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Abstract

We provide a deterministic polynomial time algorithm for multicasting in a linear deterministic relay network proposed by Avestimehr, Diggavi and Tse (2011). The running time of our algorithm matches the complexity of unicast computations for each sinks, i.e., our algorithm is optimal in this sense. Our approach is based on the polylinking flow model of Goemans, Iwata and Zenklusen (2012), and the mixed matrix completion technique of Harvey, Karger and Murota (2005).

1 Introduction

Determining the capacity of a wireless information channel is a longstanding open problem. The difficulty comes from the two essential features of a wireless information channel: *broadcast* and *superposition*. In wireless communication, signals are sent to multiple users in the network, and superposition makes it hard to recover original signals. By these obstacles, even a simple wireless channel, such as network with a single source, a single relay, and a single sink, has not been fully characterized. This is in contrast to classic wired information channels, whose capacity can be characterized by the Ford-Fulkerson max-flow min-cut theorem.

A *linear deterministic relay network (LDRN)* has been introduced by Avestimehr, Diggavi and Tse [3], to study the capacity of a wireless information channel. It captures the two main features of wireless communication, and they have shown that a linear deterministic relay network approximates a wireless information channel in additive constant factor. Furthermore, the capacity of a linear deterministic relay network can be characterized in terms of "source-destination cuts", which generalize the concept of cuts in a wired network. Although the original proof in [3] employed the *probabilistic method* and therefore algorithmic aspects were not studied, deterministic algorithms to compute the capacity has been developed subsequently [2, 10].

Multicast algorithms in a linear deterministic relay network also have been developed. Ebrahimi and Fragouli [5, 6] have presented the first deterministic multicast algorithm working on an arbitrary linear deterministic relay network. The fastest known algorithm has been proposed by Yazdi and Savari [18]. The running time[1] is $O(dq((nr)^3 \log(nr) + n^2r^4))$, where d is the number of sinks, q is the number of layers in the network, n is the maximum number of nodes in each layer and r is the capacity of a node. Their method uses an efficient algorithm of Goemans, Iwata and Zenklusen [10] to compute a unicast flow for each sink, and then determines a linear coding scheme with the unicast flow information. All of these multicast algorithm assume that the field size is larger than the number of sinks.

1.1 Our Contribution

In this paper, we develop a faster deterministic algorithm for constructing a multicast scheme in a linear deterministic relay network. Using the unicast algorithm of Goemans, Iwata and Zenklusen [10], our algorithm runs in $O(dq(nr)^3 \log(nr))$ time, while the running time of Yazdi and Savari [18] is $O(dq((nr)^3 \log(nr) + n^2r^4))$. Our algorithm is *optimal* in the sense that unicast computations for d sinks already take $O(dq(nr)^3 \log(nr))$ time, as long as we use the unicast algorithm of Goemans, Iwata and Zenklusen [10]. Note that by the multicast theorem shown by Avestimehr, Diggavi and Tse [3], the unicast capacity for each sink can be obtained immediately once we solve the multicast problem.

We also compare the running time excluding the complexity of unicast flow computations. Our algorithm requires $O(dqn^3r^3)$ time, while the algorithm of [18] requires $O(dqn^2r^4)$ time. In this comparison, our algorithm is faster than that of Yazdi and Savari when $n = o(r)$. Practically, r is the number of bits exchanged between two nodes and therefore it can be considerably greater than n, the maximum number of nodes in each layer.

While both Yazdi and Savari's algorithm and ours use the same unicast algorithm of Goemans, Iwata and Zenklusen [10], we achieve several technical improvements. The main differences of the present work are as follows:

- After finding a unicast flow for each sink, Yazdi and Savari's algorithm determines a linear coding scheme in a *node-by-node* manner, which

[1] Although the running time stated in [18] is $O(dq((nr)^3 \log(nr) + n^2r^3 + nr^4))$, there is a mistake that bounds the time for a multiplication of an $r \times n$ matrix and an $n \times r$ matrix by $O(nr)$.

increases the number of matrix operations that have to be carried out. To reduce this complexity, we introduce a *matrix completion technique* which enables us to determine a linear coding scheme of a *layer* at once.

