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Shinji ITO, Kensuke AISHIMA, Takaaki NARA and Masaaki SUGIHARA

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DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

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Orthogonal Polynomial Approach to Estimation of Poles of Meromorphic Functions from Data on Open Curves

Shinji ITO *, Kensuke AISHIMA*, Takaaki NARA†
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Abstract

This paper deals with the problem to find poles of rational functions from function values on open curves on the complex plane. For this problem, Nara and Ando have recently proposed an algorithm that reduces the problem to a linear equation through contour integration. The main aim of this paper is to analyze and improve this algorithm by giving a new interpretation to the algorithm in terms of orthogonal polynomials. It is demonstrated that the linear equation is not always uniquely solvable and that this difficulty can be remedied by doubling the number of the linear equations. Moreover, to cope with discretization errors caused by numerical integration, we introduce new polynomials similar, in spirit, to discrete orthogonal polynomials, which yield an algorithm free from discretization errors.

1 Introduction

Inverse source problems have a wide variety of applications, such as in electronic impedance tomography (EIT) [1, 2, 3, 4, 6, 11], electroencephalography (EEG) [3, 7], magnetoencephalography (MEG) [17], and so on. It is known that many of such problems can be reduced to estimating locations of poles of meromorphic functions [3, 9, 17]. In particular for 2D EIT problems, it is shown in [3] that the data, that is the difference between the electric potential on the boundary of a domain with and without small inclusions, is the real part of a meromorphic function with simple poles in the domain with some holomorphic function. See also [9] for the relationship between small inclusions in 2D EIT and poles of a rational function. For the details of EIT, the readers are referred to the books by Ammari and Kang [1, 2]. Furthermore, it is also shown in [17] that EEG and MEG, which are formulated in 3D space, can be reduced to localizing poles of rational functions. Pole identification of rational functions also plays important roles in eigenvalue problems and computing zeros of analytic functions [12, 13, 18].

For the estimation from boundary measurement, which means data on closed surfaces or curves, various algorithms have been proposed together with theoretical analysis of their behaviors [7, 10, 11, 14, 15, 19]. Estimation from data on open surfaces or curves is also important. For example, in EIT, it is not possible in many cases to place sensors for measuring the voltage on the entire boundary. Thus it is useful in practice if we can estimate the defect position from partial boundary data by some algebraic method. In

*Department of Mathematical Informatics, University of Tokyo, Tokyo 113-8656, Japan. {shinji_ito, Kensuke_Aishima}@mist.i.u-tokyo.ac.jp

†Depertment of Information Physics and Computing, University of Tokyo, Tokyo 113-8656, Japan. nara@alab.t.u-tokyo.ac.jp

‡Department of Physics and Mathematics, Aoyama Gakuin University, 5-10-1 Nobe, Chuo-ku, Kanagawa 252-5258, Japan. sugihara@gem.aoyama.ac.jp

spite of such practical needs, not much has been studied about estimation of poles from data on open curves. Among the few existing methods for such problems are Baratchart et al. [4], Ben Abda et al. [5], and Nara–Ando [16], which propose different kinds of algorithms. The algorithms of [4] and [5] involve iterative computing, which makes the error analysis harder. On the other hand, Nara–Ando's algorithm is an algebraic method that is free from iterative computing and is amenable to theoretical analysis.

The aim of this paper is to analyze and improve Nara–Ando's algorithm on the basis of a novel idea of orthogonal polynomials. The Nara–Ando's algorithm determines a polynomial corresponding to the denominator of rational functions by solving a system of linear equations. Our approach using orthogonal polynomials gives an alternative derivation of the system of linear equations, and furthermore, enables us to obtain new theorems and improved algorithms regarding unique solvability of the linear equations and numerical errors.

To be specific, our approach gives the following results:

- It is claimed in [16] that the linear equation is uniquely solvable. We disprove this claim by giving counterexamples.
- But we can remedy this situation by doubling the number of equations, which makes the linear equations uniquely solvable in all cases.
- Numerical errors caused by numerical integration can be eliminated by introducing new polynomials which have a property similar to orthogonality with respect to numerical integration.

We deal with the following problem.

Problem 1. *Find the poles $\{z_1, z_2, \ldots, z_N\}$ of a rational function f, where it is assumed that f can be represented in the form of*

$$f(z) = \sum_{k=1}^{N} \frac{\mu_k}{z - z_k}$$

with $\mu_k \neq 0$ and distinct z_k. The number N of poles is known and the values of μ_k and z_k are unknown. A simple open curve Γ is given, and the values of f at any points on Γ are available.

In practical situations, we need to consider noise in given data. In this paper, however, we assume that exact data are given and focus on theoretical analysis under this assumption. It is also noted that, once the poles $z_1, z_2, \ldots, z_N$ are identified, the values of $\mu_1, \mu_2, \ldots, \mu_N$ can be easily computed by the method of least squares.

The paper is organized as follows. In Sect. 2, we briefly describe Nara–Ando's algorithm. In Sect. 3, orthogonal polynomials are introduced and their significance for Nara–Ando's algorithm is shown. Section 4 discusses the unique solvability of the linear equation in Nara–Ando's algorithm. We first give some examples for which the linear equation is not uniquely solvable and then show that such difficulty can be remedied by doubling the number of the linear equations. In Sect. 5, we propose a new algorithm that is free from discretization errors caused by numerical integration. Moreover, it is numerically demonstrated that the proposed algorithm is indeed effective to cope with numerical errors.

2 The algorithm by Nara and Ando

We shall describe the algorithm for Problem 1 proposed by Nara and Ando [16]. In Nara–Ando's algorithm, a polynomial P of degree N is computed such that $f(z)P(z)$ is holomorphic. The poles of f can be computed as the roots of such P by solving the algebraic equation $P(z) = 0$. Here the essential problem is how to obtain the polynomial P. By associating an N-dimensional vector with a monic polynomial of degree N, the problem of finding P is translated into finding the associated N-dimensional vector. The desired vector is obtained as the solution of a linear equation in Nara–Ando's algorithm.

Let Γ be a simple open curve whose end points are -1 and 1. For a vector of complex variables $\mathbf{c} = (c_1, c_2, \ldots, c_N) \in \mathbb{C}^N$, let $P_{\mathbf{c}}$ be a polynomial of degree N defined by

$$P_{\mathbf{c}}(z) := z^N - \sum_{j=1}^{N} c_j z^{j-1},$$

and let $\mathbf{c}^*$ denote the vector satisfying

$$P_{\mathbf{c}^*}(z) = \prod_{k=1}^{N} (z - z_k) \tag{1}$$

for the solution $z_1, z_2, \ldots, z_N$ of Problem 1. In the following, we use the notation

$$\binom{n}{m} := \begin{cases} \frac{n!}{m!(n-m)!} & (n \geq m), \\ 0 & (n < m). \end{cases}$$

The vector $\mathbf{c}^*$ has the following property, which plays the key role throughout this paper. We include the original proof since we shall shed a new light on this proof from our viewpoint in Sect. 3.

Theorem 1 (Nara–Ando [16])**.**
For all nonnegative integers j, let

$$h_j := \int_{\Gamma} f(z) z^j \mathrm{d}z. \tag{2}$$

Then for all nonnegative integers n, the following equation holds:

$$\sum_{j=0}^{N} (-1)^j \binom{N}{j} \binom{n+2j}{N} (h_{n+2j} - c_N^* h_{n+2j-1} - \cdots - c_1^* h_{n+2j-N}) = 0. \tag{3}$$

Proof. Since $P_{\mathbf{c}^*}(z) f(z)$ is a polynomial of degree at most $N - 1$, for all nonnegative integers n, we have

$$\int_{\Gamma} z^n \frac{\mathrm{d}^N}{\mathrm{d}z^N} (P_{\mathbf{c}^*}(z) f(z)) \mathrm{d}z = 0. \tag{4}$$

Applying integration by parts N times on the left-hand side of (4), we obtain

$$\begin{aligned} &\left[\sum_{k=0}^{N-1} (-1)^k \frac{\mathrm{d}^k z^n}{\mathrm{d}z^k} \frac{\mathrm{d}^{N-k-1} (P_{\mathbf{c}^*}(z) f(z))}{\mathrm{d}z^{N-k-1}} \right]_{-1}^{1} \\ &+ (-1)^N N! \binom{n}{N} (h_n - c_N^* h_{n-1} - \cdots - c_1^* h_{n-N}) = 0. \end{aligned} \tag{5}$$

Hence, for all nonnegative integers n,

$$\begin{aligned} &\sum_{j=0}^{N} (-1)^j \binom{N}{j} \left[\sum_{k=0}^{N-1} (-1)^k \frac{\mathrm{d}^k z^{n+2j}}{\mathrm{d}z^k} \frac{\mathrm{d}^{N-k-1} (P_{\mathbf{c}^*}(z) f(z))}{\mathrm{d}z^{N-k-1}} \right]_{-1}^{1} \\ &+ \sum_{j=0}^{N} (-1)^{j+N} \binom{N}{j} N! \binom{n+2j}{N} (h_{n+2j} - \sum_{l=1}^{N} c_l^* h_{n+2j-N-1+l}) = 0 \end{aligned}$$

holds. Here, for the first half of the left-hand side of this expression, we have

$$\begin{aligned}
&\sum_{j=0}^{N}(-1)^j\binom{N}{j}\left[\sum_{k=0}^{N-1}(-1)^k\frac{\mathrm{d}^k z^{n+2j}}{\mathrm{d}z^k}\frac{\mathrm{d}^{N-k-1}(P_{\mathbf{c}^*}(z)f(z))}{\mathrm{d}z^{N-k-1}}\right]_{-1}^{1}\\
&=\left[\sum_{k=0}^{N-1}(-1)^k\frac{\mathrm{d}^k}{\mathrm{d}z^k}\left(\sum_{j=0}^{N}(-1)^j\binom{N}{j}z^{n+2j}\right)\frac{\mathrm{d}^{N-k-1}(P_{\mathbf{c}^*}(z)f(z))}{\mathrm{d}z^{N-k-1}}\right]_{-1}^{1}\\
&=\left[\sum_{k=0}^{N-1}(-1)^k\frac{\mathrm{d}^k}{\mathrm{d}z^k}\left(z^n(1-z^2)^N\right)\frac{\mathrm{d}^{N-k-1}(P_{\mathbf{c}^*}(z)f(z))}{\mathrm{d}z^{N-k-1}}\right]_{-1}^{1} \qquad (6)
\end{aligned}$$

and this value is equal to 0. This leads to (3). □

An essential point of the proof of Theorem 1 is that linear combinations of the left-hand side of equation (5) make the boundary terms vanish.

Theorem 1 states that $\mathbf{c}=\mathbf{c}^*$, where $\mathbf{c}^*$ is defined by (1), satisfies

$$\sum_{j=0}^{N}(-1)^j\binom{N}{j}\binom{n+2j}{N}(h_{n+2j}-c_N h_{n+2j-1}-\cdots-c_1 h_{n+2j-N})=0 \qquad (7)$$

for all nonnegative integers n. It is shown empirically in [16] that $\mathbf{c}^*$ is the unique solution of the system of N linear equations (7) with $n=0,1,\ldots,N-1$.

