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Yutaro YAMAGUCHI

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DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

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PACKING A-PATHS IN GROUP-LABELLED GRAPHS VIA LINEAR MATROID PARITY*

YUTARO YAMAGUCHI†

Abstract. Mader's disjoint $\mathcal{S}$-paths problem is a common generalization of matching and Menger's disjoint paths problems. Lovász (1980) suggested a polynomial-time algorithm for this problem through a reduction to matroid matching. A more direct reduction to the linear matroid parity problem was given later by Schrijver (2003), which leads to faster algorithms.

As a generalization of Mader's problem, Chudnovsky, Geelen, Gerards, Goddyn, Lohman, and Seymour (2006) introduced a framework of packing non-zero A-paths in group-labelled graphs, and proved a min-max theorem. Chudnovsky, Cunningham, and Geelen (2008) provided an efficient combinatorial algorithm for this generalized problem. On the other hand, Pap (2007) introduced a framework of packing non-returning A-paths as a further genaralization.

In this paper, we discuss possible extensions of Schrijver's reduction technique and the algorithm of Chudnovsky, Cunningham, and Geelen to another framework introduced by Pap (2006), under the name of the subgroup model, which apparently generalizes but in fact is equivalent to packing non-returning A-paths. We provide a necessary and sufficient condition for the groups in question to admit a reduction to the linear matroid parity problem. As a consequence, we give faster algorithms for important special cases of packing non-zero A-paths such as odd-length A-paths. In addition, it turns out that packing non-returning A-paths admits a reduction to the linear matroid parity problem, which leads to its efficient solvability, if and only if the size of the input label set is at most four.

Key words. A-path, disjoint paths, group-labelled graph, linear matroid parity

AMS subject classifications. 05B35, 05C38, 05C50, 05C70, 05C85, 20C25

1. Introduction.

Let Γ be a group. A Γ-*labelled graph* (G, ψ) is a pair of an undirected graph $G = (V, E)$ and a *label function* ψ on the edge set to Γ, which is defined below. For a directed graph $\vec{G} = (V, \vec{E})$ obtained from G by replacing each edge with a pair of arcs of opposite directions, a function $\psi : \vec{E} \to \Gamma$ is called a label function if $\psi(\bar{e}) = \psi(e)^{-1}$ holds for each $e \in \vec{E}$, where $\bar{e}$ denotes the reverse arc of e. In this paper, for $e = uv = vu \in E$ replaced with $e' = uv \in \vec{E}$ and $\bar{e'} = vu \in \vec{E}$, we will use the notation of $\psi(e, v) := \psi(e') = \psi(\bar{e'})^{-1} =: \psi(e, u)^{-1}$. For each undirected path $P = (v_0, e_1, v_1, \ldots, e_k, v_k)$ in G, where $e_i = v_{i-1}v_i \in E$ for every $1 \leq i \leq k$, we define the *label* of P as $\psi(P) := \psi(e_k, v_k) \cdots \psi(e_2, v_2) \cdot \psi(e_1, v_1)$.

For a prescribed terminal set $A \subseteq V$, an A-*path* is an undirected path between distinct terminals in A which does not intersect with A in between. In this paper, we consider the *subgroup model* of packing A-paths in group-labelled graphs introduced by Pap [11]. In this model, for a given proper subgroup Γ' of Γ, an A-path P is called *admissible* if $\psi(P) \notin \Gamma'$, and is called *non-admissible* otherwise. Our objective is to find a maximum family of (fully) vertex-disjoint admissible A-paths in a given Γ-labelled graph with terminal set A. Note that it is not necessary that G is simple.

1.1. Main result.

The subgroup model was introduced at the end of a sequence of extensions of Mader's disjoint $\mathcal{S}$-paths problem, which is known to be solvable by a reduction to matroid matching due to Lovász [6]. A more direct reduction to linear matroid parity has been presented by Schrijver [14]. In this paper, we extract some

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† Department of Mathematical Informatics, University of Tokyo, Japan. Supported by JSPS Fellowship for Young Scientists. E-mail: yutaro_yamaguchi@mist.i.u-tokyo.ac.jp

aspects of Schrijver's reduction and introduce the concept of *coherent representation*. For an instance of the subgroup model, we call a matrix a coherent representation if it satisfies Properties 2.1 and 2.2 described in Section 2.1. The main result of this paper is a characterization of the subgroup model that admits a coherent representation.

For a positive integer $n \in \mathbb{N}$ and a field $\mathbb{F}$, $\mathrm{GL}(n, \mathbb{F})$ denotes the set of all the nonsingular $n \times n$ matrices over $\mathbb{F}$, and let $\mathrm{PGL}(n, \mathbb{F}) := \mathrm{GL}(n, \mathbb{F})/\{kI_n \mid k \in \mathbb{F}\}$, where I_n is the $n \times n$ identity matrix. In this paper, each element of PGL is denoted by its representative in GL. We are now ready to state the main theorem.

THEOREM 1.1. *Let Γ be a group, Γ' be its proper subgroup, and $\mathbb{F}$ be a field. Then the following two statements are equivalent.*

- **(i)** *For any Γ-labelled graph $(G = (V, E), \psi)$ with any terminal set $A \subseteq V$, the subgroup model with respect to Γ' can be reduced to the linear matroid parity problem with a coherent representation over $\mathbb{F}$.*
- **(ii)** *There exist a homomorphism $\rho : \Gamma \to \mathrm{PGL}(2, \mathbb{F})$ and a 1-dimensional linear subspace Y of $\mathbb{F}^2$ such that $\Gamma' = \{\alpha \in \Gamma \mid \rho(\alpha)Y = Y\}$.*

1.2. Background.

1.2.1. Packing A-paths. Finding a maximum family of (fully) vertex-disjoint A-paths is a path-packing problem which includes non-bipartite matching as a special case with $A = V$. Mader [9] suggested a more generalized problem, called Mader's disjoint $\mathcal{S}$-paths problem, and showed a min-max relation. Here $\mathcal{S}$ is a partition of A and an *$\mathcal{S}$-path* is an A-path between terminals in distinct subsets in $\mathcal{S}$. Hence, for any disjoint $S, T \subseteq V$, the concept of $\mathcal{S}$-path includes that of S–T path as a special case with $\mathcal{S} = \{S, T\}$.

As a generalization of Mader's problem, Chudnovsky et al. [3] introduced a framework of packing A-paths in group-labelled graphs, called the *non-zero model* in this paper, and proved a min-max theorem which generalizes Mader's theorem. Later Pap [12] introduced a slightly more generalized model, called the *non-returning model* in this paper, and prove an extension of the min-max theorem by a substantially simpler argument. Chudnovsky et al. [2] gave an efficient combinatorial algorithm for the non-zero model, and Pap [13] suggested one for the non-returning model, whose running time bound is not known.

THEOREM 1.2 (Chudnovsky, Cunningham, and Geelen [2]). *A maximum family of vertex-disjoint non-zero A-paths can be found in $O(|V|^5)$ time.*

These frameworks include interesting special cases besides Mader's problem such as packing odd-length A-paths, and packing A-paths on surfaces under various constraints according to the homotopy class of the curve associated with each A-path. The non-returning model generalizes the non-zero model and is in fact equivalent to the subgroup model (see [11, §3.6] for a detail argument), so we mainly discuss the subgroup model in this paper.

1.2.2. Linear matroid parity. For an undirected graph $G = (V, E)$ and a matroid $\mathbf{M} = (V, \mathcal{I})$, the *matroid matching problem* is to find a maximum matching $F \subseteq E$ with $V(F) \in \mathcal{I}$. If G forms a perfect matching, then the problem is called the *matroid parity problem*. In fact, these two problems are known to be equivalent (see [8, Chapter 11]). If $\mathbf{M}$ is a linearly represented matroid, then this problem, called the *linear* matroid parity problem, is known to be efficiently solvable. Let m and n be the numbers of columns and rows, respectively, of the representation matrix of the input matroid, and let ω be the matrix multiplication exponent, which is at most 2.373.

THEOREM 1.3 (Gabow and Stallmann [5], Orlin [10]). *The linear matroid parity problem can be solved in $O(mn^3)$ time. If fast matrix multiplication is used, then the running time is improved to $O(mn^\omega)$.*

THEOREM 1.4 (Cheung, Lau, and Leung [1]). *The linear matroid parity problem can be solved with high probability in $O(mn^2)$ time. If fast matrix multiplication is used, then the running time is improved to $O(mn^{\omega-1})$.*

A direct application of these theorems to Schrijver's reduction of Mader's problem implies that a maximum family of vertex-disjoint $\mathcal{S}$-paths in an undirected graph $G = (V, E)$ can be found in $O(|E| \cdot |V|^\omega)$ time, and moreover with high probability in $O(|E| \cdot |V|^{\omega-1})$ time. Cheung et al. [1] improved the latter running time bound to $O(|V|^\omega)$ under the assumption, without loss of generality, that the input graph is simple.

1.3. Our contribution. Theorem 1.1 clarifies a necessary and sufficient condition for the groups in question to admit a reduction with a coherent representation, which leads to fast algorithms. A recent work of Tanigawa and the author [15] showed that Lovász's reduction idea to matroid matching, which implies the polynomial-time solvability by Lovász's matroid matching algorithm [7], is always extendable even when there is no coherent representation.

By our reduction, linear matroid parity algorithms can be used to solve a number of special cases of the subgroup model. Naive applications of Theorems 1.3 and 1.4 lead to deterministic $O(|E| \cdot |V|^\omega)$ and randomized $O(|E| \cdot |V|^{\omega-1})$ algorithms. One can improve the latter bound to $O(|E| + |V|^\omega)$ by the same argument as [1, §5.1.3]. Since $|\Gamma'| + 1$ edges are enough between each pair of vertices (if there are $|\Gamma'| + 1$ edges between $u, v \in V$ with different labels, then any A-path using an edge uv can be admissible), we may assume $|E| = O\left(|\Gamma'| \cdot |V|^2\right)$.

On the other hand, the algorithm of Chudnovsky et al. [2] for the non-zero model can be extended to the subgroup model. The resulting algorithm runs in $O\left((|E| + |V|^4) \cdot |V|\right)$ time. Since the original paper [2] contains some errors and omissions, we give a complete description of the extended algorithm. Compared to this running time bound, a coherent representation, if exists, leads to a much faster $O(|E| + |V|^\omega)$-time algorithm, which makes it interesting to characterize when a given subgroup model admits a coherent representation.

The rest of this paper is organized as follows. Section 2 provides a definition of coherent representation and a proof of Theorem 1.1. Section 3 discusses under which condition important special cases of the subgroup model have coherent representations. Finally, in Section 4, we present an extension of the algorithm of Chudnovsky et al. [2].

2. Reduction to Linear Matroid Parity. This section is devoted to Theorem 1.1. We introduce the concept of coherent representation in Section 2.1, and then prove Theorem 1.1 in Sections 2.2 and 2.3. Section 2.2 shows that the condition **(ii)** is sufficient for the reducibility **(i)** to the linear matroid parity problem with a coherent representation, and Section 2.3 gives a proof of the converse direction, i.e., **(ii)** is necessary for any graph to have a coherent representation.

2.1. Coherent representation. We introduce two natural properties satisfied by Schrijver's reduction of Mader's problem to the linear matroid parity problem. Let

Γ be a group and $\mathbb{F}$ be a field. For a Γ-labelled graph $(G = (V, E), \psi)$ with terminal set $A \subseteq V$, we consider constructing a representation matrix $Z \in \mathbb{F}^{2n\times 2m}$ which defines an instance of the linear matroid parity problem, where $|V| = n$, $|E| = m$. For the simplicity of description, we assume $V = [n] := \{1, 2, \dots, n\}$ and $E = [m]$. The representation is desired to be based on the incidence matrix of G.

