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Kazuo MUROTA and Yu YOKOI

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DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

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Kazuo MUROTA and Yu YOKOI

Department of Mathematical Informatics
Graduate School of Information Science and Technology
The University of Tokyo
{murota,yu_yokoi}@mist.i.u-tokyo.ac.jp

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Abstract

The stable allocation model is a many-to-many matching model in which each pair's partnership is represented by a nonnegative integer. This paper establishes a link between two different formulations of this model; the choice function model studied thoroughly by Alkan and Gale and the discrete-concave (M$^\natural$-concave) value function model introduced by Eguchi, Fujishige and Tamura. We show that the choice functions induced from M$^\natural$-concave value functions are endowed with consistency, persistence and size-monotonicity. This implies, by the result of Alkan and Gale, that the stable allocations for M$^\natural$-concave value functions form a distributive lattice with several significant properties such as polarity, complementarity, and uni-size property. Furthermore, we point out that these results can be extended for quasi M$^\natural$-concave value functions.

1 Introduction

Since the pioneering work of Gale and Shapley [15], the (two-sided) stable matching model has been generalized in many different ways [17, 20, 21]. Among them is the stable allocation model (schedule matching) of Alkan–Gale [3]. This is a many-to-many matching model where each pair's partnership is represented by a nonnegative integer. A thorough study of this model was made by Alkan and Gale [3] in terms of choice functions. A value function approach for the stable allocation model was initiated by Eguchi, Fujishige and Tamura [8, 13] by utilizing concepts and results from discrete convex analysis (Fujishige [11], Murota [23, 24]). Specifically, discrete concavity called $\mathrm{M}^\natural$-concavity plays the primary role as the property of value functions.

The objective of this paper is to establish a substantial connection from the value function approach to the choice function approach, with particular interest in the following questions about the $\mathrm{M}^\natural$-concave value function model:

- Whether stable allocations exist or not?
- Whether stable allocations form a lattice or not?
- Whether stable allocations form a distributive lattice or not?

Stable matching model In the college admissions problem considered by Gale–Shapley [15], each student (college) has a strict preference ordering on colleges (students). Additionally, each college has a quota, the maximum number of students it can admit. This is the stable matching problem in its original form.

Blair [6] generalized this model to a great extent. In his model, agents on each side can have multiple partners and preferences are given by path-independent choice functions. A choice function represents a preference on combinations of agents, not on individuals. He showed that the set of (pairwise) stable matchings is nonempty and forms a lattice. However, the lattice operations are not simple and the lattice is not necessarily distributive.

Fleiner [10] pointed out that the nonemptiness and the lattice structure of Blair's model can be shown by using Tarski's fixed point theorem [31]. The fundamental observation in this approach is that stable matchings correspond to fixed points of a certain monotone function and the deferred acceptance algorithm of Gale–Shapley can be regarded as an iteration of this function. Moreover, Fleiner found that if the choice functions are "w-increasing" (beyond being path-independent), the lattice operations for stable matchings become simpler and the lattice of stable matchings is distributive.

Independently, Alkan [2] obtained a similar result: if the path-independent choice functions are "cardinal-monotone," which is a special case of Fleiner's w-increasingness, the lattice operations for stable matchings are simple and the lattice of stable matchings is distributive. Furthermore, despite the absence of quotas, each agent matches the same number of partners in any stable matching, which is a generalization of the rural hospital theorem. Hatfield–Milgrom [18] studied one-to-many matching in terms of contracts signed between doctors and hospitals, and obtained results similar to Fleiner's [10] or Alkan's [2]. It was shown that if hospitals' choice functions satisfy "law of aggregate demand," which is similar to cardinal-monotonicity, the strategy proofness holds for the deferred acceptance algorithm with doctors proposing.

Stable allocation model The stable matching model has been extended to stable allocation model. This is an extension from $\{0, 1\}$-variables to integer-valued (or real-valued) variables. In this model, we determine how much time each pair spend together, whereas in the matching model we determine whether or not each pair takes partnership.

Baïou and Balinski [4] were the first to consider this extension and defined the generalized (pairwise) stability for stable allocation model. In their model, each agent's preference is represented by a strict ordering on the opposite agents and each agent has capacity constraint.

Alkan and Gale [3] considered the stable allocation model with more general preferences by extending Alkan's choice function model [2] to vectors. It was found that Alkan's results can naturally be extended to their vector versions. That is, if the choice functions have consistency and persistence, which can be regarded as a vector version path-independence, the set of stable allocations is nonempty and forms a lattice. Moreover, if the choice functions additionally have "size-monotonicity," which is a vector version of cardinal monotonicity, the lattice of stable allocations is distributive and has several significant properties which they called polarity, complementarity, and uni-size property.

Discrete-concave value function model When variables take integers or reals, it may be more natural or convenient to assume that agents' preferences are represented by value functions rather than by choice functions.

Eguchi, Fujishige and Tamura [8] proposed a stable allocation model where allocations are integer vectors and each agent has his value function to evaluate the desirability of allocations for him. It was found that a stable allocation always exists if the value functions are $\mathrm{M}^{\natural}$-concave. $\mathrm{M}^{\natural}$-concavity is a kind of concavity for discrete functions with relevance in mathematical economics, to be described in Section 2.

Assuming that agent's choice is to maximize his value function, we can define choice functions from value functions. Then the value function model can be transformed to a choice function model. If there are no ties in the maximum of the value function, the maximizer is uniquely determined and the choice function returns a single vector as the possible choice. If this is the case, we say that the value function is unique-selecting.

In this paper, we show that if the value functions are unique-selecting and $\mathrm{M}^\natural$-concave, the choice functions induced from them have consistency, persistence and size-monotonicity, which are the properties highlighted in Alkan–Gale [3] (Lemmas 3.8 and 4.9). A combination of these facts with the results of [3] shows that the stable allocations for $\mathrm{M}^\natural$-concave value functions form a distributive lattice with several significant properties such as polarity, complementarity, and uni-size property (Theorems 4.11 and 4.12).

Furthermore, we point out that these results can be extended for quasi $\mathrm{M}^\natural$-concave value functions (Theorems 5.6 and 5.7). Quasi M-concavity is defined by ordinal relationship of function values, and not by values themselves. Therefore, the extensibility to quasi M-concave functions means that the desirable structure of stable allocations is guaranteed solely by the concave-like ordering of values. This agrees with the fact that the stability of allocations is defined not by function values themselves but by their ordinal relationship.

2 M$^\natural$-concave Functions

In this section we introduce the concept of M$^\natural$-concave functions, which plays a central role in discrete convex analysis (see Murota [23] for details).

2.1 Definition

Let S be a nonempty finite set, and $\mathbf{Z}$ and $\mathbf{R}$ be the sets of integers and reals, respectively. We define the positive support and the negative support of $x = (x(e) \mid e \in S) \in \mathbf{Z}^S$, respectively, by

$$\mathrm{supp}^+(x) = \{\, e \in S \mid x(e) > 0 \,\}, \qquad \mathrm{supp}^-(x) = \{\, e \in S \mid x(e) < 0 \,\}.$$

For any $x, y \in \mathbf{Z}^S$, the vectors $x \wedge y$ and $x \vee y$ in $\mathbf{Z}^S$ are defined by

$$(x \wedge y)(e) = \min\{x(e), y(e)\}, \qquad (x \vee y)(e) = \max\{x(e), y(e)\} \quad (e \in S).$$

For each $e \in S$, we define χ_e as the vector whose e-component is 1 and other components are 0. For a function $f : \mathbf{Z}^S \to \mathbf{R} \cup \{-\infty\}$, we define the effective domain of f by

$$\mathrm{dom} f = \{\, x \in \mathbf{Z}^S \mid f(x) \neq -\infty \,\}.$$

A function $f : \mathbf{Z}^S \to \mathbf{R} \cup \{-\infty\}$ with $\mathrm{dom} f \neq \emptyset$ is called M$^\natural$-concave[1] if it satisfies

(M$^\natural$) $\forall x, y \in \mathrm{dom} f,\ \forall e \in \mathrm{supp}^+(x - y),\ \exists e' \in \mathrm{supp}^-(x - y) \cup \{0\}$:

$$f(x) + f(y) \leq f(x - \chi_e + \chi_{e'}) + f(y + \chi_e - \chi_{e'}),$$

where χ_0 is a zero vector.

