MATHEMATICAL ENGINEERING
TECHNICAL REPORTS

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METR 2013–36 December 2013

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Agglomeration mechanism of long narrow economy: comparison with racetrack economy [1]

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Abstract

The mechanism of spatial agglomeration of a system of cities in a long narrow economy is elucidated through a comparative study with that in a racetrack economy, which serves as an idealized homogeneous space. Agglomeration economies are described by an analytically solvable core–periphery model, and spatial agglomerations are investigated by bifurcation theory and comparative static analysis with respect to transportation cost. When agglomeration forces are relatively high, both the long narrow and racetrack economies are shown to have a common fundamental mechanism of agglomeration: a spatial period doubling cascade, followed by concentration to a megalopolis and re-dispersion thereafter. The value of transport cost at the occurrence of the cascade of the long narrow economy is shown to be analytically predictable.

JEL: R12, R13, C65, F12

Keywords: bifurcation, core–periphery model, long narrow economy, racetrack economy, spatial agglomeration, spatial period doubling cascade

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[1]Support of Scientific Research (B) 19360227/21360240/ 24360202 from JSPS is appreciated.

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Preprint submitted to METR *December 23, 2013*

1. Introduction

Spatial concentration of economic activities has been the cradle of development and prosperity engendering industrial clusters and megalopolises worldwide. A mechanism of the evolution of spatial agglomeration was elucidated by the core–periphery model of Krugman (1991) [22], which provided a new framework to explain migration of population between places that occurs as a consequence of factor mobility and the trade-off between increasing returns at the level of firms and transport costs. Models of "Spatial economy" of various kinds have subsequently been developed in NEG (New Economic Geography).[6]

In the rise of NEG models, the two-city economy was extensively employed as a spatial platform, which is endowed with analytical tractability, to observe the core–periphery pattern of agglomeration triggered by bifurcation. Yet the space of this economy is too degenerated and, therefore, would be insufficient as a platform of spatial economy. In fact, a criticism has been raised that economic agglomerations, in reality, would take place at more than two locations.[7] In this manner, the importance of a proper spatial platform for core–periphery models has come to be acknowledged. More realistic spatial platforms that are common in the literature are

- A racetrack economy on the circumference of a circle.
- A long narrow economy (city) on a one-dimensional segment or a one-dimensional infinite space.

The racetrack economy presents a homogeneous trade space by realizing equal competition between places. Krugman (1993) [23] conducted a numerical analysis with 12 places to observe spatial agglomerations of central places roughly evenly spread across the landscape. By virtue of its homogeneity, the racetrack economy is endowed with much desired analytical tractability of the spatial agglomeration mechanism. Mossay (2003) [27] and Picard and Tabuchi (2010) [32] demonstrated the emergence of discrete agglomeration out of the uniformity. As

[6]For overviews of these models, see Brakman et al. (2001) [8] and Combes et al. (2008) [10].

[7]This criticism was stated by Behrens and Thisse (2007) [6], and was empirically evidenced by Bosker et al., 2010 [7].

a most characteristic and systematic course of agglomeration, the spatial period doubling cascade has been reported.[8]

For a long narrow economy, the evolution of cities around a mono-center in association with an increase of the economy's population size was demonstrated by Fujita and Mori (1997) [16] and Fujita, Krugman, and Mori (1999) [14]. A highly regular central place system *a la* Christaller and Lösch[9] was observed. Mori (1997) [26] showed "a formation of a megalopolis which consists of large core cities that are connected by *an industrial belt*, i.e., *a continuum of cities* associated with lower transport costs."

In the agglomeration patterns observed up to now in the two kinds of economies, difference[10] and diversity were emphasized. Nonetheless, there is an underlying belief that both economies would tend to display similarity for a large number of places. In fact, a similarity can be observed in the existence of spatial patterns with high spatial regularity in both economies (Mori (1997) [26] and Footnote 8).

That said, the objective of this paper is to disclose the similarity between these two kinds of economy that lies behind such diversify in their spatial patterns, and to clarify the agglomeration mechanism of the long narrow economy through a comparison with that of the racetrack economy. For this purpose, spatial agglomerations of these two economies are investigated theoretically, numerically, and comparatively using an analytically solvable core–periphery model by Forslid and Ottaviano (2003) [12]. Locations of places are discretized, and a bounded space on a line segment is employed for the long narrow economy. This long narrow economy is not endowed with homogeneity, as the places at a boundary are con-

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[8]Period doubling bifurcation from the uniformly distributed population engenders a state in which concentrating cities and extinguishing cities alternate along the circle for the racetrack economy with 2^k places (k is a positive integer). A repeated occurrence of such bifurcation is called period doubling bifurcation cascade. See, e.g., Tabuchi and Thisse (2011) [34], Ikeda, Akamatsu, and Kono (2012) [18], and Akamatsu, Takayama, and Ikeda (2012) [2].

[9]For central place theory, see Christaller (1933) [9] and Lösch (1940) [24].

[10]Such difference was emphasized in a comparative study of the two types of economies for Beckman's urban model (1976) [5] with social interactions that was conducted by Mossay and Picard (2011) [29]. It was concluded that a single city emerges in a long narrow economy, and multiple equilibria with an odd number of cities arise in the racetrack economy.

nected to a single neighboring place, and, hence, places near boundaries have a disadvantage in the transportation of goods. Nonetheless, this economy is closer to a realistic situation, and a study of agglomeration of this economy would give a hint at an economic implication of *agglomeration shadow*.[11] There is a trade-off in that the racetrack economy is not realistic but analytically tractable, while the long narrow economy is realistic but its investigation relies on numerical analysis. Theoretical and numerical information on agglomeration of the racetrack economy is employed to gauge the agglomeration of the long narrow economy.

As an essential contribution of this paper, "spatial period doubling cascade" is advanced as a common mechanism governing the spatial agglomerations of the two economies. This cascade can be clearly seen in the racetrack economy, but only for relatively high transport costs in the long narrow economy. The value of the transport cost at the onset of the first period doubling is analytically predictable, whereas the values of the second and further doublings are empirically deductible from the spatial agglomeration of the racetrack economy. In this sense, the former economy serves as an idealization of the latter economy. In addition, model dependence, as well as parameter value dependence, of spatial agglomerations is advanced by pointing out differences with studies in the literature, such as that of Mossay and Picard (2011) [29].

This paper is organized as follows. The governing equation for the analytically solvable core–periphery model is presented in Section 2. Bifurcations of the long narrow economy and the racetrack economy are described in Section 3. Spatial agglomerations of these two economies are investigated in Section 4. A comparative study of these behaviors is given in in Section 5. Influence of parameter values is studied in Section 6.

[11] Arthur (1990) [4] stated: "Locations with large numbers of firms therefore cast an 'agglomeration shadow' in which little or no settlement takes place. This causes separation of the industry." See also Fujita, Krugman, and Venables (1999) [15], Ioannides and Overman (2004) [21], and Fujita and Mori (2005) [17].

2. Core–Periphery Model

As a typical example of the core–periphery model, let us consider the analytically solvable core–periphery model by Forslid and Ottaviano (2003) [12] that replaces the production function of Krugman with that of Flam and Helpman (1987) [11]. The fundamental logic and governing equation of this model are presented in a simplified form here, whereas they are presented formally in Appendix A. The analytical solvability of this model plays a pivotal role in the study of break bifurcations (Section 5.3).

2.1. Basic assumptions

The economy of this model is composed of K places (labeled $i = 1, \dots, K$), two factors of production (skilled and unskilled labor), and two sectors (manufacturing, M, and agriculture, A). Both H skilled and L unskilled workers consume two final goods: manufacturing sector goods and agricultural sector goods. Workers supply one unit of each type of labor inelastically. Skilled workers are mobile among places, and the number of skilled workers in place i is denoted by λ_i ($\sum_{i=1}^{K} \lambda_i = H$). Unskilled workers are immobile and equally distributed across all places with unit density (i.e., $L = 1 \times K$).

Preferences U over the M- and A-sector goods are identical across individuals. The utility of an individual in place i is

$$U(C_i^{\mathrm{M}}, C_i^{\mathrm{A}}) = \mu \ln C_i^{\mathrm{M}} + (1 - \mu) \ln C_i^{\mathrm{A}} \qquad (0 < \mu < 1), \tag{1}$$

where μ is a constant parameter expressing the expenditure share of manufacturing sector goods, C_i^{A} is the consumption of the A-sector product in place i, and C_i^{M} is the manufacturing aggregate in place i, which is defined as

$$C_i^{\mathrm{M}} \equiv \left(\sum_j \int_0^{n_j} q_{ji}(\ell)^{(\sigma-1)/\sigma} d\ell \right)^{\sigma/(\sigma-1)},$$

where $q_{ji}(\ell)$ is the consumption in place i of a variety $\ell \in [0, n_j]$ produced in place j, n_j is the continuum range of varieties produced in place j, often called the number of available varieties, and $\sigma > 1$ is the constant elasticity of substitution between any two varieties. The budget constraint is given as

$$p_i^{\mathrm{A}} C_i^{\mathrm{A}} + \sum_j \int_0^{n_j} p_{ji}(\ell) q_{ji}(\ell) d\ell = Y_i, \tag{2}$$

where p_i^{A} is the price of A-sector goods in place i, $p_{ji}(\ell)$ is the price of a variety ℓ in place i produced in place j and Y_i is the income of an individual in place i. The incomes (wages) of skilled workers and unskilled workers are represented, respectively, by w_i and w_i^{L}.

The A-sector is perfectly competitive and produces homogeneous goods under constant-returns-to-scale technology, whereas the M-sector output is produced under increasing-returns-to-scale technology and Dixit-Stiglitz monopolistic competition. The transportation costs for M-sector goods are assumed to take the iceberg form. That is, for each unit of M-sector goods transported from place i to place j ($\neq i$), only a fraction $1/\phi_{ij} < 1$ arrives. More concretely, the transport cost ϕ_{ij} between places i and j is defined as $\phi_{ij} = \exp(\tau D_{ij})$, where τ is the transport cost parameter and D_{ij} represents the shortest transportation distance between places i and j. (We define $\phi_{ii} = 1$.)

The core–periphery model follows two stages of equilibria: (i) market (short-run) equilibrium that is defined as the economic state in which workers are assumed to be immobile between places, and (ii) spatial (long-run) equilibrium of the economic state for mobile workers.

2.2. Market equilibrium

In the short run, skilled workers are immobile between places, i.e., their spatial distribution $\boldsymbol{\lambda} = (\lambda_i)$ is assumed to be given. The market equilibrium conditions consist of the M-sector goods market clearing condition and the zero-profit condition because of the free entry and exit of firms.

