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Abstract

We present a special variant of the local discontinuous Galerkin (LDG) method for time-dependent partial differential equations with certain variational structures and associated conservation or dissipation properties. The method provides a way to construct fully-discrete LDG schemes that retain discrete counterparts of the conservation or dissipation properties. Numerical results confirm the accuracy and effectiveness of the method.

1 Introduction

Recently it has been widely accepted that when solving partial differential equations (PDEs) certain specialized numerical methods which maintain characteristic features of the original PDEs are more efficient than general-purpose methods, and enthusiastic attention has been paid to the development of such methods. The so-called "structure-preserving methods" are one strong branch of them (see, for example, Hairer–Lubich–Wanner [9]).

In this paper, along the line of these studies, we are particularly interested in certain variational structures of PDEs and their associated conservation and dissipation properties. One successful method for them is the discrete variational derivative method (DVDM) [7, 8] and its variants [11, 12], which for example targets the following PDEs. One is the conservative PDEs of the form

$$u_t = \left(\frac{\partial}{\partial x}\right)^{2s-1} \frac{\delta G}{\delta u}, \quad s = 1, 2, 3, \ldots, \quad x \in \Omega := [0, L], \quad t > 0, \tag{1}$$

where $\delta G/\delta u$ is the variational derivative of $G(u, u_x)$ with respect to $u(t, x)$. The subscript $(\,\cdot\,)_t$ (and $(\,\cdot\,)_x$) denotes the time (and space, resp.) derivatives. Under appropriate boundary conditions they become conservative:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega G(u, u_x)\,\mathrm{d}x = 0.$$

The Korteweg–de Vries (KdV) equation:

$$u_t = 6uu_x + u_{xxx} \tag{2}$$

is an example of them with $s = 1$ and $G(u, u_x) = u^3 - u_x^2/2$. The second class is the dissipative PDEs of the form

$$u_t = (-1)^{s+1}\left(\frac{\partial}{\partial x}\right)^{2s}\frac{\delta G}{\delta u}, \quad s = 0, 1, 2, \dots, \quad x \in \Omega, \quad t > 0, \tag{3}$$

which is, again under appropriate boundary conditions, dissipative:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega G(u, u_x)\,\mathrm{d}x \le 0.$$

The Cahn–Hilliard equation:

$$u_t = (\alpha u + \gamma u^3 + \beta u_{xx})_{xx} \tag{4}$$

is an example with $s = 1$ and $G(u, u_x) = \alpha u^2/2 + \gamma u^4/4 - \beta u_x^2/2$.

DVDM was first proposed in the seminal paper by Furihata [7], where he tried to preserve certain variational structures of PDEs in fully-discretized scheme, so that the desired conservative or dissipative properties are kept even after the discretization. It was then extended in various ways and proved that it actually works beautifully in many practical applications (see [8] and the references therein). But as the method had matured, one drawback became more and more apparent; since the method was constructed in the framework of finite difference method, it was difficult to consider non-uniform meshes or complex domains in 2D or 3D applications. In order to overcome this difficulty some trials were done; for example Yaguchi–Matsuo–Sugihara [19] tried a mapping technique to allow the use of non-uniform meshes. Matsuo [11] took a completely different approach; he considered a Galerkin (i.e. finite element) version of DVDM, which should more naturally allow flexible meshes and complex domains. Then the idea was furnished with a complete framework in Miyatake–Matsuo [12], where by introducing the concept of L^2 projection, a method for constructing conservative/dissipative Galerkin schemes, which at the same time automatically finds underlying conservative/dissipative weak forms, was given for almost all PDEs with the targeted variational structure.

Our goal in this paper is to construct a *local discontinuous Galerkin* version of the Galerkin DVDM. The discontinuous Galerkin (DG) method is a variant of finite element method that uses discontinuous piecewise polynomial spaces for test and trial functions. It can be regarded as something between finite element and finite volume methods, and thanks to the discontinuity of functions, it has favorable features that it is easy to increase the order of accuracy, and also that the resulting schemes are highly parallelizable when the schemes are explicit. DG was first introduced by Reed–Hill [14] for solving hyperbolic equations. It was then extended by Bassi–Rebay [1] for an elliptic problem, i.e., such that higher-order derivatives can be also handled. Encouraged by this success Cockburn–Shu [5] developed a generalization called local discontinuous Galerkin (LDG) method. The basic idea of the LDG method is to rewrite higher-order derivatives into first-order derivatives by employing intermediate variables. Above history of DG can be found in the book by Cockburn–Karniadakis–Shu [4] with further detailed information. DG has various applications. Below we list some examples. Yan–Shu [20] applied LDG to KdV equation. Other DG studies on nonlinear waves include those for Camassa–Holm equation [18] and nonlinear Schrödinger equation [17]. Xia–Xu–Shu [15] applied LDG to Cahn–Hilliard equation. There are also some structure-preserving DG studies. Xing–Chou–Shu [16] introduced an energy-conservative LDG scheme for linear wave equations and gave an error estimate. Bona–Chen–Karakashian–Xing [2] proposed a DG scheme for generalized KdV equation which preserves the L^2 invariant. A similar work can be found in Yi–Huang–Liu [21], where they employed a variant of the DG method, called the direct DG method.

In viewing such history of DG, it is natural to raise a question that if it is possible to construct a generic DG version of the Galerkin DVDM, by which we can automatically construct energy-preserving or -dissipative DG schemes for wide variety of variational PDEs. If such a method exists, it should give a flexible and parallelizable structure-preserving method for 2D or 3D PDEs. This is our goal as mentioned above, and we will show that it actually exists.

This paper is organized as follows. In Section 2, we introduce target PDEs with variational structure and show their weak forms which preserve the variational structures (and accordingly the associated energy-preservation or -dissipation properties). In Section 3, we describe brief introduction of DG, and show the proposed method. To confirm the effectiveness and accuracy of proposed scheme, we demonstrate the method by some numerical results in section 4. Finally we conclude this paper and give some future plans of this work.

As mentioned above, there are several variants in the DG methods. In this paper, we consider only the standard LDG method for simplicity.

2 Target PDEs and their energy-preserving/dissipative H^1 weak forms

In this section, we introduce target PDEs with variational structure and show their weak forms and associated properties. Just for simplicity, we restrict our consideration to a conservative PDE (1) with $s = 1$ and a dissipative PDE (3) with $s = 1$. We also make an assumption that $G(u, u_x)$ is separable: $G(u, u_x) = G_1(u) + G_2(u_x)$ for some functions G_1, G_2, and $G_2(u_x)$ is a quadratic function. This greatly simplifies the following discussion, and still covers many practical PDEs such as KdV (2) and Cahn–Hilliard (4) equations. For more general cases, see Remark 1. Below, we show weak forms of the above PDEs, which are based on first-order systems with appropriate intermediate variables, and explicitly express energy-preservation/dissipation properties.