PROPERTY 2.1. *For each $e = uv \in E$, there exists exactly one pair of two corresponding columns z_{2e-1}, z_{2e} of Z, each of which has at most four nonzero entries at $2u - 1, 2u, 2v - 1, 2v$-th rows. In other words,*

$$z_{2e-1}, z_{2e} \in \{a_{2u-1}\vec{e}_{2u-1}+a_{2u}\vec{e}_{2u}+a_{2v-1}\vec{e}_{2v-1}+a_{2v}\vec{e}_{2v} \mid a_i \in \mathbb{F}\ (i = 2u-1, 2u, 2v-1, 2v)\},$$

where $\vec{e}_i \in \mathbb{F}^{2n}$ denote i-th unit vectors for $i \in [n]$.

For a vector $z \in \mathbb{F}^{2n}$ and a vertex $v \in V$, let $z(v) \in \mathbb{F}^2$ denote the 2-dimensional vector whose first and second entries are equal to $(2v - 1)$-th and $2v$-th entries of z, respectively. For each vertex $v \in V$ and each edge $e \in E$, let us denote the corresponding 2×2 submatrix of Z by $Z_{v,e} := (z_{2e-1}(v), z_{2e}(v))$. Then Z can be regarded as $Z = (Z_{v,e})_{1\le v\le n, 1\le e\le m}$, and its nonzero entries (2×2 matrices) are only at pairs v, e such that e is incident to v, by Property 2.1.

Let us call an edge set $F \subseteq E$ *feasible* if the set $\{z_{2e-1}, z_{2e} \mid e \in F\}$ of all corresponging vectors is linearly independent. The following property guarantees natural relation between the subgroup model and the linear matroid parity problem.

PROPERTY 2.2. *For each A-path P in G, its edge set $E(P)$ is feasible if and only if P is admissible.*

2.2. Proof of sufficiency. In this section, we prove that the condition **(ii)** in Theorem 1.1 is sufficient for **(i)**. Fix a field $\mathbb{F}$, a projective representation $\rho : \Gamma \to \mathrm{PGL}(2, \mathbb{F})$, and a 1-dimensional subspace Y of $\mathbb{F}^2$ which satisfy **(ii)**. Furthermore, fix an arbitrary Γ-labelled graph $(G = (V, E), \psi)$ and an arbitrary terminal set $A \subseteq V$. Let us show how to construct a representation matrix for the linear matroid parity problem.

Associate each edge $e = uw \in E$ to a 2-dimensional linear subspace

$$L_e := \{x \in (\mathbb{F}^2)^V \mid \rho(\psi(e, w))x(u) + x(w) = \mathbf{0},\ x(v) = \mathbf{0}\ (v \in V \setminus \{u, w\})\}$$

of $(\mathbb{F}^2)^V$. For each terminal $v \in A$, let

$$Q_v := \{x \in (\mathbb{F}^2)^V \mid x(v) \in Y,\ x(u) = \mathbf{0}\ (u \in V \setminus \{v\})\}.$$

Let $Q_U := \sum_{v\in U} Q_v$ for each $U \subseteq A$, and let $Q := Q_A$.

Let $\mathcal{E} := \{L_e/Q \mid e \in E\}$. Note that $\dim(L_e/Q) = 2$ for every edge $e \in E$, since we may assume that no edge labelled with an element in Γ' connects two terminals. Let us construct a representation matrix $Z \in \mathbb{F}^{2|V|\times 2|E|}$ associated with $\mathcal{E}$ by enumerating the bases of L_e/Q for all $e \in E$. Then each edge set $F \subseteq E$ is feasible if and only if $\dim(L_F/Q) = 2|F|$, where $L_F := \sum_{e\in F} L_e$.

LEMMA 2.3. *Let $\nu(\mathcal{E})$ denote the cardinality of a maximum feasible edge set, and let $\mu(G, \psi, A)$ denote the maximum number of vertex-disjoint admissible A-paths in G with respect to $\psi : E \to \Gamma$. If G is connected and $A \neq \emptyset$, then $\nu(\mathcal{E}) = |V| - |A| + \mu(G, \psi, A)$.*

We prepare two lemmas in order to prove Lemma 2.3.

LEMMA 2.4. *If an edge set $F \subseteq E$ is feasible, then each subset $F' \subseteq F$ forming a connected component of the subgraph (V, F) satisfies one of the following conditions:*

- $|V(F') \cap A| = 0$ *and* F' *contains at most one cycle,*
- $|V(F') \cap A| = 1$ *and* F' *contains no cycle,*
- $|V(F') \cap A| = 2$, F' *contains no cycle and the A-path between the two terminals is admissible.*

Proof. Let F_1, F_2 form different connected components in F. Then the corresponding subspaces L_{F_1} and L_{F_2} have the trivial intersection $\{\mathbf{0}\}$. Therefore, we may assume that F is connected, and hence $|F| \geq |V(F)| - 1$.

Let $X_F := L_F + Q_{A(F)}$, where $A(F) := V(F) \cap A$. Since every $x \in X_F$ has at most $2|V(F)|$ nonzero entries, $\dim(X_F) \leq 2|V(F)| \leq 2|F| + 2$. By $L_F/Q = L_F/Q_{A(F)}$ and $\dim(Q_{A(F)}) = |V(F) \cap A|$, we have

$$\begin{aligned}\dim(L_F/Q) &= \dim(L_F) - \dim(L_F \cap Q_{A(F)}) \\ &= \dim(X_F) - \dim(Q_{A(F)}) \\ &\leq 2|F| + 2 - |V(F) \cap A|.\end{aligned}$$

Since $\dim(L_F/Q) = 2|F|$, this inequality implies $|V(F) \cap A| \leq 2$.

Case 1. Suppose that F contains a cycle. By $|V(F)| \leq |F|$, we have $\dim(X_F) \leq 2|V(F)| \leq 2|F|$, which implies $|V(F) \cap A| \leq 0$. Therefore, F contains exactly one cycle C and $|V(F) \cap A| = 0$.

Case 2. Suppose that F contains no cycle and $|V(F) \cap A| = 2$. Let $P = (v_0, e_1, v_1, \ldots, e_k, v_k)$ be the unique A-path contained in F. Since F is feasible, there is no set of vectors $x_i \in L_{e_i}$ $(1 \leq i \leq k)$ such that at least one of them is a nonzero vector and

$$\sum_{i=1}^{k} x_i(v_j) \in \begin{cases} Y & j = 0, k \\ \{\mathbf{0}\} & 1 \leq j \leq k-1. \end{cases} \tag{2.1}$$

If $\psi(P) \in \Gamma'$, then $\rho(\psi(P))Y = Y$, and hence it is easily seen that $x_1(v_0) := y \in Y \setminus \{\mathbf{0}\}$ makes such a set of vectors, a contradiction, since $x_k(v_k) = -\rho(\psi(P))x_1(v_0) \in Y$. □

Lemma 2.5. *An edge set* $F \subseteq E$ *is feasible if*

- F *contains no cycle, and*
- *for each subset* $F' \subseteq F$ *forming a connected component of the subgraph* (V, F), *either* $|V(F') \cap A| \leq 1$, *or* $|V(F') \cap A| = 2$ *and the A-path between the two terminals is admissible.*

Proof. Similarly to the proof of Lemma 2.4, we may assume that F is connected. Suppose that there exists an infeasible edge set F satisfying the two conditions. Let us take such a minimal edge set F.

Suppose that F has a non-terminal leaf $v \in V \setminus A$. Let $e = vw \in F$ be the incident edge. Since L_e has two degrees of freedom at v-th entry, $\dim(L_F/Q) = \dim(L_{F-e}/Q) + 2$, contradicting the choice of F.

Thus every leaf is a terminal. Recall that $|V(F) \cap A| \leq 2$. Since a tree has at least two leaves, we have $|V(F) \cap A| = 2$, and hence F forms a admissible A-path $P = (v_0, e_1, v_1, \ldots, e_k, v_k)$. Since F is infeasible, there exists a set of vectors $x_i \in L_{e_i}$ $(1 \leq i \leq k)$ such that at least one of them is a nonzero vector and (2.1) holds. By the second part of (2.1) and the definition of the subspaces L_e, we have $x_{i+1}(v_i) = -x_i(v_i) = \rho(\psi(e_i, v_i))x_i(v_{i-1})$ $(1 \leq i \leq k-1)$, and hence $x_k(v_k) = -\rho(\psi(P))x_1(v_0)$. By the first part of (2.1), we have $x_1(v_0), x_k(v_k) \in Y$, but $\rho(\psi(P))Y \neq Y$ by $\psi(P) \notin \Gamma'$, a contradiction. □

Proof of Lemma 2.3. Let us simply denote $\nu := \nu(\mathcal{E})$ and $\mu := \mu(G, \psi, A)$. Let F be a maximum feasible edge set. We denote by c_i the number of connected components of (V, F) containing exactly i terminals, where each isolated terminal contributes c_1. By Lemma 2.4 and the maximality of F, we may assume $c_2 \leq \mu$ and that each connected component of (V, F) contains one or two terminals. Note that, if there exists an edge set F' forming a connected component of (V, F) with no terminal, and hence including a cycle C by the maximality of F, then F remains feasible after replacing an edge $e \in E(C)$ with a bridge connecting $(V, F'-e)$ and another connected component of (V, F). Therefore, $\nu = |F| = |V| - (c_1 + c_2) = |V| - |A| + c_2 \leq |V| - |A| + \mu$.

The converse direction can be easily seen. Let F be the edge set of a maximum family of vertex-disjoint admissible A-paths. By Lemma 2.5, F is feasible and can be extended to feasible F' such that each connected component of (V, F') contains one or two terminals, and hence $|V| - |A| + \mu \leq |F'| \leq \nu$. ◻

This proof implies that one can construct a maximum family of vertex-disjoint admissible A-paths from a maximum feasible edge set by the depth first search from each terminal, in linear time. Thus we conclude the proof of the sufficiency.

2.3. Proof of necessity. Let Γ/Γ' denote the left cosets $\{\alpha\Gamma' \mid \alpha \in \Gamma\}$. For each $i \in \{1, 2, 3\}$, let $G_i = (V_i, E_i)$ be a star with $2|\Gamma/\Gamma'|$ leaves, and let $v_i \in V_i$ be its center vertex. Let $G = (V, E)$ be a graph obtained by connecting each two of the center vertices by parallel $|\Gamma|$ edges. Let $A \subseteq V$ be the set of all leaves in the three stars. Let ψ assign each representative of Γ/Γ' to two edges in each star (directed from leaves to the center vertex) and each element of Γ to each one of each parallel edges. Suppose that, for the Γ-labelled graph (G, ψ) with terminal set A, there exists $Z \in \mathbb{F}^{2n \times 2m}$ satisfying Properties 2.1 and 2.2.

For each edge $e \in E$, let b_e and c_e denote the corresponding column vectors z_{2e-1} and z_{2e} of Z, respectively.

Claim 2.6. *Let $i \in \{1, 2, 3\}$. For each $e = uv_i \in E_i$, we may assume that $b_e(u) = \mathbf{0} \neq c_e(u)$. Moreover, for two edges $e_1, e_2 \in E_i$, $\{b_{e_1}(v_i), b_{e_2}(v_i)\}$ is linearly dependent if and only if $\psi(e_1, v_i) = \psi(e_2, v_i)$.*

Proof. Without loss of generality, choose $e_1, e_2, e_3 \in E_1$ with $\alpha := \psi(e_1, v_1) = \psi(e_2, v_1) \neq \psi(e_3, v_1) =: \beta$ and let $e_i = u_i v_1$ $(i = 1, 2, 3)$. Then two A-paths $(u_i, e_i, v_1, e_3, u_3)$ $(i = 1, 2)$ are admissible and an A-path $(u_1, e_1, v_1, e_2, u_2)$ is non-admissible, so two edge sets $\{e_i, e_3\}$ $(i = 1, 2)$ are feasible and an edge set $\{e_1, e_2\}$ is infeasible by Property 2.2.