M$^\natural$-concavity for a set function f is also defined by (M$^\natural$), where $f : 2^S \to \mathbf{R} \cup \{-\infty\}$ is identified with $\hat{f} : \mathbf{Z}^S \to \mathbf{R} \cup \{-\infty\}$ defined by

$$\hat{f}(x) = \begin{cases} f(X) & \text{if } x = \chi_X \text{ for some } X \in 2^S, \\ -\infty & \text{otherwise.} \end{cases}$$

Here χ_X is the characteristic vector of X, i.e., $\chi_X(e) = 1$ if $e \in X$ and $\chi_X(e) = 0$ otherwise.

The condition (M$^\natural$) is originated from the exchange axiom in matroid theory. Despite its seemingly complicated definition, M$^\natural$-concave functions include many functions familiar to us; see Appendix A.

[1] M$^\natural$-concave functions are defined by Murota–Shioura [25] as a variant of M-convex functions introduced by Murota [22].

2.2 Properties

M$^\natural$-concave functions have nice features from the point of view of mathematical economics.

A value function (or utility function) is usually assumed to be concave in economics. For any M$^\natural$-concave function $f : \mathbf{Z}^S \to \mathbf{R} \cup \{-\infty\}$, there exists a concave function $\bar{f} : \mathbf{R}^S \to \mathbf{R} \cup \{-\infty\}$ with $\bar{f}(x) = f(x)$ for any $x \in \mathbf{Z}^S$. That is, an M$^\natural$-concave function on $\mathbf{Z}^S$ has a concave extension on $\mathbf{R}^S$.

Also a value function is usually assumed to have decreasing marginal returns, which is equivalent to submodularity in the discrete case. An M$^\natural$-concave function $f : \mathbf{Z}^S \to \mathbf{R} \cup \{-\infty\}$ satisfies

$$f(x) + f(y) \geq f(x \vee y) + f(x \wedge y) \quad (x, y \in \mathrm{dom} f).$$

Kelso–Crawford [19] introduced *gross substitutes* condition for a set function $U : 2^S \to \mathbf{R} \cup \{-\infty\}$:

(GS) For any $p, q \in \mathbf{R}^S$ with $p \leq q$ and any $X \in 2^S$ which maximizes the value of $U[-p]$, there exists $Y \in 2^S$ such that Y maximizes $U[-q]$ and $Y \supseteq X \cap \{ e \in S \mid p(e) = q(e) \}$,

where $U[-p] : 2^S \to \mathbf{R} \cup \{-\infty\}$ is defined by $U[-p](X) = U(X) - \sum_{e \in X} p(e)$. This property (GS) is widely accepted as an important property of value function in demand theory. If we interpret $X \in 2^S$ as a set of commodities, $U(X)$ as a monetary valuation for X, and $p \in \mathbf{R}^S$ as prices, then the above condition says: when each price increases or remains the same, the consumer still wants commodities which are chosen before and whose prices remain the same.

A gross substitute function U is M$^\natural$-concave. The converse is also true. That is, a function f with $\mathrm{dom} f \subseteq \{0, 1\}^S$ is M$^\natural$-concave if and only if the associated U is gross substitute [14, 16, 28]. Furthermore, it is known [7, 27] that M$^\natural$-concavity for functions on $\mathbf{Z}^S$ can be characterized by generalized versions of (GS) under certain natural assumptions.

3 Stable Allocation Model

We now consider two finite sets of agents I and J. We interpret I as workers and J as firms. Each worker $i \in I$ can work at multiple firms, and each firm $j \in J$ can employ multiple workers. Each firm employs each worker with multi-units of labor time . An *allocation* is an $I \times J$ matrix of nonnegative integers. The (i, j)-entry represents the number of units of labor time for which $i \in I$ works for $j \in J$. In the stable allocation model, each agent has a preference on his labor allocation and the main theme is the stability of allocation that we define later.

Let $E = I \times J$, which is the set of all pairs of workers $i \in I$ and firms $j \in J$. Then an allocation X is an element of $\mathbf{Z}_+^E$, i.e., an $I \times J$ matrix $(x(i,j) \mid i \in I, j \in J)$. Let $E_i = \{i\} \times J$ for $i \in I$ and $E_j = I \times \{j\}$ for $j \in J$. We denote the i-th row of X by x_i, i.e., $x_i = (x(i,j) \mid j \in J) \in \mathbf{Z}_+^{E_i} \simeq \mathbf{Z}_+^{|J|}$ and the j-th column of X by x_j, i.e., $x_j = (x(i,j) \mid i \in I) \in \mathbf{Z}_+^{E_j} \simeq \mathbf{Z}_+^{|I|}$. Each agent $k \in I \cup J$ has a preference on allocations. We assume that k's preference for $X \in \mathbf{Z}_+^E$ depends only on x_k.

To represent agents' preferences mathematically, several ways can be conceived. In Sections 3.1 and 3.2, we introduce two different submodels: value function model and choice function model. The difference between the two is how to represent agents' preferences. In Section 3.3, we explain the relation between these two submodels.

3.1 Value function model

We describe the value function model of Eguchi–Fujishige–Tamura [8] and Fujishige–Tamura [13]. In this model, each agent $k \in I \cup J$ has a value function $f_k : \mathbf{Z}_+^{E_k} \to \mathbf{R} \cup \{-\infty\}$ to evaluate the desirability of allocations for k. We assume that for each $k \in I \cup J$, $\mathrm{dom} f_k$ is bounded and has $\mathbf{0} \in \mathbf{Z}_+^{E_k}$ as the minimum point.

Let $u_k \in \mathbf{Z}_+^{E_k}$ be $u_k = \sup(\mathrm{dom} f_k)$, the upper bound of $\mathrm{dom} f_k$, for all $k \in I \cup J$. We define the set of maximizers of f_k subject to a capacity $x \in \mathbf{Z}_+^{E_k}$ by

$$\arg\max \{ f_k(y) \mid y \le x \} = \{ z \in \mathbf{Z}_+^{E_k} \mid z \le x \text{ and } [\forall y \le x,\ f(z) \ge f(y)] \} .$$

Then the (pairwise) stability notion is formalized as follows[2]:

Definition 3.1. (Stability in terms of value functions) An allocation $X \in \mathbf{Z}_+^E$ is *stable* with respect to $\{f_k\}_{k \in I \cup J}$ if it satisfies the following two conditions:

1. For every $k \in I \cup J$, $\quad x_k \in \arg\max \{ f_k(y) \mid y \le x_k \}$.

[2]This is a special case of the Fujishige–Tamura model [13] found in the exposition in Tamura [30].

2. There is no blocking pair, where a *blocking pair* for X is a pair $(i, j) \in E$ such that for some $y_i \in \mathrm{dom} f_i$, $y_j \in \mathrm{dom} f_j$, the following hold:

$$\begin{aligned} f_i(x_i) < f_i(y_i), &\ \ y_i \le (x_i \vee u_i(j)\chi_j), \\ f_j(x_j) < f_j(y_j), &\ \ y_j \le (x_j \vee u_j(i)\chi_i), \\ y_i(j) =& y_j(i). \end{aligned}$$

Here it should be clear that $x_i, \chi_j \in \mathbf{Z}_+^{E_i} \simeq \mathbf{Z}_+^{|J|}$ and $x_j, \chi_i \in \mathbf{Z}_+^{E_j} \simeq \mathbf{Z}_+^{|I|}$. ■

If the condition 1 fails, some agent $k \in I \cup J$ prefers to decrease his labor time, and he tries to deviate from the allocation.

The inequality $y_i \le (x_i \vee u_i(j)\chi_j)$ in the condition 2 means that y_i's j-component is virtually unbounded while the other components are bounded by x_i. The inequality $y_j \le (x_j \vee u_j(i)\chi_i)$ can be interpreted similarly. Then if the condition 2 fails under the condition 1, there exists some pair (i, j) such that both i and j prefer to increase the labor time between them and their demands coincide, and they try to increase the allocation on (i, j) together.

It is known[3] that when $\{f_k\}_{k \in I \cup J}$ are all M$^\natural$-concave functions, the last equation in the above definition can be removed without changing the meaning of stability. Therefore under the assumption of M$^\natural$-concavity, the above definition can be rewritten in the following form:

Lemma 3.2. Assume that value functions $\{f_k\}_{k \in I \cup J}$ are all M$^\natural$-concave. An allocation $X \in \mathbf{Z}_+^E$ is stable with respect to $\{f_k\}_{k \in I \cup J}$ if and only if it satisfies the following two conditions:

1. For every $k \in I \cup J$, $\quad x_k \in \arg\max \{ f_k(y) \mid y \le x_k \}$.
2. For every pair $(i, j) \in E$, $\quad x_i \in \arg\max \{ f_i(y) \mid y \le (x_i \vee u_i(j)\chi_j) \}$ or $x_j \in \arg\max \{ f_j(y) \mid y \le (x_j \vee u_j(i)\chi_i) \}$.