First, we consider the conservative case.

Weak form 1. Find $u(t, \cdot), p, q \in H^1(\Omega)$ such that, for any $v_1, v_2, v_3 \in H^1(\Omega)$,

$$(u_t, v_1)_\Omega = (p_x, v_1)_\Omega\,, \tag{5}$$

$$(p, v_2)_\Omega = \left(\frac{\partial G}{\partial u}, v_2\right)_\Omega - \left(\partial_x \frac{\partial G}{\partial q}, v_2\right)_\Omega, \tag{6}$$

$$(q, v_3)_\Omega = (u_x, v_3)_\Omega\,, \tag{7}$$

where the inner product is defined by $(f, g)_\Omega = \int_\Omega f g \,\mathrm{d}x$.

We usually restrict function spaces to appropriate subspaces of $H^1(\Omega)$ corresponding to boundary conditions. Nevertheless to make the discussion in Section 3 clear and avoid confusing notation, we leave this issue to later consideration, and simply assume the existence of the solution.

Theorem 2.1. Assume that $u_t \in H^1(\Omega)$, and the boundary conditions satisfy

$$\left[\frac{\partial G}{\partial q} u_t\right]_\Omega = 0, \quad \left[p^2\right]_\Omega = 0 \tag{8}$$

($[f]_\Omega = f|_{x=L} - f|_{x=0}$). Then the solution of Weak form 1 satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega G(u, q) \,\mathrm{d}x = 0.$$

Proof.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega G(u, q) \,\mathrm{d}x &= \left(\frac{\partial G}{\partial u}, u_t\right)_\Omega + \left(\frac{\partial G}{\partial q}, q_t\right)_\Omega = \left(\frac{\partial G}{\partial u}, u_t\right)_\Omega + \left(\frac{\partial G}{\partial q}, u_{xt}\right)_\Omega \\ &= \left(\frac{\partial G}{\partial u}, u_t\right)_\Omega - \left(\partial_x \frac{\partial G}{\partial q}, u_t\right)_\Omega + \left[\frac{\partial G}{\partial q} u_t\right]_\Omega = (p, u_t)_\Omega \\ &= (p, p_x)_\Omega = -\left(p_x, p\right)_\Omega + \left[p^2\right]_\Omega = 0. \end{aligned}$$

The first equality is a simple application of the chain rule. Temporally differentiating (7) and substituting $v_3 = \partial G/\partial q$, we obtain the second equality. This procedure is allowed by the assumption that $\partial G/\partial q \propto q$ belongs to the same space as q. The third equality is obtained by integration-by-parts. The fourth and fifth equalities follow from (6) with $v_2 = u_t$ and (5) with $v_1 = p$, respectively. In the calculation, the boundary terms are eliminated due to the boundary conditions (8). □

Remark 1. It is also possible to consider more general PDEs based on the fact that the method [12] automatically finds appropriate weak forms. For example, if we drop the restriction on $G(u, u_x)$ (i.e. $G(u, u_x)$ is not necessarily separable and quadratic with respect to u_x), we find the following weak form: Find $u(t, \cdot), p, q, w \in H^1(\Omega)$ such that, for any $v_1, v_2, v_3, v_4 \in H^1(\Omega)$,

$$
\begin{aligned}
(u_t, v_1)_\Omega &= (p_x, v_1)_\Omega\,, \\
(p, v_2)_\Omega &= \left(\frac{\partial G}{\partial u}, v_2\right)_\Omega - (w_x, v_2)_\Omega\,, \\
(w, v_3)_\Omega &= \left(\frac{\partial G}{\partial q}, v_3\right)_\Omega\,, \\
(q, v_4)_\Omega &= (u_x, v_4)_\Omega\,.
\end{aligned}
$$

Then we can carry out the same procedure below, but the discussion would become slightly cumbersome due to the additional intermediate variable w.

Remark 2. For the conservative case, the periodic boundary condition is most typical among those satisfying (8). There are, however, other possibilities; for example, the Dirichlet (and Neumann) conditions satisfy (8) when $G(u, u_x) = u^3$ (and $G(u, u_x) = {u_x}^2$, resp.).

Second, we consider the dissipative case.

Weak form 2. Find $u(t, \cdot), p, q, r \in H^1(\Omega)$ such that, for any $v_1, v_2, v_3, v_4 \in H^1(\Omega)$,

$$(u_t, v_1)_\Omega = (r_x, v_1)_\Omega\,, \tag{9}$$

$$(r, v_2)_\Omega = (p_x, v_2)_\Omega\,, \tag{10}$$

$$(p, v_3)_\Omega = \left(\frac{\partial G}{\partial u}, v_3\right)_\Omega - \left(\partial_x \frac{\partial G}{\partial q}, v_3\right)_\Omega\,, \tag{11}$$

$$(q, v_4)_\Omega = (u_x, v_4)_\Omega\,. \tag{12}$$

Theorem 2.2. Assume that $u_t \in H^1(\Omega)$, and the boundary conditions satisfy

$$\left[\frac{\partial G}{\partial q} u_t\right]_\Omega = 0, \quad [rp]_\Omega = 0. \tag{13}$$

Then the solution of Weak form 2 satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega G(u,q)\,\mathrm{d}x \le 0.$$

Proof.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega G(u,q)\,\mathrm{d}x &= \left(\frac{\partial G}{\partial u}, u_t\right)_\Omega + \left(\frac{\partial G}{\partial q}, q_t\right)_\Omega = \left(\frac{\partial G}{\partial u}, u_t\right)_\Omega + \left(\frac{\partial G}{\partial q}, u_{xt}\right)_\Omega \\
&= \left(\frac{\partial G}{\partial u}, u_t\right)_\Omega - \left(\partial_x\frac{\partial G}{\partial q}, u_t\right)_\Omega + \left[\frac{\partial G}{\partial q}u_t\right]_\Omega = (p, u_t)_\Omega = (r_x, p)_\Omega \\
&= -(r, p_x)_\Omega + [rp]_\Omega = -(r,r)_\Omega = -\|r\|^2_{L^2(\Omega)} \le 0.
\end{aligned}$$

The second equality is obtained by (12) with $v_4 = \partial G/\partial q$. The Forth, fifth and seventh equalities follow from (11) with $v_3 = u_t$, (9) with $v_1 = p$ and (10) with $v_2 = r$, respectively. □

3 Proposed method: Deriving energy-preserving/dissipative LDG schemes

In this section, we propose a new method to derive energy-preserving/dissipative LDG schemes. Although we demonstrate the procedure for simple problems (1) and (3) with $s = 1$, the procedure embraces a variety of PDEs with variational structure.