The latter implies that $Z_i := Z_{u_i, e_i}$ $(i \in \{1, 2\})$ is singular. Since there exists an edge $e_4 \in E_1$ such that $e_4 \neq e_3$ and $\psi(e_4, v_1) = \beta$, this holds also for $i = 3$, i.e., $Z_3 := Z_{u_3, e_3}$ is also singular. The former implies that Z_i $(i \in \{1, 2\})$ is not the zero matrix, since otherwise the set $\{b_{e_j}, c_{e_j} \mid j = i, 3\}$ of corresponding vectors is linearly dependent, a contradiction. Thus we have $\mathrm{rank}(Z_i) = 1$ $(i = 1, 2, 3)$, and hence we may assume $b_{e_i}(u_i) = \mathbf{0} \neq c_{e_i}(u_i)$.

By this assumption and Property 2.1, the infeasibility of each two-edge set $\{e, e'\}$ in the same star G_i is equivalent to the linear dependence of $\{b_e(v_i), b_{e'}(v_i)\}$. By Property 2.2, $\{e, e'\}$ is infeasible if and only if the two-edge A-path is non-admissible. ◻

Claim 2.7. *For each parallel edge $e = v_i v_j \in E$, both $Z_{v_i, e}$ and $Z_{v_j, e}$ are nonsingular.*

Proof. Without loss of generality, choose $e_{i1}, e_{i2} \in E_i$ $(i = 1, 2)$ and $e = v_1 v_2 \in E$

with $\psi(e, v_2) = \beta$ such that $\alpha_1 := \psi(e_{11}, v_1) \neq \psi(e_{12}, v_1) =: \alpha_2$, and $\psi(e_{21}, v_2) \sim \beta\alpha_1 \nsim \beta\alpha_2 \sim \psi(e_{22}, v_2)$, where $\sim$ denotes the left equivalence with respect to Γ'. There exist four A-paths consisting of some of the above edges and traversing e. Two of these are admissible, and the other two are non-admissible.

By Claim 2.6, $\{b_{e_{i1}}(v_i), b_{e_{i2}}(v_i)\}$ is linearly independent for each $i \in \{1, 2\}$. The infeasibility of $\{e_{1j}, e, e_{2j}\}$ and the feasibility of $\{e_{1j}, e\}$ and $\{e, e_{2j}\}$ for $j = 1, 2$ require $b_{e_{i1}}(v_i), b_{e_{i2}}(v_i) \in \langle b_e(v_i), c_e(v_i)\rangle$ for $i = 1, 2$, where $\langle\cdot\rangle$ denotes the spanning subspace. Therefore $\{b_e(v_i), c_e(v_i)\}$ is linearly independent, and hence $Z_{v_i,e}$ is nonsingular. ◻

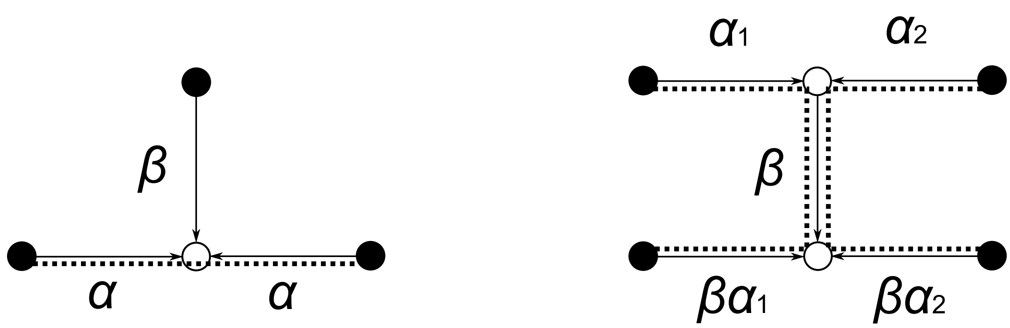

FIG. 2.1. *Observed parts in Claims 2.6 and 2.7 (dotted: non-admissible A-paths)*

By Claim 2.7, for every triplet of parallel edges $e_1 = v_1v_2$, $e_2 = v_1v_3$ and $e_3 = v_3v_2$, we may assume that $Z_{v_1,e_1} = Z_{v_1,e_2} = Z_{v_3,e_3} = I_2$, and moreover that $Z_{v_2,e_1} \sim Z_{v_3,e_2} \sim Z_{v_2,e_3}$ if $\psi(e_1, v_2) = \psi(e_2, v_3) = \psi(e_3, v_2)$. Here, for $Z_1, Z_2 \in \mathbb{F}^{2\times 2}$, $Z_1 \sim Z_2 \Leftrightarrow Z_1 = kZ_2$ for some $k \in \mathbb{F}$. This can be seen as follows.

Choose $e_1 = v_1v_2 \in E$ with $\psi(e_1) = 1_\Gamma$, and let $B := Z_{v_2,e_1}$. Redefine $b_e(v_2) := B^{-1}b_e(v_2)$ for each $e \in E_2$ and $Z_{v_2,e} := B^{-1}Z_{v_2,e}$ for each $e = v_iv_2 \in E$ ($i \in \{1, 3\}$), then we have $Z_{v_2,e_1} = I_2$. This redefinition leads the linear dependence of $\{b_{f_1}(v_1), b_{f_2}(v_2)\}$ for each pair of edges $f_1 \in E_1$ and $f_2 \in E_2$ with $\psi(f_1, v_1) = \psi(f_2, v_2)$. Then, by the similar redefinition around v_3, each two-vector subset of $\{b_{f_i}(v_i) \mid i = 1, 2, 3\}$ is linearly dependent for $f_i \in E_i$ ($i = 1, 2, 3$) with $\psi(f_1, v_1) = \psi(f_2, v_2) = \psi(f_3, v_3)$.

The following claim concludes the proof of the necessity.

CLAIM 2.8. *For each parallel edge $e = v_1v_2 \in E$, let $\rho(\psi(e, v_2)) := Z_{v_2,e}$. Then $\rho : \Gamma \to \mathrm{PGL}(2, \mathbb{F})$ is homomorphic. Moreover, for $f = uv_1 \in E_1$ with $\psi(f) = 1_\Gamma$, $Y := \langle b_f(v_1)\rangle$ satisfies $\Gamma' = \{\alpha \in \Gamma \mid \rho(\alpha)Y = Y\}$.*

Proof. Choose $e_{i1}, e_{i2} \in E_i$ ($i = 1, 2$), $e_1 = v_1v_2$, $e_2 = v_1v_3$, and $e_3 = v_3v_2$ such that $\beta_1 = \beta_3\beta_2$, $\alpha_1 := \psi(e_{11}, v_1) \neq \psi(e_{12}, v_2) =: \alpha_2$, and $\psi(e_{21}, v_2) \sim \beta_1\alpha_1 \nsim \beta_1\alpha_2 \sim \psi(e_{22}, v_2)$, where $\beta_i := \psi(e_i)$ ($i = 1, 2, 3$).

For each $j \in \{1, 2\}$, since A-paths formed by $\{e_{1j}, e_1, e_{2j}\}$ and $\{e_{1j}, e_2, e_3, e_{2j}\}$ are non-admissible, we have $b_{e_{2j}}(v_2) \in \langle \rho(\beta_1)b_{e_{1j}}(v_1)\rangle$ and $b_{e_{2j}}(v_2) \in \langle \rho(\beta_3)\rho(\beta_2)b_{e_{1j}}(v_1)\rangle$. Since $\langle b_{e_{i1}}(v_i)\rangle \neq \langle b_{e_{i2}}(v_i)\rangle$ ($i = 1, 2$) by Claim 2.6, $\rho(\beta_1) \sim \rho(\beta_3)\rho(\beta_2)$ holds.

Suppose $\alpha_1 = 1_\Gamma$ and let $Y := \langle b_{e_{11}}(v_1)\rangle$. Then $\beta_1 \in \Gamma' \Leftrightarrow \beta_1\alpha_1 \in \Gamma' \Leftrightarrow \psi(e_{21}, v_2) \sim 1_\Gamma \Leftrightarrow b_{e_{21}}(v_2) \in Y$ by Claim 2.6. The infeasibility of $\{e_{11}, e_1, e_{21}\}$ implies $b_{e_{21}}(v_2) \in \rho(\beta_1)Y$. Since Y and $\rho(\beta_1)Y$ are 1-dimensional subspaces and $b_{e_{21}}(v_2) \neq \mathbf{0}$, $b_{e_{21}}(v_2) \in Y \Leftrightarrow Y = \rho(\beta_1)Y$. ◻

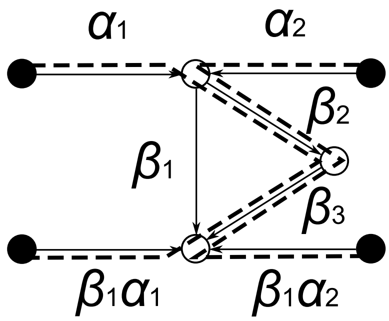

FIG. 2.2. *Each observed part in Claim 2.8 (dashed: roundabout non-admissible A-paths)*

3. Applications. In this section, we present a number of important special cases of the subgroup model. Almost all following cases satisfies $|\Gamma'| = O(1)$, and hence the running time of the linear matroid parity algorithm of Cheung et al. [1] is bounded by $O(|V|^\omega)$, which is much better than $O(|V|^5)$ derived from the algorithm of Chudnovsky et al. [2]. Without loss of generality, Y is fixed to $\langle \vec{e}_1 \rangle$ where $\vec{e}_1 \in \mathbb{F}^2$ denotes the first unit vector over each field $\mathbb{F}$.

3.1. Mader's $\mathcal{S}$-paths. Let $\mathcal{S} = \{A_1, \ldots, A_k\}$ be a partition of the terminal set A. Then Mader's problem is a special case of the subgroup model: $\Gamma = (\mathbb{Z}, +)$, $\Gamma' = \{0\}$ and $\psi(e) = i - j$ for each $e = uv \in \vec{E}$ with $u \in A_i$ and $v \in A_j$, where $A_0 := V \setminus A$. In this case, ρ defined as follows leads to the same coherent representation over $\mathbb{Q}$ as Schrijver's one with appropriate base transformations:

$$\rho(i) \ := \ \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \quad (i \in \mathbb{Z}).$$

3.2. Odd-length A-paths. To find a maximum family of vertex-disjoint odd-length A-paths is a special case of the subgroup model: $\Gamma = (\{1, -1\}, \times) \simeq \mathbb{Z}/2\mathbb{Z}$, $\Gamma' = \{1\}$, and $\psi(e) = -1$ for each $e \in \vec{E}$. In this case, ρ defined as follows leads to a coherent representation over an arbitrary field:

$$\rho(1) \ := \ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(-1) \ := \ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

3.3. Non-zero model. The non-zero model dealt with in [3, 2] is a simple special case of the subgroup model with Γ' trivial. The previous two examples are included in this model. Here we describe a more general claim than in the previous section.

For any $d \in \mathbb{N}$, let Γ be the cyclic group $C_d \simeq \mathbb{Z}/d\mathbb{Z}$ of order d generated by α. In this case, ρ defined as follows leads to a coherent representation over $\mathbb{R}$:

(3.1)
$$\rho(\alpha^k) \ := \ \begin{pmatrix} \cos \dfrac{k\pi}{d} & -\sin \dfrac{k\pi}{d} \\ \sin \dfrac{k\pi}{d} & \cos \dfrac{k\pi}{d} \end{pmatrix} \quad (0 \le k \le d-1).$$

Note that this ρ is not a general linear representation but a projective representation.

If Γ' is normal, then the subgroup model can be reduced the non-zero model by considering the quotient group Γ/Γ'. In the following two sections, let us see examples with Γ' not normal.