- The method of Ebrahimi and Fragouli [5, 6] reduces multicasting in a linear deterministic relay network to a modified network coding problem and then applies the matrix completion technique to the modified problem. However, their approach does not consider the layered structure of linear deterministic relay networks, and therefore needs to handle a relatively large matrix. Having considered the layered nature, we designed an algorithm that only needs to handle smaller matrices.

1.2 Related Works

Combinatorial properties of linear deterministic relay networks have been studied in the literature. Goemans, Iwata and Zenklusen [10] have studied more general flow model called a *polylinking network*, and have shown that a unicast flow in a polylinking network can be found by solving the *submodular flow problem* [7]. Subsequently, Fujishige [9] has slightly extended the polylinking network model and shown that the extended flow model is equivalent to the *neoflow* problems [8], which include the submodular flow problem and other variants. Note that a similar result has been obtained independently by Yazdi and Savari [17].

Similarities between linear deterministic relay networks and *network coding* [1] have been pointed out by several authors. In the original paper of Avestimehr, Diggavi and Tse [3], it is shown that the multicast capacity of a linear deterministic relay network equals the minimum unicast capacity for each sink. This result can be compared to the famous result for the multicast capacity achievable with network coding [1]. Ebrahimi and Fragouli [6] have devised a multicast algorithm for a linear deterministic relay network based on reduction to the modified network coding problem, in which a network has nodes performing predetermined linear operation. Such extended network coding problems have been studied by Király and Kovács [12, 13].

The *matrix completion problem*, more specially, the *maximum rank matrix completion problem* has been studied in the area of combinatorial optimization, and has rich applications to network coding [11]. In the problem, we are given a *matrix with indeterminates*, a matrix whose entries may contain indeterminates. The objective is to find a value assignment for the indeterminates maximizing the rank of the matrix obtained by substitution. The outstanding work in matrix completion algorithms has been achieved by Harvey, Karger and Murota [11]. They have devised an efficient completion algorithm for *mixed matrices*, matrices with indeterminates such that each indeterminate appears only once. Their algorithm is based on clever use of the combinatorial structure of mixed matrices [14]. Their mixed matrix completion algorithm has been generalized to more complicated matrix completion problems in [15].

1.3 Organization of this paper

The rest of this paper is organized as follows. Section 2 provides a summary of results of the mixed matrix theory and mixed matrix completion. The flow model for a linear deterministic relay network is described in Section 3. The main section of this paper is Section 4, which presents our new algorithm for multicasting in a linear deterministic network and analyzes its complexity. In Section 5, we refine the analysis of Section 4 and estimate the complexity of our algorithm excluding the complexity for unicast computations. Finally, we conclude this paper in Section 6.

2 Preliminaries

This section provides a brief summary of the mixed matrix theory and mixed matrix completion. For further details of the mixed matrix theory, the reader is referred to a monograph of Murota [14]. We also provide a useful formula from linear algebra. For the sake of simplicity, we introduce some notations. For a matrix A, we denote by $\mathrm{Row}(A)$ the set of row indices of A. Similarly, we define $\mathrm{Col}(A)$ as the set of column indices of A.

2.1 Mixed Matrix

Let $\mathbf{F}$ be a field. A *matrix with indeterminate* is a matrix whose entry may contain indeterminates. A matrix is said to be *generic* if the set of its nonzero entries is algebraically independent over $\mathbf{F}$. A matrix with indeterminate is called a *mixed matrix* if each indeterminate appears only once. Equivalently, a matrix A is a mixed matrix if $A = Q + T$, where Q is a "constant" matrix over $\mathbf{F}$ and T is a generic matrix. A mixed matrix $A = Q + T$ is called a *layered mixed matrix*, or, an *LM-matrix* for short, if the set of nonzero rows in Q and that in T are disjoint. In other words, an LM-matrix is a mixed matrix in the form of $A = \left[\begin{smallmatrix} Q \\ T \end{smallmatrix}\right]$. The following is basic in the mixed matrix theory.