As worked out in Appendix A.2, the market equilibrium wage $w_i(\boldsymbol{\lambda}, \tau)$ is determined from the equation

$$w_i(\boldsymbol{\lambda}, \tau) = \frac{\mu}{\sigma} \sum_{j=1}^{K} \frac{d_{ij}}{\Delta_j(\boldsymbol{\lambda}, \tau)} (w_j(\boldsymbol{\lambda}, \tau)\lambda_j + 1), \tag{3}$$

where $d_{ij} = \phi_{ij}^{1-\sigma}$ is a spatial discounting factor between places j and i, and $\Delta_j(\boldsymbol{\lambda}, \tau) = \sum_{k=1}^{K} d_{kj}\lambda_k$ denotes the market size of the M-sector in place j. The indirect utility $v_i(\boldsymbol{\lambda}, \tau)$, given the spatial distribution of the skilled workers, is obtained as

$$v_i(\boldsymbol{\lambda}, \tau) = S_i(\boldsymbol{\lambda}, \tau) + \ln[w_i(\boldsymbol{\lambda}, \tau)], \tag{4}$$

where $S_i(\boldsymbol{\lambda}, \tau) \equiv \mu(\sigma - 1)^{-1} \ln \Delta_i(\boldsymbol{\lambda}, \tau)$.

2.3. General form of spatial equilibrium conditions

Although diverse core–periphery models have been developed on the basis of an ensemble of economic principles and assumptions, it is possible to present a general form of spatial equilibrium.

Population λ_i of skilled workers at the ith place is chosen as an independent variable, and vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_K)^\top$ is defined. As is customary in comparative static analysis, the transport cost parameter τ is chosen as the main parameter.

In the description of the spatial equilibrium in core–periphery models, the adjustment dynamics

$$\frac{d\boldsymbol{\lambda}(t)}{dt} = \boldsymbol{F}(\boldsymbol{\lambda}(t), \tau) \tag{5}$$

is considered with some appropriate function $\boldsymbol{F}(\boldsymbol{\lambda}, \tau)$. A stationary point of this adjustment dynamics (5) is defined as $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\tau)$ that satisfies the spatial equilibrium condition

$$\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = \boldsymbol{0}. \tag{6}$$

The stability of a solution $\boldsymbol{\lambda}$ to (6) can be defined in relation to the associated dynamical system (5), and the solution is termed linearly stable if every eigenvalue of the Jacobian matrix $J(\boldsymbol{\lambda}, \tau) = \partial \boldsymbol{F}/\partial \boldsymbol{\lambda}$ has a negative real part, and linearly unstable if at least one eigenvalue has a positive real part.

As a specific functional form of $\boldsymbol{F}(\boldsymbol{\lambda}, \tau)$, we employ

$$\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = H\boldsymbol{P}(\boldsymbol{v}(\boldsymbol{\lambda}, \tau)) - \boldsymbol{\lambda}. \tag{7}$$

Here H is the total sum of the mobile population and $\boldsymbol{P}(\boldsymbol{v}) = (P_1, \ldots, P_K)^\top$ is the choice function vector, which is a function of the indirect utility function vector $\boldsymbol{v} = (v_1, \ldots, v_K)^\top$ and $\sum_{i=1}^K P_i = 1$.

For the equilibrium $(\boldsymbol{\lambda}, \tau)$ of (6), the conservation law

$$H = \sum_{i=1}^{K} \lambda_i \tag{8}$$

is satisfied. In the comparative static analysis conducted in this paper, the total number H of skilled workers is normalized as $H = 1$.

We employ the logit choice function $P_i = P_i(\boldsymbol{v})$ given by

$$P_i(\boldsymbol{v}) = \frac{\exp[\theta v_i]}{\sum_{j=1}^K \exp[\theta v_j]}, \tag{9}$$

where $\theta \in (0, \infty)$ is a positive parameter.[12] The adjustment process described by (5) and (7) with (9) is the logit dynamics.[13]

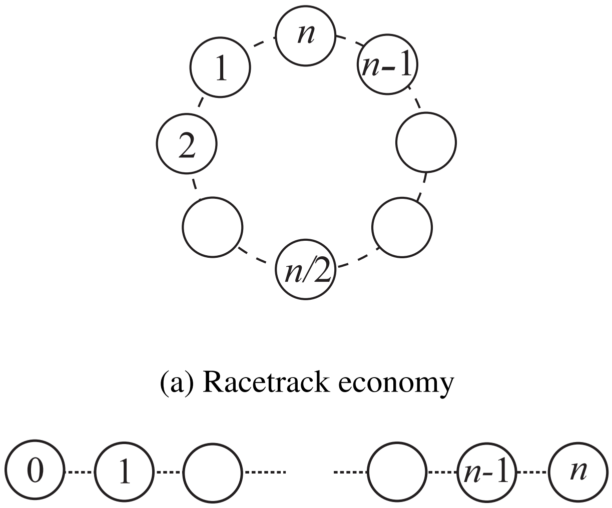
[12] In the spatial equilibrium, the skilled workers are assumed to be heterogeneous in their preferences for location choice; see, e.g., Tabuchi and Thisse (2002) [33], Murata (2003) [30], and Akamatsu, Takayama, and Ikeda (2012) [2]. The parameter θ in (9) denotes the inverse of variance of the idiosyncratic taste, which is assumed to follow the Gumbel distribution that is identical across places (e.g., McFadden, 1974 [25]; Anderson, de Palma, and Thisse, 1992 [3]).

[13] The logit dynamics has been studied in evolutionary game theory (e.g., Fudenberg and Levine, 1998 [13]).

3. Bifurcation of long narrow economy and racetrack economy

Bifurcation is the most characteristic phenomenon of the spatial economy, by which a set of identical places produces non-uniformly distributed places associated with a reduction of the transport cost. The geometry and associated bifurcation rule for a long narrow economy and a racetrack economy are presented as fundamentals of the study of agglomeration in the following sections. These two economies have different geometrical configurations, and, therefore, undergo different kinds of bifurcations.

3.1. Geometry of economic activity space



(a) Racetrack economy

(b) Long narrow economy

Figure 1: Spatial platforms for the economy

First, as a homogeneous counterpart of the long narrow economy, the economic space of the *racetrack economy* with n places (labeled $i = 1, \dots, n$) spread equally on the circumference of a circle in Fig. 1(a) is considered. Neighboring places are connected by a road of length $d = 1/n$, and the whole length of the road of the economy is given by 1. The number n of places is assumed to be even. All places are given identical economic environments and the same opportunity. In this sense, the racetrack economy serves as a homogeneous space for economic activities and is suitable for theoretical treatment by bifurcation theory.

Next, the economic space of the *long narrow economy* in Fig. 1(b) with $n + 1$ places (labeled $i = 0, 1, \ldots, n$) spread equally on a line segment is considered.[14] Neighboring places are connected by a road of length $d = 1/n$, and the whole length of the road of the economy is given by 1. A place at a boundary has an unfavorable transportation environment as this place is connected only to one place, whereas places inside are connected to two places. Nonetheless such non-uniformity would produce characteristic spatial agglomerations closer to a real situation.

3.2. Bifurcation and agglomeration of the racetrack economy

The racetrack economy has a larger symmetry, comprising bilateral and cyclic symmetries, than does the long narrow economy. The spatial agglomeration of the racetrack economy with this symmetry has been studied theoretically in the literature (Ikeda, Akamatsu, and Kono, 2012 [18]; Akamatsu, Takayama, and Ikeda, 2012 [2]; Ikeda and Murota, 2014 [20]), the theoretical result being introduced below. Consistently with numerical examples in Sections 4 and 5, the number n of places is assumed to be even.

The flat earth equilibrium (i.e., uniformly distributed state)

$$\boldsymbol{\lambda}^* = (1/n, \ldots, 1/n)^\top \tag{10}$$

exists, for any values of the transport cost parameter τ, as a pre-bifurcation state of equilibrium. The spatial period L is equal to $L = 1/n$, i.e., $L/d = 1$.

The agglomeration from the flat earth equilibrium proceeds only via bifurcation that breaks partial symmetry of the economy. Bifurcation takes place when one or two eigenvalues of the Jacobian matrix $J = J(\boldsymbol{\lambda}^*, \tau)$ become zero. The associated bifurcation point with a single zero eigenvalue is called a simple bifurcation point and that with two zero eigenvalues is called a double bifurcation point (see Ikeda and Murota, 2010 [19], 2014 [20] for the double point). Spatial agglomerations engendered by bifurcations are shown in Fig. 2, for example, for $n = 4$.

[14] This long narrow economy has the same number n of inter-place roads as the racetrack economy in Fig. 1(a). By preliminary numerical analyses for several number of places, it was found that long narrow and racetrack economies display similar spatial agglomerations when they have the same number of inter-place roads.

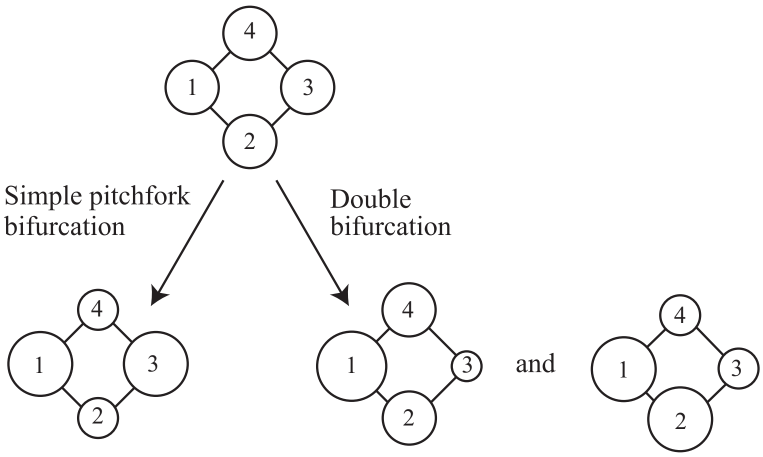


Figure 2: Agglomeration mechanism in racetrack economy for four places (size of the area represents population size)

3.2.1. *Simple pitchfork bifurcation point*

At a simple pitchfork[15] bifurcation point with n even (see the left of Fig. 2), a bifurcating equilibrium path exists in the direction of the critical eigenvector

$$\frac{1}{\sqrt{n}}(1, -1, \dots, 1, -1)^\top \tag{11}$$

associated with the single zero eigenvalue. This leads to an agglomeration to every other place corresponding to the spatial period doubling, i.e., $L/d = 1 \longrightarrow 2$.