3.4. Dihedral groups. Let Γ be the dihedral group D_d of degree $d \geq 2$ generated by the minimum rotation r and the reflection R, and let Γ' be its proper subgroup. If $r^k R \notin \Gamma'$ for any $0 \leq k \leq d-1$, then Γ' is generated only by r^q for some divisor q of d (possibly $q = d$). In this case, Γ' is normal and Γ/Γ' is isomorphic to the dihedral group D_q of degree q. Hence we may assume that Γ' is trivial (i.e., $q = d$), and this setting admits a coherent representation over $\mathbb{R}$ by ρ defined as follows: $\rho(r^k)$ are the same as (3.1) with $\alpha = r$, and

$$\rho(R) \ := \begin{pmatrix} \cos\dfrac{\pi}{d+1} & \sin\dfrac{\pi}{d+1} \\ \sin\dfrac{\pi}{d+1} & -\cos\dfrac{\pi}{d+1} \end{pmatrix}.$$

Otherwise, suppose $R \in \Gamma'$ without loss of generality, and then Γ' is generated by R and r^q for some divisor q of d (possibly $q = d$). Let us take q minimum. Note that Γ' is not normal unless $q = 2$. In this case, ρ defined as follows leads to a coherent representation over $\mathbb{R}$:

$$\rho(r^k R^l) \ := \begin{pmatrix} \cos\dfrac{k\pi}{q} & -\sin\dfrac{k\pi}{q} \\ \sin\dfrac{k\pi}{q} & \cos\dfrac{k\pi}{q} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^l \qquad \begin{pmatrix} 0 \leq k \leq d-1 \\ l \in \{0,1\} \end{pmatrix}.$$

Note that, in the latter case, the size of Γ' is not necessary to be constant and hence the running time bound of the algorithm of Cheung et al. depends on $|\Gamma'| = 2d/q$.

3.5. Non-returning model. First we describe the definition of the non-returning model dealt with in [12, 13]. Let Π be a finite set, $\omega : A \to \Pi$ be a map on the terminal set, and $\pi : \vec{E} \to S(\Pi)$ be a map on the edge set to the permutations on Π with reference orientation. In this model, an A-path $(v_0, e_1, v_1, \ldots, e_k, v_k)$ is admissible if and only if $\omega(v_k) \neq \pi(e_k, v_k) \circ \cdots \circ \pi(e_1, v_1)(\omega(v_0))$ holds. Let $d := |\Omega| \geq 2$.

This model is equivalent to the subgroup model, and in particular it reduces to the following setting. Let Γ be the symmetric group S_d of degree d, and $\Gamma' := \{\sigma \in \Gamma \mid \sigma(d) = d\} = S_{d-1}$. If $d \geq 3$, Γ' is not a normal subgroup of Γ. If $d = 2, 3$, this problem reduces to privious examples by $S_2 \simeq \mathbb{Z}/2\mathbb{Z}$ and $S_3 \simeq D_3$. We shall clarify the case of $d \geq 4$.

Theorem 3.1. *The subgroup model reduced from the non-returning model with the label set Ω admits a coherent representation if and only if $|\Omega| \leq 4$.*

Proof. Suppose $d = |\Omega| \geq 4$ and that there exist desired ρ and Y. Then ρ is faithful.

To see this, suppose to the contrary that the kernel of ρ is not trivial, i.e., $\Gamma'' := \{\sigma \in \Gamma \mid \rho(\sigma) = I_2\}$ contains a non-identity permutation. By the basic fact of group theory, the kernel Γ'' is a normal subgroup of Γ. On the other hand, S_{d-1} is not a normal subgroup of S_d if $d \geq 3$, since S_{d-1} has a fixed point d and $(k\ d)S_{d-1}(k\ d)$ does not fix d for any $k \in [d-1]$.

Since we assume $Y = \langle \vec{e}_1 \rangle$ without loss of generality, for every $\sigma \in \Gamma'$, ρ has the following form:

$$\rho(\sigma) \;=\; \begin{pmatrix} 1 & a_\sigma \\ 0 & b_\sigma \end{pmatrix},$$

where $a_\sigma, b_\sigma \in \mathbb{F}$. Then it is seen as follows that the characteristic of $\mathbb{F}$ is 3.

Let $p := a_{(1\ 2)},\ q := b_{(1\ 2)},\ r := a_{(1\ 2\ 3)},\ s := b_{(1\ 2\ 3)}$. Because of $(1\ 2)(1\ 2\ 3) = (2\ 3)$, $(1\ 2\ 3)(1\ 2) = (1\ 3)$, $(1\ 2\ 3)^2 = (1\ 3\ 2)$ and $(1\ 2)^2 = (2\ 3)^2 = (1\ 3)^2 = (1\ 2\ 3)^3 = (1\ 3\ 2)^3 = \mathrm{id}$, we have

$$p(q+1) = (r+ps)(1+qs) = (p+qr)(1+qs) = r(s^2+s+1) = 0,$$

$$q^2 = q^2 s^2 = s^3 = 1. \tag{3.2}$$

Hence we have $s = 1$, and $r(1+1+1) = 0$. If $r = 0$, then $\rho\big((1\ 2\ 3)\big) = I_2$, contradicting the faithfulness of ρ. Thus we have $1+1+1 = 0$.

By (3.2), we have $q = \pm 1$. Suppose $q = 1$. Then $2p = 0$, which implies $p = 0$ and $\rho\big((1\ 2)\big) = I_2$, contradicting the faithfulness of ρ. Thus we have $q = -1 = 2$.

Now we have the following representation:

$$\begin{aligned}
\rho\big((1\ 2)\big) \;&=\; \begin{pmatrix} 1 & p \\ 0 & -1 \end{pmatrix}, & \rho(\mathrm{id}) \;&=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\rho\big((1\ 3)\big) \;&=\; \begin{pmatrix} 1 & p-r \\ 0 & -1 \end{pmatrix}, & \rho\big((1\ 2\ 3)\big) \;&=\; \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \\
\rho\big((2\ 3)\big) \;&=\; \begin{pmatrix} 1 & p+r \\ 0 & -1 \end{pmatrix}, & \rho\big((1\ 3\ 2)\big) \;&=\; \begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix}, \\
\rho\big((1\ 4)\big) \;&:=\; \begin{pmatrix} w_1 & x_1 \\ y_1 & z_1 \end{pmatrix}, & \rho\big((2\ 4)\big) \;&:=\; \begin{pmatrix} w_2 & x_2 \\ y_2 & z_2 \end{pmatrix}.
\end{aligned}$$

Since ρ is faithful, we have $p \neq p-r \neq p+r \neq p$, and hence at least two of $p_1 := p+r,\ p_2 := p-r,\ p_3 := p$ are nonzero. By symmetry, without loss of generality, we may assume $p_1 \neq 0 \neq p_2$.

For each $i \in \{1,2\}$, since $(j\ k)(i\ 4) = (i\ 4)(j\ k)$ ($\{j,k\} = \{1,2,3\} - i$) and $(i\ 4)^2 = \mathrm{id}$,

$$\begin{aligned}
\begin{pmatrix} w_i + p_i y_i & x_i + p_i z_i \\ -y_i & -z_i \end{pmatrix} \;&=\; l_{i1} \begin{pmatrix} w_i & p_i w_i - x_i \\ y_i & p_i y_i - z_i \end{pmatrix}, \\
\begin{pmatrix} w_i^2 + x_i y_i & x_i(w_i + z_i) \\ y_i(w_i + z_i) & x_i y_i + z_i^2 \end{pmatrix} \;&=\; l_{i2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\end{aligned}$$

where $l_{i1}, l_{i2} \in \mathbb{F}$. If $d \geq 5$, then we have $y_i = 0 \neq w_i$ by $(i\ 4) \in \Gamma'$, and hence $l_{i1} = 1$. If $x_i = 0$, then $w_i = z_i$ by $p_i \neq 0$, i.e., $\rho\big((i\ 4)\big) = w_i I_2 \sim \rho(\mathrm{id})$, a contradiction. Otherwise, by $x_i(w_i + z_i) = 0$, we have $z_i = -w_i$, and hence $x_i - p_i w_i = p_i w_i - x_i$. This implies $x_i - p_i w_i = 0$, i.e., $\rho\big((j\ k)(i\ 4)\big) = w_i I_2 \sim \rho(\mathrm{id})$, a contradiction. Thus we have proved that the non-returning model does not admit a coherent representation if $d \geq 5$.

If $d = 4$, then we have $y_i \neq 0$ by $(i\ 4) \notin \Gamma'$, and hence $l_{i1} = -1$ and $z_i = -w_i$. Therefore, we have the following equations:

$$w_i + p_i y_i = -w_i, \quad x_i - p_i w_i = -p_i w_i + x_i, \quad w_i = -p_i y_i - w_i.$$

The first and third equations imply $w_i = p_i y_i$, and the second holds obviously.

By patient but straightforward calculation, we get a desired projective representation ρ over $\mathbb{F} := \mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$, which is an isomorphism to $\mathrm{PGL}(2, \mathbb{F}_3)$, as follows:

$$
\begin{aligned}
\rho(\mathrm{id}) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \rho\big((1\ 2\ 3)\big) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & \rho\big((1\ 3\ 2)\big) &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \\
\rho\big((1\ 2)\big) &= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, & \rho\big((2\ 3)\big) &= \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, & \rho\big((1\ 3)\big) &= \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \\
\rho\big((1\ 4)\big) &= \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}, & \rho\big((2\ 4)\big) &= \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, & \rho\big((3\ 4)\big) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\rho\big((1\ 2)(3\ 4)\big) &= \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, & \rho\big((1\ 3)(2\ 4)\big) &= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, & \rho\big((1\ 4)(2\ 3)\big) &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \\
\rho\big((1\ 2\ 4)\big) &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \rho\big((1\ 4\ 2)\big) &= \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}, & \rho\big((1\ 3\ 4)\big) &= \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \\
\rho\big((1\ 4\ 3)\big) &= \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}, & \rho\big((2\ 3\ 4)\big) &= \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}, & \rho\big((2\ 4\ 3)\big) &= \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}, \\
\rho\big((1\ 3\ 2\ 4)\big) &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, & \rho\big((1\ 4\ 2\ 3)\big) &= \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}, & \rho\big((1\ 2\ 3\ 4)\big) &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \\
\rho\big((1\ 4\ 3\ 2)\big) &= \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}, & \rho\big((1\ 3\ 4\ 2)\big) &= \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, & \rho\big((1\ 2\ 4\ 3)\big) &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.
\end{aligned}
$$

The correctness can be easily confirmed by checking $A^2 = B^2 = C^2 = I_2$, $AC = CA$, $ABA = BAB$, $BCB = CBC$ where $A := \rho\big((1\ 2)\big)$, $B := \rho\big((2\ 3)\big)$, and $C := \rho\big((3\ 4)\big)$. ∎

4. Extension of Combinatorial Algorithm. In this section, we present an extension of the algorithm of Chudnovsky et al. [2] for the non-zero model to the subgroup model. We first introduce necessary definitions and notations in Section 4.1, and describe min-max and structure theorems in Section 4.2, which are automatically proved by the correctness of our extended algorithm. Section 4.3 presents the overview of our algorithm, whose main procedures are guaranteed by the lemmas shown in Section 4.4. Finally, we analyze its running time bound in Section 4.5.

4.1. Preliminaries. Let $G = (V, E)$ be an undirected graph. As basic notations, for $U \subseteq V$, let $\delta_G(U) := \{e = uv \in E \mid |U \cap \{u, v\}| = 1\}$ and $N_G(U) := \{v \in V \setminus U \mid \exists e = uv \in \delta_G(U) \text{ s.t. } u \in U\}$, and omit the braces $\{\}$ for a singleton $U = \{v\}$. For $U \subseteq V$ and $F \subseteq E$, let $G - U$ denote a graph obtained from G by removing the vertices in U and the edges that have at least one end in U, and $G - F$ denote one obtained by removing the edges in F.

Fix a group Γ and its proper subgroup Γ' in Section 4. A path starting in $A \subseteq V$ is called a *half A-path*. We call a family of vertex-disjoint admissible A-paths an *A-packing*, and a family of vertex-disjoint admissible A-paths and half A-paths covering A an *A-collection*. We use letters P, Q for paths, $\mathcal{P}, \mathcal{Q}$ for A-packings and A-collections, and Π for a family of A-collections. For a path P, let $P[u : v]$ denote its subpath from $u \in V(P)$ to $v \in V(P)$, and moreover the omission of u (or v) means u (or v) is the start (end) of P.