Lemma 2.1 (Murota [14])**.** *For a square LM-matrix $A = \left[\begin{smallmatrix} Q \\ T \end{smallmatrix}\right]$, let $C := \mathrm{Col}(A)$, $R_Q := \mathrm{Row}(Q)$ and $R_T := \mathrm{Row}(T)$. Then A is nonsingular if and only if there exists a column subset $J \subseteq C$ such that both $Q[R_Q, C \setminus J]$ and $T[R_T, J]$ are nonsingular.*

In fact, the set J described in the lemma can be found by solving the *independent matching problem* [16], a generalization of bipartite matching. Formally, the independent matching problem is defined as follows. Let $G = (V^+, V^-; E)$ be a bipartite graph with vertex set $V^+ \dot\cup V^-$ and edge set E.

Let $\mathbf{M}^+$ and $\mathbf{M}^-$ be matroids on V^+ and V^-, respectively. A matching M in G is said to be *independent* if the sets of vertices in V^+ and V^- incident to M are independent in $\mathbf{M}^+$ and in $\mathbf{M}^-$, respectively. The independent matching problem is to find an independent matching of maximum size.

In the mixed matrix theory, the independent matching problem is often used in the following form. Define a bipartite graph $G = (V^+, V^-; E)$ with $V^+ := C_Q \dot{\cup} R_T$ and $V^- := C$, where C_Q is a copy of $C = \mathrm{Col}(A)$. Let E be the set $\{ij : i \in R_T,\ j \in C \text{ and } T_{ij} \neq 0\} \cup \{j_Q j : j_Q \text{ is the copy of } j\}$. Define $\mathbf{M}^+$ to be the direct sum of the vector matroid of Q and the free matroid on R_T. Let $\mathbf{M}^-$ be the free matroid on V^-. Then, a matching M is independent if and only if $\partial^+ M \cap C_Q$ is an independent set in the vector matroid of Q, where $\partial^+ M$ is the set of vertices in V^+ incident to M. Then the set J described in Lemma 2.1 coincides with the set of vertices matched to R_T by a maximum independent matching.

Example 2.2. The bipartite graph G corresponding to the LM-matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ x & y & z & 0 \\ w & 0 & 0 & t \end{bmatrix} \tag{1}$$

is shown in Figure 1. An independent matching is shown in thick edges, and the corresponding subset $J \subseteq C$ is shown in a shaded box.

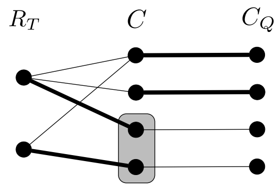

Figure 1: The corresponding bipartite graph of the LM-matrix A in Example 2.2.

Conversely, if we have a column subset $J \subseteq C$ such that both $Q[R_Q, C \setminus J]$ and $T[R_T, J]$ are nonsingular, the corresponding independent matching can be found as follows. Since $T[R_T, J]$ is nonsingular, $G[R_T \dot{\cup} J]$ has a bipartite matching N such that $J \subseteq \partial N$, where $G[R_T \dot{\cup} J]$ is the subgraph of G induced by $R_T \dot{\cup} J$. Then $M := N \dot{\cup} \{j_Q j : j \in C \setminus J\}$ is a maximum matching in G and the set of vertices matched to R_T by M coincides with J.

2.2 Mixed Matrix Completion

In the *mixed matrix completion* problem, we are given a mixed matrix and the objective is to find values for the indeterminates that maximize the rank of the resulting matrix obtained by substitution. Harvey, Karger and Murota [11] have shown that a solution of a mixed matrix completion can be found in $O(n^{2.77})$ time, where n is the size of an input mixed matrix.

We need a more general matrix completion problem in this paper: *simultaneous mixed matrix completion*, in which we are given a *collection* of mixed matrices and some indeterminates may appear in more than one of these matrices. The objective is to find a value assignment of the indeterminates that maximizes the rank of *every* resulting matrix obtained by substitution. Again, Harvey, Karger and Murota [11] have devised an efficient algorithm to find a solution of this problem under a certain condition on the field size. Here we present an overview of their approach.