3.2.2. *Spatial period doubling bifurcation cascade*

The racetrack economy with $n = 2^k$ places (k is some positive integer) can undergo a cascade of simple pitchfork bifurcations (Tabuchi and Thisse, 2011 [34]; Ikeda, Akamatsu, and Kono, 2012 [18]; Akamatsu, Takayama, and Ikeda, 2012 [2]) leading to successive doublings of the spatial period:

$$\frac{L}{d} = 1 \longrightarrow 2 \longrightarrow 4 \longrightarrow \cdots \longrightarrow n. \tag{12}$$

This is called *spatial period doubling bifurcation cascade*. Figure 3 depicts this cascade for four places.

[15]A simple bifurcation point of the racetrack economy is necessarily a pitchfork bifurcation point, which is either subcritical or supercritical. A subcritical one corresponds to the break bifurcation point for the two-place economy (Fujita, Krugman, and Venables, 1999 [15]), whereas supercritical ones are observed in the present analysis in Section 4.1.

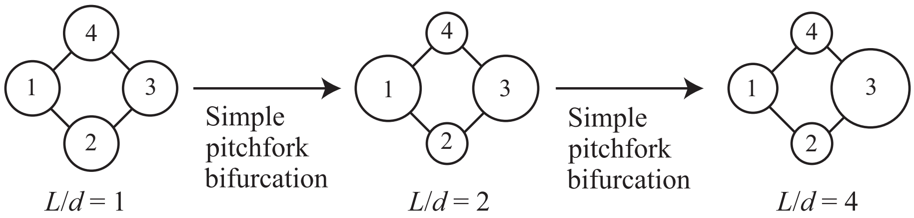


Figure 3: Spatial period doubling bifurcation cascade in racetrack economy for four places (size of an area represents population size)

3.3. Bifurcation and agglomeration of the long narrow economy

A symmetric system is known to lose its symmetry due to bifurcation. As a simple but famous example of such a phenomenon, we first refer to break bifurcation of the two-place economy studied by Krugman (1991) [22]. This economy serves as a special case of the long narrow economy with $n = 1$. A state $\boldsymbol{\lambda} = (1/2, 1/2)^\top$ of two identical places has bilateral symmetry in that the two places are exchangeable. This state is stable when the transport cost parameter τ is large. Such symmetry is broken by bifurcation to engender an unstable state of

$$\boldsymbol{\lambda} = (1/2 + \delta, 1/2 - \delta)^\top, \quad 0 < \delta < 1/2,$$

en route to a completely agglomerated state $\boldsymbol{\lambda} = (1, 0)^\top$ for a core–periphery pattern.

Next, let us consider the agglomeration in the long narrow economy with $n \geq 2$. As we see in numerical examples, the flat earth equilibrium $\boldsymbol{\lambda}^* = (1/n, \ldots, 1/n)^\top$ in (10) exists as a stable equilibrium in the limit of $\tau \to \infty$. The spatial period L among places with the same population is $L = 1/n$.

There are two theoretically possible courses of agglomeration thereafter.

- Migration of population among places that retains bilateral symmetry can proceed without undergoing bifurcation, as shown at the left of Fig. 4 for five places ($n = 4$).
- Bifurcation that breaks the bilateral symmetry, as shown at the right of Fig. 4.

In the numerical analysis in this paper, the agglomeration of population progressed without undergoing such bifurcation; accordingly, the bifurcation played literally no role.[16]

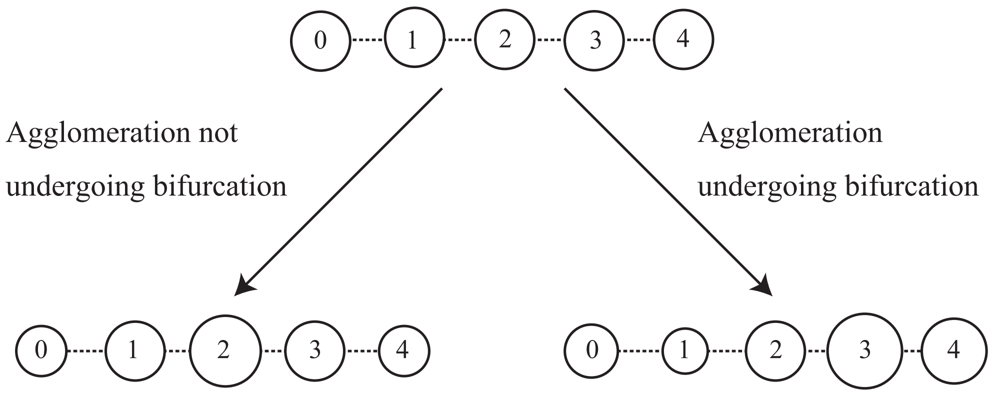


Figure 4: Agglomeration mechanism for long narrow economy with 5 places (size of the area represents population size)

[16]Such bifurcation, however, is theoretically possible and was observed for three places for a core–periphery model that assumes a quadratic utility function with linear transport costs (Ago, Isono, and Tabuchi, 2006 [1]).

4. Spatial Agglomerations of long narrow and racetrack economies

Spatial agglomerations of the long narrow and racetrack economies are observed by comparative static analysis for the core–periphery model of Forslid and Ottaviano (2003) [12] (Section 2). This section is devoted to a detailed investigation of the agglomerations in individual economies as a prelude to the comparative study of agglomerations of these economies in Section 5.

The agglomerations are dependent on the setting of the parameters of the core–periphery model as explained in detail in Section 6. The standard values of constant μ expressing the expenditure share of manufacturing sector goods, the constant elasticity σ of substitution between any two varieties, and the inverse θ of variance of the idiosyncratic taste in (9) are chosen to be

$$(\mu, \sigma, \theta) = (0.4, 10.0, 10000). \tag{13}$$

These parameter values satisfy the so-called no-black-hole condition (Fujita, Krugman, and Venables, 1999 [15]): $(\sigma - 1)/\sigma = 0.9 > \mu = 0.4$. Robustness analysis for some parameter values is presented in Section 6 and Appendix D.

4.1. Spatial agglomeration of the racetrack economy

The spatial agglomeration of the racetrack economy with 16 places is here investigated.[17] This economy has the flat earth equilibrium $\boldsymbol{\lambda}^* = (1/16, \ldots, 1/16)^\top$ in (10) for any values of the transport cost parameter τ as a pre-bifurcation equilibrium.

Figure 5(a) shows the curves of population $\lambda_{n/2}$ for place $n/2$ (= 8) plotted against transport cost parameter τ obtained by numerical analysis. The solid curves express stable equilibria and the dotted ones unstable ones.[18] The curve OAF of the flat earth equilibrium has several bifurcation points denoted by (■) for simple bifurcation and by (○) for double bifurcation. The curves of bifurcated equilibria AB, BC, and CD are shown in this figure and associated population distributions are presented in Fig. 5(b).

[17]A comparative study of this agglomeration with that for the long narrow economy with 17 places is conducted in Section 5.

[18]The stability of these equilibria is classified in accordance with the eigenvalues of the Jacobian matrix $J(\boldsymbol{\lambda}, \tau)$ (see Section 2.3).

Among a plethora of bifurcating equilibria, a stable progress of spatial agglomeration that takes place in association with a decrease of the transport cost parameter τ is of economic interest. The flat earth equilibrium OA (with an evenly distributed state) is stable when the transport cost parameter τ is sufficiently large. At the simple pitchfork bifurcation point A on this flat earth equilibrium, the spatial period doubling bifurcation ($L/d = 1 \longrightarrow 2$) takes place to engender the stable bifurcated equilibria AB with an agglomeration to every other place.[19]

As τ further decreases, another spatial period doubling is encountered at the simple bifurcation point B, which engenders the stable equilibria BC associated with an agglomeration to four equidistant places with identical populations. These successive bifurcations are the spatial period doubling bifurcation cascade in (12) for $n = 16$ that entails stable successive elongation[20] of spatial period as

$$\begin{array}{ccccccccc} \dfrac{L}{d} & = & 1 & \rightarrow & 2 & \rightarrow & 4 & \rightarrow & 8. \\ & & \text{OA} & \text{A} & \text{AB} & \text{B} & \text{BC} & \text{C} & \text{CD} \end{array}$$

As τ is further reduced beyond the point D, no continuation of stable equilibria exists, and a dynamical shift[21] to E leading to a complete agglomeration is expected to take place. In this stable progress of agglomeration, simple bifurcations play a pivotal role, whereas double bifurcations play literally no role.

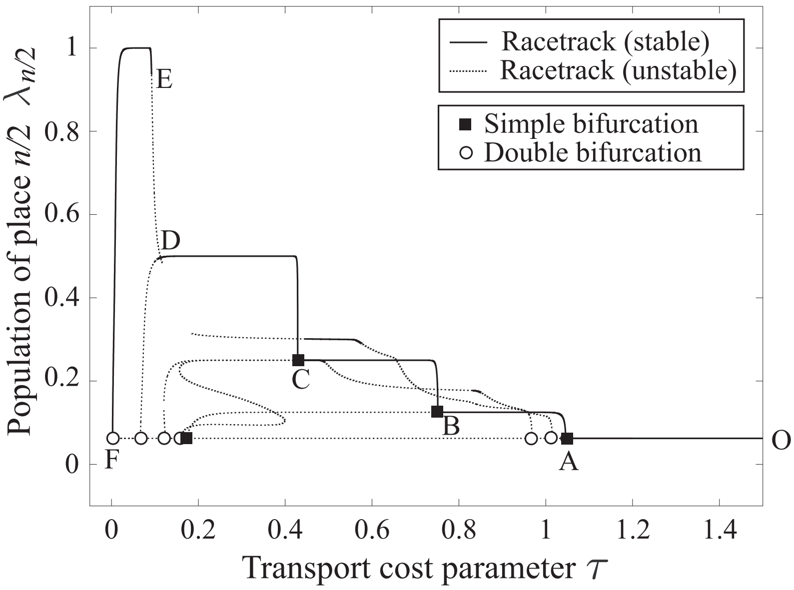
Thus, the stable spatial period doubling bifurcation cascade,[22] followed by a dynamical shift, is observed as a course of spatial agglomeration. Such course can be observed for other values of n (see Section 5.1 for $n = 64$) and, hence, can be

[19]It should be noted that the stability of equilibria observed above is dependent on modeling. For the agglomeration of the racetrack economy of the Krugman model, the first bifurcation was unstable (subcritical pitchfork or tomahawk) and the cascade did not progress stably en route to catastrophic change of agglomeration via dynamical shifts (Ikeda, Akamatsu, and Kono, 2012 [18]). By contrast, in the agglomeration observed here for the Forslid and Ottaviano model, the first bifurcation at point A is stable (supercritical pitchfork) and the cascade progresses stably.