For an A-collection $\mathcal{P}$, we define its *value* and *reachability* as follows:

$$
\begin{aligned}
\mathrm{val}(\mathcal{P}, \psi, A) &:= |\{P \in \mathcal{P} \mid P : \text{admissible } A\text{-path}\}|, \\
R(\mathcal{P}, \psi, A) &:= \{(v, \alpha) \mid \exists P \in \mathcal{P} : \text{half } A\text{-path ending at } v \text{ with } \psi(P) = \alpha\}.
\end{aligned}
$$

Each element of $R(\mathcal{P}, \psi, A)$ is called a *reachable pair*. Two elements $\alpha, \beta \in \Gamma$ are said to be *left equivalent* if $\beta^{-1}\alpha \in \Gamma'$. Similarly to this, we say that a reachable pair (u, α) is *equivalent* to (v, β) if $u = v$ and α is left equivalent to β. We use the symbol $\sim$ to represent these two equivalences.

We provide several notations for the rest of Section 4. Let $\Pi(G, \psi, A)$ denote the set of all A-collection in G, and fix an arbitrary $\Pi \in \Pi(G, \psi, A)$. Recall that $\mu(G, \psi, A)$ denotes the maximum size of an A-packing. Similarly, let $\mu(\Pi, \psi, A)$ denote the maximum value of an A-collection in Π. The set of all reachable pair w.r.t. Π is denoted by $\mathcal{R}(\Pi, \psi, A)$, and that w.r.t. all A-collections with the maximum value by $\mathcal{R}(G, \psi, A)$. Let $\star$ be Π or G below. For a vertex $v \in V$, the set of reachable left cosets of v is defined by $\Gamma(\star, \psi, A; v) := \{\alpha\Gamma' \mid (v, \beta) \in \mathcal{R}(\star, \psi, A),\ \beta \in \alpha\Gamma'\}$. Let $D_1(\star, \psi, A)$ denote the set of uniquely reachable vertices, i.e., $D_1(\star, \psi, A) = \{v \in V \mid |\Gamma(\star, \psi, A; v)| = 1\}$, and D_2 that of multiply reachable vertices, i.e., $D_2(\star, \psi, A) = \{v \in V \mid |\Gamma(\star, \psi, A; v)| > 1\}$. Define $D(\star, \psi, A) := D_1(\star, \psi, A) \cup D_2(\star, \psi, A)$ and $\varphi(\Pi, \psi, A) := |D_1(\Pi, \psi, A)| + 2|D_2(\Pi, \psi, A)|$.

4.2. Min-max relation and Edmonds-Gallai type structure. It is known that the following min-max relation holds for the subgroup model for any group Γ and any proper subgroup Γ' of Γ.

THEOREM 4.1 (Pap [11, 12]). *For any Γ-labelled graph $(G = (V, E), \psi)$ with terminal set $A \subseteq V$, $\mu(G, \psi, A)$ is equal to the minimum value of the maximum number of vertex disjoint A_F-paths in $G - F$ taken over all edge subsets $F \subseteq E$ that contain no admissible A-path, where $A_F := A \cup V(F)$.*

By Gallai's min-max theorem [4] (see [14, §73]) for vertex-disjoint A-paths, the latter value is equal to the minimum value of

$$t(G, A; X, F) \ := \ |X| + \sum_{i=1}^{k} \left\lfloor \frac{1}{2}|X_i \cap A_F| \right\rfloor \tag{4.1}$$

taken over all vertex subsets $X \subseteq V$ and all edge subsets $F \subseteq E$ that contain no admissible A-path, where $X_1, \ldots, X_k$ are the vertex sets of the connected components of $G - X - F$. For each equality, the inequality of max $\leq$ min is obvious.

Note that, in this problem, it is important only which A-paths are admissible. For two Γ-labelled graph (G, ψ) and (G, ψ') with the same terminal set A, we say that they (or their label functions) are *A-equivalent* if, for every A-path P in G, we have $\psi(P) \notin \Gamma' \ \Leftrightarrow\ \psi'(P) \notin \Gamma'$. We say that a label function ψ' is obtained from a label function ψ by *shifting* at $v \in V$ with $\alpha \in \Gamma$ if $\psi'(e, v) = \alpha \cdot \psi(e, v)$ $(\forall e \in \delta_G(v))$ and $\psi'(e) = \psi(e)$ $(\forall e \in E \setminus \delta_G(v))$ hold. Note that shifting at a non-terminal always results in an A-equivalent label function.

LEMMA 4.2. *Let $(G = (V, E), \psi)$ be a Γ-labelled graph with terminal set $A \subseteq V$. Then there exists a Γ-labelled graph (G, ψ') A-equivalent to (G, ψ) such that:*

- *for each $v \in D_1(G, \psi, A)$, we have $\Gamma(G, \psi', A; v) = \{\Gamma'\}$, and*
- *for each $u \in N_G(D(G, \psi, A)) \setminus A$, there exists $e = uv \in E$ such that $\psi'(e, v)\Gamma' \in \Gamma(G, \psi', A; v)$.*

Proof. Take $v \in D_1(G, \psi, A)$ with $\Gamma(G, \psi, A; v) = \{\alpha\Gamma'\}$. If $v \in A$, then $\alpha = 1_\Gamma$. On the other hand, if $v \notin A$ and $\alpha \neq 1_\Gamma$, then by shifting ψ at v with α^{-1}, we obtain A-equivalent ψ' such that $\Gamma(G, \psi', A; v) = \{\Gamma'\}$.

Take $e = uv \in E$ with $u \notin A \cup D(G, \psi, A)$ and $v \in D(G, \psi, A)$. For $\alpha\Gamma' \in$

$\Gamma(G,\psi,A;v)$, by shifting ψ at u with $\alpha^{-1}\cdot\psi(e,v)$, we get A-equivalent ψ' such that $\psi'(e,v)=\alpha$. Moreover, these two types of shifting do not interfere with each other. □

Chudnovsky et al. [2] showed a structure theorem for the non-zero model, and we show its generalized form and prove it by our extended combinatorial algorithm. Here we define an *odd (even) component* of a graph G with a terminal set as a connected component of G that contains odd (even) number of terminals.

THEOREM 4.3. *Suppose that a Γ-labelled graph (G,ψ) satisfies the two conditions in Lemma* 4.2. *Let $A' := A\cup D_1(G,\psi,A)\cup N_G(D(G,\psi,A))$, $F := \{e=uv\in E \mid u,v\in A',\ \psi(e)\in\Gamma'\}$, and $X := N_{G-F}(D(G,\psi,A))$. Then $X\subseteq V$ and $F\subseteq E$ attain the minimum of* (4.1)*, $D(G,\psi,A)$ is the union of the odd components of $G-X-F$, and $V\setminus(D(G,\psi,A)\cup X)$ is the union of the even components of $G-X-F$.*

4.3. Algorithm description. The main routine of the extended algorithm is Algorithm 1. For simple description, important procedures are separated to Section 4.3.1 as subroutines, which are based on key lemmas in Section 4.4. In each iteration step of the main routine, we check whether the optimality condition holds (lines 3–8), and if it does not hold then we construct a new A-collection with greater value or a new effective reachable pair contributing φ (lines 10–33).

4.3.1. Subroutines.

Sub. 1. Take $v\in A'\setminus\tilde{A}$, and let $\tilde{A}\leftarrow\tilde{A}+v$, $\tilde{G}\leftarrow\tilde{G}-\{e=uv\in E\mid u,v\in\tilde{A},\ \tilde{\psi}(e)\in\Gamma'\}$. For each old admissible $\tilde{A}$-path in old $\tilde{G}$, there exists at least one new admissible $\tilde{A}$-path in new $\tilde{G}$ as its subpath. Moreover, each old half $\tilde{A}$-path includes either a new admissible $\tilde{A}$-path or a new half $\tilde{A}$-path that has an equivalent label to the old one as its subpath. Construct new $\tilde{\Pi}$ by replaceing each path in each old $\tilde{A}$-collection $\mathcal{P}\in\tilde{\Pi}$ intersecting v by one of such paths and adding the trivial path from the terminal in new $\tilde{A}$ that is not used. Note that $\mu(\tilde{\Pi},\tilde{\psi},\tilde{A})$ does not decrease and that, if it does not increase, then $D_i(\tilde{\Pi},\tilde{\psi},\tilde{A})$ does not reduce for each $i\in\{1,2\}$.

Sub. 2. Take $v\in X\setminus\tilde{X}$. Let $\tilde{X}\leftarrow\tilde{X}+v$, $\tilde{A}\leftarrow\tilde{A}-v$, and $\tilde{G}\leftarrow\tilde{G}-\{v\}$. For each old $\tilde{A}$-collection in $\tilde{\Pi}$, removing v affects at most one path that intersects v. Construct new $\tilde{\Pi}$ by removing such paths that appear in old $\tilde{\Pi}$. Note that $\mu(\tilde{\Pi},\tilde{\psi},\tilde{A})$ decreases at most one and that, if it decreases by one, then $D_i(\tilde{\Pi},\tilde{\psi},\tilde{A})$ does not change for each $i\in\{1,2\}$.

Sub. 3. Because of Lemma 4.9 and the optimality condition (not holding in line 6), there exists a connected component K of $\tilde{G}=G-X-F$ with $|V(K)\cap\tilde{A}|-2\mu(\tilde{\Pi}_K,\tilde{\psi},\tilde{A})>1$, $V(K)\cap\tilde{A}=D_1(\tilde{\Pi}_K,\tilde{\psi},\tilde{A})$, and $V(K)\setminus\tilde{A}=D_2(\tilde{\Pi}_K,\tilde{\psi},\tilde{A})$, where $\tilde{\Pi}_K$ denotes the restriction of $\tilde{\Pi}$ to K. According to Lemma 4.8, find $\tilde{\mathcal{P}}_K\in\Pi(K,\tilde{\psi},\tilde{A})$ with $\mathrm{val}(\tilde{\mathcal{P}}_K,\tilde{\psi},\tilde{A})=\mu(\tilde{\Pi}_K,\tilde{\psi},\tilde{A})+1$. Take $\tilde{\mathcal{P}}'\in\tilde{\Pi}$ and replace it with $\tilde{\mathcal{P}}:=\tilde{\mathcal{P}}_K\cup(\tilde{\mathcal{P}}'\setminus K)$. Note that new $\tilde{\Pi}$ with $\tilde{\mathcal{P}}$ satisfies $\mathrm{val}(\tilde{\mathcal{P}},\tilde{\psi},\tilde{A})\geq\mathrm{val}(\tilde{\mathcal{P}}',\tilde{\psi},\tilde{A})+1\geq\mu(\Pi,\psi,A)-|X|+1$.

Sub. 4. Take $v\in\tilde{X}$ that was latest added to $\tilde{X}$ by Sub. 2. Suppose that adding v to $\tilde{X}$ makes $\tilde{G}'\to\tilde{G}$, $\tilde{A}'\to\tilde{A}$, and $\tilde{\Pi}'\to\tilde{\Pi}$. Then $\mu(\tilde{\Pi}',\tilde{\psi},\tilde{A}')\leq\mu(\tilde{\Pi},\tilde{\psi},\tilde{A})+1\leq\mathrm{val}(\tilde{\mathcal{P}},\tilde{\psi},\tilde{A})+1$ and $D_i(\tilde{\Pi}',\tilde{\psi},\tilde{A}')=D_i(\Pi,\psi,A)$ $(i=1,2)$. Note that we have $\mathrm{val}(\tilde{\mathcal{P}},\tilde{\psi},\tilde{A})\geq\mu(\tilde{\Pi}',\tilde{\psi},\tilde{A}')$ or $(u,\gamma)\in R(\tilde{\mathcal{P}},\tilde{\psi},\tilde{A})$ with $u\in D_i(\tilde{\Pi}\cup\{\mathcal{P}\},\tilde{\psi},\tilde{A})\setminus D_i(\Pi,\psi,A)$ for some i. According to Lemma 4.6, find $\tilde{\mathcal{Q}}\in\Pi(\tilde{G}',\tilde{\psi},\tilde{A}')$ with $\mathrm{val}(\tilde{\mathcal{Q}},\tilde{\psi},\tilde{A}')\geq\mu(\tilde{\Pi}',\tilde{\psi},\tilde{A}')$ and $D_i(\tilde{\Pi}'\cup\{\tilde{\mathcal{Q}}\},\tilde{\psi},\tilde{A}')\setminus D_i(\tilde{\Pi}',\tilde{\psi},\tilde{A}')\neq\emptyset$ for some i. Let $\tilde{X}\leftarrow\tilde{X}-v$, $\tilde{G}\leftarrow\tilde{G}'$, $\tilde{A}\leftarrow\tilde{A}'=\tilde{A}+v$, and $\tilde{\mathcal{P}}\leftarrow\tilde{\mathcal{Q}}$.