The above facts lead to Nara–Ando's pole estimation algorithm, which reads as follows.

Algorithm 1 ([16]).
(1) From the value of $f(z)$ *on* Γ*, compute* h_j *in equation* (2) *for* $j=0,1,\ldots,3N-1$.
(2) Solve the system of N *linear equations* (7)*, where* $n=0,1,\ldots,N-1$*, and let* $\mathbf{c}^*$ *be the solution.*
(3) Solve the algebraic equation in $z^N-c_N^*z^{N-1}-\cdots-c_1^*=0$ *in* z *and output the solutions as* $z_1,z_2,\ldots,z_N$.

3 Proposed approach with orthogonal polynomials

In this section, we present an alternative derivation of the system of linear equations in Theorem 1 by introducing a system of orthogonal polynomials suitable for our theoretical analysis.

Motivated by the expression (6), let us define polynomials

$$q_n^{(N)}(z):=\frac{\mathrm{d}^N}{\mathrm{d}z^N}\left(z^n\left(1-z^2\right)^N\right) \qquad (8)$$

for nonnegative integers n. Note an alternative expression

$$q_n^{(N)}(z)=N!\sum_{k=1}^{N}\binom{N}{k}\binom{2k+n}{N}(-1)^k z^{2k+n-N}, \qquad (9)$$

which shows that $q_n^{(N)}$ is a polynomial of degree $N+n$. The following orthogonality property of this polynomial plays the key role in this paper.

Lemma 1.
For all polynomials p *of degree at most* $N-1$ *and for all nonnegative integers* n*, we have*

$$\int_\Gamma q_n^{(N)}(z)p(z)\mathrm{d}z=0. \qquad (10)$$

Proof. For $j \in \{0, 1, \ldots, N-1\}$, we have

$$\frac{\mathrm{d}^j}{\mathrm{d}z^j}\left(z^n\left(1-z^2\right)^N\right)\bigg|_{z=1} = \frac{\mathrm{d}^j}{\mathrm{d}z^j}\left(z^n\left(1-z^2\right)^N\right)\bigg|_{z=-1} = 0.$$

From this equation, the left-hand side of (10) is rewritten as follows:

$$\begin{aligned}
&\int_\Gamma q_n^{(N)}(z)p(z)\mathrm{d}z \\
&= \int_\Gamma \frac{\mathrm{d}^N}{\mathrm{d}z^N}\left(z^n\left(1-z^2\right)^N\right)p(z)\mathrm{d}z \\
&= \left[\frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}}\left(z^n\left(1-z^2\right)^N\right)p(z)\right]_{-1}^{1} - \int_\Gamma \frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}}\left(z^n\left(1-z^2\right)^N\right)\frac{\mathrm{d}}{\mathrm{d}z}p(z)\mathrm{d}z \\
&= -\int_\Gamma \frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}}\left(z^n\left(1-z^2\right)^N\right)\frac{\mathrm{d}}{\mathrm{d}z}p(z)\mathrm{d}z \\
&= \cdots = (-1)^N \int_\Gamma z^n\left(1-z^2\right)^N \frac{\mathrm{d}^N}{\mathrm{d}z^N}p(z)\mathrm{d}z.
\end{aligned}$$

Since p is a polynomial of degree at most $N-1$, we have $\frac{\mathrm{d}^N}{\mathrm{d}z^N}p(z) = 0$, and hence the above integral vanishes. □

For $i \in \{1, 2, \ldots\}$ and $j \in \{1, 2, \ldots, N\}$, let a_{ij} and b_i be defined by

$$a_{ij} := \int_\Gamma q_{i-1}^{(N)}(z)f(z)z^{j-1}\mathrm{d}z, \quad b_i := \int_\Gamma q_{i-1}^{(N)}(z)f(z)z^N\mathrm{d}z. \tag{11}$$

Note that a_{ij} and b_i can be computed from the values of f on the curve Γ. Then the following theorem provides a system of linear equations for $\mathbf{c}^*$.

Theorem 2.
For all $i \in \{1, 2, \ldots\}$, the equation

$$\sum_{j=1}^{N} a_{ij}c_j^* = b_j \tag{12}$$

holds.

Proof. It is easily seen that

$$\begin{aligned}
b_i - \sum_{j=1}^{N} a_{ij}c_j^* &= \int_\Gamma q_{i-1}^{(N)}(z)f(z)z^N\mathrm{d}z - \sum_{j=1}^{N}\left(c_j^*\int_\Gamma q_{i-1}^{(N)}(z)f(z)z^{j-1}\mathrm{d}z\right) \\
&= \int_\Gamma q_{i-1}^{(N)}(z)f(z)\left(z^N - \sum_{j=1}^{N} c_j^* z^{j-1}\right)\mathrm{d}z \\
&= \int_\Gamma q_{i-1}^{(N)}(z)f(z)P_{\mathbf{c}^*}(z)\mathrm{d}z \\
&= \int_\Gamma q_{i-1}^{(N)}(z)f(z)\prod_{k=1}^{N}(z-z_k)\mathrm{d}z.
\end{aligned} \tag{13}$$

This value is equal to 0 since $f(z)\prod_{k=1}^{N}(z-z_k)$ is a polynomial of degree at most $N-1$. □

Here we define $A \in \mathbb{C}^{N\times N}$ and $\mathbf{b} \in \mathbb{C}^N$ by

$$A := (a_{ij})_{1\le i,j\le N}, \quad \mathbf{b} := (b_i)_{1\le i\le N}. \tag{14}$$

Then Theorem 2 shows that the system $A\mathbf{c} = \mathbf{b}$ of N linear equations in $\mathbf{c}$ has $\mathbf{c}^*$ as a solution. Furthermore, this linear equation coincides with the linear equation in Nara–Ando's algorithm, as is verified as follows. By (9) we obtain

$$\begin{aligned}
&b_i - \sum_{j=1}^{N} a_{ij} c_i \\
&= N! \sum_{k=1}^{N} \binom{N}{k} \binom{2k+i-1}{N} (-1)^k \int_\Gamma \left(z^{2k+i-2} - \sum_{j=1}^{N} c_j z^{2k+i+j-2-N} \right) \mathrm{d}z \\
&= N! \sum_{k=1}^{N} \binom{N}{k} \binom{2k+i-1}{N} (-1)^k \left(h_{2k+i-1} - \sum_{j=1}^{N} c_j h_{2k+i+j-2-N} \right).
\end{aligned}$$

This expression is equivalent to the left-hand side of the equation (7), where $n = i - 1$.

We have thus demonstrated an alternative derivation of the key system of equations by means of our approach based on polynomials $q_n^{(N)}$. In the following sections, we pursue this approach to discuss unique solvability of the system of equations in Nara–Ando's algorithm and to devise an algorithm free from numerical errors.

4 Unique solvability of the linear equations in the algorithm

In this section, we shall discuss unique solvability of the linear equation $A\mathbf{c} = \mathbf{b}$. In [16], no examples with singular A were observed. As is shown in Sect. 4.1, actually, the linear equation $A\mathbf{c} = \mathbf{b}$ has a unique solution in most cases. However, the fact is that there exist examples with singular A, and we shall prove this in terms of specific examples. Furthermore, we show that doubling the number of equations always yields a unique solvable system.

4.1 Unique solvability in most cases

We shall show that the linear equation $A\mathbf{c} = \mathbf{b}$ defined by (11) has a unique solution in most cases. Let $B := (b_{ij}) \in \mathbb{C}^{N\times N}$ be a matrix with entries b_{ij} $(1 \le i, j \le N)$ defined by

$$b_{ij} := \int_\Gamma q_{i-1}^{(N)}(z) \frac{1}{z - z_j} \mathrm{d}z \tag{15}$$

using the polynomial $q_{i-1}^{(N)}(z)$ introduced in (8). For $f(z) = \sum_{k=1}^{N} \frac{\mu_k}{z - z_k}$ the entry a_{ij} in (11) is calculated as

$$\begin{aligned}
a_{ij} &= \sum_{k=1}^{N} \int_\Gamma q_{i-1}^{(N)}(z) \frac{\mu_k z^{j-1}}{z - z_k} \mathrm{d}z \\
&= \sum_{k=1}^{N} \mu_k \int_\Gamma q_{i-1}^{(N)}(z) \left(\frac{z_k^{j-1}}{z - z_k} + (\text{a polynomial of degree at most } N-1) \right) \mathrm{d}z \\
&= \sum_{k=1}^{N} b_{ik} \mu_k z_k^{j-1},
\end{aligned}$$

where the orthogonality property (10) in Lemma 1 is used. Therefore, we have

$$A = BDV, \quad \mathbf{b} = BD\mathbf{v}, \tag{16}$$

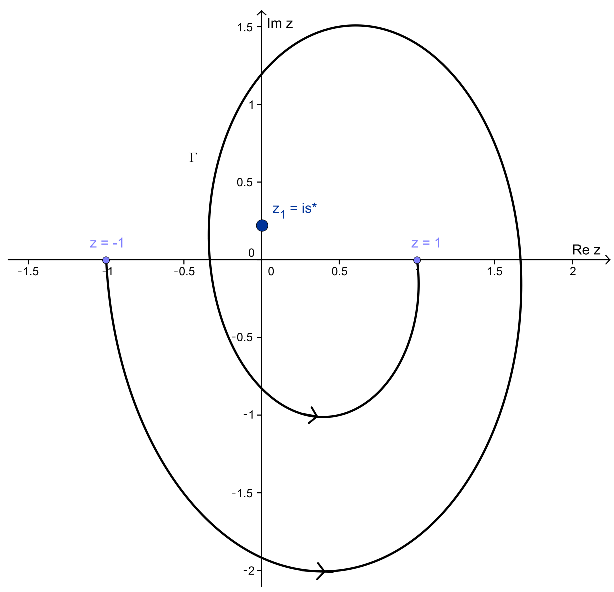


Figure 1: An example which gives a singular equation for $N = 1$

where

$$D := \begin{pmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_N \end{pmatrix}, \quad V := \begin{pmatrix} z_1^0 & z_1^1 & \dots & z_1^{N-1} \\ z_2^0 & z_2^1 & \dots & z_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ z_N^0 & z_N^1 & \dots & z_N^{N-1} \end{pmatrix}, \quad \mathbf{v} := \begin{pmatrix} z_1^N \\ z_2^N \\ \vdots \\ z_N^N \end{pmatrix}.$$

Since the ith component of $\mathbf{v} - V\mathbf{c}$ is equal to $P_{\mathbf{c}}(z_i)$ for $i = 1, 2, \dots, N$, it holds that $\mathbf{v} - V\mathbf{c}^* = \mathbf{0}$. Hence $\mathbf{b} - A\mathbf{c}^* = BD(\mathbf{v} - V\mathbf{c}^*) = \mathbf{0}$, which shows that $\mathbf{c} = \mathbf{c}^*$ is a solution of $A\mathbf{c} = \mathbf{b}$.