We divide the computational domain $\Omega = [0, L]$ into N intervals

$$0 = x_{1/2} < \cdots < x_{j-1/2} < x_{j+1/2} < \cdots < x_{N+1/2} = L.$$

We denote the computational cell by $I_j = (x_{j-1/2}, x_{j+1/2})$ for $j = 1, \ldots, N$. We denote by $u^+_{j+1/2}$ and $u^-_{j+1/2}$ the values of u at $x_{j+1/2}$, from the right cell I_{j+1} and from the left cell I_j. This rule applies also to other variables and functions. We define the piecewise polynomial space V_h as the space of polynomials of degree up to k in each cell I_j, i.e.,

$$V_h = \{v : v \in P^k(I_j) \text{ for } x \in I_j, j = 1, \ldots, N\}.$$

3.1 Conservative cases

In this subsection, we shall derive an energy-preserving LDG scheme. We first show a semi-discrete scheme, and then summarize the essential idea of our method. Finally, we derive a fully-discrete scheme. We start the derivation with the following abstract form of a semi-discrete LDG scheme, which is obtained from Weak form 1.

Semi-discrete scheme 1.

Find $u(t,\cdot), p, q \in V_h$ such that, for any $v_1, v_2, v_3 \in V_h$ and for $j = 1, \dots, N$,

$$(u_t, v_1)_{I_j} = -(p, (v_1)_x)_{I_j} + [\hat{p}v_1]_{I_j}, \tag{14}$$

$$(p, v_2)_{I_j} = \left(\frac{\partial G}{\partial u}, v_2\right)_{I_j} + \left(\frac{\partial G}{\partial q}, (v_2)_x\right)_{I_j} - \left[\widehat{\frac{\partial G}{\partial q}} v_2\right]_{I_j}, \tag{15}$$

$$(q, v_3)_{I_j} = -(u, (v_3)_x)_{I_j} + [\hat{u}v_3]_{I_j}, \tag{16}$$

$$\text{where } \left[\hat{f}v\right]_{I_j} = \hat{f}_{j+1/2}v_{j+1/2} - \hat{f}_{j-1/2}v_{j-1/2}.$$

The "hat" terms, called numerical fluxes, result from integration-by-parts in each cell, and are single valued functions defined on the edges. In the standard LDG theory, these terms are introduced to ensure the numerical stability and reflect boundary conditions. Here we show that there is another choice such that the semi-discrete scheme become energy-preserving. In what follows, we call fluxes at $x = x_{1/2},\ x_{N+1/2}$ boundary fluxes, and others internal fluxes.

We assume that internal fluxes are given by, for $j = 1, \dots, N-1$,

$$\hat{p}_{j+1/2} = \frac{1}{2}(p^+_{j+1/2} + p^-_{j+1/2}), \tag{17}$$

$$\widehat{\frac{\partial G}{\partial q}}_{j+1/2} = \lambda \frac{\partial G^+}{\partial q}_{j+1/2} + (1-\lambda)\frac{\partial G^-}{\partial q}_{j+1/2}, \tag{18}$$

$$\hat{u}_{j+1/2} = (1-\lambda)u^+_{j+1/2} + \lambda u^-_{j+1/2}, \tag{19}$$

with a real parameter λ, and boundary fluxes are set to satisfy

$$\left(\frac{1}{2}p^-_{N+1/2} - \hat{p}_{N+1/2}\right)p^-_{N+1/2} - \left(\frac{1}{2}p^+_{1/2} - \hat{p}_{1/2}\right)p^+_{1/2} = 0. \tag{20}$$

$$\begin{aligned}\left(u^-_{N+1/2}\right)_t \frac{\partial G^-}{\partial q}_{N+1/2} - \left(\hat{u}_{N+1/2}\right)_t \frac{\partial G^-}{\partial q}_{N+1/2} - \left(u^-_{N+1/2}\right)_t \widehat{\frac{\partial G}{\partial q}}_{N+1/2} \\ - \left(u^+_{1/2}\right)_t \frac{\partial G^+}{\partial q}_{1/2} + \left(\hat{u}_{1/2}\right)_t \frac{\partial G^+}{\partial q}_{1/2} + \left(u^+_{1/2}\right)_t \widehat{\frac{\partial G}{\partial q}}_{1/2} = 0.\end{aligned} \tag{21}$$

Obviously, the conditions (20) and (21) corresponds to $[p^2]_0^L = 0$ and $[\frac{\partial G}{\partial q}u_t]_0^L = 0$, respectively. We will discuss the derivation of the above *energy-preserving* fluxes after seeing the following theorem and its proof.

Theorem 3.1. If the fluxes are set to (17), (18), (19), and set such that (20), (21) hold, the solution of Semi-discrete scheme 1 satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega G(u,q)\,\mathrm{d}x = 0.$$

Proof. First, we note that for Semi-discrete scheme 1 the following holds.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega G(u,q)\,\mathrm{d}x = -\Theta_{p^2} - \Theta_{uq},$$
$$\text{where } \Theta_{p^2} = \sum_{j=1}^{N}\left[\frac{1}{2}p^2 - \hat{p}p\right]_{I_j}, \quad \Theta_{uq} = \sum_{j=1}^{N}\left[u_t\frac{\partial G}{\partial q} - \hat{u}_t\frac{\partial G}{\partial q} - u_t\widehat{\frac{\partial G}{\partial q}}\right]_{I_j}, \tag{22}$$

independently of the choice of fluxes. This can be checked as follows.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega G(u,q)\,\mathrm{d}x \\
&= \sum_{j=1}^{N}\left\{\left(\frac{\partial G}{\partial u}, u_t\right)_{I_j} + \left(\frac{\partial G}{\partial q}, q_t\right)_{I_j}\right\} \\
&= \sum_{j=1}^{N}\left\{\left(\frac{\partial G}{\partial u}, u_t\right)_{I_j} - \left(\left(\frac{\partial G}{\partial q}\right)_x, u_t\right)_{I_j} + \left[\frac{\partial G}{\partial q}\hat{u}_t\right]_{I_j}\right\} \\
&= \sum_{j=1}^{N}\left\{(p, u_t)_{I_j} - \left(\frac{\partial G}{\partial q}, u_{xt}\right)_{I_j} - \left(\left(\frac{\partial G}{\partial q}\right)_x, u_t\right)_{I_j} + \left[\frac{\partial G}{\partial q}\hat{u}_t + \widehat{\frac{\partial G}{\partial q}}u_t\right]_{I_j}\right\} \\
&= \sum_{j=1}^{N}\left\{-(p, p_x)_{I_j} + [\hat{p}p]_{I_j} - \left[\frac{\partial G}{\partial q}u_t - \frac{\partial G}{\partial q}\hat{u}_t - \widehat{\frac{\partial G}{\partial q}}u_t\right]_{I_j}\right\} \\
&= \sum_{j=1}^{N}\left\{-\left[\frac{1}{2}p^2 - \hat{p}p\right]_{I_j} - \left[\frac{\partial G}{\partial q}u_t - \frac{\partial G}{\partial q}\hat{u}_t - \widehat{\frac{\partial G}{\partial q}}u_t\right]_{I_j}\right\} \\
&= -\Theta_{p^2} - \Theta_{uq}.
\end{aligned} \tag{23}$$