For each mixed matrix $A = Q + T$ in the collection, we define the corresponding LM-matrix $\tilde{A}$ by

$$\tilde{A} = \begin{bmatrix} I & Q \\ Z & ZT \end{bmatrix}, \tag{2}$$

where Z is a nonsingular diagonal generic matrix. Note that $\operatorname{rank} \tilde{A} = n + \operatorname{rank} A$, where we assume that A is an $n \times n$ matrix. Let M be a maximum independent matching in the bipartite graph G corresponding to the LM-matrix $\tilde{A}$. Roughly speaking, the independent matching M captures a combinatorial structure of the LM-matrix $\tilde{A}$. Harvey, Karger and Murota [11] have shown that we can determine a value of each indeterminate with clever use of the combinatorial structure. The complexity of their algorithm is $O(|\mathcal{A}| \cdot (\mathrm{IM}(n) + kn^2))$ time, where $\mathrm{IM}(n)$ is the complexity of solving the independent matching problem for a single mixed matrix and k is the number of indeterminates in $\mathcal{A}$. For example, in a standard augmenting path algorithm for the independent matching problem, we have $\mathrm{IM}(n) = O(n^3 \log n)$ [4].

Furthermore, this running time can be improved if the collection admits a certain structure. A collection of mixed matrices is said to be *column-compatible* if the following condition holds: for arbitrary two indeterminates, if some matrix contains them in the same column, then no matrix in the collection contains them in distinct columns.

Theorem 2.3 (Harvey, Karger and Murota [11]). *Let* $\mathbf{F}$ *be a field and let* $\mathcal{A}$ *be a column-compatible collection of* $n \times n$ *mixed matrices. If* $|\mathbf{F}| > |\mathcal{A}|$*, then there exists a solution of the simultaneous matrix completion for* $\mathcal{A}$ *and it can be found in* $O(|\mathcal{A}| \cdot (\mathrm{IM}(n) + kn + n^3))$ *time, where* k *is the number of indeterminates in* $\mathcal{A}$*. Using a standard independent matching algorithm, the algorithm can be implemented to run in* $O(|\mathcal{A}| \cdot n^3 \log n)$ *time.*

2.3 Cauchy-Binet Formula

The Cauchy-Binet formula describes the terms in the determinant of the product of two rectangular matrices.

Lemma 2.4 (Cauchy-Binet Formula)**.** *For an $n \times r$ matrix A and an $r \times n$ matrix B with $n \leq r$, we have*

$$\det AB = \sum_{J:|J|=n} \det A[R, J] \cdot \det B[J, C], \tag{3}$$

where $R := \mathrm{Row}(A)$, $C := \mathrm{Col}(B)$ and the sum is taken over all subsets J of $\mathrm{Col}(A) = \mathrm{Row}(B)$ such that $|J| = n$.

3 Flow Model

This section provides the flow model for multicast in a linear deterministic relay network. The model is based on the linking network model of Goemans, Iwata and Zenklusen [10].

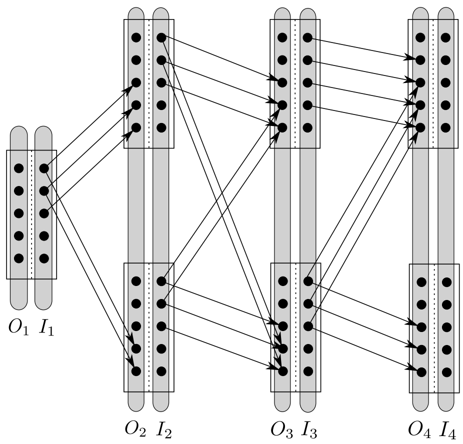

Figure 2: An example of a linear deterministic relay network

A linear deterministic relay network is a layered network with node set $V = V_1 \dot\cup \ldots \dot\cup V_q$, where V_i is the set of nodes in the ith layer for $i = 1, \ldots, q$. Each node consists of r *input* vertices and r *output* vertices, for some global constant $r \in \mathbf{Z}_+$. For each i, let I_i and O_i be the set of all inputs and the set of all outputs in the layer V_i, respectively. We may assume that the first layer V_1 consists of a single node s called the *source node*.

Signals are modeled as elements of a finite field $\mathbf{F}$ and are sent as follows. For each i, let z_i denotes the vector consisting of signals at the inputs in the layer V_i and let y_{i+1} denotes the vector of signals received in the outputs of the next layer V_{i+1}. Then y_{i+1} is equal to $M_i z_i$, where M_i is a given $O_{i+1} \times I_i$ matrix over $\mathbf{F}$ representing a "connection" between V_i and V_{i+1}. In the original paper [3] of a linear deterministic relay network, M_i is represented as a block matrix of *shift* matrices. That is, for $u \in V_i$ and $v \in V_{i+1}$, $M_i[u, v] = S^{r-n_{uv}}$ for some integer n_{uv}, where S is the following $r \times r$

matrix:

$$S = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix}. \tag{4}$$

However, we consider more general settings in this paper, i.e., we assume that M_i can be an arbitrary matrix.