Furthermore, since D and V are nonsingular matrices, (16) means that $A\mathbf{c} = \mathbf{b}$ is uniquely solvable if and only if B is nonsingular. The next proposition implies that the equation $A\mathbf{c} = \mathbf{b}$ is uniquely solvable in most cases.

Proposition 1. *Let S be the set of all $(z_1, z_2, \dots, z_N) \in (\mathbb{C} \setminus \Gamma)^N$ for which B defined by* (15) *is singular. Then S is a closed subset of $(\mathbb{C} \setminus \Gamma)^N$ with empty interior.*

Proof. This is proved in Appendix A. □

4.2 Problem instances with singular systems of equations

We prove that there exist problem instances for which the system of N equations $A\mathbf{c} = \mathbf{b}$ is not uniquely solvable. We begin with $N = 1$ and then proceed to general N.

The following theorem deals with the case of $N = 1$.

Proposition 2.
For $N = 1$, there exist f and Γ for which A is singular.

Proof. Let Γ be the curve shown in Fig. 1. Since $N = 1$ and $q_0^{(1)}(z) = -2z$ by (8), we have $A = (a_{11})$ with

$$a_{11} = -2\mu_1 \int_\Gamma \frac{z}{z - z_1} \mathrm{d}z.$$

For $s \in [0,1]$, we define $\psi(s)$ to be this integral with $z_1 = \mathrm{i}s$, i.e.,

$$\psi(s) = \int_\Gamma \frac{z}{z - \mathrm{i}s} \mathrm{d}z.$$

Then ψ is clearly continuous. As is proven below, $\psi(s)$ is real-valued and $\psi(0)\psi(1) < 0$. It then follows that there exists $s^* \in [0,1]$ satisfying $\psi(\mathrm{i}s^*) = 0$. Defining $f(z) = \frac{\mu_1}{z - \mathrm{i}s^*}$ with this s^*, we have $a_{11} = 0$, and thus the desired f and Γ are constructed.

What is left is to show that $\psi(s)$ is real-valued and $\psi(0)\psi(1) < 0$. For each $s \in [0,1]$,

$$\psi(s) = \int_\Gamma \frac{z}{z - \mathrm{i}s} \mathrm{d}z = \int_\Gamma \frac{\mathrm{i}s}{z - \mathrm{i}s} \mathrm{d}z + 2. \tag{17}$$

Let $\gamma : [0,1] \to \Gamma$ be the homeomorphism defining Γ, and express $\gamma(t)$ as

$$\gamma(t) = \mathrm{i}s + r_s(t) \exp(\mathrm{i}\theta_s(t)), \quad t \in [0,1],$$

using continuous functions $r_s : [0,1] \to \mathbb{R}$ and $\theta_s : [0,1] \to \mathbb{R}$, where $r_s(t) > 0$ for all $t \in [0,1]$, and i is the imaginary unit. Then, since $r_s(0) = |-1 - \mathrm{i}s| = |1 - \mathrm{i}s| = r_s(1)$, we have

$$\begin{aligned} \int_\Gamma \frac{1}{z - \mathrm{i}s} \mathrm{d}z &= \int_0^1 \left(\frac{r_s'(t)}{r_s(t)} + \mathrm{i}\theta_s'(t) \right) \mathrm{d}t \\ &= \log \left(\frac{r_s(1)}{r_s(0)} \right) + \mathrm{i} \left(\theta_s(1) - \theta_s(0) \right) \\ &= \mathrm{i} \left(\theta_s(1) - \theta_s(0) \right). \end{aligned} \tag{18}$$

It follows from (17) and (18) that

$$\psi(s) = -s(\theta_s(1) - \theta_s(0)) + 2 \in \mathbb{R}.$$

From this expression, one can easily verify that $\psi(0) = 2 > 0$ and $\psi(1) = -\frac{5}{2}\pi + 2 < 0$. $\square$

The above proposition implies that there exists an example for which Nara–Ando's algorithm breaks down. Furthermore, the following proposition shows that for all positive integers N, there exist examples such that the linear equation $A\mathbf{c} = \mathbf{b}$ is not uniquely solvable.

Proposition 3.
For each positive integer N, there exist f and Γ for which A is singular.

Proof. As is discussed in Sect. 4.1, it follows from (16) that A is nonsingular or singular, according as B defined by (15) is nonsingular or singular. We shall construct f and Γ such that the first or second row of B vanishes depending on the parity of N.

We first consider the case of odd N. For $s > 0$, it follows from residue theorem that

$$\int_\Gamma q_0^{(N)}(z) \frac{1}{z - \mathrm{i}s} \mathrm{d}z = \int_{-1}^1 q_0^{(N)}(z) \frac{1}{z - \mathrm{i}s} \mathrm{d}z + m(s, \Gamma) \times 2\pi \mathrm{i} q_0^{(N)}(\mathrm{i}s), \tag{19}$$

where $m(s, \Gamma)$ is an integer determined by the position of is relative to Γ. For example, $m(s^*, \Gamma) = 1$ in Fig. 1. By setting

$$\rho(s) := \int_{-1}^1 q_0^{(N)}(z) \frac{1}{z - \mathrm{i}s} \mathrm{d}z, \quad \xi(s) := 2\pi \mathrm{i} q_0^{(N)}(\mathrm{i}s), \tag{20}$$

we can express (19) as

$$\int_\Gamma q_0^{(N)}(z) \frac{1}{z - \mathrm{i}s} \mathrm{d}z = \rho(s) + m(s, \Gamma)\xi(s). \tag{21}$$

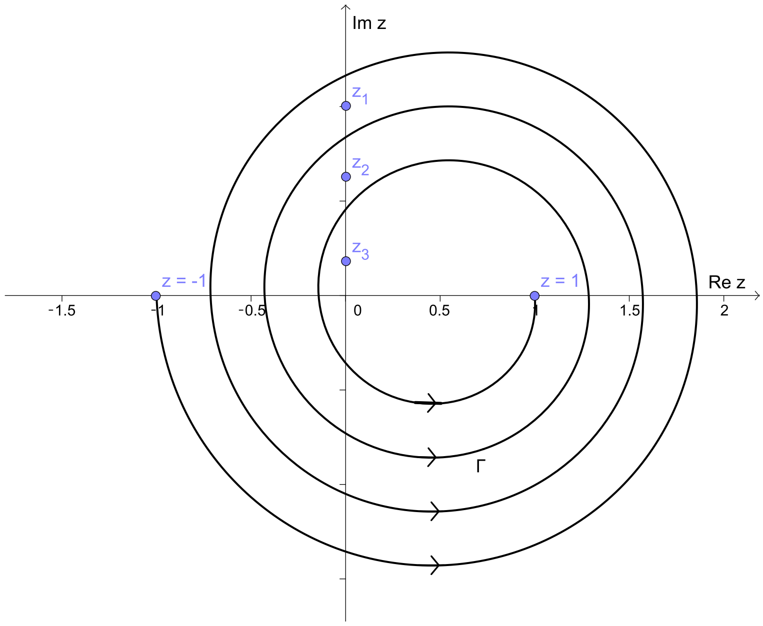


Figure 2: An example which gives a singular equation for $N = 3$

It will be shown below that:

$$\text{There exist } s_1 > \cdots > s_N > 0 \text{ s.t. } \rho(s_k) + k\xi(s_k) = 0 \ (k = 1, 2, \dots, N). \tag{22}$$

For these $s_1, s_2, \dots, s_N$, we set $z_k := \mathrm{i}s_k$ and take Γ for which $m(s_k, \Gamma) = k$ for $k = 1, 2, \dots, N$, as shown in Fig. 2. Then we have $b_{1k} = \rho(s_k) + m(s_k, \Gamma)\xi(s) = 0$ for all k and hence the first row of B vanishes, implying that A is singular.

In order to show (22), it is sufficient to show that the following three properties:

- $\rho(s)$ and $\xi(s)$ are real-valued for $s > 0$. (23)
- $\xi(s)$ is not equal to 0 for all $s > 0$. (24)
- $\lim_{s\to\infty} \frac{\rho(s)}{\xi(s)} = 0$ and $\lim_{s\downarrow 0} \frac{\rho(s)}{\xi(s)} = -\infty$. (25)

It follows from these three properties that there exist $s_1 > s_2 > \cdots > s_N > 0$ such that $\frac{\rho(s)}{\xi(s)} = -k$ for $k = 1, 2, \dots, N$, which proves (22). By expanding $q_0^{(N)}$, we obtain

$$\begin{aligned} q_0^{(N)}(z) &= \frac{\mathrm{d}^N}{\mathrm{d}z^N}\left(\left(1 - z^2\right)^N\right) \\ &= \frac{\mathrm{d}^N}{\mathrm{d}z^N}\left(\sum_{k=0}^{N}\binom{N}{k}(-1)^k z^{2k}\right) \\ &= N!(-1)^{\frac{N+1}{2}}\sum_{l=0}^{\frac{N-1}{2}}\binom{N}{l+\frac{N+1}{2}}\binom{2l+N+1}{N}(-1)^l z^{2l+1}. \end{aligned} \tag{26}$$

Since N is odd, $q_0^{(N)}(z)$ is an odd function. Then $\xi(s)$ in (20) is given by

$$\begin{aligned} \xi(s) &= 2\pi\mathrm{i}N!(-1)^{\frac{N+1}{2}}\sum_{l=0}^{\frac{N-1}{2}}\binom{N}{l+\frac{N+1}{2}}\binom{2l+N+1}{N}(-1)^l(\mathrm{i}s)^{2l+1} \\ &= 2\pi N!(-1)^{\frac{N-1}{2}}s\sum_{l=0}^{\frac{N-1}{2}}\binom{N}{l+\frac{N+1}{2}}\binom{2l+N+1}{N}s^{2l}. \end{aligned} \tag{27}$$

Hence $\xi(s)$ is real-valued and has the same sign as $(-1)^{\frac{N-1}{2}}$ for all positive s. In particular, $\xi(s)$ is not equal to 0 for any $s>0$. Next, $\rho(s)$ defined in (20) can be expressed by

$$\rho(s)=\int_{-1}^{1}\frac{q_0^{(N)}(z)}{z-\mathrm{i}s}\mathrm{d}z=\int_{-1}^{1}\frac{zq_0^{(N)}(z)}{z^2+s^2}\mathrm{d}z+\mathrm{i}s\int_{-1}^{1}\frac{q_0^{(N)}(z)}{z^2+s^2}\mathrm{d}z.$$

Since $q_0^{(N)}$ is an odd function, the last integral in the above equation vanishes, and hence we have

$$\rho(s)=\int_{-1}^{1}\frac{zq_0^{(N)}(z)}{z^2+s^2}\mathrm{d}z.$$

This value is real since $q_0^{(N)}$ is a polynomial with real coefficients, and hence the first two properties (23) and (24) are shown.