■

3.2 Choice function model

Let $b \in \mathbf{Z}_+^S$ be an upper bound vector and $\mathcal{B} = \{ x \in \mathbf{Z}_+^S \mid x \le b \}$ be a feasible vectors set[4]. A function $C : \mathcal{B} \to \mathcal{B}$ is called a *choice function* if $C(x) \le x$ for all $x \in \mathcal{B}$.

In this subsection, we explain the choice function model. Let each agent $k \in I \cup J$ have an upper bound vector $b_k \in \mathbf{Z}_+^{E_k}$ and a choice function $C_k : \mathcal{B}_k \to \mathcal{B}_k$, where $\mathcal{B}_k = \{ x \in \mathbf{Z}_+^{E_k} \mid x \le b_k \}$. For $x \in \mathcal{B}_k$, we interpret $C_k(x) \in \mathcal{B}_k$ as k's choice subject to the capacity x, i.e., $C_k(x)$ is k's (unique) most desirable allocation among $\{ z \in \mathcal{B}_k \mid z \le x \}$.

Then the (pairwise) stability notion due to [3] is formalized as follows:

[3]See Lemma 3.1 (i) in Fujishige–Tamura [13].

[4]In Alkan–Gale [3], $\mathcal{B}$ is defined more generally

Definition 3.3. (Stability in terms of choice functions) An allocation $X \in \mathbf{Z}_+^E$ is *stable* with respect to $\{C_k\}_{k \in I \cup J}$ if it satisfies the following two conditions:

1. For every $k \in I \cup J, \quad x_k = C_k(x_k)$,
2. For every pair $(i, j) \in E$, x_i is j-satiated or x_j is i-satiated (or both).

Here we say that x_i is *j-satiated*, if $C_i(y)(j) \le x_i(j)$ holds for all $y \ge x_i$. We say that x_j is *i-satiated* similarly. ■

In Alkan–Gale [3], choice functions are assumed to possess two natural properties below.

Definition 3.4. C is *consistent* if $[\ C(x) \le y \le x \implies C(y) = C(x)\ ]$. ■

Definition 3.5. C is *persistent* if $[\ x \ge y \implies y \wedge C(x) \le C(y)\ ]$. ■

Consistency is quite a reasonable property since $C(x)$ means the most desirable allocation among $\{\, z \in \mathcal{B} \mid z \le x \,\}$.

Persistence is a generalization of the substitutability that has widely been used in ordinary matching models since Roth [29]. The condition of persistence is equivalent to the following: for each $y \in \mathcal{B}$ and $e \in S$, $[\ C(y)(e) < y(e) \implies C(x)(e) \le C(y)(e)\ (\forall x \ge y)\ ]$. This says that if an agent wants an item e strictly less than the capacity $y(e)$, he does not increase his demand on e when the capacity is enlarged to x.

When $\{C_k\}_{k \in I \cup J}$ are all consistent and persistent, the definition of stability in Definition 3.3 can be rewritten in the following form:

Lemma 3.6. Assume that choice functions $\{C_k\}_{k \in I \cup J}$ are all consistent and persistent. An allocation $X \in \mathbf{Z}_+^E$ is stable with respect to $\{C_k\}_{k \in I \cup J}$ if and only if it satisfies the following two conditions:

1. For every $k \in I \cup J, \quad x_k = C_k(x_k)$,
2. For every pair $(i, j) \in E$, $x_i = C_i(x_i \vee b_i(j)\chi_j)$ or $x_j = C_j(x_j \vee b_j(i)\chi_i)$.

■

Remark 3.7. For a choice function $C : 2^S \to 2^S$, it is known that path-independence condition

$$C(C(X) \cup Y) = C(X \cup Y) \quad (X, Y \in 2^S)$$

is equivalent to the combination of consistence and persistence (substitutability) [1]. Hence for a choice function $C : \mathcal{B} \to \mathcal{B}$, consistence and persistence in conjunction can be regarded as a vector version of path-independence. ■

3.3 From value function model to choice function model

Let us say that a value function $f : \mathbf{Z}_+^S \to \mathbf{R} \cup \{-\infty\}$ is *unique-selecting* if for any $x \in \mathbf{Z}_+^S$, $\arg\max \{ f(y) \mid y \leq x \}$ is a singleton. Then we can define a choice function $C : \mathcal{B} \to \mathcal{B}$ with $\mathcal{B} = \{ x \in \mathbf{Z}_+^S \mid x \leq \sup(\mathrm{dom} f) \}$ by

$$C(x) = \arg\max \{ f(y) \mid y \leq x \} \quad (x \in \mathcal{B}). \tag{3.1}$$

We say that C is *induced* from f.

The following lemma asserts that M$^\natural$-concavity of value function f implies persistence.

Lemma 3.8. For unique-selecting M$^\natural$-concave value function f, the choice function C induced from f is consistent and persistent.

Proof. Consistency of C is obvious by the definition (3.1) of induction. Persistence can be proved by using Lemma 5.2 of [12], which is valid for a general M$^\natural$-concave function. For a unique-selecting M$^\natural$-concave function, however, the following simpler proof is possible.

To prove by contradiction, suppose that there exist $x, y \in \mathbf{Z}_+^S$ such that $x \geq y$ holds and $y \wedge C(x) \leq C(y)$ fails. Set $x' = C(x)$, $y' = C(y)$. Since $y \wedge x' \leq y'$ fails, there is some $e \in S$ such that $y(e) \wedge x'(e) > y'(e)$. Then $e \in \mathrm{supp}^+(x' - y')$, so we can apply the exchange axiom (M$^\natural$) to x', y' and e. Then for some $e' \in \mathrm{supp}^-(x' - y') \cup \{0\}$ the following inequality holds:

$$f(x') + f(y') \leq f(x' - \chi_e + \chi_{e'}) + f(y' + \chi_e - \chi_{e'}). \tag{3.2}$$

In (3.2), we have two cases: $e' \in \mathrm{supp}^-(x' - y')$ or $e' = 0$. In the case of $e' \in \mathrm{supp}^-(x' - y')$, we have $x'(e') < y'(e') \leq y(e') \leq x(e')$, and therefore $x' - \chi_e + \chi_{e'} \leq x$, which is also true in the other case of $e' = 0$. Since $x' = C(x)$ is the unique maximizer of f in $\{ z \in \mathbf{Z}_+^S \mid z \leq x \}$, it holds that $f(x' - \chi_e + \chi_{e'}) < f(x')$. Similarly, we have $f(y' + \chi_e - \chi_{e'}) < f(y')$, since $y' + \chi_e - \chi_{e'} \leq y$ by $y'(e) < y(e) \wedge x'(e) \leq y(e)$. These two strict inequalities contradict (3.2). □

By Lemmas 3.2, 3.6, and 3.8 we obtain the following fact.

Theorem 3.9. Assume that $\{f_k\}_{k \in I \cup J}$ are all unique-selecting M$^\natural$-concave functions, and let C_k be the choice function induced from f_k for each $k \in I \cup J$. Then $X \in \mathbf{Z}_+^E$ is stable with respect to $\{f_k\}_{k \in I \cup J}$ if and only if it is stable with respect to $\{C_k\}_{k \in I \cup J}$. ■

Note that when $\{C_k\}_{k \in I \cup J}$ and $\{\mathcal{B}_k\}_{k \in I \cup J}$ are defined from $\{f_k\}$ as above, the vector $u_k \in \mathbf{Z}_+^{E_k}$ in the value function model serves as the vector $b_k \in \mathbf{Z}_+^{E_k}$ in the choice function model, i.e., $\mathcal{B}_k = \{x \in \mathbf{Z}_+^{E_k} | x \leq u_k\}$ for all $k \in I \cup J$ since $\sup(\mathrm{dom} f_k) = u_k$. Therefore we can identify b_k in Lemma 3.6 with u_k in Lemma 3.2.

4 Strong Lattice Structure of Stable Allocations

4.1 Choice function model

Alkan–Gale [3] showed that if choice functions are consistent and persistent, there always exists a stable allocation.