Let t be a node in the layer V_k. An *s-t flow* F in the linear deterministic network is a subset of V such that:

1. For each node N, $|F \cap N \cap O| = |F \cap N \cap I|$, i.e., F contains outputs as many as inputs contained in F.

2. For each $i = 1, \dots, k-1$, $M_i[F \cap O_{i+1}, F \cap I_i]$ is nonsingular.

3. The outputs of sink t contains $F \cap O_k$.

The *rate* of an s-t flow F is the value $|F \cap O_1|$.

The *unicast* problem in a linear deterministic relay network is a problem to find an s-t flow for a single s-t pair. Goemans, Iwata and Zenklusen [10] have shown that the unicast problem can be solved with *matroid partition.*

Theorem 3.1 (Goemans, Iwata and Zenklusen [10])**.** *An s-t flow in a linear deterministic relay network can be found in $O(d(nr)^3 \log(nr))$ time, where d is the number of layers and n is the maximum number of nodes in each layer.*

In the paper of Goemans, Iwata and Zenklusen [10], a node can only send its receiving signals. However it is natural to introduce linear coding operation in nodes, i.e., we assume that a node can send a linear combination of its receiving signals. Formally, for each i, the input vector $z_i \in \mathbf{F}^{I_i}$ is determined as

$$z_i = X_i y_i, \tag{5}$$

where y_i is the output vector of layer V_i and X_i is a block diagonal matrix each of whose block size is r. We can easily see that the output vector y_{i+1} of the next layer V_{i+1} is determined by

$$y_{i+1} = M_i X_i y_i. \tag{6}$$

The *multicast* problem in a linear deterministic relay network is defined as follows. Givens are a linear deterministic relay network $\mathcal{N}$ and a set T of sink nodes in $\mathcal{N}$. We consider the source node s as an information source generating a *message* $w \in \mathbf{F}^r$. The objective is to design coding matrices $X_1, \dots, X_{q-1}$ so that each sink t in T can decode w from its receiving signals for an *arbitrary* w.

4 Algorithm

In this section, we describe a new algorithm for finding a multicast scheme in a linear deterministic relay network. For simplicity of description, we assume that there exists an s-t flow with rate r for each sink t. Let $w \in \mathbf{F}^r$ denote the message vector, T denote the set of sinks and let d denote the number of sinks.

Our algorithm splits into two parts. In the first part, we compute an s-t flow F_t in the linear deterministic relay network for each sink t in T. This part can be done efficiently using the unicast algorithm of Goemans, Iwata and Zenklusen [10]. In the second part, we determine linear coding coefficients from the first layer to the last layer. More precisely, we determine the entries of linear coding coefficients matrix X_i so that the following condition is satisfied: the original message vector w can be recovered from subvector $y_{i+1}[F_t \cap O_{i+1}]$ for each sink t.

Suppose that sink t is in layer V_k for some k. Then t can recover the original message w because the set of outputs of s contains $F_t \cap O_k$. The following lemma summarizes the above arguments.

Lemma 4.1. *If $|\mathbf{F}| > d$, there exist matrices $X_1, \ldots, X_{q-1}$ such that w can be recovered from $y_i[F_t \cap O_i]$ for $i = 1, \ldots, q$ and for each $t \in T$.*

We will prove the lemma by induction on i. For $i = 1$, w can be trivially recovered from $y_1[F_t \cap O_1] = y_1$ because we have $y_1 = w$ and $F_t \cap O_1 = O_1$. Suppose that $i > 1$. Let P_i be the matrix satisfying that $y_i = P_i w$. By the induction hypothesis, $P_i[F_t \cap O_i, O_1]$ is nonsingular. Then, we can easily see that the original message w can be recovered from subvector $y_{i+1}[F_t \cap O_{i+1}]$ if and only if $M_i[F_t \cap O_{i+1}, I_i] X_i P_i$ is nonsingular.