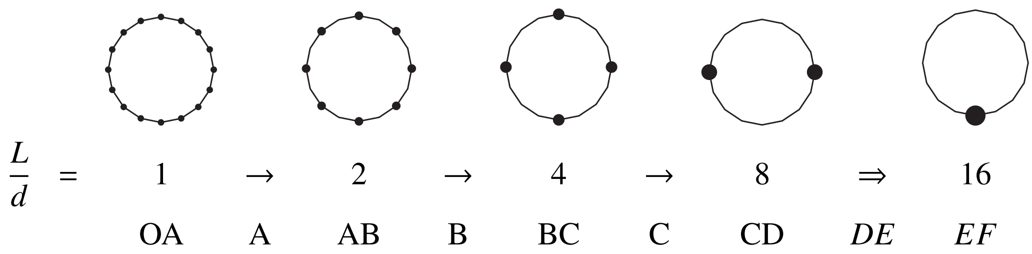
[20]The shift to a spatial agglomeration with a lower spatial frequency associated with the transport cost reduction was stated also in Proposition 2 in Mossay (2013) [28].

[21]When a stable equilibrium path becomes unstable at a critical point where a stable bifurcating equilibrium does not exist, the stable path often shifts dynamically to another stable path. This is called *dynamical shift*.

[22]Such a stable spatial period doubling bifurcation cascade was predicted by Akamatsu, Takayama, and Ikeda (2012) [2].



(a) The curves of population $\lambda_{n/2}$ for place $n/2$ (= 8) plotted against transport cost parameter τ



(b) Change of population distribution

Figure 5: Spatial agglomeration of the racetrack economy with 16 places (solid curves: stable; dotted curves: unstable; →: bifurcation; ⇒: dynamical shift)

advanced as a fundamental mechanism of spatial agglomeration. It is noteworthy that period doubling can be seen more clearly than in the long narrow economy dealt with in Section 4.2.

During the course of the cascade, even-numbered agglomerated places are observed, except for the mono-center. In contrast, the existence of an odd number of agglomerated places was declared in Lemma 4 of Mossay and Picard (2011) [29], which was conducted for Beckman's CBD formation model (1976) [5] for a continuous space without investigating the stability of equilibria.

4.2. Spatial agglomeration of the long narrow economy

The spatial agglomeration of a long narrow economy with 17 places is obtained by solving the nonlinear governing equation (6) of the core–periphery model. The change of the population distribution observed with a decrease of τ is expressed in the bar charts in Fig. 6. As τ decreases, the spatial agglomeration progresses without undergoing bifurcation as below.

- Uniformly distributed state ($1.5 < \tau \leq 5.0$): The population is almost evenly distributed in all places and the spatial period among these places is $L/d = 1$. Thereafter ($\tau \approx 1.5$), the population at the boundary places 0 and 16 migrate to their neighboring places 1 and 15, respectively.

- Spatial period doubling state I ($\tau \approx 0.8$): The population increases at seven places $2, 4, \ldots, 14$ and decreases at seven places $1, 3, \ldots, 15$. At $\tau = 0.8$, the population at places $1, 3, \ldots, 15$ almost disappears, and the spatial period between the agglomerated places is doubled to $L/d = 2$.

- Agglomeration to five places ($\tau \approx 0.6$): The population is agglomerated to five unevenly spread places.

- Spatial period doubling state II ($\tau \approx 0.4$): The population is agglomerated to a triplet of places, i.e., places 4, 8, and 12 and the spatial period between agglomerated places is doubled to $L/d = 4$ in comparison with that in the period doubling state I.

- Completely agglomerated state ($\tau \approx 0.3$): The population is completely agglomerated to the mono-center at place 8.

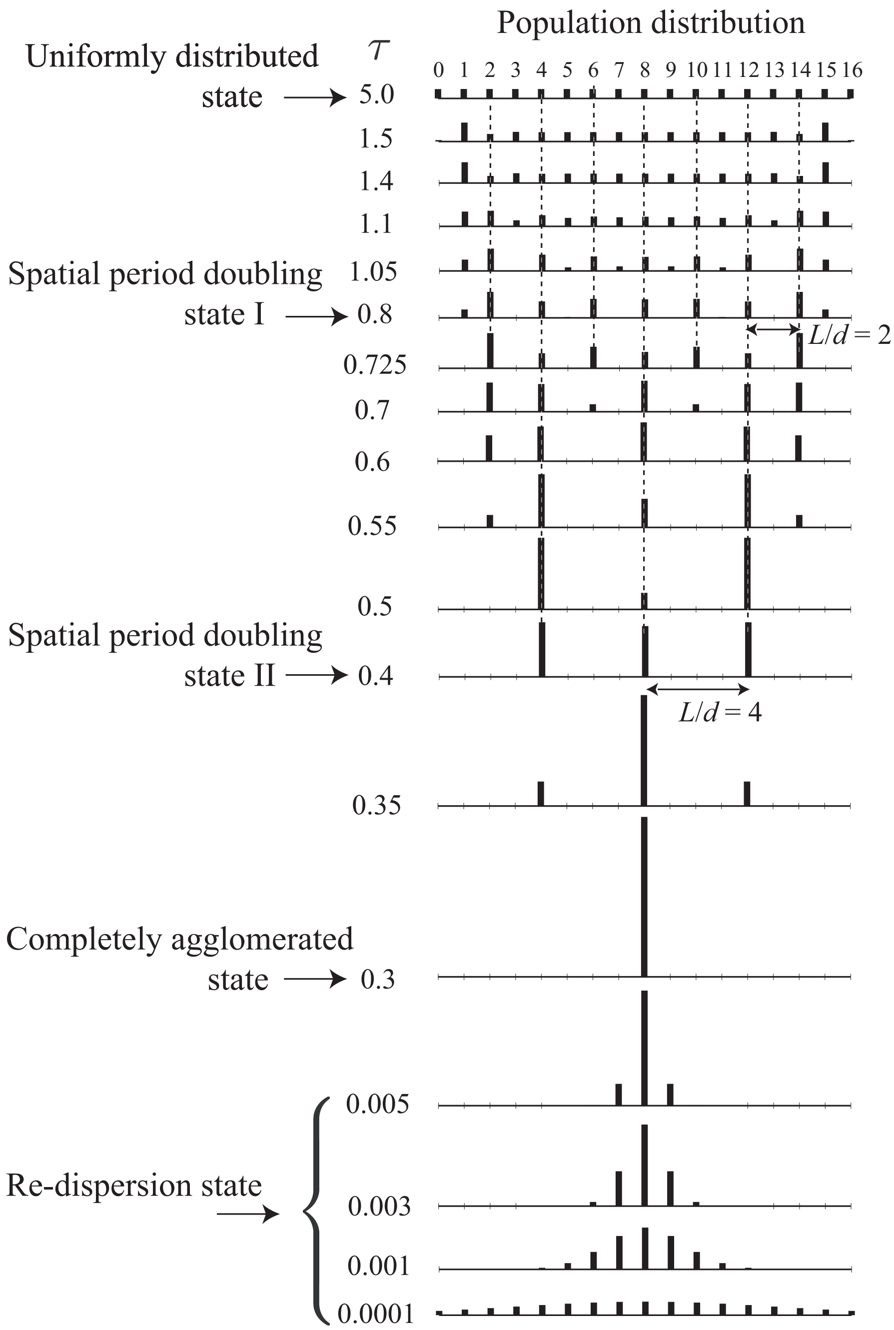


Figure 6: Spatial agglomeration of the long narrow economy with 17 places observed in association with the decrease of the transport cost parameter τ

- Re-dispersion state ($0.0001 \leq \tau \leq 0.005$): The peak becomes flatter by re-dispersion and arrives at an almost uniform state at $\tau \approx 0.0001$.

To sum up, the number of agglomerated places decreases as $7 \longrightarrow 5 \longrightarrow 3 \longrightarrow 1$ en route to complete agglomeration at a mono-center. Repeated spatial period doublings $L/d = 1 \longrightarrow 2 \longrightarrow 4$ are thus observed. Such doublings are also observed for 65 places (see Section 5.2) and, hence, are advanced as a fundamental mechanism of spatial agglomeration.

This is in line with the study of Fujita, Krugman, and Mori (1999) [14], in which "a highly regular hierarchical system *a la* Christaller" for a core–periphery model related to the model of Krugman (1991) [22] was found. Such system was seen clearly in a two-dimensional economic space in Ikeda and Murota (2014) [20].

The spatial agglomeration observed above indicates the existence of multiple cities, as well as a mono-center. This is in sharp contrast with the study of Mossay and Picard (2011) [29], in which the existence of multiple cities was denied without resort to the stability of equilibria. It is, therefore, vital in the study of the pattern of spatial agglomeration to acknowledge its dependence on modeling.

5. Comparative study of spatial agglomerations of two economies

By a series of preliminary numerical analyses for different numbers of places, it was found that the spatial agglomeration of a long narrow economy with $n + 1$ places is comparable to that of a racetrack economy with n places; note that these two economies have the same number n of roads connecting neighboring places. In this section, a comparative study of these two economies is conducted for $n = 16$ and 64, for which the spatial period doubling bifurcation cascade is predominant in the racetrack economy. Although these two economies have different kinds of bifurcation mechanisms (Section 3), their geometrical configurations are alike in that places are located equidistantly or continuously and display similar agglomerations sufficiently away from boundaries.

5.1. Case 1: $n = 16$

The spatial agglomeration of the long narrow economy with 17 places is investigated in comparison with that of the racetrack economy with 16 places; recall Section 4 for the agglomerations of individual economies. Figure 7(a) shows the curves of population $\lambda_{n/2}$ for place $n/2$ (= 8) plotted against transport cost parameter τ.