Algorithm 1 Finding a maximum A-packing

Input: A Γ-labelled graph $(G = (V, E), \psi)$ with terminal set $A \subseteq V$ and a proper subgroup Γ' of Γ.
Output: An A-packing of size $\mu(G, \psi, A)$.

Set $\Pi \leftarrow \{\mathcal{P}_0\}$, where $\mathcal{P}_0 := \{(a) \mid a \in A\}$ is a trivial A-collection.
while `true` **do**
  Compute $D_1(\Pi, \psi, A)$ and $D_2(\Pi, \psi, A)$.
  By shifting according to Lemma 4.2, get $\tilde{\psi}$ A-equivalent to ψ.
  Set $A' \leftarrow A \cup D_1(\Pi, \psi, A) \cup N_G(D(\Pi, \psi, A))$, $F \leftarrow \{e = uv \in E \mid u, v \in A', \ \tilde{\psi}(e) \in \Gamma'\}$, and $X \leftarrow N_{G-F}(D(\Pi, \psi, A))$.
  if $\mu(\Pi, \psi, A) = t(G, A; X, F)$ **then**
    return the maximum A-packing included in some $\mathcal{P} \in \Pi$.
  end if
  Set $\tilde{G} \leftarrow G$, $\tilde{A} \leftarrow A$, $\tilde{X} \leftarrow \emptyset$, and $\tilde{\Pi} \leftarrow \Pi$.
  while $\mu(\tilde{\Pi}, \tilde{\psi}, \tilde{A}) = \mu(\Pi, \psi, A) - |\tilde{X}|$, $D_i(\tilde{\Pi}, \tilde{\psi}, \tilde{A}) = D_i(\Pi, \psi, A)$ $(i = 1, 2)$, and $[\tilde{A} \neq A'$ or $\tilde{X} \neq X]$ **do**
    if $\tilde{A} \neq A'$ **then**
      Update $\tilde{G}$, $\tilde{A}$, and $\tilde{\Pi}$ by Sub. 1.
    else
      Update $\tilde{G}$, $\tilde{A}$, $\tilde{X}$, and $\tilde{\Pi}$ by Sub. 2.
    end if
  end while
  if $\mu(\tilde{\Pi}, \tilde{\psi}, \tilde{A}) = \mu(\Pi, \psi, A) - |\tilde{X}|$ and $D_i(\tilde{\Pi}, \tilde{\psi}, \tilde{A}) = D_i(\Pi, \psi, A)$ $(i = 1, 2)$ **then**
    Find $\tilde{\mathcal{P}} \in \Pi(\tilde{G}, \tilde{\psi}, \tilde{A})$ with $\mathrm{val}(\tilde{\mathcal{P}}, \tilde{\psi}, \tilde{A}) > \mu(\tilde{\Pi}, \tilde{\psi}, \tilde{A})$ by Sub. 3 and add to $\tilde{\Pi}$.
  else
    Choose $\tilde{\mathcal{P}} \in \tilde{\Pi}$ that violates at least one of the conditions.
  end if
  while $\tilde{X} \neq \emptyset$ **do**
    Update $\tilde{G}$, $\tilde{A}$, $\tilde{X}$, and $\tilde{\mathcal{P}}$ by Sub. 4.
  end while
  while $\tilde{A} \neq A$ **do**
    Update $\tilde{G}$, $\tilde{A}$, and $\tilde{\mathcal{P}}$ by Sub. 5.
  end while
  if $\mathrm{val}(\tilde{P}, \psi, A) \geq \mu(\Pi, \psi, A) + 1$ **then**
    Let $\Pi \leftarrow \{\tilde{\mathcal{P}}\}$.
  else
    Let $\Pi \leftarrow \Pi \cup \{\tilde{\mathcal{P}}\}$.
  end if
end while

Sub. 5. Take $v \in \tilde{A} \setminus A$ that was latest added to $\tilde{A}$ by Sub. 1. Suppose that adding v to $\tilde{A}$ makes $\tilde{G}' \to \tilde{G}$, $\tilde{A}' \to \tilde{A}$, and $\tilde{\Pi}' \to \tilde{\Pi}$. Then $\mu(\tilde{\Pi}', \tilde{\psi}, \tilde{A}') \leq \mu(\tilde{\Pi}, \tilde{\psi}, \tilde{A}) \leq \mathrm{val}(\tilde{\mathcal{P}}, \tilde{\psi}, \tilde{A})$ and $D_i(\tilde{\Pi}', \tilde{\psi}, \tilde{A}') = D_i(\Pi, \psi, A)$ $(i = 1, 2)$. Note that we have $\mathrm{val}(\tilde{\mathcal{P}}, \tilde{\psi}, \tilde{A}) \geq \mu(\tilde{\Pi}', \tilde{\psi}, \tilde{A}') + 1$ or $(u, \gamma) \in R(\tilde{\mathcal{P}}, \tilde{\psi}, \tilde{A})$ with $u \in D_i(\tilde{\Pi} \cup \{\mathcal{P}\}, \tilde{\psi}, \tilde{A}) \setminus D_i(\Pi, \psi, A)$ for some i. According to Lemma 4.7, find $\tilde{\mathcal{Q}} \in \Pi(\tilde{G}', \tilde{\psi}, \tilde{A}')$ with either $\mathrm{val}(\tilde{\mathcal{Q}}, \tilde{\psi}, \tilde{A}') \geq \mu(\tilde{\Pi}', \tilde{\psi}, \tilde{A}') + 1$, or $\mathrm{val}(\tilde{\mathcal{Q}}, \tilde{\psi}, \tilde{A}') \geq \mu(\tilde{\Pi}', \tilde{\psi}, \tilde{A}')$ and $D_i(\tilde{\Pi}' \cup \{\tilde{\mathcal{Q}}\}, \tilde{\psi}, \tilde{A}') \setminus D_i(\tilde{\Pi}', \tilde{\psi}, \tilde{A}') \neq \emptyset$ for some i. Let $\tilde{G} \leftarrow \tilde{G}'$, $\tilde{A} \leftarrow \tilde{A}' = \tilde{A} - v$, and $\tilde{\mathcal{P}} \leftarrow \tilde{\mathcal{Q}}$.

Remark. Since the original paper [2] has deliberate omissions and careless mistakes mainly in Section 5, we have completed them in our algorithm description. For example, we describe concrete procedures Subs. 1 and 2 corresponding to an omitted proof of [2, Lemma 5.1]. These subroutines add and remove a terminal, respectively, and construct a family of new A-collections in the resulting graph from a family of old ones in the original graph without drastic changes.

More significant problems appear in [2, Lemmas 5.2 and 5.3]. The left-hand sides $\nu_{A''}(\mathcal{P}')$ and $\nu_{A'-X'}(\mathcal{P}')$ of inequalities and equations should be replaced by $\mathrm{val}_{A''}(\Pi')$ and $\mathrm{val}_{A'-X'}(\Pi')$, respectively, in the notation of [2]. Moreover, the assumption of [2, Lemma 5.3] is too strong. Correctly, the right-hand sides of the inequality in (i) and the equation in (ii) need an additional term $-|X'|$. Similar modifications are necessary in the subsequent paragraph and in an omitted proof of [2, Lemma 5.3].

Finally, it should be noted that the procedures Subs 3, 4 and 5 correspond to [2, Lemma 4.5 and 5.4], [2, Lemma 5.3] and [2, Lemma 5.2], respectively.

4.4. Key Lemmas. This section shows important lemmas for our algorithm. Let $(G = (V, E), \psi)$ be a Γ-labelled graph with terminal set $A \subseteq V$.

LEMMA 4.4. *Let $\mathcal{P}_1, \mathcal{P}_2 \in \Pi(G, \psi, A)$ satisfy* $\mathrm{val}(\mathcal{P}_1, \psi, A) = \mathrm{val}(\mathcal{P}_2, \psi, A) - 1$ *and fix $e = uv \in E$. If there exist $p, (u, \alpha) \in R(\mathcal{P}_1, \psi, A)$ and $(v, \beta) \in R(\mathcal{P}_2, \psi, A)$ with $\beta \not\sim \psi(e, v) \cdot \alpha$, then one can find, in $O(|V|^2)$ time, $\mathcal{P}_3 \in \Pi(G, \psi, A)$ such that* $\mathrm{val}(\mathcal{P}_3, \psi, A) = \mathrm{val}(\mathcal{P}_2, \psi, A)$ *and at least one of p' and (u, α') is in $R(\mathcal{P}_3, \psi, A)$ for some $p' \sim p$ and $\alpha' \sim \alpha$.*

Proof. The proof is done by reverse induction on $|E(\mathcal{P}_1) \cup E(\mathcal{P}_2)| < 2|V|$. In each induction step, we just trace a constant number of paths in G, which takes $O(|V|)$ time. Thus the running time is $O(|V|^2)$. Let $P_1, P_2 \in \mathcal{P}_1$, $Q_1 \in \mathcal{P}_2$ be the half A-paths which make $p, (u, \alpha) \in R(\mathcal{P}_1, \psi, A)$, $(v, \beta) \in R(\mathcal{P}_2, \psi, A)$, respectively.

Because of $2\mathrm{val}(\mathcal{P}_1, \psi, A) + 2 = 2\mathrm{val}(\mathcal{P}_2, \psi, A)$, there exists a teminal $a \in A$ at which Q_1 or an admissible A-path (or its reversed path) in $\mathcal{P}_2$ starts, say $Q \in \mathcal{P}_2$, such that P_1, P_2 and all admissible A-paths in $\mathcal{P}_1$ are disjoint from a. We may assume that the half A-path in $\mathcal{P}_1$ starting at a is the trivial one $(a) \in \mathcal{P}_1$, since it does not affect the assumption of the lemma. If Q is disjoint from $\mathcal{P}_1 \setminus \{(a)\}$, then $Q = Q_1$ and hence we get desired $\mathcal{P}_3 := \mathcal{P}_1 \Delta \{(a), P_2, P'\}$ where $P' := (P_2, e, Q_1)$ is admissible A-path since $\beta \not\sim \psi(e, v) \cdot \alpha$ implies $\psi(P') = \beta^{-1} \cdot \psi(e, v) \cdot \alpha \notin \Gamma'$. Otherwise Q intersects some paths in $\mathcal{P}_1 \setminus \{(a)\}$, let $P \in \mathcal{P} \setminus \{(a)\}$ be the first one in walking along Q, and let w be the shared vertex.

Case 1. Suppose that P is a half A-path. Let P' be the A-path obtained by connecting $P[\ : w]$ and $\bar{Q}[w :\ ]$, and then P' is disjoint from each path in $\mathcal{P}_1 \setminus \{P, (a)\}$. If $\psi(P') \notin \Gamma'$, then we get desired $\mathcal{P}_3 := \mathcal{P}_1 \Delta \{P, (a), P'\}$.

Otherwise, let P'' be the half A-path obtained by connecting $Q[\ : w]$ and $P[w :\ ]$, and let $\mathcal{P}_1 := \mathcal{P}_1 \Delta \{P, P''\}$. We then have $p', (u, \alpha') \in R(\mathcal{P}_1, \psi, A)$ with respect to new $\mathcal{P}_1$ for some $p' \sim p$ and $\alpha' \sim \alpha$, since $\psi(P') \in \Gamma'$ implies $\psi(P'') \sim \psi(P)$. Note that the value of $|E(\mathcal{P}_1) \cup E(\mathcal{P}_2)|$ decreases since each new edge which appears in $E(\mathcal{P}_1)$ is in $E(\mathcal{P}_2)$ and at least one edge in the subpath $P[\ : w]$ is removed from $E(\mathcal{P}_1)$ and does not belong to $E(\mathcal{P}_2)$.