This calculation is quite similar to that in the proof of Theorem 2.1. The first equality is a simple application of the chain rule. The second follows from (16) with $v_3 = \partial G/\partial q$. The third and fourth are obtained from (15) with $v_2 = u_t$ and (14) with $v_1 = p$, respectively.

Next, we show that $\Theta_{p^2} = \Theta_{uq} = 0$. Since Θ_{p^2} is rewritten as

$$\begin{aligned}
\Theta_{p^2} &= \sum_{j=1}^{N} \left\{ \frac{1}{2}\left(p^-_{j+1/2}\right)^2 - \frac{1}{2}\left(p^+_{j-1/2}\right)^2 - \hat{p}_{j+1/2}p^-_{j+1/2} + \hat{p}_{j-1/2}p^+_{j-1/2} \right\} \\
&= \sum_{j=1}^{N-1} \left\{ \frac{1}{2}\left(\left(p^-_{j+1/2}\right)^2 - \left(p^+_{j+1/2}\right)^2 \right) - \hat{p}_{j+1/2}\left(p^-_{j+1/2} - p^+_{j+1/2}\right) \right\} \\
&\quad + \frac{1}{2}\left(p^-_{N+1/2}\right)^2 - \hat{p}_{N+1/2}p^-_{N+1/2} - \frac{1}{2}\left(p^+_{1/2}\right)^2 + \hat{p}_{1/2}p^+_{1/2} \\
&= \sum_{j=1}^{N-1} \left\{ \frac{1}{2}\left(p^-_{j+1/2} + p^+_{j+1/2}\right) - \hat{p}_{j+1/2} \right\} \left(p^-_{j+1/2} - p^+_{j+1/2}\right) \\
&\quad + \left(\frac{1}{2}p^-_{N+1/2} - \hat{p}_{N+1/2}\right) p^-_{N+1/2} - \left(\frac{1}{2}p^+_{1/2} - \hat{p}_{1/2}\right) p^+_{1/2},
\end{aligned}$$

$\Theta_{p^2} = 0$ holds under the assumptions (17) and (20). Similarly, since Θ_{uq} is rewritten as

$$\begin{aligned}
\Theta_{uq} &= \sum_{j=1}^{N} \Bigg\{ \left(u^-_{j+1/2}\right)_t \frac{\partial G^-}{\partial q}_{j+1/2} - \left(\hat{u}_{j+1/2}\right)_t \frac{\partial G^-}{\partial q}_{j+1/2} - \left(u^-_{j+1/2}\right)_t \widehat{\frac{\partial G}{\partial q}}_{j+1/2} \\
&\quad - \left(u^+_{j-1/2}\right)_t \frac{\partial G^+}{\partial q}_{j-1/2} + \left(\hat{u}_{j-1/2}\right)_t \frac{\partial G^+}{\partial q}_{j-1/2} + \left(u^+_{j-1/2}\right)_t \widehat{\frac{\partial G}{\partial q}}_{j-1/2} \Bigg\} \\
&= \sum_{j=1}^{N-1} \Bigg\{ \left(\left(u^-_{j+1/2}\right)_t \frac{\partial G^-}{\partial q}_{j+1/2} - \left(u^+_{j+1/2}\right)_t \frac{\partial G^+}{\partial q}_{j+1/2} \right) \\
&\quad - \left(\hat{u}_{j+1/2}\right)_t \left(\frac{\partial G^-}{\partial q}_{j+1/2} - \frac{\partial G^+}{\partial q}_{j+1/2} \right) - \left(\left(u^-_{j+1/2}\right)_t - \left(u^+_{j+1/2}\right)_t \right) \widehat{\frac{\partial G}{\partial q}}_{j+1/2} \Bigg\} \\
&\quad + \left(u^-_{N+1/2}\right)_t \frac{\partial G^-}{\partial q}_{N+1/2} - \left(\hat{u}_{N+1/2}\right)_t \frac{\partial G^-}{\partial q}_{N+1/2} - \left(u^-_{N+1/2}\right)_t \widehat{\frac{\partial G}{\partial q}}_{N+1/2} \\
&\quad - \left(u^+_{1/2}\right)_t \frac{\partial G^+}{\partial q}_{1/2} + \left(\hat{u}_{1/2}\right)_t \frac{\partial G^+}{\partial q}_{1/2} + \left(u^+_{1/2}\right)_t \widehat{\frac{\partial G}{\partial q}}_{1/2},
\end{aligned}$$

$\Theta_{uq} = 0$ holds under the assumptions (18), (19) and (21). This completes the proof. □

Here we summarize the procedure to find energy-preserving fluxes. Note that the calculation (23) is standard in the LDG context, whereas the terms Θ_{p^2} and Θ_{uq} are intrinsic to the *discontinuous* case—i.e., they essentially do not appear in the standard *continuous* Galerkin context. Thus, it is natural to demand that these terms vanish $\Theta_{p^2} = \Theta_{uq} = 0$ by choosing special fluxes. In order to find such fluxes, we separate Θ_{p^2} and Θ_{uq} into the internal and boundary terms, and first choose internal fluxes such that the terms Θ_{p^2} and Θ_{uq} are cancelled out in internal edges. Then we confirm the remaining terms successfully correspond to the original boundary conditions, so that we can set appropriate discrete boundary conditions. Note also that, in the standard DG, the strategy is different in that they are set such that $\Theta_{p^2}, \Theta_{uq} \geq 0$ hold, which often implies "energy stability."