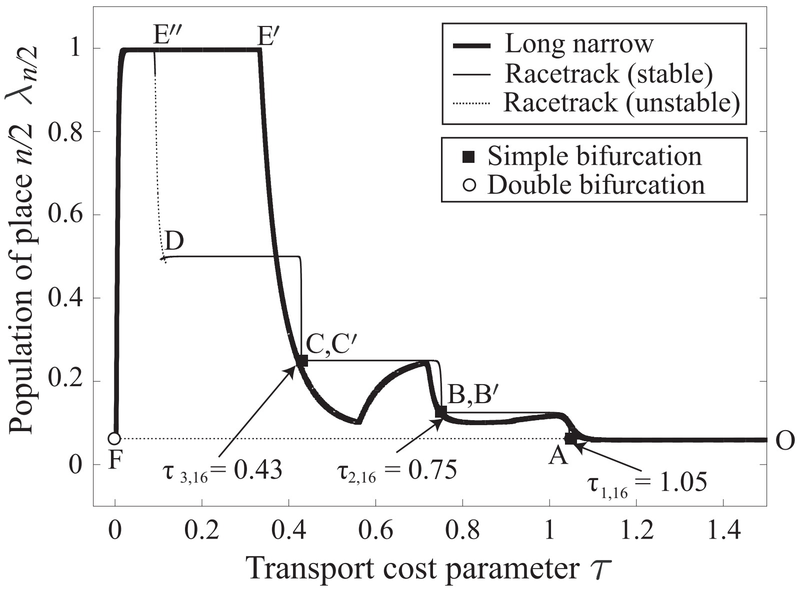
In the first stage, the bold curve for the long narrow economy OB′C′ closely follows thin curve OAB and point C of the racetrack economy; here, the points A, B, and C correspond to pitchfork bifurcation points of the racetrack economy, at which the spatial period L is doubled successively (see Section 3.2.2 for theoretical outline). Moreover, spatial agglomerations of both economies, which are illustrated comparatively for several values of the transport cost parameter τ in Fig. 7(b), display the occurrence of a spatial period doubling cascade:

$$\frac{L}{d} \;=\; 1 \;\rightarrow\; 2 \;\rightarrow\; 4.$$

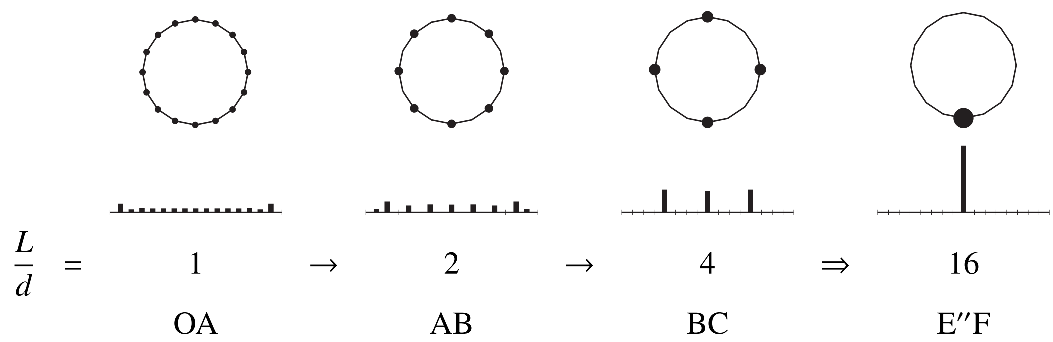
This shows a similarity to the spatial agglomerations of these economies, and, therefore, the role of the racetrack economy as an idealization of the long narrow economy.

Let us observe the value, which is termed *break point*,[23] of the transport cost parameter at the onset of the spatial period doublings. Its value at the mth b-

[23] For the two-place economy, a decrease of the transport cost to the *break point*, at which



(a) The curves of population $\lambda_{n/2}$ for place $n/2$ (= 8) plotted against transport cost parameter τ



(b) Change of population distribution

Figure 7: Comparison of the spatial agglomerations of the long narrow economy with 17 places and of the racetrack economy with 16 places (thin curves: racetrack economy; bold curves: long narrow economy)

ifurcation in this cascade for n places is denoted by $\tau_{m,n}$. In the present case, $\tau = \tau_{1,16} = 1.05$ at the first bifurcation point A and $\tau = \tau_{2,16} = 0.75$ at the second bifurcation point B in the racetrack economy can be employed as indexes for the initiation of agglomeration behavior change in the long narrow economy. Moreover, these values are analytically predictable for the present analytically solvable model, as explained in Section 5.3.

In the intermediate stage (C′E′E″), these two economies display significantly different kinds of agglomerations as evident from the quite different behaviors of the bold and thin curves in Fig. 7(a). In the last stage (E″F), the two economies again display similar agglomerations associated with the formation of a mono-center and re-dispersion thereafter.

5.2. Case 2: $n = 64$

The similarity between the agglomerations of the two kinds of economies can be seen more clearly in the comparison of the long narrow economy with 65 places and the racetrack economy with 64 places shown in Fig. 8 ($n = 64$).

The thin curve of the racetrack economy traces a stable course OGHIJKL of agglomeration via a spatial period doubling bifurcation cascade occurring at pitchfork bifurcation points G, H, I, J, and K. This cascade is followed by a dynamical shift to a stable state at point L to M leading to complete agglomeration thereafter and re-dispersion at N.

The bold curve of the long narrow economy accurately traces the curve OGHII′ of the racetrack economy. Moreover, spatial distributions of both economies in Fig. 8(a) display the occurrence of spatial period doubling cascade

$$\frac{L}{d} = 1 \rightarrow 2 \rightarrow 4 \rightarrow 8.$$

As shown in an enlarged view in Fig. 8(b), the break points $\tau_{1,64}$, $\tau_{2,64}$, and $\tau_{3,64}$ serve as excellent indexes for the prediction of the phase changes in the long narrow economy.

In the intermediate stage between I′M′, the spatial agglomerations of the two

symmetric places change catastrophically into a core–periphery pattern, is highlighted as a key concept (Fujita, Krugman, and Venables, 1999 [15]).

economies are quite different.[24] In the stage of formation of a mono-center and re-dispersion (M′N), the two economies display similar spatial agglomerations. Thus the similarity of the agglomerations of the two economies is enhanced as the number of places increases.

5.3. Laws for break points

As explained in Section 5.1, the values $\tau_{m,n}$ of break bifurcations for the race-track economy with n even serve as characteristic indexes for the progress of the spatial agglomeration in the long narrow economy. The value $\tau_{1,n}$ at the occurrence of the first period doubling is analytically predictable by the following law (see Appendix C for derivation)

$$\tau_{1,n} = c_1 n \tag{14}$$

with a constant

$$c_1 = \frac{1}{2\pi(\sigma - 1)} \log\left(\frac{1 + \sqrt{\alpha^*}}{1 - \sqrt{\alpha^*}}\right)$$

that is expressed implicitly as a nonlinear function in μ, σ, and θ.

For the parameter values $(\mu, \sigma, \theta) = (0.4, 10.0, 10000)$ in (13), the constant c_1 becomes

$$c_1 = 0.065.$$

The use of this value in (14) gives theoretically predicted values as

$$\tau_{1,16} = 1.05, \quad \tau_{1,32} = 2.11, \quad \tau_{1,64} = 4.22. \tag{15}$$

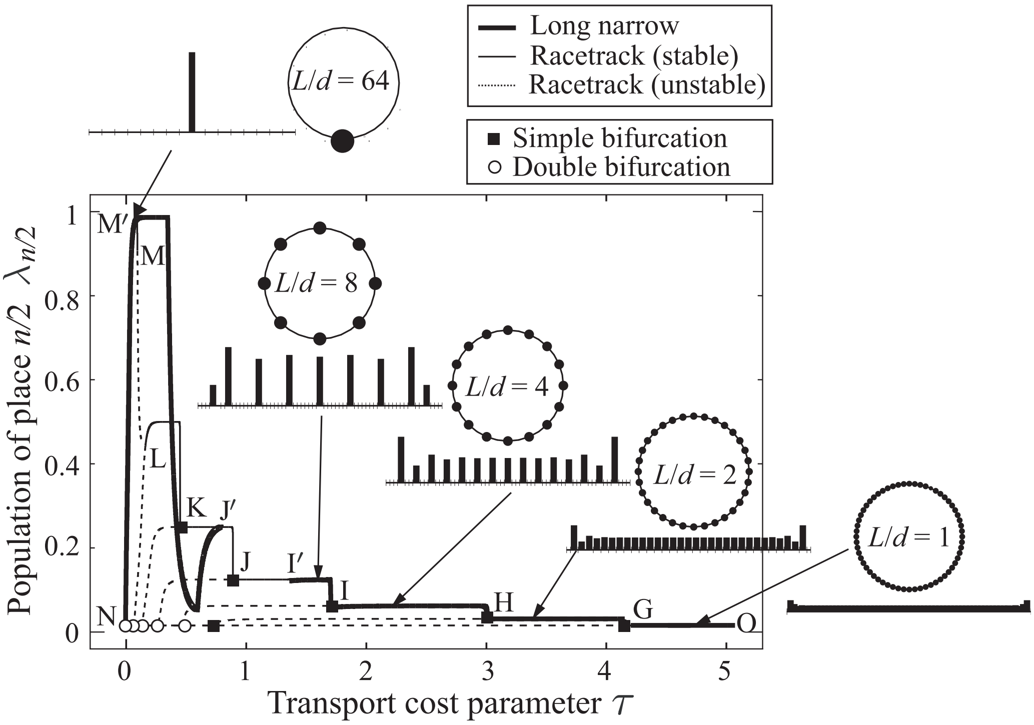
As the number n is doubled successively, the value of $\tau_{1,n}$ is doubled accordingly.

For comparison, the computationally obtained values

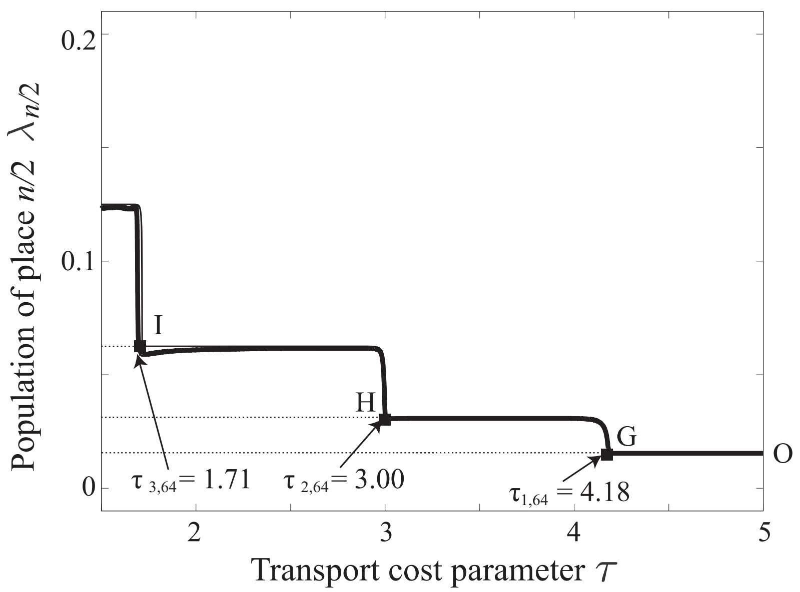
$$\tau_{1,16} = 1.05, \quad \tau_{1,32} = 2.09, \quad \tau_{1,64} = 4.18$$

are very close to the theoretical ones in (15). This suffices to ensure the validity of the law (14).