We now turn to (25). Since $\frac{q_0^{(N)}(z)}{z}$ is a polynomial by (26), the set $\{|\frac{q_0^{(N)}(z)}{z}| : z\in[-1,1]\}$ is bounded above. Let us denote by M an upper bound of this set. Then we have

$$\left|\frac{zq_0^{(N)}(z)}{z^2+s^2}\right|\le\left|\frac{zq_0^{(N)}(z)}{z^2}\right|\le M$$

for all $z\in[-1,1]$ and $s>0$. Thus $|\rho(s)|\le 2M$ holds and

$$\lim_{s\downarrow 0}\rho(s)=\int_{-1}^{1}\lim_{s\downarrow 0}\left(\frac{zq_0^{(N)}(z)}{z^2+s^2}\right)\mathrm{d}z=\int_{-1}^{1}\frac{q_0^{(N)}(z)}{z}\mathrm{d}z$$

follows from Lebesgue's dominated convergence theorem. Let us denote the value of this integral by α. By using the expression (9) and the orthogonality (10) in Lemma 1, we have

$$\begin{aligned}
&\int_{-1}^{1}\left(\frac{q_0^{(N)}(z)}{z}\right)^2\mathrm{d}z\\
&=N!(-1)^{\frac{N+1}{2}}\sum_{l=0}^{\frac{N-1}{2}}\binom{N}{l+\frac{N+1}{2}}\binom{2l+N+1}{N}(-1)^l\int_{-1}^{1}\frac{q_0^{(N)}(z)}{z}z^{2l}\mathrm{d}z\\
&=N!(-1)^{\frac{N+1}{2}}\binom{N}{\frac{N+1}{2}}\binom{N+1}{N}\int_{-1}^{1}\frac{q_0^{(N)}(z)}{z}\mathrm{d}z\\
&=N!(-1)^{\frac{N+1}{2}}\binom{N}{\frac{N+1}{2}}\binom{N+1}{N}\alpha.
\end{aligned}$$

This shows that α is a non-zero real number with the same sign as $(-1)^{\frac{N+1}{2}}$, which means that α has the same sign as $-\xi(s)$ with $s>0$. Since $\lim_{s\downarrow 0}\xi(s)=0$ by (26), we have

$$\lim_{s\downarrow 0}\frac{\rho(s)}{\xi(s)}=\lim_{s\downarrow 0}\frac{\alpha}{\xi(s)}=-\infty.$$

On the other hand, since it is also clear by (27) that $\lim_{s\to\infty}|\xi(s)|=\infty$, we obtain

$$0\le\lim_{s\to\infty}\left|\frac{\rho(s)}{\xi(s)}\right|\le\lim_{s\to\infty}\frac{2M}{|\xi(s)|}=0.$$

Thus the third property (25) is shown to hold, which completes the proof for odd N.

For even N, $q_1^{(N)}(z)$ is an odd function. By replacing $q_0^{(N)}(z)$ by $q_1^{(N)}(z)$ in the above argument, we can construct f and Γ for which the second row of B vanishes. □

4.3 Proposed method to guarantee nonsingular systems of equations

Proposition 3 in Section 4.1 points out a difficulty of Nara–Ando's algorithm (Algorithm 1). This difficulty, however, can be resolved by doubling the number of linear equations. In this section, we shall show that the system of linear equations (7) for $n = 0, 1, \ldots, 2N-1$, which has twice as many equations as the variables, is uniquely solvable for arbitrary poles and open curves.

Let A_M denote the matrix with M rows and N columns whose (i, j) entry is given by a_{ij} in (11). Similarly, let $\mathbf{b}_M$ denote the M-dimensional column vector whose ith element is given by b_i in (11):

$$A_M := (a_{ij})_{1\le i\le M, 1\le j\le N} \in \mathbb{C}^{M\times N}, \quad \mathbf{b}_M := (b_i)_{1\le i\le M} \in \mathbb{C}^M. \tag{28}$$

Note that $A = A_N$ and $\mathbf{b} = \mathbf{b}_N$ for the matrix A and vector $\mathbf{b}$ in (14). Then Theorem 2 means that $\mathbf{c}^*$ in (1) satisfies

$$A_M\mathbf{c}^* = \mathbf{b}_M$$

for all positive integers M. This is equivalent to the fact that the linear equation

$$A_M\mathbf{c} = \mathbf{b}_M \tag{29}$$

has a solution $\mathbf{c}^*$ for all M.

The linear equation (29) with $M = N$ used in Nara–Ando's algorithm (Algorithm 1) is not always uniquely solvable. It would be natural to expect that $A_M\mathbf{c} = \mathbf{b}_M$ for some M greater than N is uniquely solvable. Accordingly, we prove that $A_M\mathbf{c} = \mathbf{b}_M$ is uniquely solvable when $M = 2N$, i.e., when the number of equations is twice the number of variables.

Definition 1.
We define V to be the complex vector space of all complex polynomials. For each polynomial $p = p_1 + \mathrm{i}p_2$, where p_1 and p_2 are polynomials with real coefficients, $\overline{p}$ is defined by

$$\overline{p}(z) = p_1(z) - \mathrm{i}p_2(z).$$

We define the inner product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ by

$$\langle p, q\rangle := \int_\Gamma p(z)\overline{q}(z)\mathrm{d}z = \int_{-1}^{1} p(z)\overline{q(z)}\mathrm{d}z.$$

For all positive integers n, we define V_n to be the vector space of all complex polynomials of degree at most $n-1$:

$$V_n := \mathrm{span}_{\mathbb{C}}(\{1, z, \ldots, z^{n-1}\}).$$

We define $\{p_n\}_{n=0,1,2,\ldots}$ to be the normalized Legendre polynomials:

$$p_n(z) := \frac{\sqrt{n+\frac{1}{2}}}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left((z^2-1)^n\right). \tag{30}$$

As is well known, these polynomials $\{p_n\}_{n=0,1,2,\ldots}$ are orthonormal with respect to the inner product $\langle\cdot,\cdot\rangle$, i.e.,

$$\langle p_m, p_n\rangle = \delta_{mn} := \begin{cases} 1 & (m = n) \\ 0 & (m \ne n) \end{cases}, \quad m, n \in \{0, 1, \ldots\}.$$

Theorem 3. *The equation*

$$A_{2N}\mathbf{c} = \mathbf{b}_{2N} \tag{31}$$

has a unique solution.

Proof. Suppose that $A_{2N}\mathbf{c} = \mathbf{b}_{2N}$. We prove the unique solvability by deriving $\mathbf{c} = \mathbf{c}^*$ from this equation. By the definition of A_{2N} and $\mathbf{b}_{2N}$ (cf. (13)), we have

$$\int_\Gamma q_n^{(N)}(z) P_{\mathbf{c}}(z) f(z) \mathrm{d}z = 0, \quad n = 0, 1, \ldots, 2N-1. \tag{32}$$

Define $\tilde{f}$ by

$$\tilde{f}(z) := P_{\mathbf{c}}(z) f(z) - \sum_{n=0}^{N-1} \left(\int_\Gamma p_n(z) P_{\mathbf{c}}(z) f(z) \mathrm{d}z \right) p_n(z), \tag{33}$$

which is intended to be a modification of $P_{\mathbf{c}}(z)f(z)$ by lower order terms. We want to show that $\tilde{f}$ is the zero function.

From the orthonormality of $\{p_n\}$, it holds that

$$\int_\Gamma p_n(z) \tilde{f}(z) \mathrm{d}z = 0, \quad n = 0, 1, \ldots, N-1, \tag{34}$$

whereas, from the equation (32) and Lemma 1, it also holds that

$$\int_\Gamma q_n^{(N)}(z) \tilde{f}(z) \mathrm{d}z = 0, \quad n = 0, 1, \ldots, 2N-1. \tag{35}$$

Since p_n and $q_n^{(N)}$ are polynomials of degree n and $N+n$, respectively, the equations (34) and (35) imply that

$$\int_\Gamma p(z) \tilde{f}(z) \mathrm{d}z = 0, \quad p \in V_{3N}. \tag{36}$$

By the definition (33) of $\tilde{f}$, it can be expressed in the following form:

$$\tilde{f}(z) = \frac{P_2(z)}{P_1(z)},$$

where $P_1(z) = \prod_{k=1}^N (z - z_k) \in V_{N+1}$ and $P_2(z) \in V_{2N}$. Then we have

$$\langle P_2, P_2 \rangle = \int_\Gamma P_2(z)\overline{P_2}(z) \mathrm{d}z = \int_\Gamma P_1(z)\overline{P_2}(z)\tilde{f}(z) \mathrm{d}z = 0,$$

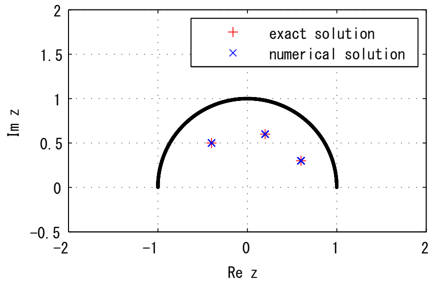
where the last equality follows from (36) and $P_1(z)\overline{P_2}(z) \in V_{3N}$. Therefore $P_2 = 0$, and hence $\tilde{f} = 0$.

Since $\tilde{f} = 0$ in (33), $P_{\mathbf{c}}(z)f(z)$ is a polynomial, which implies, in turn, that $\mathbf{c} = \mathbf{c}^*$. □

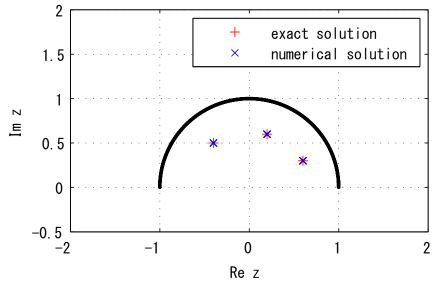
Theorem 3 means that Algorithm 2 below can successfully solve all problem instances including those constructed in the proof of Propositions 2 and 3.

Algorithm 2.
(1) From $f(z)$ on Γ, compute a_{ij} and b_i in equation (11) *for $i = 1, 2, \ldots, 2N$ and $j = 1, 2, \ldots, N$.*
(2) Solve the system $A_{2N}\mathbf{c} = \mathbf{b}_{2N}$ of $2N$ linear equations defined by (28) *and let $\mathbf{c}^*$ be a solution.*
(3) Solve the algebraic equation $z^N - c_N^* z^{N-1} - \cdots - c_1^* = 0$ in z and output the solutions as $z_1, z_2, \ldots, z_N$.

Note that the linear system $A_{2N}\mathbf{c} = \mathbf{b}_{2N}$ has twice as many equations as the variables, and hence it is solved by the method of least squares in numerical simulation.