Let A_t be the matrix defined as follows:

$$A_t := \begin{bmatrix} I & O & P_i \\ X_i & I & O \\ O & M_i[F_t \cap O_{i+1}, I_i] & O \end{bmatrix}. \tag{7}$$

The matrix A_t is nonsingular if and only if $M_i[F_t \cap O_{i+1}, I_i] X_i P_i$ is nonsingular. Therefore, if there exists a constant matrix X_i such that A_t is nonsingular for each sink t, then the lemma is proved. Considering each nonzero entry of X_i as an indeterminate, the problem is equivalent to simultaneous mixed matrix completion. In order to prove that there exists a solution for the simultaneous matrix completion, we have to check that A_i is nonsingular (as a mixed matrix). This can be verified by the following.

Lemma 4.2. *If X_i is a generic matrix, then $M_i[F_t \cap O_{i+1}, I_i] X_i P_i$ is nonsingular.*

Proof. By the Cauchy-Binet formula, we have

$$\begin{aligned}\det & M_i[F_t \cap O_{i+1}, I_i]X_iP_i \\ &= \sum_{I \subseteq I_i : |I|=r} \sum_{J \subseteq O_i : |O_i|=r} \det M_i[F_t \cap O_{i+1}, I] \det X_i[I, J] \det P_i[J, O_1]. \quad (8)\end{aligned}$$

No cancellation occurs among nonzero terms in the right-hand side because of the genericity of X_i. On the other hand, $M_i[F_t \cap O_{i+1}, F_t \cap I_i]$ and $X_i[F_t \cap I_i, F_t \cap O_i]$ are nonsingular because F_t is a flow. Furthermore, $P_i[F_t \cap O_i, O_1]$ is nonsingular by the inductive hypothesis. Thus if we take $I = F_t \cap I_i$ and $J = F_t \cap O_i$, the corresponding term is nonzero, and this implies that $\det M_i[F_t \cap O_{i+1}, I_i]X_iP_i$ is nonzero. □ □

We are now ready to prove Lemma 4.1. Let $\mathcal{A}_i$ be the collection of mixed matrix A_t for each sink t such that t is in V_j with $j > i$. Since $|\mathbf{F}| > d$ and each mixed matrix A_t is nonsingular, there exists a value assignment for X_i such that every resulting matrix is nonsingular. Therefore, w can be recovered from $y_{i+1}[F_t \cap O_{i+1}]$ for each t, which proves the lemma.

The above arguments provides an algorithm for finding a multicast encoding scheme. A pseudocode description is presented in Algorithm 1. Let us analyze the running time of the algorithm. A unicast flow can be found in $O(q(nr)^3 \log(nr))$ time for each sink, by the algorithm of Goemans, Iwata and Zenklusen [10]. Using the simultaneous mixed matrix completion algorithm of Harvey, Karger and Murota [11], linear encoding matrix X_i can be found in $O(d(nr)^3 \log(nr))$ time, for each layer i. Note that the collection of A_t's is column compatible.

Theorem 4.3. *If $|\mathbf{F}| > d$, a multicast linear encoding scheme over $\mathbf{F}$ for a linear deterministic relay network can be found in $O(dq(nr)^3 \log(nr))$ time, where d is the number of sinks, q is the number of layers, n is the maximum number of nodes in each layer and r is the capacity of a node.*

5 Complexity Excluding Unicast Computation

In this section, we estimate the complexity of our multicast algorithm excluding the complexity of unicast computations. Of course, as we have argued in the previous section, we can find a multicast encoding scheme by solving the simultaneous matrix completion repeatedly. However, this straightforward algorithm requires $O(dq(nr)^3 \log(nr))$ time. In fact, using the unicast flow information, we do not need to solve the independent matching problems in the simultaneous matrix completion. More precisely, an independent matching associated with the matrix A_t can be found in $O(1)$ time. Therefore, if we know a unicast flow for each sink, we can skip the

Algorithm 1 An algorithm for multicasting in a linear deterministic relay network

for each sink t in T **do**
  Compute an s-t flow F_t.
end for
Set $U := T$ and $P_1 := I$.
for $i = 1$ to q **do**
  Let $\mathcal{A}_i := \{A_t : t \in U\}$.
  Compute a solution of simultaneous mixed matrix completion for $\mathcal{A}_i$.
  Let $P_{i+1} := G_i \tilde{X}_i P_i$, where $\tilde{X}_i$ is the substituted matrix according to the solution of simultaneous matrix completion.
  Remove each sink t from U if t is in the ith layer.
end for

computation of independent matching in the simultaneous matrix completion algorithm.

