpi_k^2 z_k^* \widetilde{\Upsilon}_k z_k - \frac{\partial z_k^*}{\partial x_k}\widetilde{\Upsilon}_k \frac{\partial z_k}{\partial x_k} \\
&= \mathrm{tr}\left[\widetilde{\Upsilon}_k \left(\varpi_k^2 z_k z_k^* - \frac{\partial z_k}{\partial x_k}\frac{\partial z_k^*}{\partial x_k}\right)\right].
\end{aligned}$$

Substituting the above equality to (B.4) yields

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathsf{Q}_\Phi(w)\mathrm{d}x &\ge \sum_{k=1}^n \int_{\mathbb{R}^n} \mathrm{tr}\left[\widetilde{\Upsilon}_k \left(\varpi_k^2 z_k z_k^* - \frac{\partial z_k}{\partial x_k}\frac{\partial z_k^*}{\partial x_k}\right)\right] \mathrm{d}x \\
&= \sum_{k=1}^n \mathrm{tr}\left[\widetilde{\Upsilon}_k \int_{\mathbb{R}^n} \left(\varpi_k^2 z_k z_k^* - \frac{\partial z_k}{\partial x_k}\frac{\partial z_k^*}{\partial x_k}\right) \mathrm{d}x\right] \\
&= \mathrm{tr}\left[\begin{bmatrix} \widetilde{\Upsilon}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \widetilde{\Upsilon}_n \end{bmatrix} \begin{bmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_n \end{bmatrix}\right],
\end{aligned}$$

where $A_k \in \mathbb{H}^{\{\prod_{l=1}^n (N_{k,l}+1)\}m\times\{\prod_{l=1}^n (N_{k,l}+1)\}m}$ $(k = 1,\cdots,n)$ is the constant matrix defined by

$$A_k := \int_{\mathbb{R}^n} \left(\varpi_k^2 z_k z_k^* - \frac{\partial z_k}{\partial x_k}\frac{\partial z_k^*}{\partial x_k}\right) \mathrm{d}x.$$

Since $\widetilde{\Upsilon}_k \ge 0, \forall\, k = 1,\cdots,n$ holds from the definition of $\Upsilon_k(\zeta,\eta)$ and Lemma A.1, the inequality (26) holds for all $\ell \in \mathcal{D}^\infty(\mathbb{R}^n,\mathbb{C}^m)$ satisfying (B.3). This concludes the claim.

(iii)⇒(i) We give the proof of the claim by showing the contraposition.

Suppose that there exists a frequency vector $\omega' := (\omega'_1, \cdots, \omega'_n) \in \Omega_1$ such that $\omega'_k \geq 0$ $(k = 1, \cdots, n)$ and $\partial\Phi(j\omega') \ngeq 0$. We can assume $\omega'_k > 0$ because it can be proved the case where $\omega'_k = 0$ by replacing ω'_k with $\omega'_k + \varepsilon$ $(\varepsilon > 0)$ and taking the limitation $\varepsilon \to 0$. We can also assume $\omega'_k \neq \varpi_k$ by the similar reason.

Let $v \in \mathbb{C}^m$ be the eigenvector corresponding to the minimum eigenvalue of $\partial\Phi(j\omega')$. Since the eigenvalue is negative, we observe

$$\mathsf{Q}_\Phi\left(M\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)e^{j\left(\sum_{k=1}^n \omega'_k x_k\right)}v\right) = v^* M(j\omega')^* \partial\Phi(j\omega') M(j\omega') v < 0.$$