Theorem 4.1. (Alkan–Gale [3]) If C_k is consistent and persistent for each $k \in I \cup J$, there exists a stable allocation. ■

Moreover, the set of all stable allocations forms a lattice. To state this more precisely, we introduce orderings on allocations. For allocations X and Y, we write $C_I(X \vee Y)$ for the allocation whose ith row is $C_i(x_i \vee y_i)$ for all $i \in I$. Symmetrically we write $C_J(X \vee Y)$ for the allocation whose jth column is $C_j(x_j \vee y_j)$ for all $j \in J$. Then we can define an ordering $\succeq_I$ on allocations by $X \succeq_I Y \iff C_I(X \vee Y) = X$. Symmetrically, we define an ordering $\succeq_J$ by $X \succeq_J Y \iff C_J(X \vee Y) = X$. Then the following theorems hold.

Theorem 4.2. (Alkan–Gale [3]) If C_k is consistent and persistent for each $k \in I \cup J$, then for any stable allocations X and Y, the following holds:

$$X \succeq_I Y \iff X \preceq_J Y.$$

■

The following fact is easily implied by known facts in the literature [2, 3, 10, 18].

Theorem 4.3. If C_k is consistent and persistent for each $k \in I \cup J$, the set of all stable allocations forms a lattice under the orderings $\succeq_I$ and $\succeq_J$, respectively.

Proof. For completeness we provide a proof in Appendix B. □

According to the above theorem, any two stable allocations X and Y surely have a join (least upper bound) and a meet (greatest lower bound) with respect to the ordering $\succeq_I$, which we denote by $X \vee_I Y$ and $X \wedge_I Y$, respectively. However, $\vee_I$ and $\wedge_I$ do not admit simple representations (see Appendix B). It is also noted that the operations $\vee_I$ and $\wedge_I$ are not necessarily distributive.

Alkan–Gale identified a crucial property of choice functions that implies many nice properties of the lattice of stable allocations. We use notation $|x| = \sum_{e \in S} x(e)$ for $x \in \mathbf{Z}_+^S$.

Definition 4.4. C is *size-monotone* if $[\ x \geq y \implies |C(x)| \geq |C(y)|\ ]$. ■

Theorem 4.5. (Alkan–Gale [3]) If C_k is consistent, persistent, and size-monotone for each $k \in I \cup J$, the set of all stable allocations forms a distributive lattice under the ordering $\succeq_I$. Moreover, for two stable allocations X and Y, their join $X \vee_I Y$ and meet $X \wedge_I Y$ coincide with $C_I(X \vee Y)$ and $C_J(X \vee Y)$, respectively. ■

Theorem 4.6. (Alkan–Gale [3]) If C_k is consistent, persistent, and size-monotone for each $k \in I \cup J$, then for any stable allocations X and Y, the following hold:

1. $|x_k| = |y_k|$ for all $k \in I \cup J$.
2. $(X \vee_I Y) \vee (X \wedge_I Y) = X \vee Y,\ \ (X \vee_I Y) \wedge (X \wedge_I Y) = X \wedge Y$.

■

The condition 2 above is equivalent to the following: for each $(i, j) \in E$, $\{(X \vee_I Y)(i, j),\ (X \wedge_I Y)(i, j)\} = \{x(i, j),\ y(i, j)\}$.

Theorems 4.5 and 4.6 say that size-monotonicity guarantees a rich structure of the set of stable allocations.

Note that size-monotonicity is a natural extension of "cardinal monotonicity" of Alkan [2] or "increasing property" (for a uniform weight) of Fleiner [10]. Also it corresponds to "law of aggregate demand" of Hatfield–Milgrom [18].

4.2 Value function model

Next we turn to M$^\natural$-concave value functions. Our main concern is the implications of M$^\natural$-concavity in the lattice structure of the set of stable allocations. Specifically, we show that the choice function induced from an M$^\natural$-concave value function satisfies size-monotonicity, which leads to the rich structure of stable allocations by Theorems 4.5 and 4.6.

First of all, a stable allocation surely exists if each agent's value function is M$^\natural$-concave.

Theorem 4.7. (Eguchi–Fujishige–Tamura [8]) If f_k is M$^\natural$-concave for each $k \in I \cup J$, there exists a stable allocation. ■

Remark 4.8. An alternative proof of Theorem 4.7 can be obtained from a combination of Theorem 3.9 and Theorem 4.1. When $\{f_k\}_{k \in I \cup J}$ are all unique-selecting, a straightforward combination shows the existence of a stable allocation. In the general case with some f_k not being unique-selecting, we can obtain the result through appropriate perturbations of f_k. ■

Next we discuss the structure of the set of stable allocations. Here, we show that M$^\natural$-concavity implies size-monotonicity which guarantees the strong lattice structure of stable allocations. This implies, by Theorems 4.5 and 4.6, that M$^\natural$-concavity yields the rich structure of stable allocations, which is stated in Theorems 4.11 and 4.12 as the main results of this paper.

Lemma 4.9. For a unique-selecting M$^\natural$-concave value function f, the choice function C induced from f is size-monotone.

Proof. To prove by contradiction, suppose that there exist $x, y \in \mathbf{Z}_+^S$ such that $x \geq y$ and $|C(x)| < |C(y)|$. Set $x' = C(x)$, $y' = C(y)$. Then $|x'| < |y'|$.

Let e_0 denote a new element not in S and put $\hat{S} = \{e_0\} \cup S$. Let $\hat{f} : \mathbf{Z}_+^{\hat{S}} \to \mathbf{R} \cup \{-\infty\}$ be the function defined by

$$\hat{f}(z_{e_0}, z) = \begin{cases} f(z) & \text{if } z_{e_0} = -|z| \\ -\infty & \text{otherwise.} \end{cases} \tag{4.1}$$

Then $f(x') = \hat{f}(-|x'|, x')$ and $f(y') = \hat{f}(-|y'|, y')$. By the exchange property (M) in Lemma 4.10 below for $(-|x'|, x')$, $(-|y'|, y')$ and e_0, there exists some $e \in \text{supp}^-(x' - y')$ such that

$$\hat{f}(-|x'|, x') + \hat{f}(-|y'|, y') \leq \hat{f}(-|x'| - 1, x' + \chi_e) + \hat{f}(-|y'| + 1, y' - \chi_e).$$

By the definition of $\hat{f}$, the above inequality can be rephrased as follows:

$$f(x') + f(y') \leq f(x' + \chi_e) + f(y' - \chi_e). \tag{4.2}$$

On the other hand, as $x'(e) < y'(e) \leq y(e) \leq x(e)$, we have $x' + \chi_e \leq x$. Since $x' = C(x)$ is the unique maximizer of f in $\{ z \in \mathbf{Z}_+^S \mid z \leq x \}$, it holds that $f(x' + \chi_e) < f(x')$. Similarly, we have $f(y' - \chi_e) < f(y')$ since $y' - \chi_e \leq y' \leq y$. These two strict inequalities contradict (4.2). $\square$

Lemma 4.10. ([23]) Let $f : \mathbf{Z}^S \to \mathbf{R} \cup \{-\infty\}$ be an M$^\natural$-concave function and put $\hat{S} = \{e_0\} \cup S$ ($e_0 \notin S$). Then the function $\hat{f} : \mathbf{Z}^{\hat{S}} \to \mathbf{R} \cup \{-\infty\}$ defined by (4.1) satisfies the following condition[5]:

(M) $\forall x, y \in \text{dom}\hat{f},\ \forall e \in \text{supp}^+(x - y),\ \exists e' \in \text{supp}^-(x - y)$:

$$\hat{f}(x) + \hat{f}(y) \leq \hat{f}(x - \chi_e + \chi_{e'}) + \hat{f}(y + \chi_e - \chi_{e'}).$$

■

Here we define $X \vee_I Y$ as the allocation whose ith row is equal to $\arg\max \{ f_i(z) \mid z \leq (x_i \vee y_i) \}$ for all $i \in I$, and $X \wedge_I Y$ as the allocation whose jth column is equal to $\arg\max \{ f_j(z) \mid z \leq (x_j \vee y_j) \}$ for all $j \in J$. It is noted that these definitions of $\vee_I$ and $\wedge_I$ are consistent with those given in Section 4.1 in the choice function model. Indeed, Theorem 4.5 shows $X \vee_I Y = C_I(X \vee Y)$ and $X \wedge_I Y = C_J(X \vee Y)$, whereas the ith row of $C_I(X \vee Y)$ is equal to $\arg\max \{ f_i(z) \mid z \leq (x_i \vee y_i) \}$ and the jth column of $C_J(X \vee Y)$ is equal to $\arg\max \{ f_j(z) \mid z \leq (x_j \vee y_j) \}$ by (3.1).

Combining Lemmas 3.8 and 4.9, and Theorems 3.9, 4.5 and 4.6, we obtain the following theorems.