Remark 3. In this and some of the following remarks, we mention the treatment of boundary conditions (which as described before we basically ignore in the main text). First, let us consider the periodic boundary conditions. In this case, obviously the internal fluxes (17), (18) and (19) can be used throughout the domain, only with the small modification for periodicity: $u_{1/2} = u_{N+1/2}$, $p_{1/2} = p_{N+1/2}$, $q_{1/2} = q_{N+1/2}$. Then the boundary conditions (20) and (21) are automatically satisfied. Next, let us consider the Dirichlet boundary condition $u|_{x=0} = u|_{x=L} = 0$. In this case, we just take $\hat{u}_{1/2} = \hat{u}_{N+1/2} = 0$, $\hat{p}_{1/2} = \frac{1}{2}p^{+}_{1/2}, \hat{p}_{N+1/2} = \frac{1}{2}p^{-}_{N+1/2}$, $\hat{q}_{1/2} = q^{+}_{1/2}, \hat{q}_{N+1/2} = q^{-}_{N+1/2}$. The first one corresponds to the Dirichlet condition, and the rest are set such that (20) and (21) are satisfied.

Now we are in a position to consider fully-discrete scheme. Since the temporal discretization is exactly the same as the (Galerkin version of the) DVDM, which is also equivalent to the discrete gradient method (see [10, 13] for example), we omit the detailed discussion, and show only the result. With the discrete version of partial derivatives, called discrete partial derivatives (for the definition and examples, see [11, 12]), satisfying the discrete chain rule

$$\begin{aligned}
&\frac{1}{\Delta t}\int_{\Omega}(G(u^{(n+1)}, q^{(n+1)}) - G(u^{(n)}, q^{(n)}))\,\mathrm{d}x \\
&= \left(\frac{\partial G_{\mathrm{d}}}{\partial(u^{(n+1)}, u^{(n)})}, \frac{u^{(n+1)} - u^{(n)}}{\Delta t}\right)_{\Omega} + \left(\frac{\partial G_{\mathrm{d}}}{\partial(q^{(n+1)}, q^{(n)})}, \frac{q^{(n+1)} - q^{(n)}}{\Delta t}\right)_{\Omega},
\end{aligned}$$

one obtain the following fully-discrete scheme. We denote that temporal discrete function as $u^{(n)}(x) \approx u(n\Delta t, x)$.

Fully-discrete scheme 1.

Find $u^{(n+1)}, p^{(n+1/2)}, q^{(n+1/2)}, \in V_h$ such that, for any $v_1, v_2, v_3 \in V_h$, and for $j = 1, \dots, N$,

$$\frac{1}{\Delta t}\left(u^{(n+1)} - u^{(n)}, v_1\right)_{I_j} = -\left(p^{(n+1/2)}, (v_1)_x\right)_{I_j} + \left[\hat{p}^{(n+1/2)} v_1\right]_{I_j},$$

$$\left(p^{(n+1/2)}, v_2\right)_{I_j} = \left(\frac{\partial G_{\mathrm{d}}}{\partial(u^{(n+1)}, u^{(n)})}, v_2\right)_{I_j} + \left(\frac{\partial G_{\mathrm{d}}}{\partial(q^{(n+1)}, q^{(n)})}, (v_2)_x\right)_{I_j} - \left[\frac{\widehat{\partial G_{\mathrm{d}}}}{\partial(q^{(n+1)}, q^{(n)})} v_2\right]_{I_j},$$

$$\left(q^{(n+1/2)}, v_3\right)_{I_j} = -\left(\frac{u^{(n+1)} + u^{(n)}}{2}, (v_3)_x\right)_{I_j} + \left[\frac{\hat{u}^{(n+1)} + \hat{u}^{(n)}}{2} v_3\right]_{I_j}.$$

Here $p^{(n+1/2)}$ is the abbreviation for $(p^{(n+1)} + p^{(n)})/2$ (similar notation is used for other variables and in other places).

Theorem 3.2. If the fluxes are set to (17), (18), (19), and set such that (20), (21) hold, the solution of Fully-discrete scheme 1 satisfies

$$\frac{1}{\Delta t}\int_{\Omega} (G(u^{(n+1)}, q^{(n+1)}) - G(u^{(n)}, q^{(n)}))\,\mathrm{d}x = 0.$$

3.2 Dissipative cases

As was done in the previous section, we start the derivation of an energy-dissipative LDG scheme with the following abstract semi-discrete scheme.

Semi-discrete scheme 2.

Find $u(t,\cdot), p, q, r \in V_h$ such that, for any $v_1, v_2, v_3, v_4 \in V_h$ and for $j = 1, \dots, N$,

$$(u_t, v_1)_{I_j} = -(r, (v_1)_x)_{I_j} + [\hat{r} v_1]_{I_j}, \tag{24}$$

$$(r, v_2)_{I_j} = -(p, (v_2)_x)_{I_j} + [\hat{p} v_2]_{I_j}, \tag{25}$$

$$(p, v_3)_{I_j} = \left(\frac{\partial G}{\partial u}, v_3\right)_{I_j} + \left(\frac{\partial G}{\partial q}, (v_3)_x\right)_{I_j} - \left[\widehat{\frac{\partial G}{\partial q}} v_3\right]_{I_j}, \tag{26}$$

$$(q, v_4)_{I_j} = -(u, (v_4)_x)_{I_j} + [\hat{u} v_4]_{I_j}. \tag{27}$$

We then assume that the internal fluxes are given by, for $j = 1, \dots, N-1$,

$$\hat{r}_{j+1/2} = \eta r^+_{j+1/2} + (1-\eta) r^-_{j+1/2}, \tag{28}$$

$$\hat{p}_{j+1/2} = (1-\eta) p^+_{j+1/2} + \eta p^-_{j+1/2}, \tag{29}$$

$$\widehat{\frac{\partial G}{\partial q}}_{j+1/2} = \lambda \frac{\partial G^+}{\partial q}_{j+1/2} + (1-\lambda) \frac{\partial G^-}{\partial q}_{j+1/2}, \tag{30}$$

$$\hat{u}_{j+1/2} = (1-\lambda) u^+_{j+1/2} + \lambda u^-_{j+1/2}, \tag{31}$$

with real parameters η, γ, and boundary fluxes are set to satisfy

$$\begin{aligned} & r^-_{N+1/2} p^-_{N+1/2} - \hat{r}_{N+1/2} p^-_{N+1/2} - r^-_{N+1/2} \hat{p}_{N+1/2} \\ & - r^+_{1/2} p^+_{1/2} + \hat{r}_{1/2} p^+_{1/2} + r^+_{1/2} \hat{p}_{1/2} = 0, \end{aligned} \tag{32}$$

and again (21). Obviously (32) corresponds to $[rp]_\Omega = 0$.