In order to show the claim, we choose a special latent variable $\ell_h \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}^m)$, $h \in \mathbb{Z}$. This is constructed as follows. Define $\ell_{h,k} \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$ $(k = 1, \cdots, n)$ by

$$\ell_{h,k}(x_k) := \begin{cases} e^{j\omega'_k x_k} & \left(|x_k| \leq \dfrac{2\pi h}{\omega'_k}\right) \\ \widetilde{\ell}_k\left(x_k + \dfrac{2\pi h}{\omega'_k}\right) & \left(x_k < -\dfrac{2\pi h}{\omega'_k}\right) \\ \widetilde{\ell}_k\left(x_k - \dfrac{2\pi h}{\omega'_k}\right) & \left(x_k > \dfrac{2\pi h}{\omega'_k}\right) \end{cases},$$

where $\widetilde{\ell}_k \in \mathcal{D}^\infty(\mathbb{R}, \mathbb{C})$ $(k = 1, \cdots, n)$ is chosen as a function which does not depend on h and be such that ℓ_h is a smooth function for h. Then, the variable $\ell_h \in \mathcal{D}^\infty(\mathbb{R}^n, \mathbb{C}^m)$, $h \in \mathbb{Z}$ is constructed as a product of $\ell_{k,1}, \cdots, \ell_{k,n}$ by

$$\ell_h(x) = \left(\prod_{k=1}^n \ell_{h,k}(x_k)\right) v. \tag{B.5}$$

We now proceed to show the claim. For ℓ_h in (B.5), define the variable

$$z_{k,h} := Z_{N_k}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\ell_h \in \mathcal{D}^\infty(\mathbb{R}^n, \mathbb{C}^{\{\prod_{l=1}^n (N_{k,l}+1)\}m}).$$

Compute the value of the following integral

$$\int_{\mathbb{R}^n}\left(\frac{\partial z_{k,h}}{\partial x_k}\frac{\partial z^*_{k,h}}{\partial x_k} - \varpi_k^2 z_{k,h} z^*_{k,h}\right)\mathrm{d}x. \tag{B.6}$$

Note that this integral is finite, since ℓ_h has a compact support. The integral in (B.6) is rewritten by

$$\begin{aligned}\int_{\mathbb{R}^n}\left(\frac{\partial z_{k,h}}{\partial x_k}\frac{\partial z^*_{k,h}}{\partial x_k} - \varpi_k^2 z_{k,h} z^*_{k,h}\right)\mathrm{d}x &= \int_{\mathbb{R}^{n-1}}\left\{\int_{-\frac{2\pi h}{\omega'_k}}^{+\frac{2\pi h}{\omega'_k}}\left(\frac{\partial z_{k,h}}{\partial x_k}\frac{\partial z^*_{k,h}}{\partial x_k} - \varpi_k^2 z_{k,h} z^*_{k,h}\right)\mathrm{d}x_k\right\}\mathrm{d}s_k \\ &+ \int_{\mathbb{R}^{n-1}}\left\{\int_{+\frac{2\pi h}{\omega'_k}}^{+\infty}\left(\frac{\partial z_{k,h}}{\partial x_k}\frac{\partial z^*_{k,h}}{\partial x_k} - \varpi_k^2 z_{k,h} z^*_{k,h}\right)\mathrm{d}x_k\right\}\mathrm{d}s_k \\ &+ \int_{\mathbb{R}^{n-1}}\left\{\int_{-\infty}^{-\frac{2\pi h}{\omega'_k}}\left(\frac{\partial z_{k,h}}{\partial x_k}\frac{\partial z^*_{k,h}}{\partial x_k} - \varpi_k^2 z_{k,h} z^*_{k,h}\right)\mathrm{d}x_k\right\}\mathrm{d}s_k\end{aligned}$$

where $s_k := (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n) \in \mathbb{R}^{n-1}$. Finally, we can compute the integral in (B.6) as

$$\begin{aligned}&\int_{\mathbb{R}^n}\left(\frac{\partial z_{k,h}}{\partial x_k}\frac{\partial z^*_{k,h}}{\partial x_k} - \varpi_k^2 z_{k,h} z^*_{k,h}\right)\mathrm{d}x \\ &= \frac{(4\pi h)^n\left(\omega_k'^2 - \varpi_k^2\right)}{\prod_{k=1}^n \omega'_k} Z_{N_k}(j\varpi) v v^* Z_{N_k}(j\varpi)^* + h^{n-1}B_{n-1} + \cdots + hB_1 + B_0,\end{aligned}$$