Let us transform the mixed matrix A_t into the following LM-matrix:

$$\tilde{A}_t := \begin{bmatrix} I & O & P_i \\ O & M_i[F_t \cap O_{i+1}, I_i] & O \\ ZX_i & Z & O \end{bmatrix}, \tag{9}$$

where Z is a nonsingular diagonal generic matrix. Then A_t is nonsingular if and only if $\tilde{A}_t$ is nonsingular. We denote $\tilde{A}_t = \left[\begin{smallmatrix} Q \\ S \end{smallmatrix}\right]$ as usual. Let us denote $R_Q := \mathrm{Row}(Q)$, $R_S := \mathrm{Row}(S)$ and $C := \mathrm{Col}(\tilde{A}_t)$. The proof of the following lemma is almost same as that of Lemma 4.2.

Lemma 5.1. *For an s-t flow F_t, put $J := (F_t \cap O_i) \dot{\cup} (I_i \setminus F_t)$. Then $Q[R_Q, C \setminus J]$ and $S[R_S, J]$ are both nonsingular.*

Let G be the bipartite graph $G = (V^+, V^-; E)$ corresponding to the LM-matrix $\tilde{A}_t$ as defined in Section 2.1. The subset J described in Lemma 2.1 corresponds to the set of vertices in V^+ incident to some maximum independent matching, say M. Furthermore, we can find the maximum independent matching M as follows. Observe that it is sufficient to find a maximum matching in $G[R_T \dot{\cup} J]$, where $G[R_T \dot{\cup} J]$ is the subgraph of G induced by $R_T \dot{\cup} J$. Since ZX_i is block diagonal and each block has no zero entries, $G[R_T \dot{\cup} J]$ is the direct sum of complete bipartite graphs. Thus we can find a maximum matching in $G[R_T \dot{\cup} J]$ immediately.

Let us analyze the running time of our algorithm. For each i, the simultaneous matrix completion can be done in $O(dn^3r^3)$ time by the algorithm of Harvey, Karger and Murota [11] because the collection of A_t's is column-compatible and we can consider that $\mathrm{IM}(n) = O(1)$. Computing P_{i+1} can be done in $O(n^2r^3)$ time. In total, a linear coding scheme in the ith layer can be found in $O(dn^3r^3)$ time. Summing up $i = 1$ to $q - 1$, we can find an entire linear coding scheme in $O(qdn^3r^3)$ time. Summarizing the above arguments, we have the following refinement of Theorem 4.3.

Theorem 5.2. *If* $|\mathbf{F}| > d$*, a multicast encoding scheme in a linear deterministic relay network can be found in* $O(d \cdot \mathrm{UF}(n,q,r) + qdn^3r^3)$ *time if, where* $\mathrm{UF}(n,q,r)$ *is the complexity of unicast computation for a single sink,* d *is the number of sinks,* q *is the number of layers and* n *is the maximum number of nodes in each layer.*

6 Concluding Remarks

We study multicasting in a linear deterministic relay network. Using the simultaneous matrix completion, we devise the new algorithm whose complexity matches to that of unicast computations for each sinks. We also estimate the complexity of our algorithm excluding the complexity of unicast computations.

The mixed matrix A_t used in our algorithm has certain structures; in particular, A_t is a *sparse* matrix. In fact, the number of *nonzero* entries of A_t is $O(nr^2)$, while A_t has $O(n^2r^2)$ entries in total. However, our algorithm does not use the sparsity because Harvey, Karger and Murota's simultaneous mixed matrix completion algorithm is incompatible with sparsity. A possible direction for future studies is to design an algorithm compatible with the sparsity of the mixed matrix A_t. For example, can one design a faster algorithm which runs in $O(d \cdot \mathrm{UF}(n,q,r) + qdn^2r^3)$ time?

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