Case 2. Suppose that P is an admissible A-path. At least one P' of the two A-paths which are obtained by connecting $Q[\ : w]$ and $P[w :\ ]$, and $Q[\ : w]$ and $\bar{P}[w :\ ]$ is an admissible A-path disjoint from each path in $\mathcal{P}_1 \setminus \{P\}$. Hence, let $\mathcal{P}_1 := \mathcal{P}_1 \Delta \{P, P'\}$, and then the value of $\mathrm{val}(\mathcal{P}_1, \psi, A)$ does not change, $p, (u, \alpha) \in$

$R(\mathcal{P}_1, \psi, A)$ remains, and the value of $|E(\mathcal{P}_1) \cup E(\mathcal{P}_2)|$ decreases. $\square$

LEMMA 4.5. *Let* $\mathcal{P}_1, \mathcal{P}_2 \in \Pi(G, \psi, A)$ *satisfy* $\mathrm{val}(\mathcal{P}_1, \psi, A) = \mathrm{val}(\mathcal{P}_2, \psi, A)$*, and fix* $p_1 \in R(\mathcal{P}_1, \psi, A)$*,* $p_2, p_3 \in R(\mathcal{P}_2, \psi, A)$ *with* $p_1 \not\sim p_2 \not\sim p_3 \not\sim p_1$*. Then one can find, in* $O(|V|^2)$ *time,* $\mathcal{P}_3 \in \Pi(G, \psi, A)$ *such that either*

- $\mathrm{val}(\mathcal{P}_3, \psi, A) = \mathrm{val}(\mathcal{P}_1, \psi, A)$ *and* $p_1', p_2' \in R(\mathcal{P}_3, \psi, A)$ *or* $p_1', p_3' \in R(\mathcal{P}_3, \psi, A)$ *for some* $p_i' \sim p_i$ *for* $i = 1, 2, 3$*, or*
- $\mathrm{val}(\mathcal{P}_3, \psi, A) = \mathrm{val}(\mathcal{P}_1, \psi, A) + 1$.

Proof. The proof is similar to the proof of Lemma 4.4. Let $P_1 \in \mathcal{P}_1$, $Q_1, Q_2 \in \mathcal{P}_2$ be the half A-paths which make $p_1 \in R(\mathcal{P}_1, \psi, A)$, $p_2, p_3 \in R(\mathcal{P}_2, \psi, A)$, respectively.

Since $2\mathrm{val}(\mathcal{P}_1, \psi, A) = 2\mathrm{val}(\mathcal{P}_2, \psi, A)$, there exists a teminal $a \in A$ at which Q_1, Q_2 or an admissible A-path (or its reversed path) in $\mathcal{P}_2$ starts, say $Q \in \mathcal{P}_2$, such that P_1 and all admissible A-paths in $\mathcal{P}_1$ are disjoint from a. We may assume that the half A-path in $\mathcal{P}_1$ starting at a is the trivial one $(a) \in \mathcal{P}_1$, since it does not affect the assumption of the lemma. If Q is disjoint from each path in $\mathcal{P}_1 \setminus \{(a)\}$, then Q is a half A-path ($Q \in \{Q_1, Q_2\}$) and hence we get desired $\mathcal{P}_3 := \mathcal{P}_1 \Delta \{(a), Q\}$. Otherwise Q intersects some paths in $\mathcal{P}_1 \setminus \{(a)\}$, let $P \in \mathcal{P}_1 \setminus \{(a)\}$ be the first one in walking along Q, and let w be the shared vertex.

The rest is almost the same as the proof of Lemma 4.4, so we omit it. $\square$

In Lemmas 4.6–4.8, fix any family $\Pi \subseteq \Pi(G, \psi, A)$ of A-collections with $|\Pi| = O(|V|)$, suppose that ψ has been shifted as Lemma 4.2 with respect to Π, and let $F := \{e = uv \in E \mid u, v \in A,\ \psi(e) \in \Gamma'\}$.

LEMMA 4.6. *Fix* $v \in A \cap N_{G-F}(D(\Pi, \psi, A))$ *and* $\mathcal{P} \in \Pi(G - \{v\}, \psi, A - v)$ *with either* $\mathrm{val}(\mathcal{P}, \psi, A - v) \geq \mu(\Pi, \psi, A)$*, or* $\mathrm{val}(\mathcal{P}, \psi, A - v) \geq \mu(\Pi, \psi, A) - 1$ *and* $p \in R(\mathcal{P}, \psi, A - v) \setminus \mathcal{R}(\Pi, \psi, A)$*. Then one can find, in* $O(|V|^2)$ *time,* $\mathcal{Q} \in \Pi(G, \psi, A)$ *such that* $\mathrm{val}(\mathcal{Q}, \psi, A) \geq \mu(\Pi, \psi, A)$ *and at least one of* p' *and* (v, α) *is in* $R(\mathcal{Q}, \psi, A)$ *for some* $p' \sim p$ *and* $\alpha \in \Gamma'$*.*

Proof. If $\mathrm{val}(\mathcal{P}, \psi, A - v) \geq \mu(\Pi, \psi, A)$, then we have $(v, 1_\Gamma) \in R(\mathcal{P} \cup \{(v)\}, \psi, A)$. This means that $\mathcal{P} \cup \{(v)\} \in \Pi(G, \psi, A)$ is a desired A-collection.

Otherwise, we have $\mathrm{val}(\mathcal{P}, \psi, A - v) = \mu(\Pi, \psi, A) - 1$ and $p \in R(\mathcal{P}, \psi, A - v) \setminus \mathcal{R}(\Pi, \psi, A)$, and then $p, (v, 1_\Gamma) \in R(\mathcal{P} \cup \{(v)\}, \psi, A)$. Since $v \in N_{G-F}(D(\Pi, \psi, A))$, there exist $u \in N_G(v)$ and $e = uv \in E$ with $(u, \beta) \in \mathcal{R}(\Pi, \psi, A)$ and $\psi(e, u) \not\sim \beta$. Take $\mathcal{Q}' \in \Pi$ with $\mathrm{val}(\mathcal{Q}', \psi, A) = \mu(\Pi, \psi, A)$ and $(u, \beta) \in R(\mathcal{Q}', \psi, A)$, and, by applying Lemma 4.4 to $\mathcal{P} \cup \{(v)\}$ and $\mathcal{Q}'$, we get desired $\mathcal{Q}$ with $\alpha \sim 1_\Gamma$, i.e., $\alpha \in \Gamma'$. $\square$

LEMMA 4.7. *Fix* $v \in (D_1(\Pi, \psi, A) \cup N_G(D(\Pi, \psi, A))) \setminus A$ *and* $\mathcal{P} \in \Pi(G, \psi, A + v)$ *with either* $\mathrm{val}(\mathcal{P}, \psi, A + v) \geq \mu(\Pi, \psi, A) + 1$*, or* $\mathrm{val}(\mathcal{P}, \psi, A + v) \geq \mu(\Pi, \psi, A)$ *and* $p \in R(\mathcal{P}, \psi, A + v) \setminus \mathcal{R}(\Pi, \psi, A)$*, then one can find, in* $O(|V|^2)$ *time,* $\mathcal{Q} \in \Pi(G, \psi, A)$ *such that either* $\mathrm{val}(\mathcal{Q}, \psi, A) \geq \mu(\Pi, \psi, A) + 1$*, or* $\mathrm{val}(\mathcal{Q}, \psi, A) \geq \mu(\Pi, \psi, A)$ *and at least one of* p' *and* (v, α) *is in* $R(\mathcal{Q}, \psi, A)$ *for some* $p' \sim p$ *and* $\alpha \notin \Gamma'$*.*

Proof. By the choice of v, there exists $\mathcal{Q}_0' \in \Pi$ such that $\mathrm{val}(\mathcal{Q}_0', \psi, A) = \mu(\Pi, \psi, A)$ and either $(v, \gamma) \in R(\mathcal{Q}_0', \psi, A)$ with $\gamma \in \Gamma'$ (if $v \in D_1(\Pi, \psi, A)$) or $(u, \beta) \in R(\mathcal{Q}_0', \psi, A)$ with $e = uv \in E$ and $\psi(e, v) \cdot \beta \in \Gamma'$ (if $v \in N_G(\Pi, \psi, A)$) because of the two conditions in Lemma 4.2. By removing the unique edge $e = uv \in E(\mathcal{Q}_0')$ in the former case, we have an A-collection $\mathcal{Q}_0 \in \Pi(G, \psi, A)$ such that $\mathrm{val}(\mathcal{Q}_0, \psi, A) = \mu(\Pi, \psi, A)$ and $(u, \beta) \in R(\mathcal{Q}_0, \psi, A)$ for some $\beta \in \Gamma$ with $\beta \sim \psi(e, u)$, in the both cases.

Case 1. Suppose $\mathrm{val}(\mathcal{P}, \psi, A + v) \geq \mu(\Pi, \psi, A) + 1$. Let $P \in \mathcal{P}$ be the path starting at v. If P is a half A-path, then $\mathrm{val}(\mathcal{Q}, \psi, A) \geq \mu(\Pi, \psi, A) + 1$ for $\mathcal{Q} := \mathcal{P} \setminus \{P\} \in \Pi(G, \psi, A)$. Otherwise, since P is an admissible A-path, we have $\mathrm{val}(\mathcal{Q}, \psi, A) \geq \mu(\Pi, \psi, A)$ and $(v, \alpha) \in R(\mathcal{Q}, \psi, A)$ with $\alpha = \psi(\bar{P}) \notin \Gamma'$ for $\mathcal{Q} := \mathcal{P} \in \Pi(G, \psi, A)$. In the both cases, $\mathcal{Q}$ is a desired A-collection.

Case 2. Suppose $\mathrm{val}(\mathcal{P}, \psi, A+v) = \mu(\Pi, \psi, A)$ and $p \in R(\mathcal{P}, \psi, A+v) \setminus \mathcal{R}(\Pi, \psi, A)$. Let $P_1 \in \mathcal{P}$ be the half A-path which makes $p \in R(\mathcal{P}, \psi, A+v)$, and let $P \in \mathcal{P}$ be the path starting at v. If P is a half $(A+v)$-path with $P \neq P_1$, then $\mathcal{P} \setminus \{P\} \in \Pi(G, \psi, A)$ is a desired A-collection. The rest cases are as follows.

Case 2.1. Suppose that P is an admissible $(A+v)$-path. Let $\mathcal{Q}_1 := \mathcal{P} \in \Pi(G, \psi, A)$, and then $\mathrm{val}(\mathcal{Q}_1, \psi, A) = \mathrm{val}(\mathcal{P}, \psi, A) - 1 = \mu(\Pi, \psi, A) - 1$ and $p, (v, \alpha') \in R(\mathcal{Q}_1, \psi, A)$ with $\alpha' = \psi(\bar{P}) \notin \Gamma'$. Recall that there exists $e = uv \in E$ with $\psi(e, u) \sim \beta$ and $\mathcal{Q}_0 \in \Pi(G, \psi, A)$. Since $\psi(e, v) \cdot \beta \in \Gamma'$ implies $\alpha' \not\sim \psi(e, v) \cdot \beta$, by applying Lemma 4.4 to $\mathcal{Q}_1$ and $\mathcal{Q}_0$, we get desired $\mathcal{Q} \in \Pi(G, \psi, A)$.

Case 2.2. Suppose $P = P_1$.