[24] The curve between I′J′ is not shown in Fig. 8, as it forms a number of loops and is too complex (see Appendix B).



(a) The curves of population $\lambda_{n/2}$ for place $n/2$ (= 32) plotted against transport cost parameter τ



(b) Enlarged view of (a)

Figure 8: Comparison of the spatial agglomerations of the long narrow economy with 65 places and those of the racetrack economy with 64 places (thin curves: racetrack economy; bold curves: long narrow economy)

Table 1: Relation between $\tau_{m,n}$ and n obtained computationally

n	16	32	64
$\tau_{1,n}$	1.05	2.09	4.18
$\tau_{2,n}$	0.75	1.50	3.00
$\tau_{3,n}$	0.43	0.86	1.71
$\tau_{4,n}$	Non-existent	0.44	0.89

In addition, the doublings of $\tau_{m,n}$ for $n = 16$, 32, and 64 are observed empirically for $m \geq 2$ as listed in Table 1. Thus we can arrive at an extended law

$$\tau_{m,n} = c_m n, \tag{16}$$

where c_m is a constant, which is given for the present case as

$$c_2 = 0.047, \quad c_3 = 0.027, \quad c_4 = 0.014.$$

The formula (16) should be of great assistance in the prediction of the spatial agglomeration for a large number n of places with reference to the analysis of a small number of places. Thus, the values of the transport cost parameter at the second and further doublings are empirically deductible from the spatial agglomeration of the racetrack economy.

6. Influence of parameter values

The relative predominance of centripetal forces promoting agglomeration and centrifugal forces engendering dispersion depends on the values of the parameters in core–periphery models (see Section 2). Accordingly, parameter dependence of spatial agglomerations, as well as model dependence, is a vital factor in spatial agglomerations in core–periphery models.

The influence of the change of the expenditure share μ of manufactured goods on spatial agglomerations is investigated, whereas that of other parameters σ and θ is investigated in Appendix D. It is to be noted that a larger value of μ enhances the relative influence of the role of manufactured goods, thereby increasing agglomeration forces. We set $(\sigma, \theta) = (10.0, 10000)$ and investigate the influence of μ by changing its value as $\mu = 0.1$, 0.4, and 0.7 to arrive at the bar chart of the population distributions shown in Fig. 9, where $\mu = 0.4$ corresponds to the standard case in Fig 6.

For $\mu = 0.1$, population is continuously distributed among places and gradually agglomerates to the place at the center as τ decreases. Such spatial agglomeration is close to the one found by Mori (1997) [26]: "a formation of a megalopolis which consists of large core cities that are connected by *an industrial belt*, i.e., *a continuum of cities*."

On the other hand, for $\mu = 0.4$ and $\mu = 0.7$, agglomerated places are discrete, and the number of agglomerated places decrease $7 \longrightarrow 5 \longrightarrow 3 \longrightarrow 1$ en route to a complete agglomeration as τ decreases. Increased competition among places leads to the growth of several discretized places and the extinction of population at neighboring places. The distance of agglomerated places increases from $L/d = 4$ to 5 in association with the increase of μ from 0.4 to 0.7 that enhances agglomeration forces. This shows the development of an agglomeration shadow due to the increase of agglomeration forces. Such spatial agglomeration is close to the highly regular central place system *a la* Christaller found by Fujita and Mori (1997) [16] and Fujita, Krugman, and Mori (1999) [14].

As we have seen in this section, the spatial agglomeration is parameter value dependent and can encompass several different kinds of spatial agglomeration patterns in the literature. A more extensive investigation of this dependence will be a topic in the future.

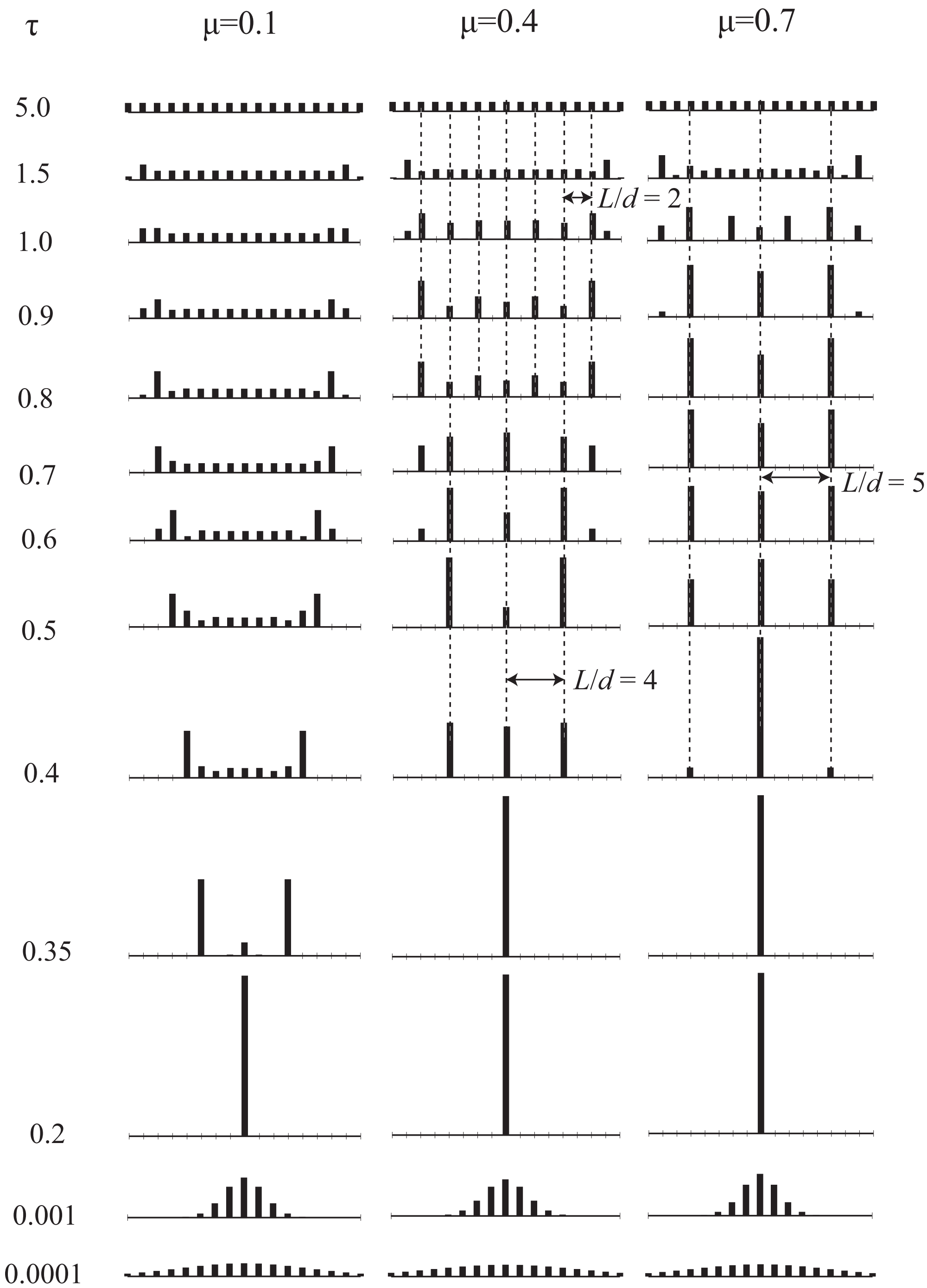


Figure 9: Influence of μ on the spatial agglomeration of the long narrow economy ($(\sigma, \theta) = (10.0, 10000)$)

7. Conclusion

The spatial agglomerations of the long narrow economy and the racetrack economy were studied by theoretical and numerical analyses. The racetrack economy is regarded as an idealized spatial platform of a homogeneous trade space. By virtue of this homogeneity, an underlying mechanism of the agglomeration of this economy can be described theoretically. On the other hand, the long narrow economy is closer to a realistic situation, but does not possess homogeneity in the strict sense and its theoretical analysis is difficult. Nonetheless, this economy with a large number of places appears to be quasi-homogeneous away from boundaries.

Early spatial agglomerations of the two economies for a large transportation cost are dominated by the common mechanism of the spatial period doubling cascade, especially for a large number of places. Such commonality makes the theoretical and empirical results for the racetrack economy applicable to the long narrow economy. The value of the transport cost at the onset of the first period doubling is analytically predictable, whereas the values of the second and further doublings are empirically deductible from the spatial agglomeration of the racetrack economy.

The spatial agglomeration of the long narrow economy was found to be dependent on the values of the parameters for the core–periphery model. When the agglomeration force is large, the spatial agglomeration is close to the one found by Mori (1997) [26]: "a formation of a megalopolis which consists of large core cities that are connected by *an industrial belt*, i.e., *a continuum of cities*." When the agglomeration force is small, the spatial agglomeration is close to the highly regular central place system *a la* Christaller found by Fujita and Mori (1997) [16] and Fujita, Krugman, and Mori (1999) [14]. Such parameter dependence, as well as model dependence, would be a vital factor to be considered in the future study of spatial agglomerations. A proper consideration of such a factor would be vital to the understanding of spatial agglomerations that display diverse aspects.

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Appendix A. Market equilibrium of core–periphery model

The core–periphery model of Forslid and Ottaviano (2003) [12] is described.