Algorithm 1, Example 1



Algorithm 2, Example 1

Figure 3: Numerical results for Example 1 by the two algorithms

Remark 1. *In the linear equation* $A_M \mathbf{c} = \mathbf{b}_M$ *in* (29), $A_M = (a_{ij})$ *and* $\mathbf{b}_M = (b_i)$ *are defined by* $\{q_0^{(N)}, q_1^{(N)}, \dots, q_{M-1}^{(N)}\}$ *as in* (11)*. It is also possible to define an equivalent system of linear equations by using Legendre polynomials* $\{p_N, p_{N+1}, \dots, p_{N+M-1}\}$*. Define* $\hat{A}_M = (\hat{a}_{ij}) \in \mathbb{C}^{M\times N}$ *and* $\hat{\mathbf{b}}_M = (\hat{b}_i) \in \mathbb{C}^M$ *by*

$$\hat{a}_{ij} := \int_\Gamma p_{N+i-1}^{(N)}(z) f(z) z^{j-1} \mathrm{d}z, \quad \hat{b}_i := \int_\Gamma p_{N+i-1}^{(N)}(z) f(z) z^N.$$

and consider $\hat{A}_M \mathbf{c} = \hat{\mathbf{b}}_M$*. As is shown below, there exists a nonsingular matrix* $C \in \mathbb{C}^{M\times M}$ *such that*

$$\hat{A}_M = C A_M, \quad \hat{\mathbf{b}}_M = C\mathbf{b}_M, \tag{37}$$

which means that these two systems of linear equations are equivalent:

$$\hat{A}_M \mathbf{c} = \hat{\mathbf{b}}_M \iff A_M \mathbf{c} = \mathbf{b}_M.$$

Such intimate connection to the Legendre polynomials gives another reason for the name of "orthogonal polynomial approach" of the present method.

The proof of (37) *is as follows. Denote the orthogonal complement of* V_N *by* $V_N^\perp$*, i.e.,*

$$V_N^\perp := \{p \in V \mid \langle p, q\rangle = 0, \quad q \in V_N\}.$$

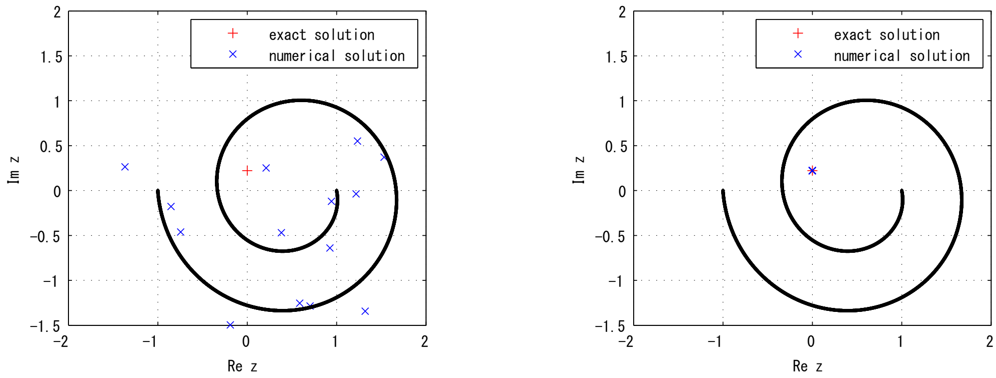
Since $V_{M+N} = V_N \oplus (V_{M+N} \cap V_N^\perp)$*, we have* $\dim(V_{M+N} \cap V_N^\perp) = M$*. Furthermore, from Lemma 1, each* $q_n^{(N)}$ *is an element of* $V_N^\perp$ *of degree* $N+n$*, which implies that* $q_0^{(N)}, q_1^{(N)}, \dots, q_{M-1}^{(N)}$ *are* M *linearly independent polynomials in* $V_{M+N} \cap V_N^\perp$*. Hence we have* $\mathrm{span}(\{q_0^{(N)}, q_1^{(N)}, \dots, q_{M-1}^{(N)}\}) = V_{M+N} \cap V_N^\perp$*. Similarly,* $\mathrm{span}(\{p_N, p_{N+1}, \dots, p_{N+M-1}\}) = V_{M+N} \cap V_N^\perp$*. Then it follows that*

$$\mathrm{span}(\{q_0^{(N)}, q_1^{(N)}, \dots, q_{M-1}^{(N)}\}) = \mathrm{span}(\{p_N, p_{N+1}, \dots, p_{N+M-1}\}). \tag{38}$$

By (38)*, all polynomials in* $\{q_0^{(N)}, q_1^{(N)}, \dots, q_{M-1}^{(N)}\}$ *can be expressed as linear combinations of* $\{p_N, p_{N+1}, \dots, p_{N+M-1}\}$ *and vice versa. This shows the existence of* C *in* (37)*.*

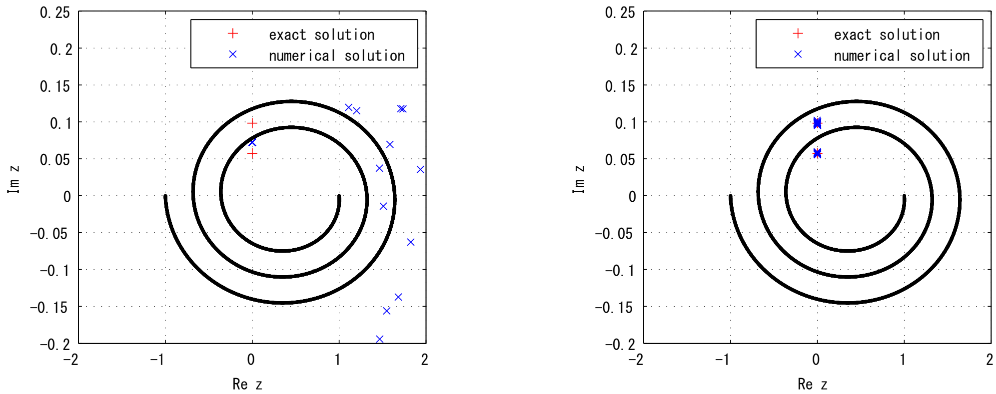
4.4 Numerical comparison of the existing and proposed methods

In this section, Algorithm 1 (Nara–Ando's algorithm) and Algorithm 2 proposed are numerically compared. In this paper, Matlab(R2008b) was used for numerical simulations. We used the data of the value of f with small noise in order to examine stability. The two algorithms were applied to the following examples. For each example, we prepared 20 data sets with different noise and applied the two algorithm to each data set.



Algorithm 1, Example 2 Algorithm 2, Example 2

Figure 4: Numerical results for Example 2 by the two algorithms



Algorithm 1, Example 3 Algorithm 2, Example 3

Figure 5: Numerical results for Example 3 by the two algorithms

(Example 1) Let f be given by

$$f(z) = \frac{\exp(\pi \mathrm{i}/4)}{z-(0.2+0.6\mathrm{i})} + \frac{\exp(3\pi \mathrm{i}/4)}{z-(-0.4+0.5\mathrm{i})} + \frac{\exp(\pi \mathrm{i}/2)}{z-(0.6+0.3\mathrm{i})},$$

where $N = 3$. Let Γ be the upper half unit circle.

This example is taken from [16]. The numerical result for this example shown in Fig. 3 seems to indicate that the two algorithm have much the same performance for this example.

(Example 2) Let $f(z) = \frac{1}{z-z_1}$ with $z_1 = \mathrm{i}s$, where s is a real number satisfying

$$s\left(\arctan\left(\frac{1}{s}\right) + \pi\right) - 1 = 0.$$

Note that $z_1 \approx 0.223\mathrm{i}$. Define $\Gamma := \gamma([0,1])$, where

$$\gamma(t) := 0.5 - (1.5-t)\exp(3\pi \mathrm{i} t).$$

This example corresponds to the case considered in Proposition 2. According to Proposition 2, when we apply Algorithm 1, a linear equation which is not uniquely solvable should appear. Figure 4 shows the numerical results of estimating poles for Example 1 by Algorithm 1 and Algorithm 2. Comparison between the two algorithms shows that the numerical solutions by Algorithm 1 are away from the exact solutions and have a greater dispersion. This situation is supposed to be caused by the singularity of the linear equation. Algorithm 2 does not show such difficulties, which implies that doubling the number of the linear equation makes the equation nonsingular. The value of A and A_{2N} computed numerically without noises is as follows:

$$A = (2.22 \times 10^{-10} - 5.50 \times 10^{-14}\mathrm{i}) \approx (0),$$
$$A_{2N} = \begin{pmatrix} 2.22 \times 10^{-10} - 5.50 \times 10^{-14}\mathrm{i} \\ 1.22 \times 10^{-13} - 8.99\mathrm{i} \end{pmatrix} \approx \begin{pmatrix} 0 \\ -8.99\mathrm{i} \end{pmatrix},$$

whereas $\mathbf{c}^* = (z_1) \approx (0.223\mathrm{i})$.

(Example 3) Let $f(z) = \frac{1}{z-z_1} + \frac{1}{z-z_2}$ with $z_1 = \mathrm{i}s_1$, $z_2 = \mathrm{i}s_2$, where s_1 and s_2 are real numbers satisfying

$$\arctan\left(\frac{1}{s_1}\right) + \pi - \frac{5s_1^2 + 4/3}{5s_1^3 + 3s_1} = 0, \quad \arctan\left(\frac{1}{s_2}\right) + 2\pi - \frac{5s_2^2 + 4/3}{5s_2^3 + 3s_2} = 0,$$

respectively. Note that $z_1 \approx 0.0982\mathrm{i}, \quad z_2 \approx 0.0574\mathrm{i}$. Define $\Gamma := \gamma([0,1])$, where

$$\gamma(t) := 0.4 - (1.4 - 0.8t)(\cos(5\pi t) + 0.11\mathrm{i}\sin(5\pi t)).$$

This example corresponds to the case of $N = 2$ in Proposition 3. The numerical results by the two algorithms are shown in Fig. 5. The value of A_{2N} computed numerically without noises is as follows:

$$A_{2N} = \begin{pmatrix} -5.09 \times 10^{-10} - 48.6\mathrm{i} & 3.52 - 8.93 \times 10^{-11}\mathrm{i} \\ 4.18 \times 10^{-9} - 1.99 \times 10^{-10}\mathrm{i} & 3.17 \times 10^{-10} + 2.20 \times 10^{-10}\mathrm{i} \\ 1.62 \times 10^{-9} + 24.3\mathrm{i} & -1.76 - 7.68 \times 10^{-11}\mathrm{i} \\ -3.17 - 4.99 \times 10^{-11}\mathrm{i} & 5.93 \times 10^{-9} - 0.246\mathrm{i} \end{pmatrix}$$
$$\approx \begin{pmatrix} -48.6\mathrm{i} & 3.52 \\ 0 & 0 \\ 24.3\mathrm{i} & -1.76 \\ -3.17 & -0.246\mathrm{i} \end{pmatrix}.$$

Since the second row of A_{2N} is nearly equal to 0, the matrix A is almost singular. The computed condition numbers of A and A_{2N} are 5.46×10^{10} and 3.24×10^3 respectively, which implies that doubling the number of equations makes the linear system solvable.

5 Exact estimation algorithm from discrete data

In Algorithm 2 (proposed algorithm), as well as in Algorithm 1 (Nara–Ando's algorithm), the dense data on Γ is needed to accurately compute the contour integrals for the coefficients a_{ij} and b_i in (11). In actual situations, however, it is required to estimate poles from data on a small number of sample points on Γ. In such cases, numerical integration must be used in place of integral, which would cause discretization errors.