[5]A function $\hat{f}$ that satisfies the condition (M) is said to be M-concave.

Theorem 4.11. If f_k is a unique-selecting M$^\natural$-concave value function for each $k \in I \cup J$, then the set of all stable allocations forms a distributive lattice with operations $\vee_I$ and $\wedge_I$. ■

Theorem 4.12. If f_k is a unique-selecting M$^\natural$-concave value function for each $k \in I \cup J$, then for any stable allocations X and Y, the following hold:

1. $|x_k| = |y_k|$ for all $k \in I \cup J$.

2. $(X \vee_I Y) \vee (X \wedge_I Y) = X \vee Y$, $(X \vee_I Y) \wedge (X \wedge_I Y) = X \wedge Y$.

■

These theorems establish an intimate connection of the M$^\natural$-concave value function model to the choice function model with size-monotonicity.

Remark 4.13. Fleiner [10] pointed out that the nonemptiness and the lattice structure of many-to-many matching model (with {0,1}-variables) can be shown by using Tarski's fixed point theorem. The fundamental observation in this approach is that "stable pairs" (which correspond to stable matchings) can be regarded as fixed points of a certain monotone function. In fact, this approach can be naturally extended to integer variables, i.e., Theorems 4.1, 4.2 and 4.3 can be obtained also by fixed point approach.

Moreover, Fleiner found that if the choice functions are w-increasing, the lattice operations of stable matchings become distributive and simply representable. Theorem 4.5 is an integer version of this result with $w = \mathbf{1}$. ■

Remark 4.14. Hatfield-Milgrom [18] studied one-to-many matching model in terms of contracts between doctors and hospitals. They showed that if hospitals preferences satisfy "law of aggregate demand" (which corresponds to size-monotonicity) and other assumptions (which correspond to consistency and persistence), the strategy proofness holds for the deferred acceptance algorithm with doctors proposing. Hence it is expected that size-monotonicity (beyond consistency and persistence) has significant implication for strategy proofness also in the stable allocation model. ■

5 Extension to Quasi-concave Model

In Sections 2 and 3, we have shown that $\mathrm{M}^\natural$-concavity in value function model leads to the strong lattice structure of stable allocations. In this section, we show that this result can be extended for quasi $\mathrm{M}^\natural$-concave model.

As described below, quasi M-concavity is defined by ordinal relationship of function values, and not by values themselves. Therefore, the extensibility to quasi M-concave functions means that the strong lattice structure of stable allocations is guaranteed solely by the concave-like ordering of values. This agrees with the fact that the stability of allocations is defined not by function values themselves but by their ordinal relationship.

5.1 Quasi $\mathrm{M}^\natural$-concavity

In this paper we have adopted the exchange property ($\mathrm{M}^\natural$) as the definition of $\mathrm{M}^\natural$-concave functions (Section 2), but the original definition [23, 25] reads: a function f is $\mathrm{M}^\natural$-concave if $\hat{f}$ in (4.1) satisfies the exchange property (M) in Lemma 4.10. Then an $\mathrm{M}^\natural$-concave function is known to satisfy ($\mathrm{M}^\natural$). That is,

$$\hat{f} \text{ satisfies (M)} \Longleftrightarrow f \text{ satisfies } (\mathrm{M}^\natural). \tag{5.1}$$

The concept of quasi M-concave function is proposed by Murota–Shioura [26] by weakening the condition (M) to

(QM) $\forall x, y \in \mathrm{dom}\hat{f},\ \forall e \in \mathrm{supp}^+(x-y),\ \exists e' \in \mathrm{supp}^-(x-y):$

$$\hat{f}(x - \chi_e + \chi_{e'}) - \hat{f}(x) \geq 0 \quad \text{or} \quad \hat{f}(y + \chi_e - \chi_{e'}) - \hat{f}(y) \geq 0. \tag{5.2}$$

Just as $\mathrm{M}^\natural$-concavity is defined (originally) in terms of M-concavity of the associated $\hat{f}$, we define quasi $\mathrm{M}^\natural$-concavity as follows.

Definition 5.1. A function $f : \mathbf{Z}^S \to \mathbf{R} \cup \{-\infty\}$ with $\mathrm{dom}f \neq \emptyset$ is called *quasi $\mathrm{M}^\natural$-concave* if $\hat{f}$ defined in (4.1) satisfies (QM). ■

Obviously, (M) implies (QM), and therefore, an $\mathrm{M}^\natural$-concave function is quasi $\mathrm{M}^\natural$-concave.

As an $\mathrm{M}^\natural$-version of (QM), we consider:

($\mathrm{QM}^\natural$) $\forall x, y \in \mathrm{dom}f,\ \forall e \in \mathrm{supp}^+(x-y),\ \exists e' \in \mathrm{supp}^-(x-y) \cup \{0\}:$

$$f(x - \chi_e + \chi_{e'}) - f(x) \geq 0 \quad \text{or} \quad f(y + \chi_e - \chi_{e'}) - f(y) \geq 0. \tag{5.3}$$

Lemma 5.2. Let $\hat{f}$ and f be associated by (4.1). Then

$$\hat{f} \text{ satisfies (QM)} \Longrightarrow f \text{ satisfies } (\mathrm{QM}^\natural). \tag{5.4}$$

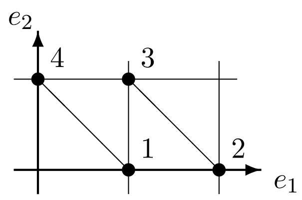

Figure 1: dom$f$ and the value of f at each point for Example 5.3

Proof. Take $x, y \in \mathrm{dom} f$, and put $S^+ = \mathrm{supp}^+(x-y)$ and $S^- = \mathrm{supp}^-(x-y)$. (QM) for $\hat{f}$ in (4.1) is translated to conditions on f as follows:

$$\begin{aligned} |x| > |y| \quad &\Rightarrow \quad \forall e \in S^+ : \exists e' \in S^- \cup \{e_0\} \quad \text{satisfies (5.3)}, \\ |x| = |y| \quad &\Rightarrow \quad \forall e \in S^+ : \exists e' \in S^- \quad \text{satisfies (5.3)}, \\ |x| < |y| \quad &\Rightarrow \quad \forall e \in S^+ \cup \{e_0\} : \exists e' \in S^- \quad \text{satisfies (5.3)}. \end{aligned}$$

This implies (QM$^\natural$). □

In contrast to (5.1), the converse of (5.4) does not hold.

Example 5.3. Here is an example of a function f that satisfies (QM$^\natural$) but the associated $\hat{f}$ does not satisfy (QM). Let $S = \{e_1, e_2\}$, $f : \mathbf{Z}^S \to \mathbf{R} \cup \{-\infty\}$ and

$$\mathrm{dom} f = \{ \ (1,0), (2,0), (1,1), (0,1) \ \},$$

$$f(1,0) = 1, \quad f(2,0) = 2, \quad f(1,1) = 3, \quad f(0,1) = 4.$$

(See Figure 1.) This f satisfies (QM$^\natural$). Define $\hat{f} : \mathbf{Z}^{\hat{S}} \to \mathbf{R} \cup \{-\infty\}$ by (4.1). Then $\hat{S} = \{e_0, e_1, e_2\}$ and

$$\mathrm{dom} \hat{f} = \{ \ (-1,1,0), (-2,2,0), (-2,1,1), (-1,0,1) \ \},$$

$$\hat{f}(-1,1,0) = 1, \quad \hat{f}(-2,2,0) = 2, \quad \hat{f}(-2,1,1) = 3, \quad \hat{f}(-1,0,1) = 4.$$

To check (QM), let $x = (-1,0,1)$, $y = (-2,2,0)$, and $e = e_0 \in \mathrm{supp}^+(x-y)$. Since $\mathrm{supp}^-(x-y) = \{e_1\}$, we can take only e_1 as $e' \in \mathrm{supp}^-(x-y)$, and then

$$\begin{aligned} \hat{f}(-2,1,1) - \hat{f}(-1,0,1) &= 3 - 4 = -1 < 0, \\ \hat{f}(-1,1,0) - \hat{f}(-2,2,0) &= 1 - 2 = -1 < 0. \end{aligned}$$

Thus, (5.2) fails for all $e' \in \mathrm{supp}^-(x-y)$. Hence $\hat{f}$ does not satisfy (QM).■

5.2 Quasi M$^\natural$-concave value function model

The choice functions induced from quasi M$^\natural$-concave functions have the same nice properties as those induced from M$^\natural$-concave functions. The proofs for quasi versions are almost the same as those for the original versions (Lemmas 3.8 and 4.9).