where $B_0, B_1, \cdots, B_{n-1} \in \mathbb{H}^{\{\prod_{l=1}^{n}(N_{k,l}+1)\}m \times \{\prod_{l=1}^{n}(N_{k,l}+1)\}m}$ are Hermitian matrices which do not depend on h. Since $\omega_k > \varpi_k$ and $\omega_k \neq 0$ holds for all $k = 1, \cdots, n$, we have the inequality

$$\int_{\mathbb{R}^n} \left(\frac{\partial z_{k,h}}{\partial x_k} \frac{\partial z_{k,h}^*}{\partial x_k} - \varpi_k^2 z_{k,h} z_{k,h}^* \right) \mathrm{d}x \leq 0$$

for sufficiently large h.

Next, we compute the value of the integral

$$\int_{\mathbb{R}^n} \mathsf{Q}_\Phi \left(M \left(\frac{\mathrm{d}}{\mathrm{d}x} \right) \ell_h \right) \mathrm{d}x$$

for $\ell_h(x)$ defined by (B.5). This integral can be decomposed to

$$\begin{aligned}\int_{\mathbb{R}^n} \mathsf{Q}_\Phi \left(M \left(\frac{\mathrm{d}}{\mathrm{d}x} \right) \ell_h \right) \mathrm{d}x = & \int_{\mathbb{R}^{n-1}} \left\{ \int_{-\frac{2\pi h}{\omega_k'}}^{+\frac{2\pi h}{\omega_k'}} \mathsf{Q}_\Phi \left(M \left(\frac{\mathrm{d}}{\mathrm{d}x} \right) \ell_h \right) \mathrm{d}x_k \right\} \mathrm{d}s_k \\ & + \int_{\mathbb{R}^{n-1}} \left\{ \int_{+\frac{2\pi h}{\omega_k'}}^{+\infty} \mathsf{Q}_\Phi \left(M \left(\frac{\mathrm{d}}{\mathrm{d}x} \right) \ell_h \right) \mathrm{d}x_k \right\} \mathrm{d}s_k \\ & + \int_{\mathbb{R}^{n-1}} \left\{ \int_{-\infty}^{-\frac{2\pi h}{\omega_k'}} \mathsf{Q}_\Phi \left(M \left(\frac{\mathrm{d}}{\mathrm{d}x} \right) \ell_h \right) \mathrm{d}x_k \right\} \mathrm{d}s_k.\end{aligned}$$

The above integral can be computed as

$$\begin{aligned}&\int_{\mathbb{R}^n} \mathsf{Q}_\Phi \left(M \left(\frac{\mathrm{d}}{\mathrm{d}x} \right) \ell_h \right) \mathrm{d}x \\ &\quad = \frac{(4\pi h)^n}{\prod_{k=1}^{n} \omega_k'} v^* M(j\omega')^* \partial\Phi(j\omega') M(j\omega') v + h^{n-1} C_{n-1} + \cdots + h C_1 + C_0,\end{aligned}$$

where $C_0, C_1, \cdots, C_{n-1} \in \mathbb{R}$ are the constants which do not depend on h. Since $v^* M(j\omega')^* \partial\Phi(j\omega') M(j\omega') v < 0$ holds, if we take h as sufficiently large in the above equality, we have

$$\int_{\mathbb{R}^n} \mathsf{Q}_\Phi \left(M \left(\frac{\mathrm{d}}{\mathrm{d}x} \right) \ell_h \right) \mathrm{d}x < 0.$$

This leads to a contradiction which completes the proof.