Fortunately, we can design, by modifying our orthogonal polynomial approach, an algorithm that gets rid of such discretization errors. Recall that Algorithm 2 is constructed by considering polynomials that are orthogonal with respect to the inner product defined by contour integral. We now consider a sequence of polynomials $\{\tilde{p}_n\}$ that have a property similar to orthogonality with respect to a sesquilinear form defined by discretized numerical integration. We may compare $\{\tilde{p}_n\}$ to discrete orthogonal polynomials. By replacing $\{q_n^{(N)}\}$ by $\{\tilde{p}_n\}$, we can construct an exact algorithm for pole estimation from data on sample points.

5.1 Construction of the algorithm

A simple open curve whose end points are -1 and 1 can be represented by a homeomorphism $\gamma : [0,1] \to \Gamma$ such that $\gamma(0) = -1, \gamma(1) = 1$. Let us assume that we can observe the value of f at K sample points $\tilde{\Gamma}$ expressed by

$$\tilde{\Gamma} = \{\gamma_1 = \gamma(s_1), \gamma_2 = \gamma(s_2), \dots, \gamma_K = \gamma(s_K)\} \subseteq \Gamma,$$

where $0 = s_1 < s_2 < \cdots < s_K = 1$. In order to compute approximate values of contour integral along Γ, we adopt the trapezoid rule with integration points $\tilde{\Gamma}$ and denote its value by $I_{\tilde{\Gamma}}(\cdot)$:

$$I_{\tilde{\Gamma}}(g) := \sum_{l=1}^{K-1} \frac{(g(\gamma_l) + g(\gamma_{l+1}))(\gamma_{l+1} - \gamma_l)}{2}.$$

Then $I_{\tilde{\Gamma}}(\cdot)$ has the following property:

$$\lim_{\Delta(\tilde{\Gamma}) \to 0} I_{\tilde{\Gamma}}(g) = \int_\Gamma g(z) \mathrm{d}z,$$

where $\Delta(\tilde{\Gamma}) := \max_{1 \le l \le K-1} \{|\gamma_{l+1} - \gamma_l|\}$.

Let a sesquilinear form $\langle \cdot, \cdot \rangle_{\tilde{\Gamma}} : V \times V \to \mathbb{C}$ be defined by

$$\langle f_1, f_2 \rangle_{\tilde{\Gamma}} := I_{\tilde{\Gamma}}(f_1 \overline{f_2}),$$

where $\overline{f_2}$ is defined in Definition 1. Note that $\langle \cdot, \cdot \rangle_{\tilde{\Gamma}}$ does not satisfy the axioms of an inner product.

This sesquilinear form $\langle \cdot, \cdot \rangle_{\tilde{\Gamma}}$ is an approximation of $\langle \cdot, \cdot \rangle$; the limit of $\langle f_1, f_2 \rangle_{\tilde{\Gamma}}$, as $\Delta(\tilde{\Gamma})$ approaches 0, is $\langle f_1, f_2 \rangle$ for all $f_1, f_2 \in V$. Hence, we assume

$$\langle p, p \rangle_{\tilde{\Gamma}} \neq 0, \quad p \in V_{3N} \setminus \{0\}, \tag{39}$$

which holds for $\tilde{\Gamma}$ with sufficient small $\Delta(\tilde{\Gamma})$; see remark below.

Remark 2. *We can verify the assumption* (39) *as follows. Define a symmetric matrix* $S_{\tilde{\Gamma},3N}$ *by*

$$S_{\tilde{\Gamma},3N} := \begin{pmatrix} \mathrm{Re}\ \langle z^0, z^0 \rangle_{\tilde{\Gamma}} & \mathrm{Re}\ \langle z^0, z^1 \rangle_{\tilde{\Gamma}} & \dots & \mathrm{Re}\ \langle z^0, z^{3N-1} \rangle_{\tilde{\Gamma}} \\ \mathrm{Re}\ \langle z^1, z^0 \rangle_{\tilde{\Gamma}} & \mathrm{Re}\ \langle z^1, z^1 \rangle_{\tilde{\Gamma}} & \dots & \mathrm{Re}\ \langle z^1, z^{3N-1} \rangle_{\tilde{\Gamma}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Re}\ \langle z^{3N-1}, z^0 \rangle_{\tilde{\Gamma}} & \mathrm{Re}\ \langle z^{3N-1}, z^1 \rangle_{\tilde{\Gamma}} & \dots & \mathrm{Re}\ \langle z^{3N-1}, z^{3N-1} \rangle_{\tilde{\Gamma}} \end{pmatrix}.$$

Then the assumption (39) *holds if* $S_{\tilde{\Gamma},3N}$ *is positive definite. This follows from the identity*

$$\mathrm{Re}\langle p, p \rangle_{\tilde{\Gamma}} = \mathbf{p}^\top S_{\tilde{\Gamma},3N} \mathbf{p} + \mathbf{q}^\top S_{\tilde{\Gamma},3N} \mathbf{q} \tag{40}$$

valid for any $p = \sum_{j=1}^{3N} (p_j + \mathrm{i} q_j) z^{j-1}$, *where* $\mathbf{p} = (p_1, p_2, \dots, p_{3N})^\top \in \mathbb{R}^{3N}$ *and* $\mathbf{q} = (q_1, q_2, \dots, q_{3N})^\top \in \mathbb{R}^{3N}$. *Note that* $\mathrm{Re}\langle p, p \rangle > 0$ *for* $p \neq 0$, *and that* $\mathrm{Re}\langle p, p \rangle_{\tilde{\Gamma}}$ *converges to* $\mathrm{Re}\langle p, p \rangle$ *as* $\Delta(\tilde{\Gamma})$ *tends to 0. Therefore, the matrix* $S_{\tilde{\Gamma},3N}$ *is positive definite for* $\tilde{\Gamma}$ *with sufficient small* $\Delta(\tilde{\Gamma})$, *and the assumption* (39) *is satisfied by such* $\tilde{\Gamma}$.

Definition 2. *Let* $\psi(z) := z$. *We define* $\{\varphi_j\}_{j=0,1,\dots,3N-1}$ *by the following recursive formula:*

$$\varphi_0(z) = 1,$$

$$\varphi_1(z) = \left(z - \overline{\left(\frac{\langle \varphi_0, \psi\varphi_0 \rangle_{\tilde{\Gamma}}}{\langle \varphi_0, \varphi_0 \rangle_{\tilde{\Gamma}}} \right)} \right) \varphi_0(z),$$

$$\varphi_n(z) = \left(z - \overline{\beta_n} \right) \varphi_{n-1}(z) - \overline{\alpha_n} \varphi_{n-2}(z), \quad n = 2, 3, \dots, 3N-1,$$

where

$$\alpha_n = \frac{\langle \varphi_{n-2}, \psi\varphi_{n-1}\rangle_{\tilde{\Gamma}}}{\langle \varphi_{n-2}, \varphi_{n-2}\rangle_{\tilde{\Gamma}}}, \quad \beta_n = \frac{\langle \varphi_{n-1}, \psi\varphi_{n-1}\rangle_{\tilde{\Gamma}} - \alpha_n \langle \varphi_{n-1}, \varphi_{n-2}\rangle_{\tilde{\Gamma}}}{\langle \varphi_{n-1}, \varphi_{n-1}\rangle_{\tilde{\Gamma}}}. \tag{41}$$

Furthermore, let $\{\tilde{p}_j\}_{j=0,1,\ldots,3N-1}$ *be defined by*

$$\tilde{p}_j(z) = \frac{\varphi_n(z)}{\sqrt{|\langle \varphi_n, \varphi_n\rangle_{\tilde{\Gamma}}|}}. \tag{42}$$

The system of polynomials $\{\tilde{p}_j\}_{j=0,1,\ldots,3N-1}$ has an orthogonality property shown in the following lemma.

Lemma 2.

$$|\langle \tilde{p}_i, \tilde{p}_n\rangle_{\tilde{\Gamma}}| = \delta_{in}, \quad 0 \le i \le n \le 3N-1.$$

Proof. It follows from the equation (42) that $|\langle \tilde{p}_n, \tilde{p}_n\rangle_{\tilde{\Gamma}}| = 1$ for $0 \le n \le 3N-1$ and that $|\langle \tilde{p}_i, \tilde{p}_n\rangle_{\tilde{\Gamma}}| = 0$ if and only if $|\langle \varphi_i, \varphi_n\rangle_{\tilde{\Gamma}}| = 0$. Hence it is sufficient to show that

$$\langle \varphi_i, \varphi_n\rangle_{\tilde{\Gamma}} = 0, \quad 0 \le i < n \le 3N-1. \tag{43}$$

We prove (43) by induction on n. For $n = 1$, the following equation shows that (43) holds:

$$\langle \varphi_0, \varphi_1\rangle_{\tilde{\Gamma}} = \langle \varphi_0, \psi\varphi_0\rangle_{\tilde{\Gamma}} - \frac{\langle \varphi_0, \psi\varphi_0\rangle_{\tilde{\Gamma}}}{\langle \varphi_0, \varphi_0\rangle_{\tilde{\Gamma}}}\langle \varphi_0, \varphi_0\rangle_{\tilde{\Gamma}} = 0.$$

For $n \ge 2$, the induction hypothesis for (43) gives the following equations:

$$\begin{aligned} \langle \varphi_{n-1}, \varphi_n\rangle_{\tilde{\Gamma}} &= \langle \varphi_{n-1}, \psi\varphi_{n-1}\rangle_{\tilde{\Gamma}} - \beta_n\langle \varphi_{n-1}, \varphi_{n-1}\rangle_{\tilde{\Gamma}} - \alpha_n\langle \varphi_{n-1}, \varphi_{n-2}\rangle_{\tilde{\Gamma}}, \\ \langle \varphi_{n-2}, \varphi_n\rangle_{\tilde{\Gamma}} &= \langle \varphi_{n-2}, \psi\varphi_{n-1}\rangle_{\tilde{\Gamma}} - \alpha_n\langle \varphi_{n-2}, \varphi_{n-2}\rangle_{\tilde{\Gamma}}. \end{aligned}$$

Substituting (41) into these equations we conclude that these values are equal to 0. For $i = 0, 1, \ldots, n-3$, from the induction hypothesis, $\langle \varphi_i, \varphi_l\rangle_{\tilde{\Gamma}}$ can be expressed as follows:

$$\begin{aligned} \langle \varphi_i, \varphi_n\rangle_{\tilde{\Gamma}} &= \langle \varphi_i, \psi\varphi_{n-1}\rangle_{\tilde{\Gamma}} - \beta_n\langle \varphi_i, \varphi_{n-1}\rangle_{\tilde{\Gamma}} - \alpha_n\langle \varphi_i, \varphi_{n-2}\rangle_{\tilde{\Gamma}} \\ &= \langle \psi\varphi_i, \varphi_{n-1}\rangle_{\tilde{\Gamma}}. \end{aligned} \tag{44}$$

Since $\psi\varphi_i \in V_{n-1}$ can be expressed by a linear combination of $\varphi_0, \varphi_1, \ldots, \varphi_{n-2}$, the value of (44) is equal to 0, and (43) is proven by induction. □

We shall propose a new algorithm by replacing $\{q_n^{(N)}\}$ by $\{\tilde{p}_n\}$, which can be computed by discrete data.