Theorem 3.3. If the fluxes are set to (28), (29), (30), (31), and are set such that (32), (21) hold, the solution of Semi-discrete scheme 2 satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega G(u,q)\,\mathrm{d}x \le 0.$$

Proof. First, we note that for Semi-discrete scheme 2 satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega G(u,q)\,\mathrm{d}x = -\|r\|^2_{L^2(\Omega)} - \Theta_{rp} - \Theta_{uq},$$

$$\text{where } \Theta_{rp} = \sum_{j=1}^N [rp - \hat{r}p - r\hat{p}]_{I_j}, \quad \Theta_{uq} = \sum_{j=1}^N \left[u_t \frac{\partial G}{\partial q} - \hat{u}_t \frac{\partial G}{\partial q} - u_t \widehat{\frac{\partial G}{\partial q}} \right]_{I_j}, \tag{33}$$

independently of the choice of fluxes. This can be checked as follows.

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} G(u,q)\,\mathrm{d}x \\
&= \sum_{j=1}^{N}\left\{\left(\frac{\partial G}{\partial u}, u_t\right)_{I_j} + \left(\frac{\partial G}{\partial q}, q_t\right)_{I_j}\right\} \\
&= \sum_{j=1}^{N}\left\{\left(\frac{\partial G}{\partial u}, u_t\right)_{I_j} - \left(\left(\frac{\partial G}{\partial q}\right)_x, u_t\right)_{I_j} + \left[\frac{\partial G}{\partial q}\hat{u}_t\right]_{I_j}\right\} \\
&= \sum_{j=1}^{N}\left\{(p, u_t)_{I_j} - \left(\frac{\partial G}{\partial q}, u_{xt}\right)_{I_j} - \left(\left(\frac{\partial G}{\partial q}\right)_x, u_t\right)_{I_j} + \left[\frac{\partial G}{\partial q}\hat{u}_t + \widehat{\frac{\partial G}{\partial q}}u_t\right]_{I_j}\right\} \\
&= \sum_{j=1}^{N}\left\{-(r, p_x)_{I_j} + [\hat{r}p]_{I_j} - \left[\frac{\partial G}{\partial q}u_t - \frac{\partial G}{\partial q}\hat{u}_t - \widehat{\frac{\partial G}{\partial q}}u_t\right]_{I_j}\right\} \\
&= \sum_{j=1}^{N}\left\{-(r, r)_{I_j} - [rp - \hat{r}p - r\hat{p}]_{I_j} - \left[\frac{\partial G}{\partial q}u_t - \frac{\partial G}{\partial q}\hat{u}_t - \widehat{\frac{\partial G}{\partial q}}u_t\right]_{I_j}\right\} \\
&= -\|r\|^2_{L^2(\Omega)} - \Theta_{rp} - \Theta_{uq}.
\end{aligned}
$$

The first equality is a simple application of the chain rule. The second follows from (27) with $v_4 = \partial G/\partial q$. Substituting $v_3 = u_t$ in (26) leads to the third equality. Integrating the second term by parts and substituting $v_1 = p$ in (24), we obtain the fourth equality. Integrating the first term by-parts and substituting $v_2 = r$ in (25), we obtain the fourth equality.

Next, we show that $\Theta_{rp} = \Theta_{uq} = 0$. Since $\Theta_{uq} = 0$ under the assumptions (30), (31), and (21) was already proved previously, we here show only $\Theta_{rp} = 0$. Since

$$
\begin{aligned}
\Theta_{rp} &= \sum_{j=1}^{N}\Big\{r^-_{j+1/2}p^-_{j+1/2} - \hat{r}_{j+1/2}p^-_{j+1/2} - r^-_{j+1/2}\hat{p}_{j+1/2} \\
&\quad -r^+_{j-1/2}p^+_{j-1/2} + \hat{r}_{j-1/2}p^+_{j-1/2} + r^+_{j-1/2}\hat{p}_{j-1/2}\Big\} \\
&= \sum_{j=1}^{N-1}\Big\{\left(r^-_{j+1/2}p^- - r^+_{j+1/2}p^+_{j+1/2}\right) \\
&\quad -\hat{r}_{j+1/2}\left(p^-_{j+1/2} - p^+_{j+1/2}\right) - \left(r^-_{j+1/2} - r^+_{j+1/2}\right)\hat{p}_{j+1/2}\Big\} \\
&\quad + r^-_{N+1/2}p^-_{N+1/2} - \hat{r}_{N+1/2}p^-_{N+1/2} - r^-_{N+1/2}\widehat{p}_{N+1/2} \\
&\quad - r^+_{1/2}p^+_{1/2} + \hat{r}_{1/2}p^+_{1/2} + r^+_{1/2}\widehat{p}_{1/2},
\end{aligned}
$$

$\Theta_{rp} = 0$ holds under the assumptions (28), (29) and (32). This completes the proof. □

Remark 4. Corresponding to Remark 3, and also in view of the Cahn–Hilliard example shown later, here we mention the choices of numerical fluxes when Neumann boundary conditions are imposed. In this case we set fluxes as follows: $\hat{r}_{1/2} = \hat{r}_{N+1/2} = 0$ and $\hat{p}_{1/2} = p^+_{1/2}, \hat{p}_{N+1/2} = p^-_{N+1/2}$. $\hat{u}_{1/2} = u^+_{1/2}, \hat{u}_{N+1/2} = u^-_{N+1/2}$ and $\hat{q}_{1/2} = \hat{q}_{N+1/2} = 0$. We used the assumption on G: $\partial G/\partial q = -q$, which in particular holds in the Cahn–Hilliard equation. It is obvious that the above choices satisfy the conditions (32) and (21).

Once one obtain numerical fluxes such that the semi-discrete scheme is energy dissipative, one can immediately derive energy dissipative fully-discrete scheme with the discrete partial derivatives.

Fully-discrete scheme 2.

Find $u^{(n+1)}, p^{(n+1/2)}, q^{(n+1/2)}, r^{(n+1/2)} \in V_h$ such that, for any $v_1, v_2, v_3, v_4 \in V_h$ and for $j = 1, \dots, N$,

$$\begin{aligned}
\frac{1}{\Delta t}\left(u^{(n+1)} - u^{(n)}, v_1\right)_{I_j} &= -\left(r^{(n+1/2)}, (v_1)_x\right)_{I_j} + \left[\hat{r}^{(n+1/2)} v_1\right]_{I_j}, \\
\left(r^{(n+1/2)}, v_2\right)_{I_j} &= -\left(p^{(n+1/2)}, (v_2)_x\right)_{I_j} + \left[\hat{p}^{(n+1/2)} v_2\right]_{I_j}, \\
\left(p^{(n+1/2)}, v_3\right)_{I_j} &= \left(\frac{\partial G_{\mathrm{d}}}{\partial(u^{(n+1)}, u^{(n)})}, v_3\right)_{I_j} + \left(\frac{\partial G_{\mathrm{d}}}{\partial(q^{(n+1)}, q^{(n)})}, (v_3)_x\right)_{I_j} \\
&\quad - \left[\frac{\widehat{\partial G_{\mathrm{d}}}}{\partial(q^{(n+1)}, q^{(n)})} v_3\right]_{I_j}, \\
\left(q^{(n+1/2)}, v_4\right)_{I_j} &= -\left(\frac{u^{(n+1)} + u^{(n)}}{2}, (v_4)_x\right)_{I_j} + \left[\frac{\hat{u}^{(n+1)} + \hat{u}^{(n)}}{2} v_4\right]_{I_j}.
\end{aligned}$$