(i)⇒(ii) Assume that the statement (ii) does not hold. This is the case if and only if there does not simultaneously exist a pair of $\Psi \in \mathbb{C}^{nm \times m}[\zeta, \eta]$ in (23) and $\Gamma \in \mathcal{G}$ satisfying

$$\mathrm{div} \mathsf{Q}_\Psi(\ell) - \mathsf{Q}_\Phi(w) + \mathsf{Q}_\Gamma(\ell) \leq \varepsilon \|\ell\|^2, \quad \forall\, \ell \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}^m)$$

for some $\varepsilon > 0$. The above inequality is equivalent to the following $2n$-variable polynomial matrix equation

$$\nabla\Psi(\zeta, \eta) - M^\star(\zeta)\Phi(\zeta, \eta)M(\eta) + \Gamma(\zeta, \eta) + \Delta(\zeta, \eta) = \varepsilon I_m \tag{B.7}$$

for some $\Delta \in \mathbb{H}^{m \times m}[\zeta, \eta]$ such that $\mathsf{Q}_\Delta(\ell) \geq 0$, $\forall\, \ell \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}^m)$. More exactly, from the definitions of $\Psi(\zeta, \eta)$ and $\Gamma(\zeta, \eta)$, the statement (ii) does not hold if and only if there exists an $\varepsilon > 0$ such that the $2n$-variable polynomial matrix equation

$$\sum_{k=1}^{n} (\zeta_k + \eta_k) \Psi_k(\zeta, \eta) - M^\star(\zeta)\Phi(\zeta, \eta)M(\eta) + \sum_{k=1}^{n} \left(\varpi_k^2 - \zeta_k \eta_k \right) \Upsilon_k(\zeta, \eta) + \Delta(\zeta, \eta) = \varepsilon I_m$$

does not have a solution $\Psi_k(\zeta,\eta)$ and $\Upsilon_k(\zeta,\eta)$ satisfying (21). This implies that there exists an $\varepsilon > 0$ such that the frequency domain equality

$$-M(j\omega)^*\partial\Phi(j\omega)M(j\omega) + \sum_{k=1}^{n}\left(\varpi_k^2 - \omega^2\right)\partial\Upsilon_k(j\omega) + \partial\Delta(j\omega) = \varepsilon I_m$$

does not have a solution $\Upsilon_k(\zeta,\eta)$ satisfying (21). This is the case if and only if the FFDI

$$M(j\omega)^*\partial\Phi(j\omega)M(j\omega) \geq \sum_{k=1}^{n}\left(\varpi_k^2 - \omega_k^2\right)\partial\Upsilon_k(j\omega), \quad \forall\, \omega \in \Omega_\mathrm{l}$$

does not have a solution $\Upsilon_k(\zeta,\eta)$ satisfying (21). Pre- and post-multiplying the above inequality by v^* and v ($v \in \mathbb{R}^m,\ v \neq 0$), respectively, we get

$$\begin{aligned}
&v^*M(j\omega)^*\partial\Phi(j\omega)M(j\omega)v\\
&\geq \sum_{k=1}^{n}\left(\varpi_k^2 - \omega_k^2\right)v^*\partial\Upsilon_k(j\omega)v\\
&= \mathrm{tr}\left(\begin{bmatrix} vv^* & & & \\ & vv^* & & \\ & & \ddots & \\ & & & vv^*\end{bmatrix}\begin{bmatrix}\left(\varpi_1^2 - \omega_1^2\right)\partial\Upsilon_1(j\omega) & & & \\ & \left(\varpi_1^2 - \omega_2^2\right)\partial\Upsilon_2(j\omega) & & \\ & & \ddots & \\ & & & \left(\varpi_n^2 - \omega_n^2\right)\partial\Upsilon_n(j\omega)\end{bmatrix}\right)\\
&= \sum_{k=1}^{n}\mathrm{tr}\left\{vv^*\left(\varpi_k^2 - \omega_k^2\right)\partial\Upsilon_k(j\omega)\right\}.
\end{aligned}$$