For $i = 1, 2, \ldots, 2N$ and $j = 1, 2, \ldots, N$, we define $\tilde{a}_{ij}$ and $\tilde{b}_i$ by

$$\tilde{a}_{ij} := I_{\tilde{\Gamma}}(z^{j-1}f(z)\overline{\tilde{p}_{N+i-1}}(z)), \quad \tilde{b}_i := I_{\tilde{\Gamma}}(z^N f(z)\overline{\tilde{p}_{N+i-1}}(z)).$$

Furthermore, let $\tilde{A}_{2N}$ and $\tilde{\mathbf{b}}_{2N}$ be as follows:

$$\tilde{A}_{2N} := (\tilde{a}_{ij})_{1\le i\le 2N, 1\le j\le N} \in \mathbb{C}^{2N\times N}, \quad \tilde{\mathbf{b}}_{2N} := (\tilde{b}_i)_{1\le i\le 2N} \in \mathbb{C}^{2N}.$$

These values can be accurately computed from $f(z)$ on the sample points $\tilde{\Gamma}$.

Theorem 4.
If $\mathbf{c} = \mathbf{c}^*$*, then* $\tilde{A}_{2N}\mathbf{c} = \tilde{\mathbf{b}}_{2N}$ *holds. The converse is also true under the assumption* (39).

Proof. From the definition of $\tilde{A}_{2N}$ and $\tilde{\mathbf{b}}_{2N}$, the ith element of $\tilde{\mathbf{b}}_{2N} - \tilde{A}_{2N}\mathbf{c}$ is expressed as follows:

$$(\tilde{\mathbf{b}}_{2N} - \tilde{A}_{2N}\mathbf{c})_i = I_{\tilde{\Gamma}}(P_{\mathbf{c}}(z)f(z)\overline{\tilde{p}_{N+i-1}}(z)), \quad i = 1, 2, \ldots, 2N.$$

If $\mathbf{c} = \mathbf{c}^*$, then $P_{\mathbf{c}}(z)f(z) \in V_N$, and hence it follows from Lemma 2 that

$$(\tilde{\mathbf{b}}_{2N} - \tilde{A}_{2N}\mathbf{c})_i = \langle P_{\mathbf{c}}(z)f(z), \tilde{p}_{N+i-1}(z)\rangle_{\tilde{\Gamma}} = 0, \quad i = 1, 2, \ldots, 2N,$$

which shows that $\mathbf{c} = \mathbf{c}^*$ implies $\tilde{A}_{2N}\mathbf{c} = \tilde{\mathbf{b}}_{2N}$.

Conversely, if $\tilde{A}_{2N}\mathbf{c} = \tilde{\mathbf{b}}_{2N}$, then

$$I_{\tilde{\Gamma}}(P_{\mathbf{c}}(z)f(z)\overline{\tilde{p}_n}(z)) = 0, \quad n = N, N+1, \ldots, 3N-1. \tag{45}$$

Let $\{f_l\}_{l=1,2,\ldots,N+1}$ be defined by

$$f_1(z) := P_{\mathbf{c}}(z)f(z),$$
$$f_{l+1}(z) := f_l(z) - \frac{I_{\tilde{\Gamma}}(f_l\overline{\tilde{p}_{N-l}})}{\langle \tilde{p}_{N-l}, \tilde{p}_{N-l}\rangle_{\tilde{\Gamma}}}\tilde{p}_{N-l}(z), \quad l = 1, 2, \ldots, N.$$

Lemma 2 and the equation (45) yield that the functions $\{f_l\}_{l=1,2,\ldots,N+1}$ satisfy

$$I_{\tilde{\Gamma}}(f_l\overline{\tilde{p}_n}) = 0, \quad n = N-l+1, N-l+2, \ldots, 3N-1.$$

Consequently, since each $\tilde{p}_n$ has degree n, it follows that

$$I_{\tilde{\Gamma}}(f_{N+1}p) = 0, \quad p \in V_{3N}. \tag{46}$$

By the definition of $\{f_l\}$, $f_{N+1} - P_{\mathbf{c}}f \in V_N$ holds, and so $f_{N+1}(z)$ can be expressed as follows:

$$f_{N+1}(z) = \frac{P_2(z)}{P_1(z)}, \tag{47}$$

where $P_1(z) = \prod_{k=1}^{N}(z - z_k) \in V_{N+1}$ and $P_2(z) \in V_{2N}$. Then the following equation holds:

$$I_{\tilde{\Gamma}}(f_{N+1}(z)P_1(z)\overline{P_2}(z)) = I_{\tilde{\Gamma}}(P_2(z)\overline{P_2}(z)) = \langle P_2, P_2\rangle_{\tilde{\Gamma}}. \tag{48}$$

Meanwhile, it follows from the equation (46) and the condition $P_1(z)\overline{P_2}(z) \in V_{3N}$ that the value of (48) is equal to 0, and therefore $P_2 = 0$ under the assumption (39). By the definition of P_2, it follows that

$$P_2(z_k) = \mu_k \prod_{1\le l\le N, l\ne k}(z_k - z_l)P_{\mathbf{c}}(z_k) = 0, \quad k = 1, 2, \ldots, N.$$

Since $\mu_k \prod_{1\le l\le N, l\ne k}(z_k - z_l) \ne 0$, we must have $P_{\mathbf{c}}(z_k) = 0$ for $k = 1, 2, \ldots, N$, which means $\mathbf{c} = \mathbf{c}^*$. This shows that $\tilde{A}_{2N}\mathbf{c} = \tilde{\mathbf{b}}_{2N}$ implies $\mathbf{c} = \mathbf{c}^*$ under the assumption (39). □

In view of this theorem, we propose the following algorithm.

Algorithm 3.
(1) From $f(z)$ on $\tilde{\Gamma}$, construct the system of $3N$ polynomials $\{\tilde{p}_n\}_{0,1,\ldots,3N-1}$ defined in Definition 2.
(2) From $f(z)$ on $\tilde{\Gamma}$, compute $\tilde{a_{ij}}$ and $\tilde{b_i}$ in equation (11) for $i = 1, 2, \ldots, 2N$, $j = 1, 2, \ldots, N$.
(3) Solve the system $\tilde{A_{2N}}\mathbf{c} = \tilde{\mathbf{b}}_{2N}$ of $2N$ linear equations defined by (28) and let $\mathbf{c}^*$ be a solution.
(4) Solve the algebraic equation $z^N - c_N^* z^{N-1} - \cdots - c_1^* = 0$ in z and output the solutions as $z_1, z_2, \ldots, z_N$.

5.2 Numerical examples for exact estimation algorithm

In this subsection it is numerically demonstrated that the proposed Algorithm 3 is free from numerical discretization error. In these numerical experiments, we use data of the values of f without noise. Algorithm 2 and Algorithm 3 are applied to the following rational function having 4 poles:

$$
\begin{aligned}
f(z) =& \frac{\pi \mathrm{i}/4}{z-(0.2+0.6\mathrm{i})} + \frac{\exp(3\pi\mathrm{i}/4)}{z-(-0.4+0.5\mathrm{i})} \\
&+ \frac{\exp(\pi\mathrm{i}/2)}{z-(0.6+0.3\mathrm{i})} + \frac{\exp(2\pi\mathrm{i}/3)}{z-(-0.7+0.2\mathrm{i})}.
\end{aligned}
$$

This function is taken from the paper [16] by Nara and Ando. Sample points were placed evenly spaced apart on Γ, where Γ is the upper half of unit circle or the straight line segment $[-1, 1]$. The results of estimation by the two algorithms are shown in Fig. 6 and 7, where K denotes the number of sample points. For all examples in Fig. 6 and 7, the assumption (39) holds, which can be verified by the criterion in Remark 2. The figure show that, in Algorithm 2, although numerical errors decrease as increasing the number K of sample points, these errors still remain. On the other hand, such errors seems to be eliminated in Algorithm 3.

6 Conclusion

This paper presented a new interpretation of Nara–Ando's algorithm for pole estimation (Problem 1) by featuring certain polynomials implicit in the algorithm and highlighting their orthogonality property. The unique solvability of the linear equation in Nara–Ando's algorithm was considered in detail. It was found that there exist an infinite family of problem instances for which the linear equation becomes singular. An improved algorithm, in which the number of the equations is doubled, was then proposed with a theoretical guarantee that the resulting linear equation is always uniquely solvable. Furthermore, in order to cope with errors caused by discretization in numerical integration, a new algorithm was proposed by introducing polynomials that have a kind of orthogonality property with respect to discretized numerical integration. The proposed algorithm is completely free from discretization errors caused by numerical integration, and this property was confirmed by numerical examples.

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Appendix A : Proof of Proposition 1

We first show that S is a closed subset of $(\mathbb{C} \setminus \Gamma)^N$. Define $\eta : (\mathbb{C} \setminus \Gamma)^N \to \mathbb{C}$ by

$$\eta(z_1, z_2, \dots, z_N) = \det B,$$

where $B = (b_{ij})_{1 \le i,j \le N}$ is the matrix defined by (15). Note that the domain $D := (\mathbb{C} \setminus \Gamma)^N$ of η is an open connected subset of $\mathbb{C}^N$. Each b_{ij}, which is dependent only on z_j, is analytic on D, and therefore η is also analytic on D; see, e.g., [8] for analytic functions in several variables.