Lemma 5.4. For unique-selecting quasi M$^\natural$-concave value function f, the choice function C induced from f is consistent and persistent.

Proof. Consistency is obvious by the definition of induction. To prove persistence by contradiction, suppose that there exist $x, y \in \mathbf{Z}_+^S$ such that $x \geq y$ holds and $y \wedge C(x) \leq C(y)$ fails. Set $x' = C(x)$, $y' = C(y)$. Since $y \wedge x' \leq y'$ fails, there is some $e \in S$ such that $y(e) \wedge x'(e) > y'(e)$. Then $e \in \mathrm{supp}^+(x' - y')$. By Lemma 5.2 we can apply the exchange axiom (QM$^\natural$) to x', y' and e. Then for some $e' \in \mathrm{supp}^-(x' - y') \cup \{0\}$ the following holds:

$$f(x' - \chi_e + \chi_{e'}) - f(x') \geq 0 \quad \text{or} \quad f(y' + \chi_e - \chi_{e'}) - f(y') \geq 0. \tag{5.5}$$

In (5.5), we have two cases: $e' \in \mathrm{supp}^-(x' - y')$ or $e' = 0$. In the case of $e' \in \mathrm{supp}^-(x' - y')$, we have $x'(e') < y'(e') \leq y(e') \leq x(e')$, and therefore $x' - \chi_e + \chi_{e'} \leq x$, which is also true in the other case of $e' = 0$. Since $x' = C(x)$ is the unique maximizer of f in $\{ z \in \mathbf{Z}_+^S \mid z \leq x \}$, it holds that $f(x' - \chi_e + \chi_{e'}) < f(x')$. Similarly, we have $f(y' + \chi_e - \chi_{e'}) < f(y')$, since $y' + \chi_e - \chi_{e'} \leq y$ by $y'(e) < y(e) \wedge x'(e) \leq y(e)$. These two strict inequalities contradict (5.5). □

Lemma 5.5. For a unique-selecting quasi M$^\natural$-concave value function f, the choice function C induced from f is size-monotone.

Proof. To prove by contradiction, suppose that there exist $x, y \in \mathbf{Z}_+^S$ such that $x \geq y$ and $|C(x)| < |C(y)|$. Set $x' = C(x)$, $y' = C(y)$. Then $|x'| < |y'|$.

Let $\hat{f} : \mathbf{Z}_+^{\hat{S}} \to \mathbf{R} \cup \{-\infty\}$ be defined by (4.1). Then $f(x') = \hat{f}(-|x'|, x')$ and $f(y') = \hat{f}(-|y'|, y')$. Since $\hat{f}$ satisfies (QM) for $(-|x'|, x')$, $(-|y'|, y')$ and e_0, there exists some $e \in \mathrm{supp}^-(x' - y')$ such that

$$\hat{f}(-|x'|-1, x'+\chi_e) - \hat{f}(-|x'|, x') \geq 0 \quad \text{or} \quad \hat{f}(-|y'|+1, y'-\chi_e) - \hat{f}(-|y'|, y') \geq 0.$$

By the definition of $\hat{f}$, this can be rephrased as follows:

$$f(x' + \chi_e) - f(x') \geq 0 \quad \text{or} \quad f(y' - \chi_e) - f(y') \geq 0. \tag{5.6}$$

On the other hand, as $x'(e) < y'(e) \leq y(e) \leq x(e)$, we have $x' + \chi_e \leq x$. Since $x' = C(x)$ is the unique maximizer of f in $\{ z \in \mathbf{Z}_+^S \mid z \leq x \}$, it holds that $f(x' + \chi_e) < f(x')$. Similarly, we have $f(y' - \chi_e) < f(y')$ since $y' - \chi_e \leq y' \leq y$. These two strict inequalities contradict (5.6). □

By Lemmas 5.4 and 5.5 as well as Theorem 4.1, we can extend Theorems 4.11 and 4.12 as follows.

Theorem 5.6. If f_k is a unique-selecting quasi M$^\natural$-concave value function for each $k \in I \cup J$, then the set of all stable allocations is nonempty and forms a distributive lattice with operations $\vee_I$ and $\wedge_I$. ■

Theorem 5.7. If f_k is a unique-selecting quasi M$^\natural$-concave value function for each $k \in I \cup J$, then for any stable allocations X and Y, the following hold:

1. $|x_k| = |y_k|$ for all $k \in I \cup J$.
2. $(X \vee_I Y) \vee (X \wedge_I Y) = X \vee Y, \ (X \vee_I Y) \wedge (X \wedge_I Y) = X \wedge Y$.

■

Remark 5.8. Suppose that f is a unique-selecting value function and satisfies (QM$^\natural$). Then, the choice function C induced from f is consistent and persistent. However, C may not be size-monotone if f is not quasi M$^\natural$-concave (i.e., if $\hat{f}$ fails to satisfy (QM)). That is, (QM$^\natural$) is sufficient for consistency and persistence, but not sufficient for size-monotonicity. ■

Example 5.9. Here is an example of a non size-monotone choice function induced from a function f that is not quasi M$^\natural$-concave but satisfies (QM$^\natural$). The function f is much the same as in Example 1, but dom$f$ contains an additional point $(0,0)$ to meet the requirement for a value function. Let $S = \{e_1, e_2\}$ and

$$\mathrm{dom} f = \{\ (0,0), (1,0), (2,0), (1,1), (0,1)\ \},$$

$$f(0,0) = 0, \quad f(1,0) = 1, \quad f(2,0) = 2, \quad f(1,1) = 3, \quad f(0,1) = 4.$$

Let C be the choice function defined by (3.1), and let $x = (2,1)$, $y = (2,0)$. Then $x \geq y$ but $|C(x)| = |(0,1)| = 1 < 2 = |(2,0)| = |C(y)|$. Hence C is not size-monotone. ■

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A M$^\natural$-concave Functions: Examples and Operations

A.1 Examples of M$^\natural$-concave functions

We give some examples of M$^\natural$-concave functions [23, 24]. We denote variable $x = (x(e) \mid e \in S) \in \mathbf{Z}^S$ and let $f : \mathbf{Z}^S \to \mathbf{R} \cup \{-\infty\}$. The inner product of $w \in \mathbf{R}^S$ and $x \in \mathbf{Z}^S$ is denoted as $\langle w, x \rangle = \sum_{e \in S} w(e) \cdot x(e)$.

Linear function: A linear (or affine) function

$$f(x) = \alpha + \langle w, x \rangle$$

with $w \in \mathbf{R}^S$ and $\alpha \in \mathbf{R}$ is M$^\natural$-concave.

Quadratic function: A quadratic function

$$f(x) = \sum_{e,e' \in S} a(e, e') \cdot x(e) \cdot x(e') = x^\top A x$$

with $a(e, e') = a(e', e) \in \mathbf{R}$ ($e, e' \in S$) is M$^\natural$-concave if and only if $a(e, e') \leq 0$ for all (e, e') and

$$a(e_1, e_2) \leq \max\{\ a(e_1, e_3),\ a(e_2, e_3)\ \} \quad \text{whenever} \quad \{e_1, e_2\} \cap \{e_3\} = \emptyset.$$

Separable convex function: A function f is called *separable concave* if it can be represented as

$$f(x) = \sum_{e \in S} \varphi_e(x(e))$$

with univariate concave functions φ_e ($e \in S$). A separable concave function f is M$^\natural$-concave.

Component-sum concave function: A function f with $\mathrm{dom} f \subseteq \mathbf{Z}^S_+$ represented as

$$f(x) = \varphi(|x|) \quad (x \in \mathbf{Z}^S_+)$$

with a univariate concave function φ is M$^\natural$-concave, where $|x|$ is the sum of all components of x.

Laminar concave function: A *laminar family* means a nonempty family $\mathcal{T}$ of subsets of S such that $T \cap T' = \emptyset$ or $T \subseteq T'$ or $T \supseteq T'$ for any $T, T' \in \mathcal{T}$. A function f is called *laminar concave* if it can be represented as

$$f(x) = \sum_{T \in \mathcal{T}} \varphi_T(x(T))$$

for a laminar family $\mathcal{T}$ and a family of univariate concave functions φ_T indexed by $T \in \mathcal{T}$, where $x(T) = \sum_{e \in T} x(e)$. A laminar concave function is M$^\natural$-concave. Actually, separable concave functions and component-sum concave functions are special cases of such functions.