We easily see that $vv^* \geq 0$ holds. Hence, if there exists a k such that $\left(\varpi_k^2 - \omega_k^2\right)\partial\Upsilon_k(j\omega) \geq 0,\ \forall\, \omega \in \Omega_\mathrm{l}$ does not hold, we do not have $vM(j\omega)^*\partial\Phi(j\omega)M(j\omega)v \geq 0,\ \forall\, \omega \in \Omega_\mathrm{l}$. This implies that

$$M(j\omega)^*\partial\Phi(j\omega)M(j\omega) \ngeq 0$$

holds for some $\omega \in \Omega_\mathrm{l}$. Hence, the statement (i) does not hold, which proves the claim. □

Appendix B.1.2 Combinatorial Frequency Case

In this subsection, we generalize the discussion in the last subsection to the case where frequency domain is given as a combination of low and high frequency domains for each frequency variable.

Define the frequency domain $\Omega_\mathrm{c} \subset \mathbb{R}^n$ as a combination of low and high frequency domains by

$$\Omega_\mathrm{c} := \prod_{k=1}^{n}\Omega_{\mathrm{c},k},\ \Omega_{\mathrm{c},k} := \{\omega \in \mathbb{R}^n \mid \tau_k\left(\omega_k - \varpi_k\right)\left(\omega_k + \varpi_k\right) \leq 0\}\ (k = 1, \cdots, n), \tag{B.8}$$

where $\varpi := (\varpi_1, \cdots, \varpi_n) \in \mathbb{R}^n$ is a given vector satisfying (16). Then, $\mathcal{G}$ in (19) becomes

$$\mathcal{G}_\mathrm{c} := \left\{\Gamma \in \mathbb{H}^{m\times m}[\zeta,\eta]\ \middle|\ \begin{array}{l}\Gamma(\zeta,\eta) := \sum_{k=1}^{n}\chi_k(\zeta,\eta)\Upsilon_k(\zeta,\eta) \text{ for some}\\ \Upsilon_k \in \mathbb{H}^{m\times m}[\zeta,\eta]\ (k = 1, \cdots, n) \text{ such that (21)}\end{array}\right\}. \tag{B.9}$$

Then, we also have

$$\partial\Gamma(j\omega) = \sum_{k=1}^{n} \tau_k \left(\varpi_k^2 - \omega_k^2\right) \cdot \tau_k \partial\Upsilon_k(j\omega)$$
$$\geq 0,\ \forall\, \omega \in \Omega_\mathrm{c}$$

holds for any $\Gamma \in \mathcal{G}_\mathrm{c}$. We have the following lemma.

Lemma B.2 *Let* $\mathfrak{B}$ *in* (4) *and* $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ *in* (8) *be given. Suppose that Assumption 1 holds. Define* Ω_c *by* (B.8) *and* $\mathcal{G}_\mathrm{c}$ *by* (B.9)*. Then, the following statements (i), (ii) and (iii) are equivalent.*

(i) The FFDI (17) *holds for all* $\omega \in \Omega_\mathrm{c}$*.*

(ii) There exist $2n$*-variable polynomial matrices* $\Psi \in \mathbb{C}^{nm\times m}[\zeta,\eta]$ *in* (23) *and* $\Gamma \in \mathcal{G}_\mathrm{c}$ *satisfying* (24) *with the image representation* (5)*.*

(iii) The inequality (26) *holds for all* $w \in \mathfrak{B}$ *with the image representation* (5) *and* $\ell \in \mathcal{D}^\infty(\mathbb{R}^n,\mathbb{C}^m)$ *satisfying*

$$\tau_k \int_{\mathbb{R}^n} \left(\frac{\partial z_k}{\partial x_k}\frac{\partial z_k^*}{\partial x_k} - \varpi_k^2 z_k z_k^*\right) \mathrm{d}x \leq 0\ (k = 1,\cdots,n),$$

where $z_k \in \mathcal{D}^\infty(\mathbb{R}^n, \mathbb{C}^{\{\prod_{l=1}^{n}(N_{k,l}+1)\}m})$ *is defined by* (28) *for some multi-index* $N_k := (N_{k,1},\cdots,N_{k,n}) \in \mathbb{Z}^n$*.*

Proof The proof is straightforward because it can be proved in a similar way to Lemma B.1. □

Appendix B.1.3 Proof of Theorem 1

As we have completed the proof of Lemma B.2, we conclude the proof of Theorem 1.