Case 2.2.1. Suppose that each path in $\mathcal{Q}_0$ is disjoint from v. Let $\mathcal{P}_0 := \mathcal{Q}_0 \cup \{(v)\}$. We then have $\mathrm{val}(\mathcal{P}_0, \psi, A+v) = \mathrm{val}(\mathcal{Q}_0, \psi, A) = \mu(\Pi, \psi, A)$ and $(u, \beta), (v, 1_\Gamma) \in R(\mathcal{P}_0, \psi, A+v)$. By applying Lemma 4.5 to $\mathcal{P}$ and $\mathcal{P}_0$, we get $\mathcal{P}_1 \in \Pi(G, \psi, A+v)$ such that either $\mathrm{val}(\mathcal{P}_1, \psi, A+v) = \mathrm{val}(\mathcal{P}, \psi, A) + 1 = \mu(\Pi, \psi, A) + 1$, or $\mathrm{val}(\mathcal{P}_1, \psi, A+v) = \mu(\Pi, \psi, A)$ and some $p' \sim p$ and one of $(u, \beta'), (v, 1_\Gamma)$ are in $R(\mathcal{P}_1, \psi, A+v)$ for some $\beta' \sim \beta$. The former case reduces to Case 1, and we have done in the latter case by $\mathcal{Q} := \mathcal{P}_1 \cup \{(v)\}$ or $\mathcal{Q} := \mathcal{P}_1 \Delta \{P_2, P_3, Q\}$ where $P_2 \in \mathcal{P}_1$ makes $(u, \beta) \in R(\mathcal{P}_1, \psi, A+v)$, $P_3 \in \mathcal{P}_1$ starts at v, and $Q := (P_2, e, P_3)$ (note $\psi(Q) = \psi(P_3)$ since $\psi(e, u) \sim \beta$).

Case 2.2.2. Suppose that $Q \in \mathcal{Q}_0$ intersects v. Let w be the vertex preceding v on Q, and let $Q_1 \in \mathcal{Q}_0$ be the path which makes $(u, \beta) \in R(\mathcal{Q}_0, \psi, A)$. If $Q = Q_1$, then either it reduces to Case 2.2.1 by replacing Q_1 with the subpath $Q_1[\ : w]$ and regarding w as new u, or $Q_1[\ : v]$ makes (v, α) with $\alpha \notin \Gamma'$ and hence $\mathcal{Q} \Delta \{Q_1, Q_1[\ : v]\}$ is a desired A-collection. Moreover, if Q is another half A-path, then it also reduces to Case 2.2.1 by replacing Q with the trivial half A-path. Suppose that Q is an admissible A-path.

Let $Q' := Q[\ : w]$ and $Q'' := \bar{Q}[\ : v]$. By reversing Q in advance if necessary, we may assume that Q'' is an admissible $(A+v)$-path. Let $\mathcal{P}_0 := \mathcal{Q}_0 \Delta \{Q, Q', Q''\}$, and then $\mathrm{val}(\mathcal{P}_0, \psi, A+v) = \mathrm{val}(\mathcal{Q}_0, \psi, A) = \mu(\Pi, \psi, A)$ and $(u, \beta), (w, \delta) \in R(\mathcal{P}_0, \psi, A+v)$ for some $\delta \in \Gamma$. By applying Lemma 4.5 to $\mathcal{P}$ and $\mathcal{P}_0$, we get $\mathcal{P}_1 \in \Pi(G, \psi, A+v)$ such that either $\mathrm{val}(\mathcal{P}_1, \psi, A+v) = \mathrm{val}(\mathcal{P}, \psi, A+v) + 1 = \mu(\Pi, \psi, A) + 1$, or $\mathrm{val}(\mathcal{P}_1, \psi, A+v) = \mu(\Pi, \psi, A)$ and p' and at least one of (u, β') and (w, δ') is in $R(\mathcal{P}_1, \psi, A+v)$ for some $p' \sim p$, $\beta' \sim \beta$ and $\delta' \sim \delta$.

The former case reduces to Case 1 and the latter case remains Case 2. Suppose that it remains Case 2.2, i.e., the path in $\mathcal{P}_1$ starting at v makes $p' \in R(\mathcal{P}_1, \psi, A+v)$. If $(u, \beta') \in R(\mathcal{P}_1, \psi, A+v)$, then by $e = uv \in E$ with $\psi(e, u) \sim \beta$ we get desired $\mathcal{Q}$ with $p' \sim p'' \in R(\mathcal{Q}, \psi, A)$ from $\mathcal{P}_1$. Otherwise, take an edge $e' = wv \in E$. If $\delta \sim \psi(e', w)$, then e' can play the same role as the above e, and if $\delta \not\sim \psi(e', w)$, then by e' we get desired $\mathcal{Q}$ with $(v, \alpha) \in R(\mathcal{Q}, \psi, A)$ for $\alpha = \psi(e', v) \cdot \delta' \notin \Gamma'$ from $\mathcal{P}_1$. □

Lemma 4.8. *Suppose that $G - F$ is connected. If $|A| - 2\mu(\Pi, \psi, A) > 1$, $V \cap A = D_1(\Pi, \psi, A)$ and $V \setminus A = D_2(\Pi, \psi, A)$, then one can find, in $O(|V|^2 \log |V|)$ time, $\mathcal{P} \in \Pi(G, \psi, A)$ with $\mathrm{val}(\mathcal{P}, \psi, A) = \mu(\Pi, \psi, A) + 1$.*

Proof. Fix $\mathcal{P}_1 \in \Pi$ with $\mathrm{val}(\mathcal{P}_1, \psi, A) = \mu(\Pi, \psi, A)$, and choose distinct vertices $u, v \in V$ such that $(u, \alpha), (v, \beta) \in R(\mathcal{P}_1, \psi, A)$ are made by half A-paths $P_1, P_2 \in \mathcal{P}_1$, respectively.

Case 1. Suppose that there exists an edge $e = uv \in E \setminus F$. If $\beta \not\sim \psi(e,v) \cdot \alpha$, then $P := (P_1, e, \bar{P}_2)$ is an admissible A-path disjoint from each path in $\mathcal{P}_1 \setminus \{P_1, P_2\}$, and hence $\mathcal{P} := \mathcal{P}_1 \Delta \{P_1, P_2, P\}$ is a desired A-collection. Otherwise, at least one of u, v is not a terminal. Suppose that u is a non-terminal and $(u, \gamma) \in \mathcal{R}(\Pi, \psi, A)$ with $\gamma \not\sim \alpha$. By Lemma 4.5, we can find $\mathcal{P}_2 \in \Pi(G, \psi, A)$ such that either $\mathrm{val}(\mathcal{P}_2, \psi, A) = \mathrm{val}(\mathcal{P}_1, \psi, A) + 1$, or $\mathrm{val}(\mathcal{P}_2, \psi, A) = \mathrm{val}(\mathcal{P}_1, \psi, A)$ and $(u, \gamma'), (v, \beta') \in R(\mathcal{P}_2, \psi, A)$ for some $\gamma' \sim \gamma$ and $\beta' \sim \beta$. Thus we have done (by the first argument in this case).

Case 2. Otherwise, choose one of the shortest paths in $G - F$ from u to v, and let w be an internal vertex. Because of $V = D(\Pi, \psi, A)$, there exists $\mathcal{P}_2 \in \Pi$ with $\mathrm{val}(\mathcal{P}_2, \psi, A) = \mu(\Pi, \psi, A)$ and $(w, \delta) \in R(\mathcal{P}_2, \psi, A)$ for some $\delta \in \Gamma$. Hence by Lemma 4.5, we can find $\mathcal{P}_3 \in \Pi(G, \psi, A)$ such that either $\mathrm{val}(\mathcal{P}_3, \psi, A) = \mathrm{val}(\mathcal{P}_1, \psi, A) + 1$, or $\mathrm{val}(\mathcal{P}_3, \psi, A) = \mathrm{val}(\mathcal{P}_1, \psi, A)$ and (w, δ') and one of $(u, \alpha'), (v, \beta')$ are in $R(\mathcal{P}_3, \psi, A)$ for some $\alpha' \sim \alpha$, $\beta' \sim \beta$, and $\delta' \sim \delta$. In the latter case, a shortest path from u or v to w is shorter than one from u to v. If we always choose w as the (nearly) middle point, then the number of this procedures is at most $\log_2 |V|$. Note that finding a shortest path can be done in $O(|V|^2)$ time by the breadth first search. □

Lemma 4.9. *If the condition of line 17 in Algorithm 1 holds, i.e., we have* $\mu(\tilde{\Pi}, \tilde{\psi}, \tilde{A}) = \mu(\Pi, \psi, A) - |X|$ *and* $D_i(\tilde{\Pi}, \tilde{\psi}, \tilde{A}) = D_i(\Pi, \psi, A)$ $(i = 1, 2)$*, then each connected component* K *of* $G - X - F$ *satisfies either*

- $2\mu(\tilde{\Pi}_K, \tilde{A}) = |V(K) \cap \tilde{A}|$*, or*
- $V(K) \cap \tilde{A} = D_1(\tilde{\Pi}_K, \tilde{A})$ *and* $V(K) \setminus \tilde{A} = D_2(\tilde{\Pi}_K, \tilde{A})$.

Proof. For each edge $e = uv \in E$ with $u \notin D(\Pi, \psi, A)$ and $v \in D(\Pi, \psi, A)$, we have either $u \in X$ or $e \in F$. Since there is no such edge in $G - X - F$, for each connected component K of $G - X - F$, we have either $V(K) \cap D(\tilde{\Pi}, \tilde{\psi}, \tilde{A}) = V(K) \cap D(\Pi, \psi, A) = \emptyset$ or $V(K) \subseteq D(\Pi, \psi, A) = D(\tilde{\Pi}, \tilde{\psi}, \tilde{A})$.

In the former case, obviously all terminals in K are covered by $\tilde{\Pi}_K$, and hence $2\mu(\tilde{\Pi}_K, \tilde{\psi}, \tilde{A}) = |V(K) \cap \tilde{A}|$. In the latter case, by the definition of $A' = \tilde{A}$, $v \in \tilde{A}$ if and only if $v \in D_1(\Pi, \psi, A) = D_1(\tilde{\Pi}, \tilde{\psi}, \tilde{A})$. Therefore, we have $V(K) \cap \tilde{A} = D_1(\tilde{\Pi}_K, \tilde{\psi}, \tilde{A})$ and $V(K) \setminus \tilde{A} = D_2(\tilde{\Pi}_K, \tilde{\psi}, \tilde{A})$. □

4.5. Running time analysis. The maximum size $\mu(G, \psi, A)$ of an A-packing is at most $|A|/2 = O(|V|)$, and the maximum value of $\varphi(\Pi, \psi, A) = |D_1(\Pi, \psi, A)| + 2|D_2(\Pi, \psi, A)| = O(|V|)$. In the main iteration (lines 2–33) of Algorithm 1, the value of $\mu(\Pi, \psi, A)$ does not decrease and that of $\varphi(\Pi, \psi, A)$ increases unless $\mu(\Pi, \psi, A)$ increases. Hence the number of executions of the main iteration is $O(|V|^2)$.

In each iteration step, Sub. 1, 2, 4 and 5 are called at most $|A' \setminus A| + |X| = O(|V|)$ times, and Sub. 3 is called exactly once. Moreover, in Sub. 1, 2, 4 and 5, Lemmas 4.4, 4.5, 4.6 and 4.7, respectively, are called once, and in Sub. 3, Lemma 4.8 is called once. Therefore, the total complexity of one iteration step is $O(|V|^3)$, which implies that the running time of the main part of this algorithm is $O(|V|^5)$.

Moreover, the algorithm has to compute the reachability w.r.t. Π, to do shifting, and to maintain A', F, and X. Computing the reachability can be done in $O(|V|^2)$ time since $|\Pi| = O(|V|)$ and each A-collection uses at most $|V| - 1$ edges. Unless $\mu(\Pi, \psi, A)$ increases, for each $v \in V \setminus A$, shifting at v is done at most twice and v is added to A' at most once, since v's state changes monotonically in the following order: in $V \setminus A'$, in $N_G(D(\Pi, \psi, A))$, in $D_1(\Pi, \psi, A)$, and in $D_2(\Pi, \psi, A)$. Therefore, maintaining A', F, and X takes totally $O(|E|)$ time (since the state of each edge

$e = uv \in E$ changes constant times) unless $\mu(\Pi, \psi, A)$ increases, and hence the running time of this maintenance part is $O(|E| \cdot |V|)$. Thus the total running time is $O\left((|E| + |V|^4) \cdot |V|\right)$.

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