Appendix A.1. Basic assumptions

An individual in place i maximizes the utility in (1) subject to (2). This yields the following demand functions:

$$C_i^{\mathrm{A}} = (1-\mu)\frac{Y_i}{p_i^{\mathrm{A}}}, \quad C_i^{\mathrm{M}} = \mu\frac{Y_i}{\rho_i}, \quad q_{ji}(\ell) = \mu\frac{\rho_i^{\sigma-1} Y_i}{p_{ji}(\ell)^{\sigma}}, \tag{A.1}$$

where ρ_i denotes the price index of the differentiated product in place i, which is

$$\rho_i = \left(\sum_j \int_0^{n_j} p_{ji}(\ell)^{1-\sigma} d\ell \right)^{1/(1-\sigma)}. \tag{A.2}$$

Since the total income and population in place i are $w_i\lambda_i + w_i^{\mathrm{L}}$ and $\lambda_i + 1$, respectively, we have the total demand $Q_{ji}(\ell)$ in place i for a variety ℓ produced in place j:

$$Q_{ji}(\ell) = \mu\frac{\rho_i^{\sigma-1}}{p_{ji}(\ell)^{\sigma}}(w_i\lambda_i + w_i^{\mathrm{L}}), \tag{A.3}$$

The A-sector is perfectly competitive and produces homogeneous goods under constant-returns-to-scale technology, which requires one unit of unskilled labor in order to produce one unit of output. For simplicity, we assume that the A-sector goods are transported between places without transportation cost and that they are chosen as the numéraire. These assumptions mean that, in equilibrium, the wage of an unskilled worker w_i^{L} is equal to the price of A-sector goods in all places (i.e., $p_i^{\mathrm{A}} = w_i^{\mathrm{L}} = 1$ for each $i = 1, \ldots, K$).

The M-sector output is produced under increasing-returns-to-scale technology and Dixit-Stiglitz monopolistic competition. A firm incurs a fixed input requirement of α units of skilled labor and a marginal input requirement of β units of unskilled labor. That is, a linear technology in terms of unskilled labor is assumed in the profit function. Given the fixed input requirement α, the skilled labor market clearing implies $n_i = \lambda_i/\alpha$ in equilibrium. An M-sector firm located in place i chooses $(p_{ij}(\ell) \mid j = 1, \ldots, K)$ that maximizes its profit

$$\Pi_i(\ell) = \sum_j p_{ij}(\ell)Q_{ij}(\ell) - (\alpha w_i + \beta x_i(\ell)),$$

where $x_i(\ell)$ is the total supply.

Recall that the transportation costs for M-sector goods are assumed to take the iceberg form. That is, for each unit of M-sector goods transported from place i to place j ($\neq i$), only a fraction $1/\phi_{ij} < 1$ arrives ($\phi_{ii} = 1$). Consequently, the total supply $x_i(\ell)$ is given as

$$x_i(\ell) = \sum_j \phi_{ij} Q_{ij}(\ell). \tag{A.4}$$

Since we have a continuum of firms, each firm is negligible in the sense that its action has no impact on the market (i.e., the price indices). Therefore, the first-order condition for profit maximization yields

$$p_{ij}(\ell) = \frac{\sigma\beta}{\sigma - 1}\phi_{ij}. \tag{A.5}$$

This expression implies that the price of the M-sector products does not depend on variety ℓ, so that $Q_{ij}(\ell)$ and $x_i(\ell)$ do not depend on ℓ. Therefore, the argument ℓ is suppressed in the sequel. Substituting (A.5) into (A.2), we have the price index

$$\rho_i = \frac{\sigma\beta}{\sigma - 1}\left(\frac{1}{\alpha}\sum_j \lambda_j d_{ji}\right)^{1/(1-\sigma)}, \tag{A.6}$$

where $d_{ji} = \phi_{ji}^{1-\sigma}$ is a spatial discounting factor between places j and i; from (A.3) and (A.6), d_{ji} is obtained as $(p_{ji}Q_{ji})/(p_{ii}Q_{ii})$, which means that d_{ji} is the ratio of total expenditure in place i for each M-sector product produced in place j to the expenditure for a domestic product.

Appendix A.2. Market equilibrium

In the short run, skilled workers are immobile between places, i.e., their spatial distribution $\boldsymbol{\lambda} = (\lambda_i)$ is assumed to be given. The market equilibrium conditions consist of the M-sector goods market clearing condition and the zero-profit condition because of the free entry and exit of firms. The former condition can be written as (A.4). The latter condition requires that the operating profit of a firm be absorbed entirely by the wage bill of its skilled workers:

$$w_i(\boldsymbol{\lambda}, \tau) = \frac{1}{\alpha}\left\{\sum_j p_{ij} Q_{ij}(\boldsymbol{\lambda}, \tau) - \beta x_i(\boldsymbol{\lambda}, \tau)\right\}. \tag{A.7}$$

Substituting (A.3), (A.4), (A.5), and (A.6) into (A.7), we have the market equilibrium wage:

$$w_i(\boldsymbol{\lambda}, \tau) = \frac{\mu}{\sigma} \sum_j \frac{d_{ij}}{\Delta_j(\boldsymbol{\lambda}, \tau)} (w_j(\boldsymbol{\lambda}, \tau)\lambda_j + 1), \tag{A.8}$$

where $\Delta_j(\boldsymbol{\lambda}, \tau) \equiv \sum_k d_{kj}\lambda_k$ denotes the market size of the M-sector in place j. Consequently, $d_{ij}/\Delta_j(\boldsymbol{\lambda}, \tau)$ defines the market share in place j of each M-sector product produced in place i.

The indirect utility $v_i(\boldsymbol{\lambda}, \tau)$, given the spatial distribution of the skilled workers, is obtained by substituting (A.1), (A.6), and (A.8) into (1) and by setting $S_i(\boldsymbol{\lambda}, \tau) \equiv \mu(\sigma - 1)^{-1} \ln \Delta_i(\boldsymbol{\lambda}, \tau)$:

$$v_i(\boldsymbol{\lambda}, \tau) = S_i(\boldsymbol{\lambda}, \tau) + \ln[w_i(\boldsymbol{\lambda}, \tau)]. \tag{A.9}$$

Appendix B. Loop behavior of long narrow economy

An interesting complicated behavior of the long narrow economy is investigated for an increased number n of places. Figure B.1 shows equilibrium paths for $n = 21$. As τ decreases from a large value, a uniformly distributed state at point A shifts into a state of spatial period doubling at point B. During $0.2 \leq \tau \leq 1.0$, these paths form several loops and become stable and unstable repeatedly.[25] Such loops are observed for $n \geq 19$, and make the behavior increasingly complicated in association with an increase of the place number n.

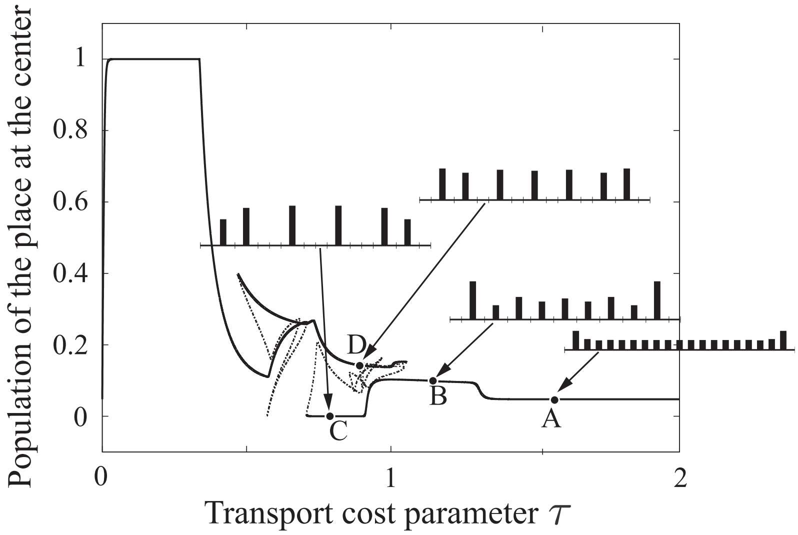

Figure B.1: Loop behavior of the long narrow economy with $n = 21$ (solid curves: stable; dotted curves: unstable)

[25]In these loops, a spatial period tripling mode with $L/d = 3$ is observed at point D.

Appendix C. Proof of the formula for break points in the racetrack economy

The formula (16) for break points for the pitchfork bifurcation in the racetrack economy with n even can be derived on the basis of Akamatsu, Takayama, and Ikeda (2012) [2], in which the value of the transport cost parameter τ at the onset of the spatial period doubling bifurcation has been studied analytically.

Define the spatial discounting matrix $D = (d_{ij})$, which represents spatial discounting for pairs of places, by

$$d_{ij} = r(\tau)^{m(i,j)},$$

where

$$\begin{aligned} r(\tau) &= \exp\left[-(\sigma - 1)\tau\frac{2\pi}{n}\right], && \text{(C.1)} \\ m(i, j) &= \min\{|i - j|,\ n - |i - j|\}. && \text{(C.2)} \end{aligned}$$

Then consider a normalized spatial discounting matrix D/d with d denoting the sum of the entries of the first row of D. This matrix D/d is a circulant matrix and the eigenvalue α for the eigenvector

$$\boldsymbol{\eta} = \frac{1}{\sqrt{n}}(1, -1, \dots, 1, -1)^{\top} \tag{C.3}$$

in (11) is evaluated in Akamatsu, Takayama, and Ikeda (2012) [2] as

$$\alpha = \left(\frac{1 - r(\tau)}{1 + r(\tau)}\right)^2. \tag{C.4}$$

From the governing equation $\boldsymbol{F}$ in (7) with $H = 1$, we have

$$\begin{aligned} \frac{\partial F_i}{\partial \lambda_j} &= \sum_{k=1}^{n} \frac{\partial F_i}{\partial v_k}\frac{\partial v_k}{\partial \lambda_j} - \delta_{ij} \\ &= -\theta \sum_{k=1}^{n} P_i P_k \frac{\partial v_k}{\partial \lambda_j} + \theta P_i \frac{\partial v_i}{\partial \lambda_j} - \delta_{ij}, \end{aligned}$$

where δ_{ij} is the Kronecker delta. Then the Jacobian matrix $\partial \boldsymbol{F}/\partial \boldsymbol{\lambda}$ is expressed as

$$\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{\lambda}}(\boldsymbol{\lambda}) = -\theta \begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix} \begin{bmatrix} P_1 \cdots P_n \end{bmatrix} \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\lambda}}(\boldsymbol{\lambda}) + \theta \begin{bmatrix} P_1 & & \\ & \ddots & \\ & & P_n \end{bmatrix} \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\lambda}}(\boldsymbol{\lambda}) - I, \tag{C.5}$$

where I is the identity matrix and $\partial \boldsymbol{v}/\partial \boldsymbol{\lambda}$ denotes the Jacobian matrix of $\boldsymbol{v}$ with respect to $\boldsymbol{\lambda}$.