Consider the following transformations of the frequency variables

$$\omega_\mathrm{g} := (\omega_{\mathrm{g},1},\cdots,\omega_{\mathrm{g},n}),\ \omega_{\mathrm{g},k} := \omega_k + \varpi_{k,+},\ \varpi_k := \varpi_{k,-}\ (k = 1,\cdots,n).$$

Then, we have $\omega_g \in \Omega$ if and only if $\omega \in \Omega_\mathrm{c}$ holds from Lemma B.2. This complete the proof of Theorem 1.

Appendix B.2 Proof of Corollary 1

(i)⇒(iii) We assume that the statement (i) holds. From Theorem 1, there exist $2n$-variable polynomial matrices $\Psi \in \mathbb{H}^{nm\times m}[\zeta,\eta]$ in (23) and $\Gamma \in \mathcal{G}$ satisfying the inequality (24) for all $w \in \mathfrak{B}$. This is the case if and only if (30) holds for some $\Delta \in \mathbb{H}^{q\times q}[\zeta,\eta]$ such that $\mathsf{Q}_\Delta(\ell) \geq 0, \forall\, w \in \mathfrak{B}$ with the image representation (5). Hence, we get the inequality

$$\mathsf{Q}_{\Delta+\Gamma}(w) = \mathsf{Q}_\Delta(w) + \mathsf{Q}_\Gamma(w) \geq 0, \quad \forall\, w \in \mathfrak{B}_\Omega.$$

Thus, the first part of (iii) follows.

In addition, by integrating (30) from $x_k = -\infty$ to $x_k = +\infty$ ($k = 1, \cdots, n$) along $\mathfrak{B}_\Omega \cap \mathcal{D}^\infty(\mathbb{R}^n, \mathbb{C}^q)$, we obtain

$$\int_{\mathbb{R}^n} \mathsf{Q}_{\Gamma+\Delta}(\ell)\mathrm{d}x = \int_{\mathbb{R}^n} \mathsf{Q}_\Phi(w)\mathrm{d}x, \quad \forall\, w \in \mathfrak{B}_\Omega \cap \mathcal{D}^\infty(\mathbb{R}^n, \mathbb{C}^q). \tag{B.10}$$

This shows that $\mathsf{Q}_{\Delta+\Gamma}(w)$ becomes a dissipation rate for $\mathfrak{B}_\Omega$ with respect to the supply rate $\mathsf{Q}_\Phi(w)$ from Definition 2 (ii). This completes the proof of (i)⇒(iii).

(iii)⇒(iv) Integrating (30) from $x_k = -\infty$ to $x_k = +\infty$ ($k = 1, \cdots, n$) along $\mathfrak{B}_\Omega \cap \mathcal{D}^\infty(\mathbb{R}^n, \mathbb{C}^q)$ yields (B.10). Since $\mathsf{Q}_{\Delta+\Gamma}(w)$ satisfies $\mathsf{Q}_{\Delta+\Gamma}(w) \geq 0, \forall\, w \in \mathfrak{B}_\Omega$, we get

$$\int_{\mathbb{R}^n} \mathsf{Q}_\Phi(w)\mathrm{d}x = \int_{\mathbb{R}^n} \mathsf{Q}_{\Gamma+\Delta}(\ell)\mathrm{d}x \geq 0, \quad \forall\, w \in \mathfrak{B}_\Omega \cap \mathcal{D}^\infty(\mathbb{R}^n, \mathbb{C}^q).$$

Hence, the statement (iv) holds.