Theorem 3.4. If the fluxes are set to (28), (29), (30), (31), and are set such that (32), (21) hold, the solution of Semi-discrete scheme 2 satisfies

$$\frac{1}{\Delta t}\int_{\Omega} (G(u^{(n+1)}, q^{(n+1)}) - G(u^{(n)}, q^{(n)}))\,\mathrm{d}x \le 0.$$

4 Application examples

In this section, we apply the proposed method to the KdV and Cahn–Hilliard equations, and show numerical results.

4.1 KdV equation

As an example of a conservative PDE, we consider the KdV equation (2) under the periodic boundary conditions. For the KdV equation, the discrete partial derivatives in Fully-discrete scheme 1 read

$$\frac{\partial G_{\mathrm{d}}}{\partial(u^{(n+1)},u^{(n)})} = \left(u^{(n+1)}\right)^2 + u^{(n+1)}u^{(n)} + \left(u^{(n)}\right)^2,$$
$$\frac{\partial G_{\mathrm{d}}}{\partial(q^{(n+1)},q^{(n)})} = -\frac{q^{(n+1)}+q^{(n)}}{2} = -q^{(n+1/2)}.$$

The energy-preserving fluxes are given by (17), (18), (19) ($j = 1,\dots,N$).

Before proceeding to its numerical example, we here mention on another integral-preserving LDG scheme. As is well known, the KdV equation is completely-integrable, and has infinitely many invariants. In particular, $\int_\Omega \frac{1}{2}u^2\,\mathrm{d}x$ is an invariant, often refereed to as norm, and it gives another Hamiltonian representation of the KdV equation:

$$u_t = \left((u\partial_x + \partial_x u) + {\partial_x}^3\right)\frac{\delta G}{\delta u}, \qquad G(u) = \frac{u^2}{2}, \tag{34}$$

where $\partial_x = \partial/\partial x$. This variational form does not belong to (1), but the proposed method can be generalized to cover this case once a conservative (i.e., norm-preserving) weak form is found. Such a weak form can be found by [12], and a norm-preserving semi-discrete LDG scheme is given as follows: Find $u, r, q \in V_h$ such that, for any $v_1, v_2, v_3 \in V_h$ and for $j = 1,\dots,N$,

$$\begin{aligned}
(u_t, v_1)_{I_j} &= -\left(3u^2, (v_1)_x\right)_{I_j} + \left[\widehat{u^2}v_1\right]_{I_j} - (r, (v_1)_x)_{I_j} + [\hat{r}v_1]_{I_j},\\
(r, v_2)_{I_j} &= -(q, (v_2)_x)_{I_j} + [\hat{q}v_2]_{I_j},\\
(q, v_3)_{I_j} &= -(u, (v_3)_x)_{I_j} + [\hat{u}v_3]_{I_j}.
\end{aligned}$$

Here $\widehat{u^2}$ is a flux arising from the nonlinear term u^2, which is not equivalent to $(\hat{u})^2$. Based on this scheme, one can obtain norm-preserving fluxes as follows: for $j = 1,\dots,N$,

$$\begin{aligned}
\widehat{u^2}_{j+1/2} &= \left(u^+_{j+1/2}\right)^2 + u^+_{j+1/2}u^-_{j+1/2} + \left(u^-_{j+1/2}\right)^2,\\
\hat{r}_{j+1/2} &= \frac{1}{2}\left(r^+_{j+1/2} + r^-_{j+1/2}\right),\\
\hat{q}_{j+1/2} &= \lambda q^+_{j+1/2} + (1-\lambda)q^-_{j+1/2},\\
\hat{u}_{j+1/2} &= (1-\lambda)u^+_{j+1/2} + \lambda u^-_{j+1/2},
\end{aligned}$$

with a real number λ. The latter three fluxes are more or less similar to those in the previous discussions, but the first one is less obvious, which essentially comes from the complex differential operator in (34). This illustrates the following two facts: (i) the proposed method can be, in principle, generalized to more generic variational PDEs, and (ii) the more the PDEs become complicated, the more the conservative (or dissipative) fluxes become nontrivial, and hard to be found without some sophisticated and automatic strategy as the proposed method.

Remark 5. Some norm-preserving DG schemes have been already proposed in [2, 21], but they are not LDG. Also, it should be mentioned that the norm-preserving H^1 weak form is not new; for example, see [3, 6]. However, here we like to emphasize that in order to establish a practical and applicable method, it is necessary to automate the process of finding such weak forms, and that can be done by the technique found in [12]. The same weak form was used to derive a LDG scheme in [20], but there the strict preservation was not considered.

We check the qualitative behaviour of the numerical solutions by the energy-preserving and norm-preserving schemes. We consider the evolution of single soliton and interaction of two solitons. The parameters were set to $x \in [0, 10]$ ($L = 10$), $\Delta x = 10/40$ ($N = 40$), $\Delta t = 0.01$ and $\lambda = 0$. We set the initial values to $u(0, x) = 2.5\,\mathrm{sech}^2(\sqrt{5}(x - 10)/2)$ for single soliton and $u(0, x) = 4\,\mathrm{sech}^2(\sqrt{2}(x - 5)) + 2\,\mathrm{sech}^2(x - 2.5)$ for two solitons. Figures 1 and 2 plot the numerical solutions obtained by the energy-preserving and norm-preserving schemes, respectively, both of which seem accurate and are qualitatively good. We also check the convergence order in terms of spatial discretizations, by utilizing P1, P2, and P3 elements. The initial value was set to $u(0, x) = 0.5\,\mathrm{sech}^2((x - 5)/2)$ with the domain $\Omega = [0, 10]$. We used the time step size $\Delta t = 0.01$. Table 1 shows the results, where the order is calculated by

$$\text{order} = \frac{\log(\text{err}(N)/\text{err}(2N))}{\log(2)}.$$

We observe that high order elements actually give more accurate numerical solutions.