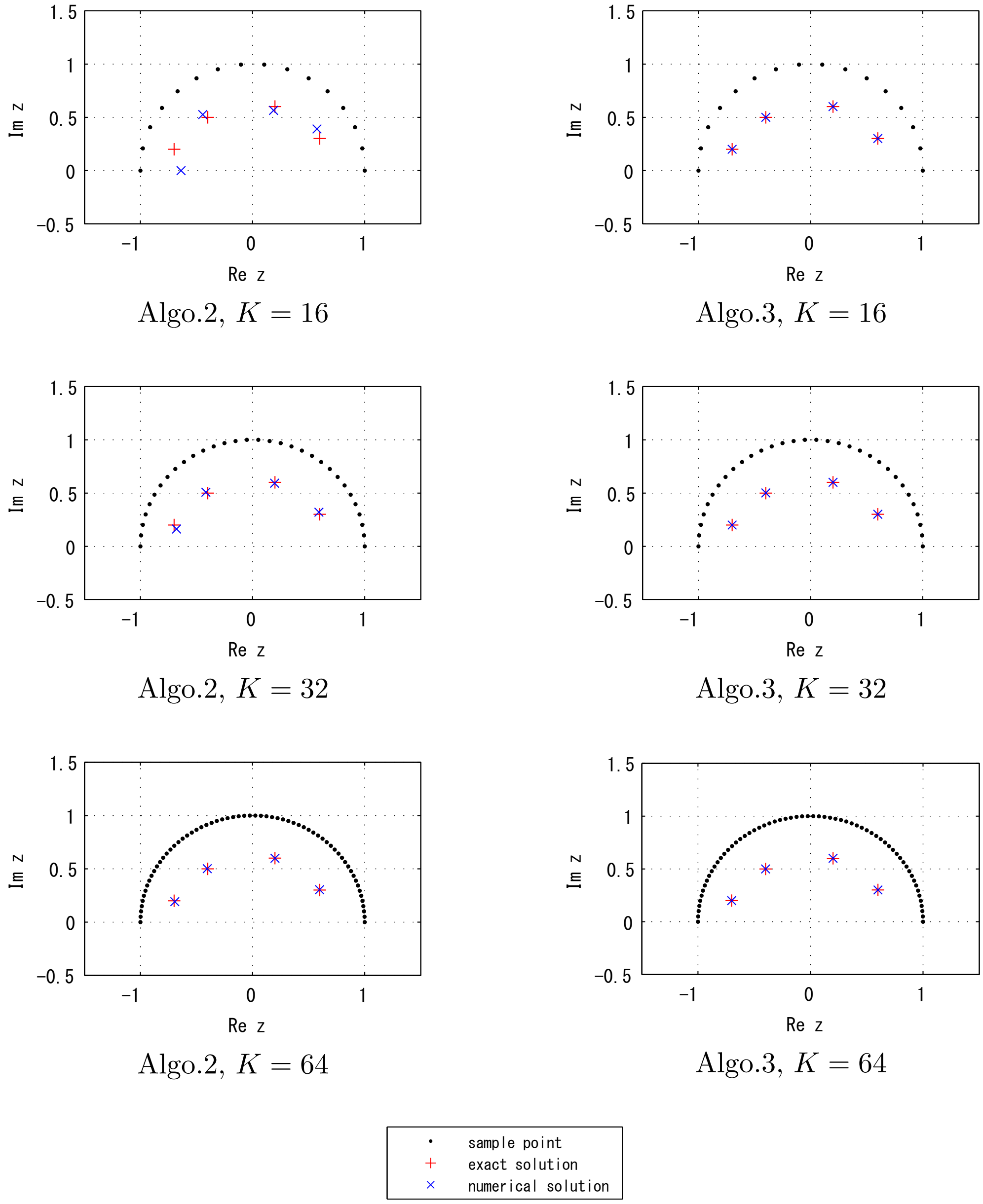

Figure 6: Comparison between Algorithm 2 and Algorithm 3, where Γ is the upper half unit circle

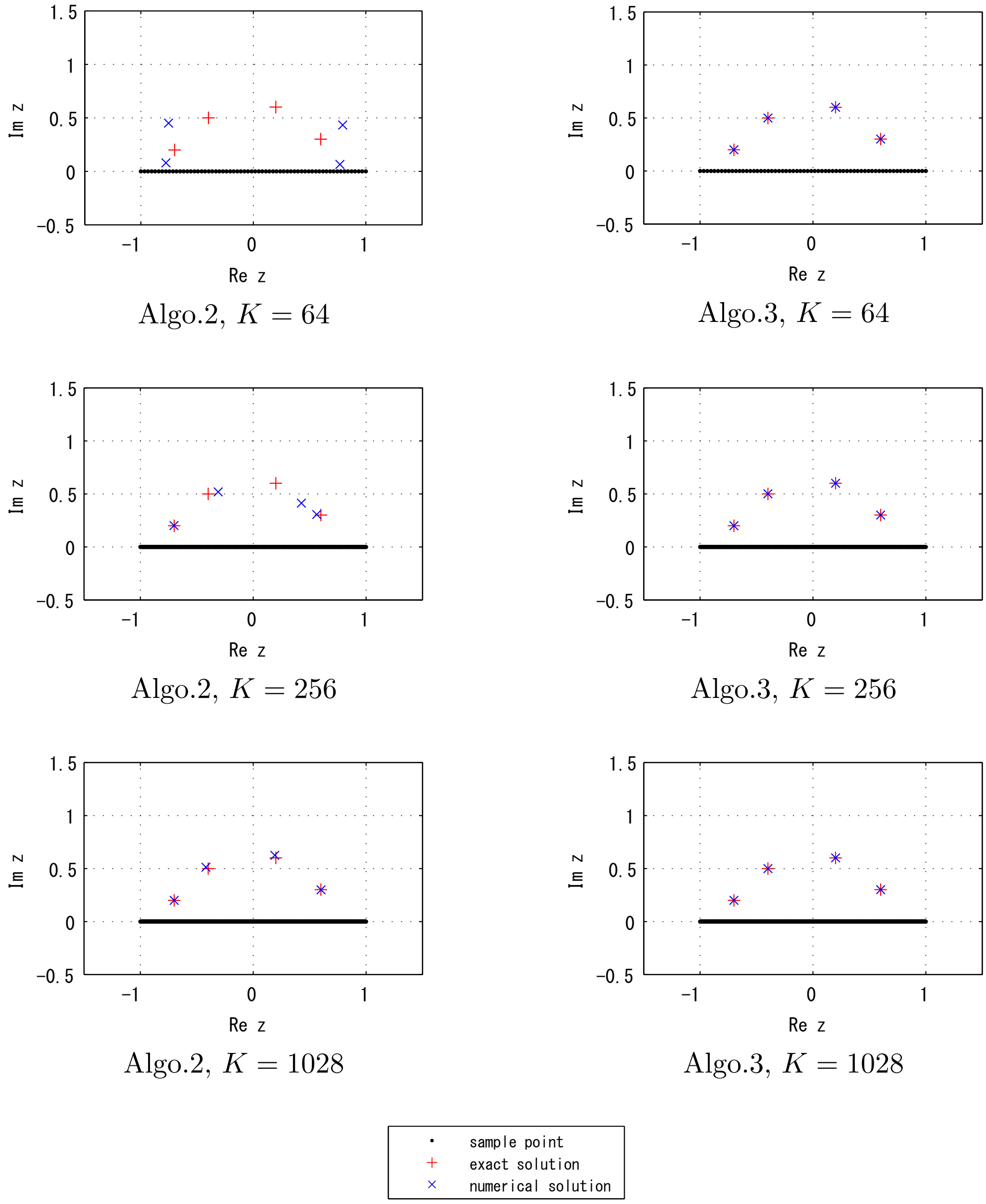

Figure 7: Comparison between Algorithm 2 and Algorithm 3, where Γ is the line segment $[-1, 1]$

The subset S of D, defined as the set of all $(z_1, z_2, \dots, z_N) \in D$ for which B is singular, can be expressed as the zeros of η:

$$S = \{(z_1, z_2, \dots, z_N) \in D \mid \eta(z_1, z_2, \dots, z_N) = 0\}.$$

Since η is a continuous function, S is a closed subset of D.

To prove, by contradiction, that S is a set with empty interior in D, we assume that S contains an open nonempty set. This assumption means that the analytic function η vanishes on an open nonempty subset of D. Then, from the identity theorem [8], η is the zero function. We can obtain a contradiction by constructing $(z_1, z_2, \dots, z_N) \in D$ for which $\eta(z_1, z_2, \dots, z_N) \neq 0$.

Let z_0 be the minimum of $\Gamma \cap \mathbb{R}$ and let $z_1, z_2, \dots, z_N$ be real numbers satisfying $z_0 > z_1 > \cdots > z_N$. Suppose that $B\mathbf{c} = \mathbf{0}$ for $\mathbf{c} = (c_1, c_2, \dots, c_N)^\top \in \mathbb{C}^N$. We derive $\mathbf{c} = \mathbf{0}$ from this equation to prove the nonsingularity of B, which means $\eta(z_1, z_2, \dots, z_N) \neq 0$.

From Cauchy's integral theorem and the definitions of z_0, each b_{ij} in (15) can be expressed as

$$b_{ij} = \int_\Gamma q_{i-1}^{(N)}(z) \frac{1}{z - z_j} \mathrm{d}z = \int_{-1}^{1} q_{i-1}^{(N)}(z) \frac{1}{z - z_j} \mathrm{d}z$$

since $z_j < z_0$ for $j = 1, 2, \dots, N$. Hence, $B\mathbf{c} = \mathbf{0}$ means that

$$\int_\Gamma q_n^{(N)}(z) \sum_{j=1}^{N} \frac{c_j}{z - z_j} \mathrm{d}z = 0, \quad n = 0, 1, \dots, N-1. \tag{49}$$

Using the normalized Legendre polynomials p_n in (30), we define $\tilde{f}$ by

$$\tilde{f}(z) := \sum_{j=1}^{N} \frac{c_j}{z - z_j} - \sum_{n=0}^{N-1} \left(\int_{-1}^{1} p_n(z) \sum_{j=1}^{N} \frac{c_j}{z - z_j} \mathrm{d}z \right) p_n(z), \tag{50}$$

which is intended to be a modification of $P_{\mathbf{c}}(z) f(z)$ by lower order terms. We want to show that $\tilde{f}$ is the zero function.

From the orthonormality of $\{p_n\}$, it holds that

$$\int_{-1}^{1} p_n(z) \tilde{f}(z) \mathrm{d}z = 0, \quad n = 0, 1, \dots, N-1, \tag{51}$$

whereas, from the equation (49) and Lemma 1, it also holds that

$$\int_{-1}^{1} q_n^{(N)}(z) \tilde{f}(z) \mathrm{d}z = 0, \quad n = 0, 1, \dots, N-1. \tag{52}$$

Since p_n and $q_n^{(N)}$ are polynomials of degree n and $N+n$, respectively, the equations (51) and (52) imply that

$$\int_{-1}^{1} p(z) \tilde{f}(z) \mathrm{d}z = 0, \quad p \in V_{2N}. \tag{53}$$

By the definition (50) of $\tilde{f}$, it can be expressed in the following form:

$$\tilde{f}(z) = \frac{P_2(z)}{P_1(z)}$$

where $P_1(z) = \prod_{k=1}^{N} (z - z_k) \in V_{N+1}$ and $P_2(z) \in V_{2N}$. Then we have

$$\int_{-1}^{1} \frac{|P_2(z)|^2}{P_1(z)} \mathrm{d}z = \int_{-1}^{1} \frac{P_2(z)\overline{P_2}(z)}{P_1(z)} \mathrm{d}z = \int_{-1}^{1} \overline{P_2}(z) \tilde{f}(z) \mathrm{d}z = 0,$$

where the last equality follows from (53) and $\overline{P_2}(z) \in V_{2N}$. Since $z_N < \cdots < z_1 < z_0 \leq -1$, we have $P_1(z) = \prod_{k=1}^{N}(z - z_k) > 0$ for all $z \in [-1, 1]$, which implies that $|P_2(z)|^2 = 0$ for all $z \in [-1, 1]$. Therefore $P_2 = 0$, and hence $\tilde{f} = 0$.

Since $\tilde{f} = 0$ in (50), $\sum_{j=1}^{N} \frac{c_j}{z - z_j}$ has no poles, which implies, in turn, that $\mathbf{c} = \mathbf{0}$. □

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