Weighted matroid: Let $\mathcal{I}$ be the family of independent sets of a matroid on S and let $w \in \mathbf{R}^S$. The function defined as

$$f(x) = \begin{cases} \langle w, x \rangle & \text{if } x = \chi_I \text{ for some } I \in \mathcal{I}, \\ -\infty & \text{otherwise} \end{cases}$$

is M$^\natural$-concave.

Linear and separable concave functions with submodular restriction: Let $\rho : 2^S \to \mathbf{Z}_+$ be a submodular function (i.e., $\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$ for all $X, Y \in 2^S$) and let $w \in \mathbf{R}^S$. The function defined as

$$f(x) = \begin{cases} \langle w, x \rangle & \text{if } x \geq \mathbf{0} \text{ and } x(X) \leq \rho(X) \text{ for all } X \in 2^S, \\ -\infty & \text{otherwise} \end{cases}$$

is M$^\natural$-concave. Moreover, f is also M$^\natural$-concave if $\langle w, x \rangle$ is replaced by a separable concave function $\sum_{e \in S} \varphi_e(x(e))$ with φ_e $(e \in S)$ being univariate concave functions.

Maximum-value function: Given $a_e \in \mathbf{R}$ for all $e \in S$, we define a set function $\mu : 2^S \to \mathbf{R} \cup \{-\infty\}$ as

$$\mu(X) = \begin{cases} \max\{ a_e \mid e \in X \} & \text{if } X \neq \emptyset, \\ a_* & \text{if } X = \emptyset \end{cases}$$

by choosing $a_* \in \mathbf{R} \cup \{-\infty\}$ such that $a_* \leq \min\{ a_e \mid e \in S \}$. Then μ is M$^\natural$-concave when identified with a function $f : \mathbf{Z}^S \to \mathbf{R} \cup \{-\infty\}$ such that $\mathrm{dom} f \subseteq \{0,1\}^S$ and $f(\chi_X) = \mu(X)$ for $X \in 2^S$. This function corresponds to *unit demand preference* [16].

A.2 Operations preserving M$^\natural$-concavity

We give some operations that preserve M$^\natural$-concavity [23].

Basic operations: For an M$^\natural$-concave function $f : \mathbf{Z}^S \to \mathbf{R} \cup \{-\infty\}$ and $\lambda \in \mathbf{R}$ and $a \in \mathbf{Z}^S$ and $w \in \mathbf{R}^S$, the function $\tilde{f} : \mathbf{Z}^S \to \mathbf{R} \cup \{-\infty\}$ defined below is M$^\natural$-concave:

$$\tilde{f}(x) = \lambda f(x + a) + \langle w, x \rangle \quad (x \in \mathbf{Z}^S).$$

Restriction and projection: For an M$^\natural$-concave function $f : \mathbf{Z}^S \to \mathbf{R} \cup \{-\infty\}$ and a subset $T \in 2^S$, the functions $f_T, f^T : \mathbf{Z}^T \to \mathbf{R} \cup \{-\infty\}$ defined below are M$^\natural$-concave:

$$f_T(x) = f(x, \mathbf{0}_{S \setminus T}) \quad (x \in \mathbf{Z}^T),$$
$$f^T(x) = \sup\{ f(x, y) \mid y \in \mathbf{Z}^{S \setminus T} \} \quad (x \in \mathbf{Z}^T).$$

Convolution: For M$^\natural$-concave functions $f_1, f_2, \dots, f_k : \mathbf{Z}^S \to \mathbf{R} \cup \{-\infty\}$, the function $\tilde{f} : \mathbf{Z}^S \to \mathbf{R} \cup \{-\infty\}$ defined by

$$\tilde{f}(x) = \sup \Big\{ \sum_{i=1}^{k} f_i(x_i) \mid x_1 + x_2 + \cdots + x_k = x \Big\} \quad (x \in \mathbf{Z}^S)$$

is M$^\natural$-concave. This operation corresponds to aggregation of utility functions.

Remark A.1. There are other operations that preserve M$^\natural$-concavity not described above. It is to be noted, however, that a sum of M$^\natural$-concave functions is not necessarily M$^\natural$-concave. ■

B Proof of Theorem 4.3

The proof of Theorem 4.3 here is based heavily on several technical results of Alkan–Gale [3]. We first introduce some definitions and lemmas.

For each $k \in I \cup J$, we define $\sigma_k : \mathcal{B}_k \to \mathcal{B}_k$ by

$$\sigma_k(x) = \bigvee \{ z \in \mathcal{B}_k \mid C_k(z) = C_k(x) \} \quad (x \in \mathcal{B}_k).$$

If C_k is consistent and persistent, $C_k(\sigma_k(x)) = C_k(x)$ holds for any $x \in \mathcal{B}_k$. We call $\sigma_k(x)$ closure of x. For an allocation X, we write $\sigma_I(X)$ for the allocation whose ith row is $\sigma_i(x_i)$ for all $i \in I$. Similarly, we write $\sigma_J(X)$ for the allocation whose jth column is $\sigma_j(x_j)$ for all $j \in J$.

We say that a vector $x \in \mathcal{B}_k$ is acceptable if $C_k(x) = x$, and we write $\mathcal{A}_k = \{ x \in \mathcal{B}_k \mid C_k(x) = x \}$. We also say that an allocation X is I-acceptable if its ith row x_i is acceptable for all $i \in I$, and J-acceptable if its jth column x_j is acceptable for all $j \in J$.

For any $k \in I \cup J$ and $x, y \in \mathcal{B}_k$, we write $x \succeq_k y$ if $C_k(x \vee y) = x$. Then $X \succeq_I Y$, which is already defined in Section 4.1, can be defined equivalently by the condition that $x_i \succeq_i y_i$ for all $i \in I$.

The following lemmas are due to Alkan–Gale [3]. Recall that $b_k \in \mathbf{Z}_+^{E_k}$ is the vector such that $\mathcal{B}_k = \{x \in \mathbf{Z}_+^{E_k} | x \le b_k\}$ for all $k \in I \cup J$.

Lemma B.1. If C_i $(i \in I)$ is consistent and persistent, the following hold:

(1) [3, Lemma 7] $x_i \in \mathcal{A}_i$ is j-satiated if and only if $\sigma_i(x_i)(j) = b_i(j)$.

(2) [3, Lemma 8(1)] $x_i \in \mathcal{A}_i$ is j-satiated if there exists y_i that is j-satiated and $x_i \succeq_i y_i$.

(3) [3, Lemma 6] $x_i \in \mathcal{A}_i$ is j-satiated if there exists y_i such that $C_i(y_i) = x_i$ and $y_i(j) > x_i(j)$.

■

Lemma B.2 ([3, Lemma 10])**.** Assume that $\{C_k\}_{k\in I\cup J}$ are all consistent and persistent, and X is a stable allocation and Y is an I-acceptable allocation. Then $Y \succeq_I X \implies Y \preceq_J X$. ■

The above lemmas also hold with (i, I) replaced by (j, J). We often use the following inequalities in the proof:

$$X \geq Y \implies C_I(X) \succeq_I C_I(Y),\ C_J(X) \succeq_J C_J(Y), \tag{B.1}$$

$$X \succeq_I Y \implies \sigma_I(X) \geq Y. \tag{B.2}$$

Now we begin the proof of Theorem 4.3. We will show below that any two stable allocations X and Y have the greatest lower bound $X \wedge_I Y$ with respect to $\succeq_I$. Then $X \wedge_J Y$ exists symmetrically, and the greatest upper bound $X \vee_I Y$ is given as $X \vee_I Y = X \wedge_J Y$ by Theorem 4.2. Thus, showing the existence of $X \wedge_I Y$ is already sufficient to establish the lattice property with respect to $\succeq_I$.

The greatest lower bound $X \wedge_I Y$ can be constructed as follows. Define sequences of allocations (B^n), (U^n), (V^n) by the following recursion rule[6] for $n = 0, 1, 2, \cdots$:

$$B^0 = \sigma_I(X) \wedge \sigma_I(Y),$$

$$U^n = C_I(B^n),$$

$$V^n = C_J(U^n),$$

and B^{n+1} is obtained from B^n as follows:

$$b^{n+1}(i,j) = b^n(i,j) \quad \text{if} \quad v^n(i,j) = u^n(i,j),$$
$$b^{n+1}(i,j) = v^n(i,j) \quad \text{if} \quad v^n(i,j) < u^n(i,j).$$

Note that (B^n) is a nonincreasing nonnegative sequence and hence converges, and then (U^n) and (V^n) also converge. Call the limits of the sequences $\widehat{B}$, $\widehat{U}$, $\widehat{V}$, respectively. We will show:

(i) $\widehat{U} = \widehat{V}$.