(iv)⇒(i) Since the statement (iv) is equivalent to the statement (iii) of Theorem 1, the proof follows immediately from Proposition 1.

(ii)⇔(iii) The proof is straightforward from Proposition 1.

Appendix B.3 Proof of Proposition 3

(i)⇒(ii) Assume that the statement (i) holds. Then, there exist $\Psi' \in \mathbb{C}^{nq\times q}[\zeta, \eta]$ and $\Gamma' \in \mathcal{G}$ satisfying (35). We can choose these matrices as $\mathfrak{B}$-canonical $2n$-variable polynomial matrices from Lemma A.3. This concludes the claim.

(ii)⇒(iii) Assume that the statement (ii) holds. Since $\Upsilon'_k(\zeta, \eta)$ $(k = 1, \cdots, n)$ is $\mathfrak{B}$-canonical, $\mathsf{Q}_{\Upsilon'_k}(w)$ is expressed as

$$\mathsf{Q}_{\Upsilon'_k}(w) = (z'_k)^*\widetilde{\Upsilon'}_k z'_k,\ \widetilde{\Upsilon}'_k \in \mathbb{H}^{\{\prod_{l=1}^n (N'_{k,l}+1)\}q\times\{\prod_{l=1}^n (N'_{k,l}+1)\}q},\ z'_k := Z_{N'_k}\left(\frac{d}{dt}\right) w$$

for $N_k = (N_{k,1}, \cdots, N_{k,n})$, $N_{k,l} \leq \deg_{\xi_l} R - 1$ $(l = 1, \cdots, n)$. Then, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathsf{Q}_\Phi(w)\mathrm{d}x &\geq \sum_{k=1}^n \int_{\mathbb{R}^n} \mathrm{tr}\left[-\widetilde{\Upsilon}'_k \left\{\mathrm{He}\left(\frac{\partial z'_k}{\partial x_k} - j\varpi_{k,1} z'_k\right)\left(\frac{\partial z'_k}{\partial x_k} - j\varpi_{k,2} z'_k\right)^*\right\}\right]\mathrm{d}x \\
&= \mathrm{tr}\left[\tau_k \widetilde{\Upsilon}'_k \sum_{k=1}^n (-\tau_k) \int_{\mathbb{R}^n} \mathrm{He}\left[\left(\frac{\partial z'_k}{\partial x_k} - j\varpi_{k,1} z'_k\right)\left(\frac{\partial z'_k}{\partial x_k} - j\varpi_{k,2} z'_k\right)^*\right]\mathrm{d}x\right] \\
&= \mathrm{tr}\left[\begin{bmatrix} \tau_1\widetilde{\Upsilon}'_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tau_n\widetilde{\Upsilon}'_n \end{bmatrix}\begin{bmatrix} \tau_1 D_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tau_n D_n \end{bmatrix}\right],
\end{aligned}$$

where $D_k \in \mathbb{H}^{\{\prod_{l=1}^n (N'_{k,l}+1)\}q\times\{\prod_{l=1}^n (N'_{k,l}+1)\}q}$ $(k = 1, \cdots, n)$ is the constant matrix defined by

$$D_k := -\int_{\mathbb{R}^n} \mathrm{He}\left[\left(\frac{\partial z'_k}{\partial x_k} - j\varpi_{k,1} z'_k\right)\left(\frac{\partial z'_k}{\partial x_k} - j\varpi_{k,2} z'_k\right)^*\right]\mathrm{d}x.$$

Since $\tau_k\widetilde{\Upsilon}'_k \geq 0, \forall\, k = 1, \cdots, n$ holds from (34) and Lemma A.1, we have (26) for all $w \in \mathfrak{B}$ satisfying (36). This concludes the claim.

(iii)⇒(i) The proof is straightforward from Theorem 1.

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