The differentiations of v_i in (A.9) with respect to λ_j and the implicit relation (A.8) for w_i with respect to λ_l yield respectively

$$\frac{\partial v_i}{\partial \lambda_j} = \kappa' \frac{d_{ji}}{\Delta_i} + \frac{1}{w_i}\frac{\partial w_i}{\partial \lambda_j}, \tag{C.6}$$

$$\frac{\partial w_i}{\partial \lambda_l} = \kappa \sum_{j=1}^{n} \frac{d_{ij}}{\Delta_j^2}\left[\left(\frac{\partial w_j}{\partial \lambda_l}\lambda_j + w_j \delta_{jl}\right)\Delta_j - (w_j\lambda_j + 1)d_{lj}\right] \tag{C.7}$$

with $\Delta_j = \sum_{k=1}^{n} d_{kj}\lambda_k$ and

$$\kappa = \frac{\mu}{\sigma}, \quad \kappa' = \frac{\mu}{\sigma - 1}.$$

We have $0 < \kappa < 1$ and $0 < \kappa' < 1$ because $\sigma > 1$, $0 < \mu < 1$, and $(\sigma - 1)/\sigma > \mu$ (no-black-hole condition).

At the flat earth equilibrium $\boldsymbol{\lambda}^* = \frac{1}{n}(1, \ldots, 1)^\top$, (C.5) gives

$$\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{\lambda}}(\boldsymbol{\lambda}^*) = -\frac{\theta}{n^2}\mathbf{1}\mathbf{1}^\top \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\lambda}}(\boldsymbol{\lambda}^*) + \frac{\theta}{n}\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\lambda}}(\boldsymbol{\lambda}^*) - I, \tag{C.8}$$

where $\mathbf{1} = (1, \ldots, 1)^\top$. The derivative $\partial \boldsymbol{v}/\partial \boldsymbol{\lambda}(\boldsymbol{\lambda}^*)$ in (C.8) can be evaluated as below. At $\boldsymbol{\lambda} = \boldsymbol{\lambda}^*$, we have

$$\Delta_i = \sum_{k=1}^{n} d_{ki}\lambda_k = \frac{d}{n},$$

and w_j is independent of j and, therefore, can be expressed as $w_j = w$. Then (A.8) becomes

$$w = \kappa \sum_{j=1}^{n} \frac{n}{d}\, d_{ij}\left(\frac{w}{n} + 1\right) = \kappa\,(w + n)\,,$$

which gives

$$w = \frac{\kappa n}{1 - \kappa}. \tag{C.9}$$

At $\boldsymbol{\lambda} = \boldsymbol{\lambda}^*$, (C.7) becomes

$$\frac{\partial w_i}{\partial \lambda_l} = \kappa \sum_{j=1}^{n} \frac{n^2}{d^2}\, d_{ij}\left[\left(\frac{1}{n}\frac{\partial w_j}{\partial \lambda_l} + w\delta_{jl}\right)\frac{d}{n} - \left(\frac{w}{n} + 1\right)d_{lj}\right],$$

which in a matrix form reads

$$W = \kappa \frac{n^2}{d^2}\, D\left[\frac{d}{n}\left(\frac{1}{n}W + wI\right) - \frac{w + n}{n}D\right]$$

with $W = (\partial w_i/\partial \lambda_l)$. With the use of (C.9), this equation can be rewritten as

$$\left(I - \kappa\frac{D}{d}\right)W = nw\,\frac{D}{d}\left(\kappa I - \frac{D}{d}\right),$$

which is further rewritten as

$$W = nw\left(I - \kappa\frac{D}{d}\right)^{-1} \cdot \frac{D}{d}\left(\kappa I - \frac{D}{d}\right).$$

Then the partial derivatives in (C.6) can be evaluated in a matrix form as

$$\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\lambda}}(\boldsymbol{\lambda}^*) = n\left[\kappa'\frac{D}{d} + \left(I - \kappa\frac{D}{d}\right)^{-1} \cdot \frac{D}{d}\left(\kappa I - \frac{D}{d}\right)\right]. \tag{C.10}$$

For the eigenvector in (C.3), we have from $(D/d)\boldsymbol{\eta} = \alpha\boldsymbol{\eta}$ and (C.10) that

$$\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\lambda}}(\boldsymbol{\lambda}^*) \cdot \boldsymbol{\eta} = \gamma\boldsymbol{\eta}$$

with

$$\gamma = n[\kappa'\alpha + (1 - \kappa\alpha)^{-1} \cdot \alpha(\kappa - \alpha)].$$

Multiplying (C.8) by the vector $\boldsymbol{\eta}$ in (C.3) from the right and using $\mathbf{1}^\top\partial\boldsymbol{v}/\partial\boldsymbol{\lambda}(\boldsymbol{\lambda}^*) \cdot \boldsymbol{\eta} = \gamma\mathbf{1}^\top\boldsymbol{\eta} = 0$, we obtain

$$\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{\lambda}}(\boldsymbol{\lambda}^*) \cdot \boldsymbol{\eta} = \left(\frac{\theta}{n} \cdot \gamma - 1\right)\boldsymbol{\eta}.$$

Then the eigenvalue β of the Jacobian matrix $\partial\boldsymbol{F}/\partial\boldsymbol{\lambda}(\boldsymbol{\lambda}^*)$ for the eigenvector $\boldsymbol{\eta}$ is expressed in terms of α as

$$\beta = \Psi(\alpha)$$

with a function Ψ defined as

$$\Psi(x) = \theta\left(\kappa' x + \frac{x(\kappa - x)}{1 - \kappa x} - \frac{1}{\theta}\right). \tag{C.11}$$

The break point $\tau_{1,n}$ is determined from the condition that the eigenvalue β for $\tau = \tau_{1,n}$ vanishes. The value α^* satisfying $\Psi(\alpha^*) = 0$ is a solution $x = \alpha^*$ of the quadratic equation

$$\theta(bx - ax^2) - 1 = 0, \tag{C.12}$$

where

$$a = \kappa\kappa_- + 1, \quad b = \kappa + \kappa_- + \theta^{-1}\kappa,$$

which are constants independent of n. Between the two solutions of (C.12), the larger is relevant, i.e.,

$$\alpha^* = \frac{b + \sqrt{b^2 - 4a\theta^{-1}}}{2a}.$$

Then by (C.4) the value $\tau = \tau_{1,n}$ for the bifurcation should satisfy

$$r(\tau_{1,n}) = \frac{1 - \sqrt{\alpha^*}}{1 + \sqrt{\alpha^*}}.$$

By (C.1), therefore, we obtain

$$\frac{\tau_{1,n}}{n} = \frac{1}{2\pi(\sigma - 1)} \log\left(\frac{1 + \sqrt{\alpha^*}}{1 - \sqrt{\alpha^*}}\right),$$

that is

$$\tau_{1,n} = c_1 n$$

with

$$c_1 = \frac{1}{2\pi(\sigma - 1)} \log\left(\frac{1 + \sqrt{\alpha^*}}{1 - \sqrt{\alpha^*}}\right).$$

Note that c_1 is a constant independent of n. This proves $\tau_{m,n} = c_m n$ in (16) for $m = 1$.

Appendix D. Robustness against parameter value variation

The robustness of agglomeration behavior against the change of parameters is here investigated. Parameters considered are the elasticity σ of substitution between any two varieties and the inverse θ of variance of the idiosyncratic taste in (9).

The change of population distribution associated with a reduction of τ for $\sigma = 5, 10, 25$ with $(\mu, \theta) = (0.4, 10000)$ is shown in Fig. D.1. As can be seen, an increase of σ leads to less agglomeration.

The change of population distribution associated with a reduction of τ for $\theta = 500$ and 2500 with $(\mu, \sigma) = (0.4, 10)$ is shown in Fig. D.2. As can be seen, an increase of θ leads to more agglomeration.

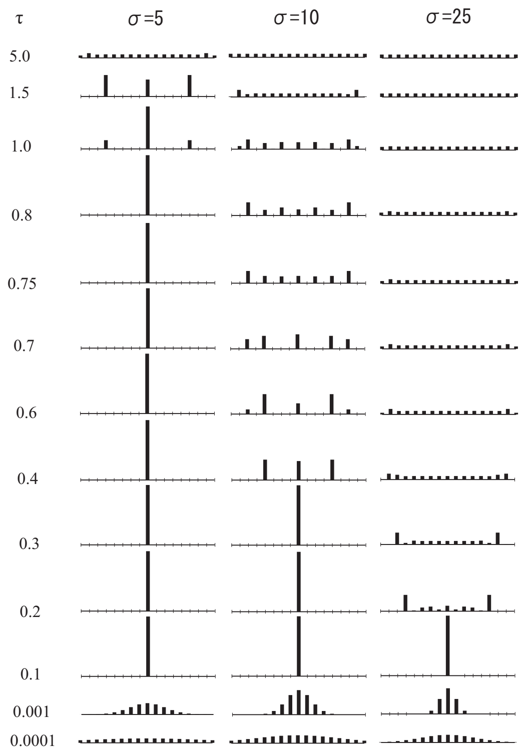

Figure D.1: Influence of σ on the agglomeration behavior $((\mu, \theta) = (0.4, 10000))$

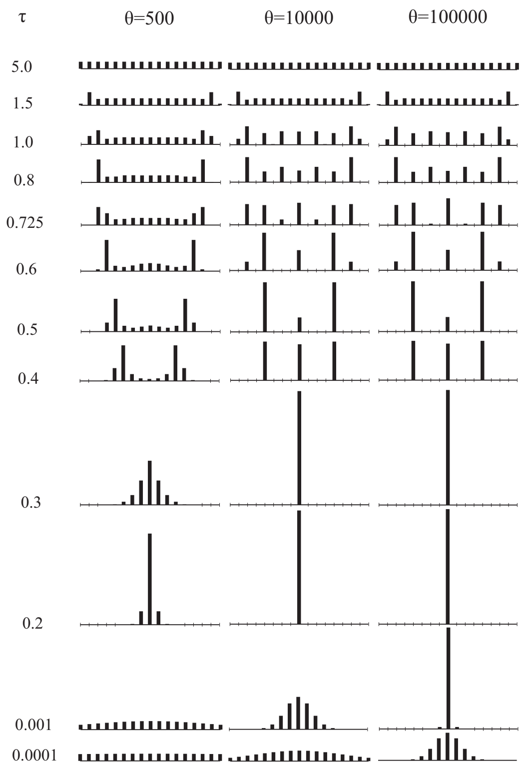

Figure D.2: Influence of θ on the agglomeration behavior $((\sigma, \mu) = (10.0, 0.4))$