(ii) $\widehat{U}$ is a stable allocation with $X \succeq_I \widehat{U}$ and $Y \succeq_I \widehat{U}$.

(iii) Any stable allocation W with $X \succeq_I W$ and $Y \succeq_I W$ satisfies $\widehat{U} \succeq_I W$.

Note that (ii) and (iii), in conjunction, mean $\widehat{U} = X \wedge_I Y$.

Before proving (i), (ii), (iii), we first show the following:

(I) $X \succeq_I U^0$, $Y \succeq_I U^0$, and $U^0 \succeq_I U^1 \succeq_I U^2 \succeq_I \cdots \succeq_I \widehat{U}$.

(II) $X \preceq_J V^0$, $Y \preceq_J V^0$, and $V^0 \preceq_J V^1 \preceq_J V^2 \preceq_J \cdots \preceq_J \widehat{V}$.

[6]This recursion rule is the same as in the proof of Theorem 1 of Alkan–Gale [3], but with a different initial value B^0.

(I) : By the definition, we have $U^0 = C_I(B^0) = C_I(\sigma_I(X) \wedge \sigma_I(Y))$. Since $\sigma_I(X) \geq \sigma_I(X) \wedge \sigma_I(Y)$, we have

$$X = C_I(\sigma_I(X)) \succeq_I C_I(\sigma_I(X) \wedge \sigma_I(Y)) = U^0$$

by using (B.1). Similarly, $Y \succeq_I U^0$. Moreover, since $U^n = C_I(B^n)$ and (B^n) is nonincreasing, we obtain from (B.1) that $U^0 \succeq_I U^1 \succeq_I U^2 \succeq_I \cdots \succeq_I \widehat{U}$. Thus (I) is proved.

(II) : First we show $X \preceq_J V^0$ and $Y \preceq_J V^0$. From $C_J(X \vee Y) \succeq_J X$ and Lemma B.2, we have $C_J(X \vee Y) \preceq_I X$, which implies $C_J(X \vee Y) \leq \sigma_I(X)$ by (B.2). Similarly, we have $C_J(X \vee Y) \leq \sigma_I(Y)$. Then

$$C_J(X \vee Y) \leq (\sigma_I(X) \wedge \sigma_I(Y)) \wedge (X \vee Y) = B^0 \wedge (X \vee Y). \tag{B.3}$$

On the other hand, from $B^0 = \sigma_I(X) \wedge \sigma_J(Y) \leq \sigma_I(X)$ and persistence, we have $B^0 \wedge X \leq C_I(B^0)$. Similarly, $B^0 \wedge Y \leq C_I(B^0)$. Therefore

$$B^0 \wedge (X \vee Y) \leq C_I(B^0) = U^0. \tag{B.4}$$

From (B.3) and (B.4), we have $C_J(X \vee Y) \leq U^0$. Hence $X = C_J(X) \preceq_J C_J(X \vee Y) \preceq_J C_J(U^0) = V^0$ by (B.1). Similarly we can obtain $Y \preceq_J V^0$.

Next we show $V^n \preceq_J V^{n+1}$. From the recursion rule, we have $V^n \leq B^{n+1}$ and $V^n \leq U^n$, i.e., $V^n \leq B^{n+1} \wedge U^n$, whereas $B^{n+1} \wedge U^n \leq U^{n+1}$ by the persistence of C_I and $B^n \geq B^{n+1}$. Hence we have $V^n \leq U^{n+1}$, from which follows $C_J(V^n) \preceq_J C_J(U^{n+1})$ by (B.1). The right-hand side $C_J(U^{n+1})$ is equal to V^{n+1} by the recursion rule, and the left-hand side is equal to V^n, since $V^n = C_J(U^n) = C_J(C_J(U^n)) = C_J(V^n)$. Thus we have obtained $V^n \preceq_J V^{n+1}$, completing the proof of (II).

Now we prove (i), (ii), and (iii).

(i) By the recursion rule, $V^n \leq U^n$ holds for any $n \geq 0$. Since (B^n) converges, there is no (i, j) such that $\widehat{v}(i, j) < \widehat{u}(i, j)$. Therefore $\widehat{U} = \widehat{V}$.

(ii) The latter half of (ii), $X \succeq_I \widehat{U}$ and $Y \succeq_I \widehat{U}$ follow directly from (I). The stability of $\widehat{U}$ can be proved as follows. By the recursion rule and (i), we have $C_I(\widehat{B}) = C_I(\widehat{U}) = \widehat{U}$ and $C_J(\widehat{U}) = \widehat{U}$. Hence the first condition of the stability: [for every $k \in I \cup J$, $C_k(\widehat{u}_k) = \widehat{u}_k$] holds. The second condition:

[for every $(i, j) \in I \times J$, $\widehat{u}_i$ is j-satiated or $\widehat{u}_j$ is i-satiated]

can be shown as follows. We consider following three cases, which exhaust all possibilities since $\widehat{b}(i, j) \leq b^0(i, j) \leq b_i(j)$.

Case 1: $\widehat{b}(i,j) = b^0(i,j) = b_i(j)$.

Since $C_i(\widehat{b}_i) = \widehat{u}_i$, it holds that $\sigma_i(\widehat{u}_i)(j) \geq \widehat{b}(i,j) = b_i(j)$ and hence $\widehat{u}_i$ is j-satiated from Lemma B.1 (1).

Case 2: $\widehat{b}(i,j) \leq b^0(i,j) < b_i(j)$.

Since $b^0(i,j) = \sigma_i(x_i)(j) \wedge \sigma_i(y_i)(j)$, $b^0(i,j) < b_i(j)$ means $\sigma_i(x_i)(j) < b_i(j)$ or $\sigma_i(y_i)(j) < b_i(j)$. By Lemma B.1 (1), we have:

$$[x_i \text{ is not } j\text{-satiated}] \ \text{ or } [y_i \text{ is not } j\text{-satiated}].$$

Since X and Y are stable, we have:

$$[x_j \text{ is } i\text{-satiated}] \ \text{ or } [y_j \text{ is } i\text{-satiated}].$$

Combining this with Lemma B.1 (2) and (II), we see that $\widehat{v}_j (= \widehat{u}_j)$ is i-satiated.

Case 3: $\widehat{b}(i,j) < b^0(i,j) = b_i(j)$.

When $\widehat{b}(i,j) < b^0(i,j)$, from the recursion rule, there exists $m \geq 0$ such that $v^m(i,j) < u^m(i,j)$. Here we have $v_j^m = C_j(u_j^m)$. Hence, v_j^m is i-satiated by Lemma B.1 (3). Combining this with Lemma B.1 (2) and (II), we see that $\widehat{v}_j (= \widehat{u}_j)$ is i-satiated.

(iii) To prove (iii), it is sufficient to show that any stable allocation W with $X \succeq_I W$ and $Y \succeq_I W$ satisfies $W \leq \widehat{B}$, since that implies $W = C_I(W) \preceq_I C_I(\widehat{B}) = \widehat{U}$ by (B.1). By induction on $n \geq 0$, we prove $W \leq B^n$, which implies $W \leq \widehat{B}$.

Because $X \succeq_I W$ and $Y \succeq_I W$, we have $W \leq \sigma_I(X)$ and $W \leq \sigma_I(Y)$ from (B.2), and hence $W \leq \sigma_I(X) \wedge \sigma_I(Y) = B^0$.

Assuming $W \leq B^n$, we prove $W \leq B^{n+1}$. Since $W \leq B^n$, it holds that $W = C_I(W) \preceq_I C_I(B^n) = U^n$ by (B.1), and therefore $W \succeq_J U^n$ by Lemma B.2. Then we have $C_J(U^n \vee W) = W$. On the other hand, from the persistence of C_J and $U^n \leq U^n \vee W$, we have $U^n \wedge C_J(U^n \vee W) \leq C_J(U^n)$. Thus $U^n \wedge W \leq V^n$ holds. This shows, by the recursion rule, that:

$$\begin{aligned} &\text{if } v^n(i,j) < u^n(i,j), \quad \text{then} \quad w(i,j) \leq v^n(i,j) = b^{n+1}(i,j), \\ &\text{if } v^n(i,j) = u^n(i,j), \quad \text{then} \quad w(i,j) \leq b^n(i,j) = b^{n+1}(i,j). \end{aligned}$$

Hence we have $W \leq B^{n+1}$.

The proof of Theorem 4.3 is completed.