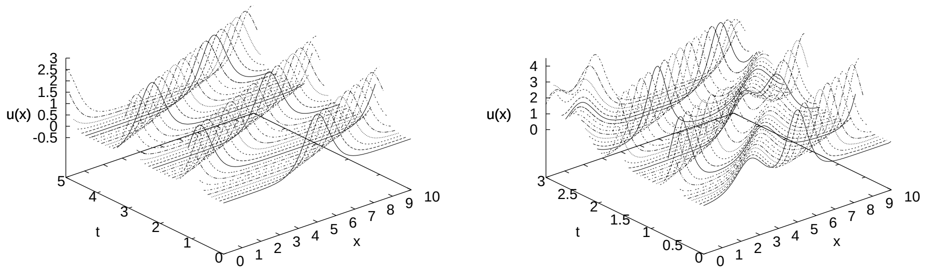

Figure 1: The numerical solutions by the energy-preserving scheme with P2-elements: (left) single soliton, (right) two solitons.

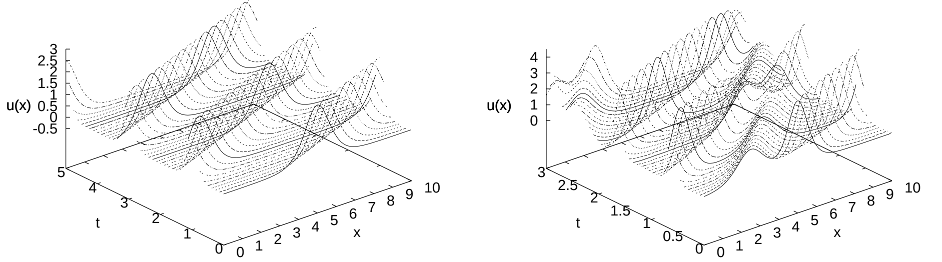

Figure 2: The numerical solutions by the norm-preserving scheme with P2-elements: (left) single soliton, (right) two solitons.

Table 1: L^2 errors of the numerical solutions at the end time $T = 0.1$ by the energy-preserving scheme with polynomial degree $k = 1, 2, 3$, on uniform mesh.

k	$N = 10$	$N = 20$		40		80	
	error	error	order	error	order	error	order
1	4.558e-03	1.663e-03	1.455	5.265e-04	1.659	1.225e-04	2.104
2	1.199e-03	9.981e-05	3.587	1.707e-05	2.548	2.771e-06	2.623
3	1.242e-03	1.882e-04	2.723	1.574e-05	3.580	1.746e-06	3.172

4.2 Cahn–Hilliard equation

As an example of a dissipative PDE, we consider the Cahn–Hilliard equation (4) with the Neumann boundary conditions $u_x|_{x=0,L} = 0, u_{xxx}|_{x=0,L} = 0$. In this case, the discrete partial derivatives read

$$\begin{aligned}\frac{\partial G_{\mathrm{d}}}{\partial(u^{(n+1)}, u^{(n)})} &= \alpha \frac{u^{(n+1)} + u^{(n)}}{2} \\ &\quad + \gamma \frac{\left(u^{(n+1)}\right)^3 + \left(u^{(n+1)}\right)^2 u^{(n)} + u^{(n+1)} \left(u^{(n)}\right)^2 + \left(u^{(n)}\right)^3}{4}, \\ \frac{\partial G_{\mathrm{d}}}{\partial(q^{(n+1)}, q^{(n)})} &= -\beta \frac{q^{(n+1)} + q^{(n)}}{2} = -\beta q^{(n+1/2)}.\end{aligned}$$

The internal fluxes are set to (28), (29), (30), (31), and the boundary fluxes are set to $\hat{q}_{1/2} = \hat{q}_{N+1/2} = 0$, $\hat{r}_{1/2} = \hat{r}_{N+1/2} = 0$ (recall that r is a variable corresponding to u_{xxx}), $\hat{p}_{1/2} = p^+_{1/2}$, $\hat{p}_{N+1/2} = p^-_{N+1/2}$, $\hat{u}_{1/2} = u^+_{1/2}$, $\hat{u}_{N+1/2} = u^-_{N+1/2}$. See also Remark 4.

The computational parameters were set to $x \in [0, 10]$, $\Delta x = 10/40$, $\Delta t = 0.01$ and $\lambda = \eta = 0$. The initial value was set to $u(0, x) = 0.2 \sin(2\pi x/L)$ with the parameters of the equation $\alpha = -1, \beta = -0.1, \gamma = 1$. Figure 3 plots the numerical solution and the evolution of the energy, both of which are fine.

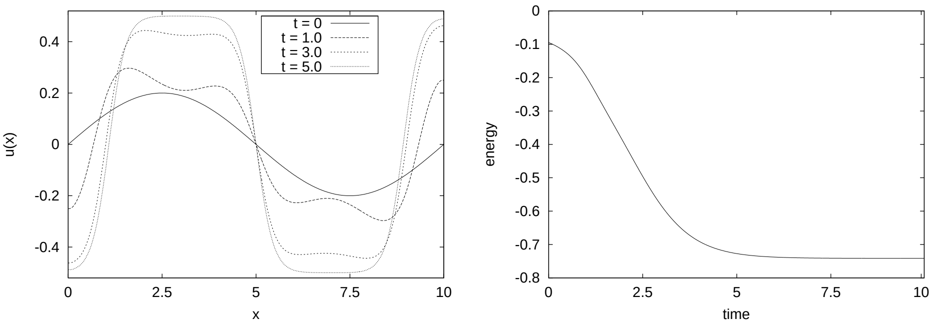

Figure 3: The numerical solution and the evolution of the energy for the Cahn–Hilliard equation obtained by the energy-dissipative scheme with P2-elements.

Remark 6. The *semi-discrete* LDG scheme for the Cahn–Hilliard equation coincides with the one already proposed in Xia–Xu–Shu [15]. In this sense, the above example demonstrates that the existing scheme can be automatically derived by the proposed method. Also, we like to note that Xia et al. then used a generic integrator for time-stepping, which generally destroys strict energy dissipation. In contrast, our fully-discrete scheme keeps the strict dissipation by construction.

5 Conclusion

In this paper we proposed a local discontinuous Galerkin method by which we can automatically derive LDG schemes inheriting energy-preservation or -dissipation properties. The keys of the derivation are:

- the use of first-order weak forms explicitly keeping the desired energy-preservation or -dissipation properties;
- the construction of energy-preserving or -dissipative numerical fluxes.

As seen in the previous sections, these steps are automatic. We also note that, as repeatedly noted above, in the first step the desired weak forms can be automatically found based on the technique devised in [12], and thus the overall procedure is actually automatic.

The proposed method is also applicable to a variety of PDEs, such as the Camassa–Holm and phase-field-crystal equations, and is easy to implement in 2D or 3D settings. Numerical and theoretical studies of them will be reported elsewhere in the near future.

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